Option pricing: A review

Article *in* Journal of Financial Economics · January 1976

DOI: 10.1016/0304-405X(76)90019-2

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Journal of Financial Economics 3 (1976) 3–51. © North-Holland Publishing Company

OPTION PRICING

A review*

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Received July 1975, revised version received July 1975

Recent advances in the general equilibrium pricing of simple put and call options lay the foundation for the development of a general theory of the valuation of contingent claims assets. This paper provides a review of: (1) the development of the general equilibrium option pricing model by Black and Scholes, and the subsequent modifications of this model by Merton and others; (2) the empirical verification of these models; and (3) applications of these models to value other contingent claim assets such as the debt and equity of a levered firm and dual purpose mutual funds.

1. Introduction

Although much interest in option pricing has been generated from the development of new options markets, such as the Chicago Board Options Exchange, the recent rapid development of theory and the application of that theory can be traced to the path-breaking paper by Fischer Black and Myron Scholes (1973). In that paper, they provide the first explicit general equilibrium solution to the option pricing problem for simple puts and calls.[1] They then suggest that this analysis could provide a basis for the general analysis of contingent claim assets, assets whose value is a non-proportional function of the value of another asset. Because puts and calls are perhaps the simplest form of a contingent claim asset, the study of these simple instruments provides keys to unlock difficult questions of other more complex contingent claims pricing situations.

*The rate of development of the body of literature on option pricing has been extremely rapid in recent times. These rapid advances themselves make the task of a survey such as this not only somewhat more difficult, but also probably more valuable. I would like to thank those authors who have kindly agreed to allow me to reference their results, which in some cases are as yet unpublished; and I apologize in advance to those authors whose work is not referenced. This paper cannot be an exhaustive survey but is meant only to convey the major thrust of the work on option pricing. An extensive list of references to books and articles on options and option pricing has been included to aid the interested reader. The author would like to thank W.A. Avera, F. Black, H.A. Latané, P. Lloyd-Davies, R.C. Merton, M. Scholes, J.L. Zimmerman, and especially M.C. Jensen and J.B. Long for comments and suggestions.

[1]A call is an option to buy a share of stock at the maturity date of the contract for a fixed amount, the exercise price. A put is an option to sell.

In deriving their model, Black and Scholes employ the following assumptions:

(1) There are no penalties for short sales.
(2) Transactions costs and taxes are zero.
(3) The market operates continuously.
(4) The risk-free interest rate is constant.
(5) The stock price is continuous.[2]
(6) The stock pays no dividends.
(7) The option can only be exercised at the terminal date of the contract.

They derive the solution to the option pricing problem as a function of only five variables:

(1) the stock price,
(2) the variance rate on the stock price,
(3) the exercise price of the option,
(4) the time to maturity of the option,
(5) the risk-free interest rate.

Two points should be noted about this list of arguments. First, variables such as the expected rate of return to the stock, or parameters denoting investor attitudes toward risk do not appear as arguments in the general equilibrium option pricing solution. Second, the only argument of the solution which is not directly observable, the variance rate, can be approximated using the sequence of past prices.

Subsequent modification of the basic Black–Scholes model by Merton (1973b, 1974, 1976) and others shows that the analysis is quite robust with respect to relaxation of the basic assumptions under which the model is derived. No single assumption seems crucial to the analysis. Thorpe (1973) examines the effects of restrictions against the use of the proceeds of short sales. Ingersoll (1976) takes explicit consideration of the effect of differential taxes on capital gains and ordinary income. Merton (1976) argues that the continuous trading solution approximates the asymptotic limit of the discrete trading solution when the stock price movement is continuous. Merton (1973b) also generalizes the model to the case of a stochastic interest rate. Thus, it appears that the relaxation of the first four assumptions involving the specification of the behavior of the capital market environment modifies the analysis in no significant way. In addition, the analytical technique developed by Black and Scholes remains valid, even if the last three assumptions dealing with the specification of the stock and option are relaxed. Merton (1976) and Cox and Ross (1976) successfully employ a Black–Scholes type analysis to examine a case in which stock price movements are discontinuous. Merton (1973b) and Thorpe (1973) modify the model to account for dividend payments on the underlying stock. Finally, Merton (1973b) shows

[2]Continuous here means 'no jumps'. Roughly speaking, in a graph of stock price against time, the stock price can be drawn without lifting the pen.

that the Black–Scholes solution for an option which can be exercised only at maturity can be appropriate to value a call option which may be exercised prior to the maturity date.

After deriving the general equilibrium option pricing equation, Black and Scholes make what may be one of the most important observations in the field of finance in the past ten years. They suggest that the equilibrium solution to the option pricing problem can be utilized to value other complex contingent claim assets, specifically the equity of a levered firm. They argue that the position of the stockholders is equivalent to that of the purchaser of a call and the position of the bondholders, to that of the writer of a call, i.e., the stockholders have the right to buy the firm back from the bondholders by paying the face value of the bonds to the bondholders. The model is also applied by Merton (1974) to analyze the effects of risk on the value of corporate debt; by Galai and Masulis (1976) to examine the effect of mergers, acquisitions, scale expansions, and spin-offs on the relative values of the debt and equity claims of the firm; by Ingersoll (1976) to value the shares of dual purpose funds; and by Black (1976) to value commodity options, forward contracts, and future contracts. This paper provides a summary of (1) the models of simple option pricing, (2) the empirical verification of these models, and (3) the applications of option pricing models to value other assets.

Section 2 below introduces the terminology employed in the trading of options and develops basic relationships between the values of options and the underlying assets, based only on the assumption that investors prefer more to less. The arguments in this section follow Merton (1973b) and offer insight into the relationships which must hold between asset values for markets to be in equilibrium. The generality of the arguments in this section prohibits a specific solution to the option pricing problem, but does define limits between which any acceptable general equilibrium solution must fall.

Sections 3 and 4 review the development of explicit solutions to the call option pricing problem.[3] To derive an explicit solution, each of the works reviewed in this section chooses a specific description of the statistical process, which describes the movement of the stock price through time. Then, given that statistical hypothesis, the equilibrium relationship between the call price, the stock price, and other variables of the economy is explored. The statistical assumptions employed are Arithmetic Brownian Motion (leading to a normally distributed stock price), Geometric Brownian Motion (leading to a log-normally distributed stock price), and the Poisson Process (or Jump Process). Section 3

[3]Option pricing models seem to fall into two categories: (1) Ad hoc models, and (2) equilibrium models. The ad hoc models generally appear in the non-academic literature and are the result of casual empiricism or curve-fitting exercises – not of maximizing behavior on the part of the market participants. Examples include Hallingby (1947), Morrison (1957), Giguere (1958), Pease (1963), Kassouf (1962, 1968a, 1969), Thorpe and Kassouf (1967), Shelton (1967b) and Turov (1973). For a summary of many of these models, see Shelton (1967a). These models will not be reviewed in this paper.

reviews selected papers written prior to the Black–Scholes paper. These papers provide an indication of the development of the thought at this early stage, but more importantly, they offer useful insight into the Black–Scholes general equilibrium call pricing solution .The Black–Scholes model and its subsequent modifications by Merton and others are discussed in section 4.

Section 5 provides an analysis of the put option pricing model. Useful relationships are defined by use of dominance arguments, including the equilibrium relationship between the price of a put option, which can only be exercised at maturity, and the value of a portfolio consisting of a call, the stock, and riskless bonds. Using this relationship, the Black–Scholes put option pricing model is derived.

Section 6 reviews empirical tests of the option pricing model and evidence on the efficiency of the option market.

Section 7 examines the application of the option pricing model to value other contingent claim assets. These applications include pricing the equity and debt of the firm, analyzing the risk structure of interest rates, and valuing the shares of a dual-purpose fund.

2. Definition of terminology and some fundamental constraints on option prices

2.1. Terminology

An option is defined as a right to buy or sell designated securities or commodities at a specified price during the period of the contract. The specified price is referred to as the exercise price, striking price, or contract price. The terminal date of the contract is called the expiration date or maturity date. Options to purchase securities which are written by individuals are termed calls; options to sell, puts.[4] If the option may be exercised before the expiration date, then it is referred to as an American option; if only on the expiration date, a European option. Options granted by corporations are termed warrants or rights. Rights are usually exercisable for very short periods of time; warrants generally for longer time periods. Contracts to purchase commodities, rather than financial instruments, are called commodity options.

Notation. In the option literature, no set of notation is regarded as standard. To facilitate comparisons between different models, one set of notation will be employed. (This will be extended to quotations.) The symbols to be used are:

t – current date,
t^* – expiration date of the option,

[4]The above terminology is employed to describe simple options. However, elements of option pricing can arise in other contexts as well. First, convertible preferred stocks and convertible bonds are essentially call options in concert with another basic security. Second, option contracts may be combined to form strips, straps, straddles or spreads. See Kruizenga (1964a).

T – time to expiration $(t^* - t)$,
B – price of a default-free pure discount bond with a face value of one dollar,
C – price of an American call option at t,
c – price of a European call option at t,
κ – expected average rate of growth in the call price $[e^{\kappa T} = E(C^*/C)]$,
P – price of an American put option at t,
p – price of a European put option at t,
r – risk-free interest rate,
S – stock price at t,
ρ – expected average rate of growth in the stock price $[e^{\rho T} = E(S^*/S)]$,
X – exercise price of the option,
V_a – value of portfolio A at t.

Starred variables such as C^*, c^*, S^*, etc., refer to prices at t^*, the expiration date of the option.

2.2. Stochastic dominance and some fundamental restrictions on call option prices

Merton (1973b) sets forth the most exhaustive set of equilibrium restrictions on call option pricing which is free of distributional assumptions. Merton makes no assumptions about the process generating the stock price over time; the restrictions he derives depend only on dominance arguments. Portfolio A is dominant over portfolio B if over some given time interval the return to A is not less than the return to B for all states of the world, and the return to A is strictly greater than the return to B for at least one state of the world. In equilibrium no dominant or dominated security can exist. If a dominant security existed, everyone would prefer to hold that security. The price would be bid up until the dominance disappeared and vice versa. The results derived through the use of dominance arguments are completely general. If the implications of a specific model based on a specific distributional description of the stock price movement violate these restrictions, then that model must be deficient in some way. The results of these dominance arguments will be used as general consistency criteria against which subsequent models may be conveniently measured.

Call prices are non-negative. From the definition of a call option exercise is voluntary. Since exercise will only be undertaken when in the best interests of the option holders,

$$\begin{aligned} &C(S, T; X) \geqq 0 \qquad \text{[American call]}, \\ &c(S, T; X) \geqq 0 \qquad \text{[European call]}. \end{aligned} \tag{1}$$

At the expiration date, t^*, the call will be priced at the maximum of either the difference between the stock price and the exercise price, $S^* - X$, or zero. When the time to expiration, T, of a call option is zero, then arbitrage is possible since the option can be converted into the common for the option plus the exercise price,

$$C(S^*, 0; X) = \text{Max}\,(0, S^* - X),$$
$$c(S^*, 0; X) = \text{Max}\,(0, S^* - X). \tag{2}$$

At any date before the maturity date an American call option must sell for at least the difference between the stock price and the exercise price. An American call can be exercised at any time before the expiration date, therefore,

$$C(S, T; X) \geqq \text{Max}\,(0, S - X). \tag{3}$$

Note that this condition cannot be deduced for a European call, since arbitrage is not possible until $T = 0$.

If two American calls differ only as to expiration date, then the one with the longer term to maturity, T_1, must sell for no less than that of the shorter term to maturity, T_2. At the expiration date of the shorter option its price will be equal to the maximum of zero and the difference between the stock price and the exercise price by (2). This then is the minimum price of the longer option by (3). Therefore, to prevent dominance,

$$C(S, T_1; X) \geqq C(S, T_2; X). \tag{4}$$

Since no correspondence to (3) exists for European calls, (4) cannot be deduced in that case by this argument.

An American call must be priced no lower than an identical European call. Since the American call confers all the rights of the European call plus the privilege of premature exercise, then, to avoid dominance,

$$C(S, T; X) \geqq c(S, T; X). \tag{5}$$

If two options differ only in exercise price then the option with the lower exercise price must sell for a price which is no less than the option with the higher exercise price to avoid dominance. This may be demonstrated by constructing two portfolios, A and B ,where portfolio A contains one call with exercise price X_2, $C(S, T; X_2)$, and portfolio B contains one call with exercise price X_1, $C(S, T; X_1)$ with $X_1 > X_2$.

Since for all possible terminal stock prices above the lower exercise price X_2, the terminal value of portfolio A, V_a^*, exceeds that of portfolio B, V_b^*, then the current price of A must be greater than or equal to the current price of B. If the current price of B were greater than that of A, the return to portfolio A would be greater than the return to B for all states of the world and B would be a dominated portfolio. Therefore,

$$C(S, T; X_1) \leqq C(S, T; X_2),$$
$$c(S, T; X_1) \leqq c(S, T; X_2), \tag{6}$$

where $X_1 > X_2$.

Table 1
Demonstration that a call with a lower exercise price, X_2, will have dollar payoffs greater than or equal to a call with a higher exercise price, X_1.[a]

Portfolio	Current value	Stock price at $T = 0$		
		$S^* \leqq X_2$	$X_2 < S^* < X_1$	$X_1 \leqq S^*$
A	$C(S, T; X_2)$	0	$S^* - X_2$	$S^* - X_2$
B	$C(S, T; X_1)$	0	0	$S^* - X_1$
Relationship between terminal values of portfolios A and B		$V_a^* = V_b^*$	$V_a^* > V_b^*$	$V_a^* > V_b^*$

[a]Terminal values of portfolios A and B for different relationships between stock price and exercise prices at the expiration date ($T = 0$) of the call.

The common stock is at least equivalent to a perpetual call (*i.e.*, $T = \infty$) *with a zero exercise price*. Then from (4) and (6) it follows that

$$S \geqq C(S, \infty; 0) \geqq C(S, T; X). \tag{7}$$

[S may exceed $C(S, \infty; 0)$ because of dividends, voting rights, etc.].[5] From (7) if a stock is worthless, i.e., $S = 0$, the option must be also

$$C(0, T; X) = c(0, T; X) = 0. \tag{8}$$

An American call on a non-dividend paying stock, will not be exercised before the expiration date. To demonstrate this, first let $B(\tau)$ be the price of a risk-free, pure discount bond, which pays one dollar τ years from now. Then, assuming interest rates are positive, at a given point in time bonds with a longer time to maturity will be priced less than bonds with a shorter time to maturity, i.e.,

$$0 = B(\infty) < B(\tau_1) < B(\tau_2) < B(\tau_3) < B(0) = 1, \tag{9}$$

where $\infty > \tau_1 > \tau_2 > \tau_3 > 0$.

[5]Bachelier's (1900) model violates this restriction. In his model, the stock does not possess the property of limited liability, i.e., that the maximum loss which can be sustained by an asset holder is limited to the size of his investment.

Now consider two portfolios A and B:

A: Purchase one European call for $c(S, T; X)$,
Purchase X bonds for $XB(T)$;

B: Purchase stock for S.

Table 2

Demonstration that a call plus discount bonds with a face value of X yield a terminal value greater than or equal to that of the respective stock if the stock pays no dividends.[a]

Portfolio	Current value	Stock price at $T=0$: $S^* < X$	Stock price at $T=0$: $X \leqq S^*$
A	$c(S, T; X)+XB(T)$	$0+X$	$(S^*-X)+X$
B	S	S^*	S^*
Relationship between terminal values of portfolios A and B		$V_a^* > V_b^*$	$V_a^* = V_b^*$

[a]Terminal values of portfolios A and B for different relationships between the stock price and the exercise price at the expiration date date ($T=0$) of the call.

Since V_a^*, the terminal value of portfolio A, is not less than the V_b^*, the current value of A must be greater than or equal to that of B to avoid dominance. Therefore, for a security paying no dividends, this restriction may be rearranged to yield

$$c(S, T; X) \geqq \text{Max}\,[0, S-XB(T)]. \tag{10}$$

Note that if dividends are paid over the life of the option, the return to portfolio B is no longer $(S^*-S)/S$. The proviso of no dividend payments can be dropped if adequate dividend protection exists for the option.[6]

From (5) and (10) it follows that (assuming no dividends and in the absence of transaction costs) the price of an American call is no less than the difference between the stock price and the present value of the exercise price,

$$C(S, T; X) \geqq c(S, T; X) \geqq \text{Max}\,[0, S-XB(T)]. \tag{11}$$

If exercised, the value of an American option is Max $[0, S-X]$ which is less than Max $[0, S-XB(T)]$ since $B(T)$ is less than 1 if T is greater than zero. Therefore, prior to expiration, the holder of an American call will always choose to sell

[6]See Merton (1973b) for a discussion of appropriate protection for options from dividend payments.

rather than exercise the option. This implies that if there are no dividends paid over the life of the option (or changes in exercise price) an American call will never be exercised before its expiration date, and therefore will have the same value as a European option. This is an important result because European calls are simpler instruments than the more familiar American calls. Further, differences between American and European calls (implying some positive probability of premature exercise) must arise from changes in exercise price or dividends, rather than from assumptions about distributions of returns or shapes of investors' utility functions.[7]

It has been generally suggested that the minimum justifiable price or the 'intrinsic value' of a call is Max $[0, S-X]$ from (3), but eq. (11) implies that the 'intrinsic value' of an option on a non-dividend paying stock should be Max $[0, S-XB(T)]$ which is greater than Max $(0, S-X)$ prior to the expiration date. Eq. (11) is a stronger condition than (3). That (3) obtains can be assured directly with arbitrage by placing simultaneous orders. Arbitrage in the usual sense does not guarantee (11) will hold; but if the condition is violated, the common stock will be a dominated security, and in equilibrium, no dominated security will exist.

A perpetual option on a non-dividend paying stock must sell for the same price as the stock. Eq. (11) reflects the fact that the exercise price need not be paid until the option is exercised at the expiration date. Therefore, $XB(T)$ represents the present value of that payment. This adjustment becomes significant as T gets large. For a perpetual option (i.e., $T = \infty$) then from (11),

$$C(S, \infty; X) \geqq \text{Max}\,[0, S-XB(\infty)]. \tag{12}$$

But the present value of one dollar paid an infinite time in the future is zero, therefore $B(\infty) = 0$, and

$$C(S, \infty; X) \geqq S. \tag{13}$$

From (7), the call price is not greater than the stock price,

$$S \geqq C(S, \infty; X).$$

Therefore, (7) and (13) imply that a perpetual option on a non-dividend paying stock must sell for the same price as the security itself,[8]

$$C(S, \infty; X) = S. \tag{14}$$

[7]Samuelson's (1965) model implies that so long as the option is more risky than the stock, there exists some positive probability of premature exercise. This implication is obviously at variance with the above result. See section 3 for an analysis of Samuelson's model.

[8]It may seem that a stock which will never pay dividends would sell for a zero price and therefore (14) would hold with $S = C = 0$. However, this is not the case. Consider two firms, identical in every respect except one has a policy of paying dividends at stated dates while at those dates the second firm takes the same number of dollars, enters the market and purchases as many shares of the firm's stock as possible at the current market price and retires the shares. Given a world with efficient markets and no taxes these two firms would have the same market value.

The call price is a convex function of the exercise price. Convexity implies that if $X_2 = \lambda X_1 + (1-\lambda)X_3$, then

$$C(S, T; X_2) \leqq \lambda\, C(S, T; X_1) + (1-\lambda)\, C(S, T; X_3), \tag{15}$$

where $X_1 \geqq X_2 \geqq X_3$ and $0 \leqq \lambda \leqq 1$.

To demonstrate this restriction, form two portfolios, A and B, where portfolio A contains λ calls with exercise price X_1 and $(1-\lambda)$ calls with exercise price X_3, and portfolio B contains one call with exercise price X_2.

Table 3

Demonstration that a convex combination of calls with different exercise prices yields a terminal value greater than or equal to that of a call with an exercise price which is the convex combination of the other two exercise prices.[a]

		Stock price at $T = 0$			
Portfolio	Current value	$S^* \leqq X_3$	$X_3 < S^* < X_2$	$X_2 < S^* < X_1$	$X_1 \leqq S^*$
A	$\lambda\, C(S, T; X_1)+$ $(1-\lambda)\, C(S, T; X_3)$	$0+$ 0	$0+$ $(1-\lambda)(S^*-X_3)$	$0+$ $(1-\lambda)(S^*-X_3)$	$\lambda(S^*-X_1)+$ $(1-\lambda)(S^*-X_3)$
B	$C(S, T; X_2)$	0	0	S^*-X_2	S^*-X_2
Relationship between the terminal values of portfolios A and B		$V_a^* = V_b^*$	$V_a^* > V_b^*$	$V_a^* > V_b^*$	$V_a^* = V_b^*$

[a]Terminal values of portfolios A and B for different relationships between the stock price and exercise prices at the expiration date ($T = 0$) of the call.

Table 3 shows that V_a^* is greater than or equal to V_b^*; therefore, to avoid dominance the current value of B must be less than or equal to that of A. This is (15).

If the call price can be expressed as a differentiable function of the exercise price, the derivative must be negative and be no larger in absolute value than the price of a pure discount bond of the same maturity. To demonstrate this, consider two portfolios A and B;

$$A: \quad c(S, T; X_1) + B(T)(X_1 - X_2),$$

$$B: \quad c(S, T; X_2).$$

Table 4 shows that V_a^* is not less than V_b^*; therefore, unless the current price of A is not less than that of B, $c(S, T; X_2)$ will be a dominated security,

$$B(T)(X_2 - X_1) \leqq c(S, T; X_1) - c(S, T; X_2) \leqq 0, \tag{16}$$

where $X_1 > X_2$.

That this is less than zero follows from (6).

If $c(S, T; X)$ is a differentiable function of X, then the partial derivative of the call price with respect to the exercise price will be no more negative than minus a bond price with the same maturity date as the option,

$$-B(T) \leqq \partial c(S, T; X)/\partial X \leqq 0. \tag{17}$$

This can be seen by dividing both sides of (16) by $(X_1 - X_2)$ and taking the limit as X_2 approaches X_1.

Table 4

Demonstration that a portfolio containing a call with a lower exercise price, X_2, yields a terminal value which is less than or equal to that of a portfolio containing a call with a higher exercise price, X_1, and pure discount bonds with a face value of $X_1 - X_2$.[a]

		Stock price at $T = 0$		
Portfolio	Current value	$S^* \leqq X_2$	$X_2 < S^* < X_1$	$X_1 \leqq S^*$
A	$c(S, T; X_1)+$ $B(T)(X_1 - X_2)$	$0+$ $(X_1 - X_2)$	$0+$ $(X_1 - X_2)$	$(S^* - X_1)+$ $(X_1 - X_2)$
B	$c(S, T; X_2)$	0	$(S^* - X_2)$	$(S^* - X_2)$
Relationship between terminal values of portfolios A and B		$V_a^* > V_b^*$	$V_a^* > V_b^*$	$V_a^* = V_b^*$

[a]Terminal values of portfolios A and B for different relationships between the stock price and exercise prices at the expiration date ($T = 0$) of the calls.

With dividend payments on the stock, premature exercise of an American call may occur. To examine the effect of dividend payments, consider a security which goes ex-dividend and pays a certain payment (D) on the expiration date of an option. Consider two portfolios A and B:

A: Buy one European call and $X + D$ bonds,

B: Buy one share of stock.

See table 5. Since the terminal value of A is not less than that of B, to prevent dominance the current price of A must not be less than B. Then,

$$c(S, T; X) \geqq \text{Max}\,[0, S - (X + D)B(T)]. \tag{18}$$

Note that the value of a pure discount bond with a face value equal to the sum of the exercise price plus the dividend may be greater or less than the exercise

Table 5

Demonstration that a portfolio containing a call plus pure discount bonds with a face value equal to the sum of the exercise price and dividend payment yields a terminal value greater than or equal to that of a portfolio containing a share of stock which pays D dollars in dividends at the expiration date of the call.[a]

		Stock price at $T = 0$	
Portfolio	Current value	$S^* < X$	$X \leqq S^*$
A	$c(S, T; X)+$ $(X+D)B(T)$	$0+$ $X+D$	S^*-X+ $X+D$
B	S	S^*+D	S^*+D
Relationship between terminal values of portfolios A and B		$V_a^* > V_b^*$	$V_a^* = V_b^*$

[a]Terminal values of portfolios A and B for different relationships between the stock price and the exercise price at the expiration date ($T = 0$) of the call.

price, therefore Max $[0, S-(X+D)B(T)] \lesseqgtr$ Max $[0, S-X]$. Then, with dividend payments, it may be advantageous to exercise an American option. before expiration

The above dominance arguments define limits between which the equilibrium call price must fall.[9] Because of the generality of dominance arguments, no specific functional relationship, can be defined between the call option price and variables such as the stock price, exercise price, time to maturity, etc. Most previous efforts to define the equilibrium option price have taken another approach to the problem. A specific description of the statistical process which describes the movement of the stock price through time is assumed. Then, given that statistical assumption, the equilibrium relationship between the call price, the stock price, and other parameters of the economy is explored. Several different hypotheses have been offered to describe the movement of the stock price.

The next two sections trace the development of the general equilibrium specification of the option pricing model. Section 3 reviews the models developed prior to the Black–Scholes model. This section not only traces the development of thought in this area, but provides useful insight into the Black–Scholes Option Pricing Model. The reader who is not primarily interested in these aspects may wish to skip to section 4, below, which discusses the first general equilibrium model by Black and Scholes and the subsequent modifications by Merton and others.

[9]See Merton (1973b) for additional restrictions requiring additional assumptions. Further restrictions are not crucial to address the major issues of this paper but the interested reader is encouraged to refer to Merton's paper for this is a useful and intuitive approach.

3. Incomplete equilibrium models of call option pricing

Prior to the Black–Scholes option pricing model, only two assumptions about the statistical process generating the stock price had been offered. Bachelier (1900) suggests an Arithmetic Brownian Motion process. This assumption leads to unacceptable general equilibrium implications.[10] The major objections to Bachelier's model may be summarized as: (1) The assumption of Arithmetic Brownian Motion in the description of expected price movements implies both a positive probability of negative prices for the security and option prices greater than their respective security prices for large T. (2) The assumption that the mean expected price change is zero suggests both zero interest rates and risk neutrality. (3) The implicit assumption that the variance is finite a priori rules out members of the stable-Paretian family other than the normal.

Geometric Brownian Motion. Since the assumption of normality seems to lead to unacceptable implications, most models have used an alternate hypothesis – that the log of the stock price follows a Wiener process (or the stock price follows Geometric Brownian Motion).[11] This hypothesis involves four assumptions:

(1) The distribution of price ratios is independent of the price level,

$$\text{Prob}\{\tilde{S}^* \leqq S^* \mid \tilde{S} = S\} = F(S^*/S;\, T), \tag{19}$$

where F is the cumulative distribution function of the stock price relatives.

(2) Price ratios are independent. Since $(S^*/S) = (S^*/S')(S'/S)$,

$$F(S^*/S;\, T) = \textstyle\int_0^\infty F(S^*/S';\, T-T')F'(S'/S;\, T')\,\mathrm{d}S', \tag{20}$$

where F' is the density function for F, S' is the stock price T' from now, and $T > T' > 0$.

(3) There is a zero probability that the stock price will become zero, therefore logarithms may be employed.

(4) The variance of the price relatives is finite – thereby ruling out other members of stable-Paretian distribution of logs.

Given these four assumptions, the only solution to (20) is the log-normal distribution,

$$\begin{aligned} F(S^*/S;\, T) &= L(S^*;\, S, \rho T, \sigma\sqrt{T}) \\ &= N\left(\frac{\ln(S^*/S) - (\rho - \sigma^2/2)T}{\sigma\sqrt{T}}\right), \end{aligned} \tag{21}$$

where L is the cumulative log-normal distribution function for S^*.

[10] Bachelier's (1900) model is explicitly discussed in appendix A below.

[11] Arithmetic Brownian Motion without drift implies that the probabilities of the stock price either rising or falling by one dollar are equal, independent of the level of the stock price. Geometric Brownian Motion without drift implies that the probabilities of the stock price either rising or falling by one percent are equal, independent of the stock price.

A useful theorem in solving integrals involving the log-normal distribution is:

Theorem.[12] *If $L'(S^*)$ is a log-normal density function with*

$$Q = \begin{cases} \lambda S^* - \gamma X, & \text{if} \quad S^* - \psi X \geqq 0, \\ 0, & \text{if} \quad S^* - \psi X < 0; \end{cases}$$

then

$$\begin{aligned} E(Q) &\equiv \int_{\psi X}^{\infty} (\lambda S^* - \gamma X) L'(S^*)\, \mathrm{d}S^* \\ &= \mathrm{e}^{\rho T} \lambda S \cdot N\left\{\frac{\ln (S/X) - \ln \psi + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \\ &\quad - \gamma X \cdot N\left\{\frac{\ln (S/X) - \ln \psi + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}, \end{aligned} \tag{22}$$

where λ, γ and ψ are arbitrary parameters, ρ is the expected average rate of growth in the stock price [$\mathrm{e}^{\rho T} \equiv E(S^*/S)$], *and N is the cumulative standard normal distribution function,*

$$N\{q\} = \int_{-\infty}^{q} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2} \mathrm{d}z.$$

The Sprenkle Model. Sprenkle (1964) partially removes the first two objections to Bachelier's formulation. Sprenkle assumes that stock prices are log-normally distributed, thus explicitly ruling out the possibility of non-positive prices for securities and removing the associated infinite prices for options. Further, he allows for drift in the random walk, thus allowing for positive interest rates and risk aversion.

The expected value of the option at the expiration date is

$$E(C^*) = \textstyle\int_X^\infty (S^* - X) L'(S^*)\, \mathrm{d}S^*, \tag{23}$$

and by the above theorem [eq. (22)] with $\lambda = \gamma = \psi = 1$,

$$\begin{aligned} E(C^*) &= \mathrm{e}^{\rho T} S \cdot N\left\{\frac{\ln (S/X) + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \\ &\quad - X \cdot N\left\{\frac{\ln (S/X) + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}. \end{aligned} \tag{24}$$

[12]The proof of this theorem follows the proof of a less general result in the appendix of Sprenkle (1964).

Sprenkle also assumes that 'it is in general not true that the investor would be willing to pay a price for the warrant exactly equal to the expected value of it to him. In fact, he would be willing to pay exactly this price only if he were neutral to risk.' Von Neumann–Morgenstern utility functions do not imply that risk-neutral individuals would be indifferent between the choice of C dollars today and a gamble with expected value $E(C^*)$ dollars at the expiration of the option. It must further be assumed that interest rates are zero. Additionally, it is self-contradictory to assume that the random walk of stock prices has a positive bias, while assuming that investors pay the expected value for options.

The final form of Sprenkle's model containing a modification for risk is

$$C = e^{\rho T} S \cdot N\left\{\frac{\ln(S/X) + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} - (1-k)X \cdot N\left\{\frac{\ln(S/X) + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}, \tag{25}$$

where k is an adjustment for the degree of market risk aversion.

Since the time value of money is ignored, this model is flawed.

The Boness Model. Boness (1964a) allows for the time value of money and thus avoids Sprenkle's error. However, his assumptions are such that different levels of risk for the stock and the options are ignored. Boness assumes that:

'(1) The market is competitive in the sense that the equilibrium price of all stocks of the same risk class imply the same expected yield on investment. *For convenience and in default of better information, all stocks on which options are traded are defined to be of the same risk class.*
(2) The probability distribution of expected percentage changes in the price of any stock is log-normal.
(3) Variance of returns is directly proportional to time, $\sigma^2 = \sigma^2 T$.
(4) Investors are indifferent to risk.'[13]

Boness expresses the expected terminal value of the option in terms of conditional expected values as

$$E(C^*) = [E(S^* \mid S^* > X) - E(X \mid S^* > X)]\,\mathrm{Prob}\,(S^* > X). \tag{26}$$

From the definition of conditional expected values, the conditional expected value of S^* given that $S^* > X$ is

$$E(S^* \mid S^* > X) = \int_X^\infty S^* L'(S^*)\,\mathrm{d}S^* \Big/ \int_X^\infty L'(S^*)\,\mathrm{d}S^*. \tag{27}$$

[13] Boness (1964a, p. 167), emphasis in original.

Since the exercise price is non-stochastic,

$$E(X \mid S^* > X) = X, \tag{28}$$

and the probability that S^* will be greater than the exercise price is

$$\text{Prob}\,(S^* > X) = \int_X^\infty L'(S^*)\,\mathrm{d}S^*. \tag{29}$$

Substituting these definitions into (26), Boness derives the expected terminal price of the option,

$$E(C^*) = \int_X^\infty (S^* - X)L'(S^*)\,\mathrm{d}S^*. \tag{30}$$

To allow for the time value of money, he then discounts the expected terminal call price back to the present using the expected rate of return to the stock, ρ,

$$C = \mathrm{e}^{-\rho T} \int_X^\infty (S^* - X)L'(S^*)\,\mathrm{d}S^*. \tag{31}$$

To solve (31), use the above theorem with $\lambda = \gamma = \mathrm{e}^{-\rho T}$ and $\psi = 1$. Then,

$$\begin{aligned} C = {} & S\cdot N\left\{\frac{\ln (S/X) + [\rho + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \\ & - \mathrm{e}^{-\rho T} X \cdot N\left\{\frac{\ln (S/X) + [\rho - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}. \end{aligned} \tag{32}$$

Boness' fourth assumption would suggest that he uses ρ as a proxy for the expected rate of return on the option, κ [$\mathrm{e}^{\kappa T} \equiv E(C^*/C)$]. However, a different use of this assumption could make his task easier. If Boness notes that assumption four also implies that in equilibrium the returns to all assets would be equal, $\rho \overset{\mathrm{e}}{=} \kappa \overset{\mathrm{e}}{=} r$, then he could use the appropriate risk-free rate and thus avoid the estimation of the expected average rate of growth in the stock price, ρ.

The Samuelson Model. Samuelson (1965) assumes stock prices follow Geometric Brownian Motion with positive drift, ρ, thus allowing for positive interest rates and risk premiums,

$$E(S^*/S) = \mathrm{e}^{\rho T}.$$

If the option price also grows at the rate κ,

$$E(C^*/C) = \mathrm{e}^{\kappa T}.$$

With the addition of the assumptions that the terminal stock price distribution is log-normal, the value of the option is

$$\begin{aligned} C &= \mathrm{e}^{-\kappa T} E(C^*) \\ &= \mathrm{e}^{-\kappa T} \int_X^\infty (S^* - X)L'(S^*)\,\mathrm{d}S^*. \end{aligned} \tag{33}$$

To solve, let $\lambda = \gamma = e^{-\kappa T}$, $\psi = 1$ and apply the theorem stated in (22), then,

$$C = e^{(\rho-\kappa)T} S \cdot N\left\{\frac{\ln(S/X)+[\rho+(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}$$
$$-e^{-\kappa T} X \cdot N\left\{\frac{\ln(S/X)+[\rho-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}. \tag{34}$$

With the additional assumption that $\rho = \kappa$, then $e^{(\rho-\kappa)T} = 1$.

Samuelson examines the more difficult question – the value of an option if the return on the option is greater than the return on the stock, $\kappa > \rho$. Samuelson suggests two situations in which $\rho < \kappa$: (1) if the stock pays a dividend at the rate δ, it would be expected that at least $\rho + \delta = \kappa$, and (2) if the market perceives the option to be more risky than the security, then investors require that $\kappa > \rho$. In the appendix to the Samuelson paper, McKean (1965) solves this problem for a perpetual option and log-normally distributed security prices. His solution is[14]

$$\frac{C}{X} = \frac{(\xi-1)^{\xi-1}}{\xi^{\xi}}\left(\frac{S}{X}\right)^{\xi}, \tag{35}$$

where $\xi \equiv (\frac{1}{2}-\rho/\sigma^2)+[(\frac{1}{2}+\rho/\sigma^2)^2+2(\kappa/\sigma^2-\rho/\sigma^2)]^{1/2}$.

Samuelson posits a biased random walk following a Geometric Brownian Motion. Further, he does not rule out the fat-tailed, infinite variance distributions. He does say that: (1) they are mathematically intractable, and (2) that he is 'inclined to believe in Merton's conjecture that a strict Levy–Pareto distribution on $\log(S^*/S)$ would lead, with $1 < \alpha < 2$, to a 5-minute warrant or call being worth 100 percent of the common'.[15]

[14]Note that if $\xi = 2$, $C/X = \frac{1}{4}(S/X)^2$, $C = (S^2/4X)$, which is the ad hoc formula Giguere (1958) assumes to define the relationship between the call and stock prices.

[15]The basis for this assertion can be demonstrated as follows: If $\ln(S^*/S)$ is distributed according to a non-normal stable-Paretian distribution, then to find the expected value of S^*/S note that $\exp[\ln(S^*/S)] = S^*/S$. Let $\alpha \equiv \ln(S^*/S)$. Then the expected value of $\ln(S^*/S)$, ρT, can be expressed as

$$\rho T = \int_0^T e^{\alpha} f(\alpha)\,d\alpha,$$

where $f(\alpha)$ is the non-normal stable-Paretian density function. This can be written as

$$\rho T = \int_0^a e^{\alpha} f(\alpha)\,d\alpha + \int_a^{\infty} e^{\alpha} f(\alpha)\,d\alpha.$$

Choose a so that e^a is greater than a^2. Then the second integral is larger than the same integral with e^{α} replaced by α^2,

$$\int_a^{\infty} \alpha^2 f(\alpha)\,d\alpha < \int_a^{\infty} e^{\alpha} f(\alpha)\,d\alpha.$$

Since the stable-Paretian is not squared summable, this new integral is infinite. Therefore, ρT is infinite. If ρT is infinite, then in equilibrium it is conjectured rT would have to be infinite also. Then (11) would imply $C = S$. I would like to thank R.C. Merton for this explanation and J.C. Cox for the demonstration that ρT is infinite.

The motivation Samuelson offers for his model is admittedly incomplete. He allows that 'a deeper theory would deduce the value of ρ [and presumably also of κ] for each category of stocks'. More pointed is the criticism by Black and Scholes (1973) that 'there seems to be no model for the pricing of securities under conditions of capital market equilibrium that would make this an appropriate procedure for determining the value of a warrant'. Without further qualification, Samuelson's procedure of assuming κ is a constant is inappropriate as a base for a theory of option pricing under capital market equilibrium.

Samuelson's arguments as to why ρ and κ might be expected to differ are general-equilibrium in origin, but the implications of this analysis are at variance with the more general restrictions of Merton. Samuelson finds with $\kappa > \rho$ that there is a positive probability of premature exercise for the option. Merton (1973b) shows that for a simple option on a stock which pays no dividends that it will never be advantageous to exercise before maturity [see (11)].

4. General equilibrium call option pricing models

4.1. The Black–Scholes call pricing model

Black and Scholes demonstrate that it is possible to create a riskless hedge by forming a portfolio containing stock and European call options. The sources of change in the value of the portfolio must be the prices, since at a point in time the quantities of the assets are fixed. If the call price is a function of the stock price and the time to maturity, then changes in the call price can be expressed as a function of the changes in the stock price and changes in the time to maturity of the option. Black and Scholes then observe that at any point in time the portfolio can be made into a riskless hedge by choosing an appropriate mixture of stock and calls, e.g., if the hedge portfolio is established with a long position in the stock and a short position in the European call and if the stock price rises, then the increase in the value of the portfolio from the profit on the long position in the stock is offset by the decrease in the value of the portfolio from the loss which the increase in the stock price generates through the short position on the option, and vice versa. If quantities of the stock and option in the hedge portfolio are continuously adjusted in the appropriate manner as the asset prices change over time, then the return to the hedge portfolio becomes riskless. Therefore, the portfolio must earn the riskless rate.

The value of the hedge portfolio, V_H, can be expressed as the stock price times the number of shares of stock plus the call price times the number of calls in the hedge,

$$V_H \equiv SQ_s + cQ_c, \tag{36}$$

where V_H is the value of the hedge portfolio, Q_s the quantity of stock and Q_c the quantity of calls (for one share each).

The change in the value of the hedge, dV_H, is the total derivative of (36),

$$dV_H = Q_s\,dS + Q_c\,dc. \tag{37}$$

Black and Scholes use stochastic calculus to express dc, the change in the call price. Itô's lemma provides a technique by which certain functions of Wiener processes may be differentiated. If it is assumed that the stock price, S, follows Geometric Brownian Motion,[16] then Itô's lemma can be employed to express dc as

$$dc = \frac{\partial c}{\partial S}\,dS + \frac{\partial c}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\,dt. \tag{38}$$

Note that the only stochastic term in the expression for dc is dS. The rest are deterministic. Substituting (38) for dc in (37) yields

$$dV_H = Q_s\,dS + Q_c\left[\frac{\partial c}{\partial S}\,dS + \frac{\partial c}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\,dt\right]. \tag{39}$$

For arbitrary quantities of stock and options the change in the value of the hedge, dV_H, is stochastic, but if the quantities of each asset are chosen so that $Q_s\,dS + Q_c\,(\partial c/\partial S)\,dS$ equals zero (i.e., so that the ratio of stock to calls, Q_s/Q_c is equal to $-\partial c/\partial S$),[17] then the return to the hedge becomes riskless. Setting $Q_s = 1$ and $Q_c = -1/(\partial c/\partial S)$ in (39) yields

$$dV_H = -\left(\frac{1}{\partial c/\partial S}\right)\left[\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\right]dt. \tag{40}$$

In equilibrium, two perfect substitutes must earn the same return; therefore, since the hedge is riskless, its return must equal the risk-free rate,

$$dV_H/V_H \stackrel{e}{=} r\,dt.^{18} \tag{41}$$

Substituting (36) and (40) into (41) defines a differential equation for the value of the option,

$$\frac{\partial c}{\partial t} = rc - rS\frac{\partial c}{\partial S} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2, \tag{42}$$

[16]It is assumed that the motion of the stock price can be described as $dS/S = \mu dt + \sigma dz$, where μ is the instantaneous expected return on S, σ^2 the instantaneous variance of return and dz is a Wiener process. For an exposition of Itô processes and Itô's lemma see Merton (1971) or McKean (1969).

[17]Note that this restriction is placed on the ratio Q_s/Q_c. It makes no difference which asset is short. If the stock were sold short instead of the call, the number of shares sold short per call should be $-\partial c/\partial S$.

[18]The equality sign $\stackrel{e}{=}$ should be read 'equal in equilibrium'. This notation is used to highlight economic interpretation of this equation which is very different from that of functional relations, $=$, or definitions, $\equiv$.

subject to the boundary condition that at the terminal date, the option price must be equal to the maximum of either the difference between the stock price and the exercise price or zero [eq. (2)],

$$c^* = \mathrm{Max}\,[0, S^* - X].$$

The differential equation (42) can be solved for the equilibrium call price. Black and Scholes transform the equation into the heat exchange equation from physics to find the solution.

A more intuitive solution technique is suggested in a paper by Cox and Ross (1975). To solve the equation, note two observations: First, whatever the solution to the differential equation, it is a function only of the variables in (42) and (2), i.e., r, S, T, σ^2, X. Second, in generating the hedge, the sole assumption involving the preferences of the individuals in the market is that two assets which are perfect substitutes must earn the same equilibrium rate of return: no assumptions involving risk are employed. This suggests that if a solution to the problem can be found which assumes one particular preference structure, it must be the solution to the differential equation for any preference structure that permits equilibrium; therefore, choose the structure which proves most tractable mathematically.

To apply this solution technique,[19] assume the market is composed only of risk-neutral investors. In that case, the equilibrium rate of return to all assets is equal, $r \stackrel{e}{=} \rho \stackrel{e}{=} \kappa$, then the option must be priced so that the current call price is the discounted expected terminal price,

$$\begin{aligned} c &= \mathrm{e}^{-rT} E(c^*) \\ &= \mathrm{e}^{-rT} \int_X^\infty (S^* - X) L'(S^*)\,\mathrm{d}S^*. \end{aligned} \tag{43}$$

Eq. (43) may be solved by applying the above theorem, eq. (22), with $\lambda = \gamma = \mathrm{e}^{-rT}$ and $\psi = 1$. Substituting r for ρ yields the general equilibrium solution to the call pricing problem,

$$\begin{aligned} c = {} & S \cdot N\left\{\frac{\ln(S/X) + [r + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \\ & - \mathrm{e}^{-rT} X \cdot N\left\{\frac{\ln(S/X) + [r - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}. \end{aligned} \tag{44}$$

[19]This solution technique, independently derived by Cox and Ross, is similar to a technique employed in an early unpublished version of the Black–Scholes paper. It also corresponds to a technique Merton has employed in lectures but not in print.

Eq. (44) is the Black–Scholes pricing equation;[20] that it satisfies the differential equation in (42) may be established by substitution.[21]

Note that the assumptions used in this derivation are essentially those employed by Boness in deriving (32). However, Boness fails to demonstrate that a riskless hedge can be created, a hedge which does not depend on the preference structure of the market – he does not justify his procedure in a general equilibrium framework.[22]

The Black–Scholes model is a function of only five variables: the stock price, the exercise price, the time to maturity of the option, the risk-free interest rate, and the instantaneous variance rate on the stock price. The first four of these variables are directly observable, only the variance rate must be estimated. Further, other unobservable variables which have appeared in earlier models do not appear as arguments in this general equilibrium solution, arguments such as the expected rate of return on the stock, the expected rate of return on the option, or a measure of market risk aversion.

[20]To derive a more intuitive understanding of this equation, consider the equilibrium call price in a world of perfect certainty. Given certainty, the terminal call price would be a positive number (or no one would have ever purchased the call) equal to the terminal stock price minus the exercise price: $C^* = S^* - X$. In equilibrium in a world of certainty, the return to all assets must be equal in equilibrium, therefore: $r \stackrel{e}{=} \rho \stackrel{e}{=} \kappa$. Then, since $S^* = S\mathrm{e}^{\rho T}$, the current call price can be expressed as $C = \mathrm{e}^{-\kappa T}(\mathrm{e}^{\rho T}S - X)$. Substituting r for ρ and κ yields $C = S - \mathrm{e}^{-rT}X$. This expression differs from (44) only in the multiplication by the cumulative standard normal terms. These terms can be viewed as probabilities reflecting the uncertainty about the terminal stock price. See footnote 22.

[21]An interesting caveat has been pointed out by John Long. In the analysis leading to the differential equation, the option pricing formula has been assumed to be twice differentiable everywhere. The economics of the option pricing problem would suggest that the solution be continuous, but there is no obvious argument that it be differentiable everywhere. If it were postulated that the solution be continuous and satisfy the differential equation only where it is differentiable, then any convex combination of the Black–Scholes equation and Max $[0, S - XB(T)]$ would satisfy these conditions. An infinite number of solutions can be generated by this process, of which the Black–Scholes solution is the only smooth solution. This point may lead to a closer analysis of the economic implications of smoothness.

[22]A comparison of Boness' model can yield a better intuitive understanding of the terms in (44). Boness shows that in a world of risk neutrality the equilibrium call price can be expressed in terms of conditional expected values as $C = \mathrm{e}^{-\kappa T}E(S^* \mid S^* > X)\,\mathrm{Prob}\,(S^* > X) - \mathrm{e}^{-\kappa T}X\,\mathrm{Prob}\,(S^* > X)$. In a risk-neutral world the equilibrium expected rates of return on all assets would be equal, therefore $r \stackrel{e}{=} \rho \stackrel{e}{=} \kappa$. Substituting r for ρ and κ in (32) yields (44), the Black–Scholes solution. Hence, in a risk-neutral economy, the two terms in (44) have natural interpretations: the first term is the discounted expected value of the terminal stock price, given the terminal stock price exceeds the exercise price, times the probability the terminal stock price is greater than the exercise price. The second term is the discounted exercise price times the probability the terminal stock price exceeds the exercise price. *Warning*: This analogy is only suggestive, as this interpretation is predicated on a world of risk neutrality. An uncritical reading of the above solution technique might suggest that this interpretation is valid for all worlds. This is not the case. The above solution technique is *only* a procedure to derive a solution to a differential equation. It suggests that where a risk-free hedge can be established, the solution is independent of the degree of risk aversion in the economy, and therefore the mathematical solutions will be identical in any economy with any degree of risk aversion which permits a solution.

The responses of the model to changes in the value of its arguments conforms to the restrictions placed on the option price by the dominance arguments of Merton:[23]

(1) As the stock price rises, so does the call price,

$$\frac{\partial c}{\partial S} = N\left\{\frac{\ln (S/X)+[r+(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} > 0. \tag{45}$$

With a log-normal distribution of stock prices, the expected terminal price is a positive function of the current price; therefore an increase in the stock price increases the expected payoff to the option.

(2) As the exercise price rises, the call price falls,

$$\frac{\partial c}{\partial X} = -e^{-rT}N\left\{\frac{\ln (S/X)+[r-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} < 0. \tag{46}$$

This conforms to the restriction derived in (18) which implies that $\partial c/\partial X$ should be between zero and the negative of the bond price, $-e^{-rT}$.

(3) As the time to expiration increases, the price of the call rises,

$$\begin{aligned}\frac{\partial c}{\partial T} = Xe^{-rT}\Bigg[\frac{\sigma}{2\sqrt{T}}\,N'\left\{\frac{\ln (S/X)+[r-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \\ +rN\left\{\frac{\ln (S/X)+[r-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}\Bigg] > 0.\end{aligned} \tag{47}$$

This reflects the fact that the present value of the exercise payment is lower if the time to expiration is greater.

(4) As the riskless rate of interest rises, the call price rises,

$$\frac{\partial c}{\partial r} = T\cdot Xe^{-rT}N\left\{\frac{\ln (S/X)+[r-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} > 0. \tag{48}$$

When the riskless rate rises the present value of the exercise price falls.

(5) As the variance rate rises, so does the call price,

$$\frac{\partial c}{\partial \sigma^2} = Xe^{-rT}N'\left\{\frac{\ln (S/X)+[r-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}\frac{\sqrt{T}}{2\sigma} > 0. \tag{49}$$

When the variance rate on the underlying stock price is higher, then the probability of a large positive price change is greater. Of course, the probability of large negative changes is also greater, but the terminal option price cannot be below zero.

[23]See Merton (1973b) and section 2.2 above.

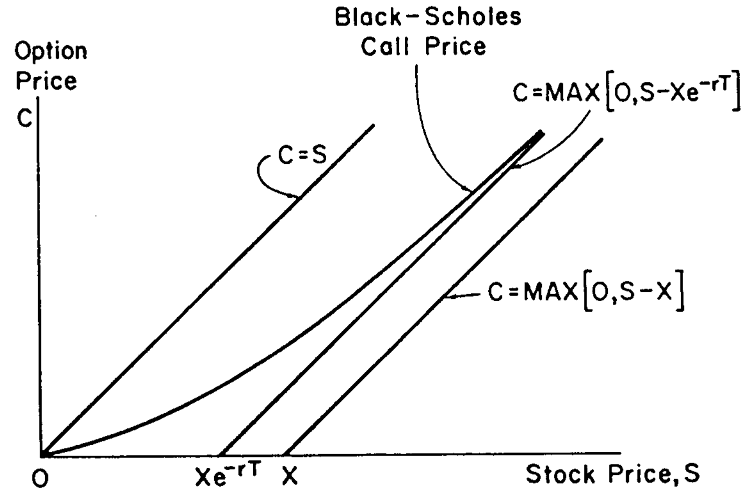

Fig. 1. Diagram of Black–Scholes call option price for different stock prices, with a given interest rate, variance rate, and time to maturity. The Black–Scholes call option price lies below the maximum possible value, $C = S$ (except where $S = 0$), and above the minimum value, $C = \text{Max}\,[0, S - Xe^{-rT}]$. Note that the curve relating the Black–Scholes call price with the stock price asymptotically approaches $C = \text{Max}\,[0, S - Xe^{-rT}]$ line.

Although the solution technique here employed assumes a general equilibrium configuration of prices, the specification of the equilibrium call price does not require the current stock price to be a general equilibrium price. A riskless hedge can be formed so long as the stock price follows an Itô process, therefore, (44) describes the equilibrium call price given the stock price.

Black and Scholes suggest that discounting the expected value of the distribution of possible values of the call when it is exercised is not an appropriate procedure. But the solution technique here employed does precisely that, and yields the same equation that Black and Scholes derived. There is nothing inappropriate in this approach itself, only in the prior implementation of that approach.

4.2. *Extensions of the Black–Scholes model*

In developing the basic Black–Scholes model, several assumptions were made. Many of these assumptions have been relaxed in the subsequent work by Merton and others. First, Merton (1973b) has shown that if a stock pays no dividends then it will not be exercised prematurely, and thus will command the same price as a European call [see (11) above]. Therefore, the Black–Scholes option pricing model may be applied to value American options on non-dividend paying stocks.

Thorpe (1973) has suggested that the assumption of no restrictions on the use of the proceeds of short sales is not necessary. If individuals hold well-diversified portfolios of securities, adding a hedge consisting of a long position in the call and a short position in the stock can be achieved by buying the call and reducing an existing long position in the stock.

The Merton Proportional Dividend Model. Merton (1973b) modifies the Black–Scholes model along several dimensions. The original model assumes no dividend payments on the stock over the life of the option. Merton relaxes this assumption for a rather special dividend policy – dividends are paid continuously such that the dividend yield, $\delta \equiv D/S$, is constant. For a European call, a hedge can be created as in (36). Now, however, the return to the stock consists of price changes plus dividend payments. Eq. (38) again describes the change in the call price. Substituting into (37) to obtain an expression for the change in the value of the hedge,

$$\mathrm{d}V_H = Q_s(\mathrm{d}S+\delta S\,\mathrm{d}t)+Q_c\left(\frac{\partial c}{\partial s}\,\mathrm{d}S+\frac{\partial c}{\partial t}\,\mathrm{d}t+\frac{1}{2}\frac{\partial^2 c}{\partial s^2}\,\sigma^2 S^2\,\mathrm{d}t\right). \tag{50}$$

Again by forming the hedge so that the ratio of the number of shares of the stock to the number of calls is equal to $-\partial c/\partial S$ the hedge becomes riskless. Setting $Q_s = \partial c/\partial S$ and $Q_c = -1$ yields

$$\mathrm{d}V_H = \frac{\partial c}{\partial S}\,\delta S\,\mathrm{d}t-\left(\frac{\partial c}{\partial t}+\frac{1}{2}\frac{\partial^2 c}{\partial s^2}\,S^2\right)\mathrm{d}t. \tag{51}$$

Again, since the hedge generates a certain return, in equilibrium it must earn the risk-free rate. Therefore, (41) again holds. Eq. (2) is still the boundary condition for the implicit differential equation defined by (41) and (51). If the above solution procedure is reapplied with $r \stackrel{c}{=} \rho+\delta \stackrel{c}{=} \kappa$, then the solution becomes

$$c = \mathrm{e}^{-\kappa T}\int_X^\infty (S^*-X)L'(S^*)\,\mathrm{d}S^*. \tag{52}$$

This can be solved by applying the above theorem, eq. (22), with $\lambda = \gamma = \mathrm{e}^{-\kappa T}$ and $\psi = 1$ and substituting r for $\rho+\delta$ and κ,[24]

$$c = \mathrm{e}^{-\delta T}S\cdot N\left\{\frac{\ln(S/X)+[r-\delta+(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}$$

$$-\mathrm{e}^{-rT}X\cdot N\left\{\frac{\ln(S/X)+[r-\delta-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}. \tag{53}$$

This is the solution to the European call option pricing problem when the underlying stock pays dividends continuously at the rate δ. Note that since dividends are paid this equation may not be applied to value American call options; Merton (1973b) has shown that there is always some probability of premature exercise of such options.

[24]This equation differs slightly from that reported in Merton (1973b, p. 171), but agrees with the solution, referenced by Merton, of Samuelson (1965).

Ingersoll Differential Tax Model. Ingersoll (1975) modifies the option pricing model to account for the effect of differential tax rates on capital gains versus ordinary income. To take the simplest case, assume that dividends are paid continuously at the rate $\delta = D/S$, that dividends and interest income are taxed at the rate τ, and that capital gains taxes are zero. A risk-free hedge can again be created as in (36), with the quantity of stock, Q_s, times the stock price, plus the quantity of calls, Q_c, times the call price,

$$V_H \equiv Q_s S + Q_c c. \tag{36}$$

But now, returns must be adjusted for the existence of taxes. The change in the stock price is composed of non-taxable capital gains and dividends taxable at the rate τ. The change in the value of the call is still described by

$$\mathrm{d}c = \frac{\partial c}{\partial S}\,\mathrm{d}S + \frac{\partial c}{\partial t}\,\mathrm{d}t + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\,\mathrm{d}t. \tag{38}$$

Therefore, the after-tax change in the value of the hedge can be expressed as

$$\mathrm{d}V_H = Q_s(\mathrm{d}S + (1-\tau)\delta S\,\mathrm{d}t) + Q_c\left(\frac{\partial c}{\partial S}\,\mathrm{d}S + \frac{\partial c}{\partial t}\,\mathrm{d}t + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\,\mathrm{d}t\right). \tag{54}$$

If the ratio of the number of shares of stock to the number of calls is again set equal to $-\partial c/\partial S$, the hedge becomes non-stochastic. Then setting $Q_s = \partial c/\partial S$ and $Q_c = -1$ yields

$$\mathrm{d}V_H = \frac{\partial c}{\partial S}\,\delta(1-\tau)S\,\mathrm{d}t - \left(\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\right)\mathrm{d}t. \tag{55}$$

Since this hedge yields a certain return, it must yield the after-tax risk-free return in equilibrium. Therefore the equilibrium condition for a world with taxes is that

$$\mathrm{d}V_H/V_H \overset{e}{=} (1-\tau)r\,\mathrm{d}t. \tag{56}$$

The boundary condition remains unchanged from (42) [eq. (2)],

$$c^* = \mathrm{Max}\,[0, S^* - X].$$

To apply the above solution technique to solve the implicit differential equation defined by (55) and (56), note that equilibrium in a risk-neutral economy would imply that the after-tax expected yield on all assets would be equal, i.e., $\rho + (1-\tau)\delta \overset{e}{=} \kappa \overset{e}{=} (1-\tau)r$. The problem then becomes

$$c = \mathrm{e}^{-\kappa T}\int_X^\infty (S^* - X)L'(S^*)\,\mathrm{d}S^*. \tag{57}$$

To solve, apply the above theorem, eq. (22), with $\lambda = \gamma = e^{-\kappa T}$, $\psi = 1$, and substitute $r(1-\tau)$ for $\rho + (1-\tau)\delta$ and κ,

$$c = e^{-\delta(1-\tau)T} S \cdot N\left\{\frac{\ln(S/X) + [(r-\delta)(1-\tau) + (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} - e^{r(1-\tau)T} X \cdot N\left\{\frac{\ln(S/X) + [(r-\delta)(1-\tau) - (\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}. \quad (58)$$

This is the solution to the option pricing problem for a world in which the stock pays dividends continuously at the rate δ, dividend and interest income are taxed at the rate τ while capital gains are not taxed.

The Merton Variable Interest Rate Model. The next model to be considered is Merton's model in which he relaxes the assumption of a constant interest rate. He assumes that the interest rate can vary over the life of the option in such a way that the return to a discount bond can be expressed as an Itô process.[25]

A hedge can again be established. Let the hedge consist of three assets – the call, the stock, and the bond,

$$V_H \equiv Q_c c + Q_s S + Q_B B. \quad (59)$$

Then the change in the value of the hedge comes from three sources – the stock price, the bond price, and the call price,

$$dV_H = Q_c dc + Q_s dS + Q_B dB. \quad (60)$$

If it is assumed that the call price can be expressed as a function of the bond price as well as the stock price and time to maturity, then applying Itô's lemma yields

$$dc = \frac{\partial c}{\partial S} dS + \frac{\partial c}{\partial B} dB + \frac{\partial c}{\partial t} dt + \frac{1}{2}\left[\frac{\partial^2 c}{\partial S^2}\sigma_s^2 S^2 + 2\frac{\partial^2 c}{\partial S \partial B}\rho_{sB}\sigma_s\sigma_B SB + \frac{\partial^2 c}{\partial B^2}\sigma_B^2 B^2\right] dt. \quad (61)$$

Substituting into (59) yields

$$dV_H = Q_c\left[\frac{\partial c}{\partial S} dS + \frac{\partial c}{\partial B} dB + \Psi dt\right] + Q_s dS + Q_B dB, \quad (62)$$

[25] Specifically, the change in the bond price is described by $dB/B(T) = \bar{r}dt + \sigma_B(T)dz(t, T)$, where $\bar{r}$ is the instantaneous expected return to the bond and $\sigma_B{}^2(T)$ is the instantaneous variance of return to the bond. Since $B(T)$ is a default risk-free bond, $B(0) = 1$. σ_B will be a function of T with $\sigma_B(0) = 0$. In the special case of a constant interest rate, $\bar{r} = r$, $\sigma_B = 0$, $B(T) = e^{-rT}$.

where

$$\Psi \equiv \left[\frac{\partial c}{\partial t}+\frac{1}{2}\left(\frac{\partial^2 c}{\partial S^2}\sigma_s^2 S^2+2\frac{\partial^2 c}{\partial S \partial B}\rho_{sB}\sigma_s\sigma_B SB+\frac{\partial^2 c}{\partial B^2}\sigma_B^2 B^2\right)\right],$$

and ρ_{sB} is the instantaneous correlation coefficient between the return to the stock and bond.

If a hedge is formed in such a way that the sum of the weights is zero, then the equilibrium return to the hedge must be zero; if there were non-zero returns, then perfectly certain returns could be generated requiring no capital – a situation obviously inconsistent with general equilibrium. But in this case it cannot be guaranteed that it will be possible to define simultaneously a riskless hedge *and* insure that the weights sum to zero. The problem has three unknowns but has four restrictions: (1) that, by construction, the value of the hedge is zero; (2) that the hedge is non-stochastic and therefore insulated from changes in the stock price; (3) that the hedge is non-stochastic and therefore insulated from unanticipated changes in the bond price; and (4) that a non-stochastic hedge employing no capital in equilibrium must generate a zero return. These restrictions can be stated as

$$\begin{aligned}
&V_H \equiv Q_c c+Q_s S+Q_B B = 0,\\
&Q_c \frac{\partial c}{\partial S}\mathrm{d}S+Q_s\mathrm{d}S = 0,\\
&Q_c \frac{\partial c}{\partial B}\mathrm{d}B+Q_B\mathrm{d}B = 0,\\
&Q_c\mathrm{d}c+Q_s\mathrm{d}S+Q_B\mathrm{d}B \stackrel{e}{=} 0.
\end{aligned} \tag{63}$$

Merton assumes that these equations are mutually consistent with a non-trivial solution and derives the implications of equilibrium.[26] These equations implicitly define two differential equations which have been assumed to hold simultaneously for the same values of Q_c, Q_s, and Q_B. Two boundary conditions are: (1) that at maturity the call sell for the maximum of either zero or the difference between the stock price and the exercise price, and (2) that a stock with a price of zero will have a zero call price,

$$\begin{aligned}
&C(S^*, 0, 1; X) = \text{Max}\,[0, S^*-X],\\
&C(0, T, B; X) = 0.
\end{aligned} \tag{64}$$

[26]The first three restrictions can be combined to yield $c-(\partial c/\partial S)S-(\partial c/\partial B)B = 0$. A sufficient condition for this equation to have a non-trivial solution is homogeneity of c in S and B. A dominance argument as in Merton (1973b) can be used to establish that if returns are independent of the level of the stock price (as in the log-normal) then the call price will be homogeneous in the stock and bond prices.

To solve this set of equations the above solution technique will again be used with two changes applied. First, since the riskless hedge now involves purchasing default-free bonds, the appropriate expression for the risk-free rate is $-\ln B(T)/T$.[27] Therefore, equilibrium in a risk-neutral economy implies $-\ln B(T)/T \stackrel{e}{=} \rho \stackrel{e}{=} \kappa$. Second, variance arises from two sources – from the stock and from the bond. Therefore, the instantaneous variance is $\hat{\sigma}^2(t) = \sigma_s^2 + \sigma_B^2(t) - 2\rho_{sB}\sigma_s\sigma_B(t)$. It follows that the variance over the life of the option will be the integral of the instantaneous variance,

$$\begin{aligned} \hat{\sigma}^2 T &\equiv \int_0^T \hat{\sigma}^2(t)\mathrm{d}t \\ &\equiv \int_0^T [\sigma_s^2 + \sigma_B^2(t) - 2\rho_{sB}\sigma_s\sigma_B(t)]\mathrm{d}t. \end{aligned} \tag{65}$$

Then the solution to this model becomes

$$c = B(T)\int_X^\infty (S^* - X)L'(S^*)\mathrm{d}S^*. \tag{66}$$

To derive the general equilibrium solution to the European call pricing problem when the interest rate is stochastic, apply the above theorem, eq. (22), with $\lambda = \gamma = B(T)$ and $\psi = 1$,

$$\begin{aligned} c = {} & S \cdot N\left\{\frac{\ln(S/X) - \ln B(T) + (\hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}\right\} \\ & - B(T)X \cdot N\left\{\frac{\ln(S/X) - \ln B(T) - (\hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}\right\}. \end{aligned} \tag{67}$$

Eq. (67) is the Merton (1973b, p. 167) solution.[28] Note that since there are no dividends, then by (11) this solution also applies to American calls.

Call Pricing With Poisson Processes. Thus far, two assumptions about stock price movements have been reviewed in this paper: the Arithmetic Brownian Motion of Bachelier, and the Geometric Brownian Motion of Sprenkle et al. One additional hypothesis is being examined, that of Poisson processes. A Poisson process or jump process offers a statistical description of the movement in the stock price which allows discontinuous changes. This process provides that with a low probability, the stock price will jump to a new level.[29] Since a

[27] In the case of a constant interest rate the bond price would be $B(T) = e^{-rt}$. Then $-(1/T)\ln B(T) = r$.

[28] Merton states his solution in terms of the error complement function (erfc). The relationship $\frac{1}{2}\operatorname{erfc}(-Q/\sqrt{2}) = N\{Q\}$ has been substituted.

[29] A simple Poisson or jump process can be described as $\mathrm{d}S/S = \mu\mathrm{d}t + (\gamma - 1)$ with probability $\lambda\mathrm{d}t$, and $\mathrm{d}S/S = \mu\mathrm{d}t + 0$ with probability $(1-\lambda)\mathrm{d}t$. Therefore, with probability $\lambda\mathrm{d}t$ the stock price will jump to γS. The parameter λ is the intensity of the process, and $(\gamma - 1)$ is the amplitude of the jump.

corollary to Itô's lemma can be used to differentiate functions of Poisson processes, the technique employed by Black and Scholes to derive the option price may be used in this context as well.

Merton (1976) examines the most general specification of the stock price movement where both the Geometric Brownian Motion and the Poisson process are present. He demonstrates that hedging both against the continuous changes and the discrete changes is not possible, and thus the risk-free hedge cannot be created. However, if the jumps are uncorrelated across securities, then the risk associated with the jump is unsystematic risk, and therefore, the risk associated with the jump is diversifiable risk which can be minimized by holding a portfolio of hedges. If the equilibrium return to a security is determined by its non-diversifiable risk (that risk which cannot be removed by simply holding additional assets), then the continuous part of the stock price movement can be hedged using Black–Scholes techniques. Since by assumption the risk associated with the jump is non-systematic, the Capital Asset Pricing Model suggests that the equilibrium rate of return on such an asset must be equal to that of the risk-free asset. Then, if the magnitude of the jump is distributed log-normally, Merton derives an expression for the option price which is a weighted average of the Black–Scholes solution conditional on the number of jumps.

Cox and Ross (1975) demonstrate that if the stochastic part of the stock price movement is defined only as a simple Poisson process, that is, that the jump is in only one direction and of a given amplitude, then a risk-free hedge, like the Black–Scholes hedge, can be created. Although this jump process is very simple, by careful specification of the parameters of the process, the movement of the stock price can be made to approximate that of Geometric Brownian Motion.

These two models suggest that the assumption of a continuous sample path for the stock price, i.e., that the stock price can only move 'locally', is not crucial to the analysis. In certain cases, the basic Black–Scholes technique can be applied to processes which allow discontinuous sample paths, i.e., jumps in the stock price. However, the solution to the option pricing problem when the stock price movement is discontinuous involves the unobservable expected return on the stock.

This section has reviewed the major models developed on call option pricing. The next section will briefly cover the much smaller body of work on put option pricing.

5. General equilibrium put option pricing models

5.1. Restrictions from stochastic dominance

Merton (1973b) develops general equilibrium restrictions on put prices using dominance arguments. *A put at expiration will be worth either the difference between the exercise price and the stock price or zero, whichever is greater,*

$$P(S^*, 0; X) = p(S^*, 0; X) = \text{Max}\,[0, X - S^*]. \tag{68}$$

This follows from the definition of a put, since the put will be exercised only if it is advantageous to do so, and from the ability to engage in arbitrage at the expiration date of the put.

When borrowing and lending rates are equal, then the price of a European put is equal to the value of a portfolio of a European call with the same terms as the put, riskless bonds with a face value of X and a short position in the stock. To demonstrate this relationship, let $B'(T)$ be the current value of one dollar payable T years from now at the borrowing rate. Now consider two portfolios A and B, where A contains one share of stock, one European put, and X dollars borrowed for T time periods; and B contains one European call with the same exercise price and maturity date as the put.

Table 6

Demonstration that a portfolio containing one share of stock, one European put and X dollars borrowed for T periods yields a terminal value equal to that of a European call with the same exercise price and time to maturity.[a]

Portfolio	Current value	Stock price at $T = 0$	
		$S^* \leqq X$	$X < S^*$
A	$S + p(S, T; X)$ $- XB'(T)$	$S^* + X - S^*$ $-X$	$S^* + 0$ $-X$
B	$c(S, T; X)$	0	$S^* - X$
Relationship between the terminal values of portfolios A and B		$V_a^* = V_b^*$	$V_a^* = V_b^*$

[a]Terminal values of portfolios A and B for different relationships between the stock price and exercise prices at the expiration date ($T = 0$) of the call.

Since $V_a^* = V_b^*$, to avoid domination the call must be priced so that

$$c(S, T; X) \leqq S + p(S, T; X) - XB'(T). \tag{69}$$

Again, consider two portfolios where A contains one European call, one share of stock sold short, and X bonds; B contains one European put with the same terms as the call. See table 7.

Since $V_a^* = V_b^*$, to avoid domination the put must be priced so that

$$p(S, T; X) \leqq c(S, T; X) - S + XB(T). \tag{70}$$

Table 7

Demonstration that a portfolio containing a European call, one share of stock sold short and discount bonds with a face value of X will yield the same terminal value as a European put.[a]

Portfolio	Current value	Stock price at $T = 0$	
		$S^* \leqq X$	$X < S^*$
A	$c(S, T; X) - S + XB(T)$	$0 - S^* + X$	$S^* - X - S^* + X$
B	$p(S, T; X)$	$X - S^*$	0
Relationship between the terminal values of portfolios A and B		$V_a^* = V_b^*$	$V_a^* = V_b^*$

[a]Terminal values of portfolios A and B for different relationships between the stock price and exercise prices at the expiration date ($T = 0$) of the call.

Therefore, if the borrowing and lending rates are equal, a European put must be priced so that

$$p(S, T; X) = c(S, T; X) - S + XB(T). \tag{71}$$

This is an extremely useful relationship, for eq. (71) can be employed with the restrictions established on call prices in the previous section to establish a number of restrictions on the price of the put. *The price of a European put must not be greater than that of a pure discount bond with a face value of X.* Since the call price is not greater than the stock price from (7),

$$p(S, T; X) \leqq XB(T). \tag{72}$$

A perpetual European put on a stock which pays no dividends must have a price of zero, since a perpetual European call would sell for the stock price S by (14),

$$\begin{aligned} p(S, \infty; X) &= c(S, \infty; X) - S + XB(\infty) \\ &= S - S + 0 = 0. \end{aligned} \tag{73}$$

The value of a European put on a stock which has a price of zero is the discounted exercise price. Since from (8), $c(0, T; X) = 0$, substituting into (71) yields

$$p(0, T; X) = 0 - 0 + XB(T). \tag{74}$$

Finally, *an American put must be priced no lower than a similar European put*, $P(S, T; X) \geqq p(S, T; X)$, since an American put contains all the features of a European put plus the right to early exercise. Merton argues that an American put will virtually always have a positive probability of premature exercise and therefore command a greater price than its European counterpart. For example, suppose the stock price falls far below the exercise price, the maximum price that the put can attain is X, the exercise price, and that price will be attained only if the stock price is zero. If the put can be exercised, then discount bonds can be purchased with the proceeds which earn the riskless rate. For stock prices far below the exercise price, since the maximum put price is bounded from above, the return to holding the put may be less than the return to exercising the put and purchasing bonds. Therefore, the put may be more valuable if exercised.

5.2. Put pricing with Geometric Brownian Motion

Use of (71) leads Black and Scholes (1973) to a direct application of their call pricing equation to derive a solution to the European put pricing problem. Substituting (44) into (71),

$$\begin{aligned} p &= S\cdot N\left\{\frac{\ln(S/X)+[r+(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \\ &\quad - Xe^{-rT}\cdot N\left\{\frac{\ln(S/X)+[r-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} - S + Xe^{-rT} \\ &= -S\cdot N\left\{\frac{-\ln(S/X)-[r+(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\} \\ &\quad + Xe^{-rT}\cdot N\left\{\frac{-\ln(S/X)-[r-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}. \end{aligned} \tag{74}$$

This is the solution to the European put pricing problem given the Black–Scholes assumptions. To modify the model to account for dividends, etc., substitute the appropriate solution from section 4.2 into (71) and solve.

In sections 2 through 5 the major thrust of the models of option pricing has been reviewed. There has been some testing of these equilibrium statements. Section 6 reviews these empirical tests.

6. Empirical tests of option pricing

6.1. Tests of the level of option prices

Little empirical testing of the general equilibrium restrictions on option pricing has appeared in print. Black and Scholes (1972) test several implications of their model. They create a hedge with zero market value involving the option, the stock, and risk-free bonds, employing the Black–Scholes Option Pricing Model to establish the quantities of the respective assets which minimize risk of the hedge,

$$V_H = Q_c c + Q_s S + Q_B B = 0, \tag{75}$$

with

$$Q_s / Q_c = -\partial c / \partial S,$$

and

$$[Q_c c + Q_s S]/B = Q_B.$$

As an approximation to continuous rebalancing, they rebalance the hedge daily. Since the proportions are not adjusted continuously, the hedge generates an uncertain return. However, Black and Scholes argue that this return will be uncorrelated with the market; and therefore, this uncertainty is diversifiable risk which can be minimized by holding a portfolio of hedges. If the market prices assets such that an investor is only compensated for accepting non-diversifiable risk, then the expected change in the value of the hedge over the single-day holding periods is zero, i.e.,

$$\Delta V_H \equiv \Delta c - \frac{\partial c}{\partial S}\Delta S - \left[c - \frac{\partial c}{\partial S} S\right] r \Delta T \overset{e}{=} 0. \tag{76}$$

Black and Scholes obtained the diaries of an option broker from 1966 to 1969, and recorded the six-month calls and straddles written on New York Stock Exchange securities. Their sample consists of 2,039 calls and 3,052 straddles. After estimating the instantaneous variance from the historical series of daily stock prices, they compute the theoretical option price using the Black–Scholes pricing equation (44). If the market price is greater than the model price they refer to the option as 'overvalued', and conversely.

Four portfolios are created using four different strategies:

(1) Buy all calls at model prices.
(2) Buy all calls at market prices.
(3) Buy undervalued calls and sell overvalued calls at model prices.
(4) Buy undervalued calls and sell overvalued calls at market prices.

The first two of these strategies test if the market or model prices are on average too high or too low. If abnormal positive returns are generated by either strategy, then the price used by that strategy (model or market) would be on average too low, and conversely. The third and fourth strategies expand the information set to include knowledge of the relationship between the market and model prices. These strategies test if the respective prices efficiently incorporate this additional information. The third strategy is another, more powerful, test of whether the model can be used to price options. If the hedge is formed by purchasing options when they are undervalued and selling options short when they are overvalued, and if this strategy generates abnormal negative returns then the model price is too high for undervalued options and too low for overvalued options. This would suggest that the market contains information not fully reflected in the model prices. The fourth strategy tests for profit opportunities over the sample period. If this strategy generates abnormal positive returns, then the market price is too low for undervalued options and too high for overvalued options. This would suggest that the Black–Scholes option pricing model contains information which is not being efficiently incorporated into the market price.

To test if the risk of the hedge actually is uncorrelated with the market, Black and Scholes regress the daily returns from the portfolios against the excess

Table 8
Measures of the average return to portfolios in dollars per contract per day adjusted for systematic risk, and measures of significance for four portfolio strategies constructed using estimates of the variance derived from the sequence of past stock prices.[a]

Portfolio employing strategy number	α[b]	$t-\alpha$	α_τ[c]	$t-\alpha_\tau$
1	−0.10	−2.02	−0.12	−0.68
2	−0.06	−1.18	−0.08	−0.47
3	−0.56	−11.60	−0.56	−5.45
4	+0.56	11.73	+0.56	6.64

[a]Adapted from Black and Scholes (1972).
[b]α = regression intercept (in dollars per contract per day).
[c]α_τ = average intercept for the 10 subperiods.

returns on the market. The slope of the regression, β, measures the systematic risk of the hedge, and the intercept, α, measures the average return to the hedge adjusted for systematic risk. Black and Scholes find no significant slope coefficients either over the entire period or for ten subperiods of approximately 75 days each. Since β is uniformly insignificant, only the intercept, α, is reported. The results of the regressions testing the above four strategies appear in table 8.

Whether bought at market or model prices, the returns to either of the first two strategies over the entire period are not significant; therefore, neither model nor market prices appear to be consistently above or below the correct level. But both the portfolios constructed by buying undervalued and selling overvalued options yield returns significantly different from zero. The third strategy generates significantly negative returns while the fourth strategy yields significant positive returns. This result could occur in several different ways, but one hypothesis which explains the observed results is that the estimate of the variance actually employed by the market is too narrow, and the historical estimates of the variance include an attenuation bias, i.e., the spread of the estimates is greater than the spread of the true variance. This would imply that for securities with a relatively high variance, the market would underestimate the variance, while using the historical price series would overestimate the variance; and conversely for relatively low variance securities. To test this possible hypothesis, Black and Scholes group the options into four categories based only on the estimate of the variance derived from past prices.

The results in table 9 indicate that in forming portfolios of hedges using model prices, significant positive returns are generated using low variance securities. This indicates that the model price is too low and therefore, the estimate of the variance is too low. The use of high variance securities generates significant negative returns, indicating that those prices and variance estimates are too high. The results employing the market prices are just the reverse. The results in

Table 9

Measures of the average return to portfolios in dollars per contract per day adjusted for systematic risk and measures of significance for four portfolios formed by ranking securities using estimates of the variance derived from the sequence of past stock prices.[a]

Portfolio (low to high variance)	Buy at model prices		Buy at market prices	
	$\alpha(\$)$	$t-\alpha$	$\alpha(\$)$	$t-\alpha$
1	0.15	2.57	−0.43	−7.47
2	0.06	0.87	−0.04	−0.56
3	−0.35	−4.39	−0.10	−1.76
4	−0.36	−5.57	0.17	2.60

[a]Adapted from Black and Scholes (1972).

table 9 are consistent with the hypothesis that the model overprices high variance securities and the market underprices them. This also suggests market inefficiency in the absence of transactions costs.

To test whether the model values securities correctly when the true variance rate of the stock is known, Black and Scholes estimate the variance rate from the actual price history over the life of the options. Since future information is required, this strategy cannot be implemented in practice but will indicate the efficacy of the approach. The previous test suggests that there may be error in the estimate of the variance. If this is the case, the above analysis mixes a test of the model with a test of the estimate of the variance. By employing the actual variance which occurs over the life of the option, a test of the model alone is derived, a test which is independent of errors in the estimate of the variance of the stock returns.

The results in table 10 indicate that if the subsequent variance is known, the Black–Scholes model performs very well.

Black and Scholes then proceed to the question of option market efficiency. Their tests establish that options prices do not appear to conform to a general equilibrium configuration: The results of table 9 indicate that higher returns are available to individuals who establish riskless hedges on high variance securities over the returns available to riskless hedges involving low variance securities. Market efficiency is a weaker condition than that of general equilibrium. A market is termed efficient if, after transactions costs, no abnormal returns are available. Black and Scholes estimate the transactions costs incurred by individual traders and conclude that these costs are higher than the abnormal returns

Table 10

Measures of the average return to portfolios in dollars per contract per day adjusted for systematic risk and measures of significance for four portfolio strategies constructed using estimates of the variance derived from the sequence of stock prices over the life of the option.[a]

Portfolio employing strategy number	α[b]	$t-\alpha$	α_τ[c]	$t-\alpha_\tau$
1	−0.04	−1.04	−0.06	−0.75
2	−0.09	−2.00	−0.11	−0.68
3	−0.06	−1.19	−0.06	−1.03
4	1.11	21.64	1.11	11.05

[a]Adapted from Black and Scholes (1972).
[b]α = regression intercept (in dollars per contract per day).
[c]α_τ = average intercept for the 10 subperiods.

Black and Scholes generate, excluding transactions costs. They therefore conclude that this market is efficient. However, market efficiency is usually stated in terms of the lowest cost trader – usually a member of the exchange. Although individuals do not appear to be able to generate abnormal returns including transactions costs, the Black–Scholes evidence is not sufficient to conclude that exchange members cannot generate abnormal returns after including their lower transactions costs.

Two final points should be noted about the procedures employed by Black and Scholes in dealing with Put and Call Dealers Association options: (1) market prices are observed only on the first day of the contract, and (2) stock prices used are closing quotes. Since only initial market prices are observed, subsequent days' prices had to be created artificially.[30]

It seems unlikely that this artificial series would correspond to the time path of option prices if a secondary market existed. Furthermore, given that stock price quotes are ending quotes, differences between market and model prices could arise from changes in stock prices between the time the option was written and the close that day. The effects of these points can be minimized by utilizing data from the Chicago Options Board Exchange or the American Options Exchange with their active secondary markets.

Galai (1975) repeats the Black–Scholes tests and extends their analysis along several dimensions using data from the first seven months of trading on the Chicago Board Options Exchange. In replicating the earlier tests and in the extensions of these tests, the implications of the analysis basically support the earlier Black–Scholes results.

Galai constructs tests that would be feasible to implement. A trading rule, considered by Galai, is formulated based on the closing prices for day t, and the transactions take place at the closing prices for day $t+1$. The returns to this strategy are generally positive but usually statistically insignificant. Furthermore, the inclusion of minimal transactions costs (less than one percent) eliminates these abnormal positive returns.

6.2. *Tests of relative option prices*

While Black and Scholes (1972) and Galai (1975) test the absolute level of the observed call prices; Stoll (1969) and Gould and Galai (1974) test the relative spreads between put and call prices. For European options with the exercise price equal to the current stock price, (71) can be rewritten as

$$p(S, T; S)+S-c(S, T; S) = SB(T). \tag{77}$$

The left-hand side of (77) generates a certain, terminal payoff of S, therefore, over the life of the options, the return to the left-hand side must equal the T-period risk-free rate (R),

$$\frac{S}{p(S, T; S)+S-c(S, T; S)}-1 = R. \tag{78}$$

[30] To calculate the 'market' price for days after the first Black and Scholes used the following procedure: [1] At the initial date, compute the Black–Scholes model price from (44). [2] Take the difference between the model price and the market price. [3] Amortize that difference over the life of the option. [4] Then for the 'market' price on day $t+1$, take the 'market' price on day t and add the appropriate amortization factor plus the change in option price implied by the Black–Scholes model (reflecting changes in time to maturity of the option plus changes in stock prices from t to $t+1$).

Stoll (1969) tested (78) using data on American options. Again, this restriction does not hold for American options, because there is a positive probability of premature exercise for an American put option. For American options, a weaker condition can be established. Since the value of an American put is no less than that of a European put, the American counterpart of (77) can be expressed as

$$C(S, T; S) - P(S, T; S) - S(1 - B(T)) \leqq 0. \tag{79}$$

Gould and Galai (1974) test (79) using both Black and Scholes' and Stoll's data. To examine market efficiency, Gould and Galai demonstrate that the existence of taxes cannot change the direction of the inequality in (79). Next, they take an indirect approach to account for transaction costs. If transaction costs are incurred in trading, market efficiency implies that

$$C(S, T; S) - P(S, T; S) - S(1 - B(T)) \leqq TC, \tag{80}$$

where TC is the transactions costs incurred by the lowest cost trader.

Table 11

Cases of potential market inefficiency in relative put and call pricing for 190-day options using data from 1969.[a]

Stock price	Lowest observation	Lowest quartile	Median	Upper quartile	Highest observation	Number of observations	
< \$20	*	\$7.02	\$25.51	\$39.05	\$64.04	131	(L)
	*	2.03	14.95	30.72	51.90	89	(H)
\$20–\$40	*	*	8.10	20.41	100.15	234	(L)
	*	*	3.94	15.52	87.85	240	(H)
\$40–\$70	*	*	*	*	66.27	73	(L)
	*	*	*	*	56.58	140	(H)
> \$70	*	*	*	*	388.82	29	(L)
	*	*	*	*	3329.86	34	(H)

[a]The quantity $[C(S, T; S) - P(S, T; S) - S(1 - B(T))]$ is calculated for every pair of puts and calls. Negative magnitudes are consistent with market efficiency and therefore not reported, while positive magnitudes are consistent with efficiency only if they are less than transactions costs. Asterisk indicates 'less than zero'. Adapted from Gould and Galai (1974, table 4, p. 116). The available data does not specify the stock price when the options were written. Gould and Galai compute the figures in table 11 using both the low (L) and high (H) stock price for the week in which the options were written.

Gould and Galai are unable to measure accurately the actual magnitude of exchange members' transaction costs, therefore they compute the left-hand side of (80) to see how large the transaction costs would have to be to imply market efficiency. Their results are summarized in table 11.

Gould and Galai estimate individual trader's transactions costs to be generally greater than the magnitudes in table 11. However, some of the figures are above reasonable estimates of the transaction costs incurred by exchange members. Since the magnitudes which appear in table 11 are observable ex ante, only those hedges which have total payoffs in excess of total costs need be created. Since abnormal positive returns can be generated by employing a strategy which only requires knowledge of current prices, what Roberts (1967) has termed the weak form of the efficiency markets hypothesis seems to have been violated in trading in the Put and Call Dealers Association options market.

7. Other applications of the option pricing model

7.1. The equity of a levered firm

Black and Scholes (1973) suggest that the Option Pricing Model can be used to price other contingent claim assets. They suggest viewing the equity of a levered firm as an option purchased from the bondholders. Assume a firm issues pure discount bonds which prohibit any dividend payments until after the bonds are paid off. If the bonds mature at the end of T periods at which time the firm is liquidated, the bondholders are paid (if possible), and the residual is paid to the stockholders, then the Black–Scholes Option Pricing Model can be applied to value the equity and debt of the firm. In essence, this situation can be thought of as one in which the stockholders sell the firm to the bondholders for the proceeds of the bond issue, and the stockholders have the option to buy back the firm from the bondholders at the maturity date of the bonds for an amount equal to the face value of the bonds. Applying the Black–Scholes solution, the value of the equity of the firm is

$$\begin{aligned}\hat{S} = {} & V \cdot N\left\{\frac{\ln(V/\hat{B}^*) + [r + (\sigma_v^2/2)]T}{\sigma_v\sqrt{T}}\right\} \\ & - e^{-rT}\hat{B}^* \cdot N\left\{\frac{\ln(V/\hat{B}^*) + [r - (\sigma_v^2/2)]T}{\sigma_v\sqrt{T}}\right\},\end{aligned} \tag{81}$$

and the value of the debt is

$$\hat{B} = V - \hat{S}, \tag{82}$$

where $\hat{S}$ is the total value of the stock (equity), V is the total value of the firm, $\hat{B}^*$ is the total face value of the bonds, $\hat{B}$ is the total current value of the bonds, σ_v^2 is the variance rate on the total value of the firm V.

Black and Scholes then briefly sketch the applicability of this analysis for important issues in corporate finance and managerial economics. They suggest that: (1) The discount due to default risk in corporate bonds may be measured by subtracting the value of the bonds given by their formula from the value of risk-free bonds of the same maturity and face value. (2) Changes in capital structure such as issuing bonds and using the proceeds to retire common stock will affect the distribution of the value of the firm between the stockholders and bondholders. (3) Dividend payments are capable of affecting the division of the total value of the firm between the stock and the bonds. (4) It is conjectured that these possibilities for effecting a change in the value of the claims represented by the stock and bonds frequently make it optimal to include restrictions in bond indentures prohibiting these activities. (5) Coupon bonds are like compound options. By paying the last coupon, the stockholders buy the option to purchase the firm by paying the bondholders the face value of a debt. At the time of the next-to-last interest payment, the stockholders have an option on an option on an option. (6) Callable bonds add another option to the analysis. With a call provision, the firm can either make the coupon payment or retire the bonds. Many applications of the option pricing model examine and expand the analysis in this list of suggested uses.

7.2. *The risk structure of interest rates*

Merton (1974) examines the risk structure of interest rates on corporate debt. He assumes the total value of the firm is unaffected by capital structure. Long (1974) emphasizes the point that to apply stochastic calculus it must be assumed that the process describing the total value of the firm can be fully specified without reference to the value of the firm's bonds or stock, or equivalently, that the total value of the firm is independent of its capital structure. Therefore, this analysis applies to a Miller–Modigliani (1958) world with no taxes or transactions costs of bankruptcy. Jensen and Meckling (1975) suggest that the existence of any agency cost[31] would cause the total value of the firm to be a function of the debt/equity ratio and therefore would invalidate the specific conclusions of this analysis.

Merton substitutes (81) into (82) to derive

$$\hat{B} = V\cdot N\left\{\frac{-\ln(V/\hat{B}^*)-[r+(\sigma_v^2/2)]T}{\sigma_v\sqrt{T}}\right\}$$
$$+\hat{B}^*\mathrm{e}^{-rT}\cdot N\left\{\frac{\ln(V/\hat{B}^*)+(r-(\sigma_v^2/2))T}{\sigma_v\sqrt{T}}\right\}. \tag{83}$$

The results of the examination of the partial effects on European calls can be used in this context to find the functional relationship between the value of the

[31] Agency costs are taken by Jensen and Meckling (1975) to mean any cost born by one class of owners of the firm which are imposed by managers of the firm or by another class of owners. These costs include costs arising from restrictive covenants within bond contracts, auditing costs, costs of monitoring managers activities, etc.

firm's debt and the value of the firm, the face value of the bonds, the time to maturity of the bonds, the variance rate on the value of the firm, and the risk-free rate.[32] Therefore this technique allows valuation of the equity and debt, given the total value of the firm.

Merton then suggests that since discussions of bond pricing frequently employ yields rather than bond prices, it is convenient to transform (83) into an excess return by using the transformation

$$e^{-r_B(T)T} \equiv \hat{B}/\hat{B}^*, \tag{84}$$

where $r_B(T)$ expresses the yield to maturity of a risky corporate bond with T periods to maturity, provided that it does not default.

Then the risk premium on risky corporate debt can be measured by

$$r_B(T) - r \equiv -\ln(\hat{B}/\hat{B}^*)T - r. \tag{85}$$

This implicitly defines a risk structure of interest rates. Because of the relationship between the bond price and the other variables in the system, $\hat{B} = \hat{B}[V, \hat{B}^*, T, \sigma^2, r]$, the risk structure can also be expressed as a function of these variables.

7.3. The effects of corporate policy

Galai and Masulis (1976) extend the Black–Scholes analysis of the effect on the distribution of ownership between the stockholders and bondholders to include various changes in investment policy of the firm. They assume that both the capital asset pricing model and the option pricing model simultaneously hold. They then explore the implications of this joint hypothesis.

The capital asset pricing model implies that the equilibrium rate of return to an asset at every point in time is

$$\bar{r}_j = r + \beta_j(\bar{r}_m - r), \tag{86}$$

where $\bar{r}_j$ is the instantaneous expected return to asset j, $\bar{r}_m$ is the instantaneous expected return to the market, $\beta_j[\equiv \text{cov}(\tilde{r}_j, \tilde{r}_m)/\sigma^2(\tilde{r}_m)]$ measures the systematic, non-diversifiable risk of a security.

Galai and Masulis demonstrate that if the systematic risk of the firm, β_v, is constant over time, the instantaneous risk of the equity, β_s, will not be stable.

[32]From (45)–(49): $0 < \partial\hat{S}/\partial V < 1$, $\partial\hat{S}/\partial\hat{B}^* < 0$, $\partial\hat{S}/\partial T > 0$, $\partial\hat{S}/\partial\sigma^2 > 0$, $\partial\hat{S}/\partial r > 0$. Since it has been assumed that $\hat{B} = V + \hat{S}$, these relationships can be employed to derive $0 < \partial\hat{B}/\partial V = (1 - \partial\hat{S}/\partial V) < 1$, $\partial\hat{B}/\partial\hat{B}^* = -(\partial\hat{S}/\partial\hat{B}^*) > 0$, $\partial\hat{B}/\partial T = -(\partial\hat{S}/\partial T) < 0$, $\partial\hat{B}/\partial\sigma^2 = -(\partial\hat{S}/\partial\sigma^2) < 0$, $\partial\hat{B}/\partial r = -(\partial\hat{S}/\partial r) < 0$.

By (39) the change in the total value of the stock, $\hat{S}$, is given by

$$\mathrm{d}\hat{S} = \frac{\partial \hat{S}}{\partial V}\mathrm{d}V + \Psi\,\mathrm{d}t, \tag{87}$$

where

$$\Psi = \frac{\partial \hat{S}}{\partial t} + \frac{1}{2}\frac{\partial^2 \hat{S}}{\partial V^2}\sigma_v^2 V^2,$$

and the instantaneous return to the stockholders can be expressed as

$$\tilde{r}_S \equiv \frac{\mathrm{d}\hat{S}}{\hat{S}} = \frac{\partial \hat{S}}{\partial V}\frac{V}{\hat{S}}\frac{\mathrm{d}V}{V} + \frac{\Psi}{\hat{S}}\mathrm{d}t \equiv \frac{\partial \hat{S}}{\partial V}\frac{V}{\hat{S}}\tilde{r}_v + \frac{\Psi}{\hat{S}}\mathrm{d}t. \tag{88}$$

Substituting into the definition of the systematic risk of the equity, β_s, yields[33]

$$\beta_s \equiv \frac{\operatorname{cov}(\tilde{r}_s, \tilde{r}_m)}{\sigma^2(\tilde{r}_m)} = \frac{\partial \hat{S}}{\partial V}\frac{V}{\hat{S}}\frac{\operatorname{cov}(\tilde{r}_v, \tilde{r}_m)}{\sigma^2(\tilde{r}_m)} \equiv \frac{\partial \hat{S}}{\partial V}\frac{V}{\hat{S}}\beta_v. \tag{89}$$

Therefore, the beta of the stock can be expressed as the elasticity of the value of the stock with respect to the value of the firm, $\varepsilon(\hat{S}, V)$, times the beta of the firm,

$$\beta_s = \varepsilon(\hat{S}, V)\beta_v, \tag{90}$$

where

$$\varepsilon(\hat{S}, V) \equiv \frac{\partial \hat{S}}{\partial V}\frac{V}{\hat{S}}.$$

Since $\varepsilon(\hat{S}, V)$ is greater than 1,[34] the systematic risk of the stock is greater than

[33] Again, note that Ψ is non-stochastic, see (39).

[34] From (83), (76) and (46),

$$\varepsilon(\hat{S}, V) = \frac{\partial \hat{S}}{\partial V}\frac{V}{\hat{S}} = N\left\{\frac{\ln(V/\hat{B}^*) + [r + (\sigma_v^2/2)]T}{\sigma_v\sqrt{T}}\right\}\frac{V}{\hat{S}}$$

$$= \frac{\left(V \cdot N\left\{\frac{\ln(V/\hat{B}^*) + [r + (\sigma_v^2/2)]T}{\sigma_v\sqrt{T}}\right\}\right)}{\left(V \cdot N\left\{\frac{\ln(V/\hat{B}^*) + [r + (\sigma_v^2/2)]T}{\sigma_v\sqrt{T}}\right\} - \hat{B}^* \mathrm{e}^{-rT} \cdot N\left\{\frac{\ln(V/\hat{B}^*) + [r - (\sigma_v^2/2)]T}{\sigma_v\sqrt{T}}\right\}\right)} > 1$$

Note that the denominator is equal to the numerator minus a positive magnitude, the elasticity is therefore greater than 1.

the systematic risk of the total firm.[35] Because $\varepsilon(\hat{S}, V)$ is a function of V, $\hat{B}^*$, r, σ_v^2, and T; if β_v is stationary, β_s will be non-stationary.

Galai and Masulis then demonstrate that if the value of the firm is unaffected by the capital structure, i.e., $V = \hat{S}+\hat{B}$, then from (81) and (83) the value of the equity and debt is a function of (1) the value of the firm; (2) the face value of the debt; (3) the risk-free rate; (4) the time to maturity of the debt; and (5) the variance rate of the value of the firm. Therefore, unanticipated changes in any of these variables can affect the market value of the stockholders' and bondholders' claims. This descriptive power makes the option pricing model extremely useful in analyzing changes in corporate investment policy. They employ comparative statics analysis to examine the effect of different policies. Given appropriate assumptions, Galai and Masulis show that: (1) Acquisitions which increase the variance rate of the firm will increase the value of the equity and reduce the value of the debt. (2) Conglomerate mergers which reduce σ_v^2 increase the value of the debt and decrease the value of the equity. (3) Only increases in the scale of operations which are financed by proportional increases in debt and equity cause no redistributions of ownership. (4) Spin-offs where assets are distributed only to stockholders reduce the value of the debt.

7.4. *Dual purpose funds*

Ingersoll (1976) applies the option pricing model to the analysis of dual purpose funds. Dual purpose funds are closed-end funds which issue two classes of shares: income shares which have the rights to all income earned by the fund, subject to a stated mimimum cumulative dividend, plus a fixed payment at the maturity of the fund; and capital shares which pay no dividends and are redeemable at net asset value at the maturity date.

The capital shares of a dual purpose fund are analogous to a European option on a dividend-paying stock, with an exercise price equal to the required payment to the holders of the income shares. Therefore, in the simplest case where the interest rate is constant over time, dividends and management fees are paid continuously at a fixed fraction of the asset value of the fund, and taxes are zero, (53) can be applied to value the capital shares,

$$c = e^{-\delta T} S \cdot N\left\{\frac{\ln (S/X)+[r-\delta+(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}$$
$$-e^{-rT} X \cdot N\left\{\frac{\ln (S/X)+[r-\delta-(\sigma^2/2)]T}{\sigma\sqrt{T}}\right\}, \tag{91}$$

where c is the value of the capital shares, $\delta \equiv \delta_1+\delta_2$ (δ_1 being the dividend pay-

[35] For $\beta_v > 0$.

ment [$D \equiv \delta_1 S$] and δ_2 the management fee [$M \equiv \delta_2 S$]), S the net asset value of the fund, X the payment to the income shareholders, σ^2 the variance rate on the net asset value.

Ingersoll derives the value of the income shares as[36]

$$I = S\left[e^{-\delta T}\left(1-\frac{\delta_1}{\delta_1+\delta_2}\right)+\frac{\delta_1}{\delta_1+\delta_2}\right]-c. \tag{92}$$

First, note that the market value of the fund (or any closed-end fund) will typically be less than the net asset value of the fund because of the management fees incurred by the holders of the shares of the closed-end fund. The difference between the net asset value of the fund and the market value of the fund's shares can be expressed as

$$S-(c+I) = S\frac{\delta^2}{\delta}(1-e^{-\delta T}). \tag{93}$$

Second, because there are no provisions for liquidation of the fund prior to the maturity date of the fund, the payments of dividends and management fees are capable of making the capital shares sell below their 'intrinsic value', the difference between the net asset value of the fund and the fixed payment owed to the holders of the income shares. This is analagous to a European option on a dividend paying stock selling below the difference between the stock price and exercise price.

Ingersoll tests this model using market data on the seven dual purpose funds. His tests suggest that the option pricing model describes fluctuations in the capital shares quite well.

8. Conclusion

The Black–Scholes Option Pricing Model provides an explicit general equilibrium solution to the problem of valuing simple puts and calls. Further, the model

[36]To gain a more intuitive understanding of this equation, it can also be derived in case of perfect certainty. In that case, the terminal value of the fund would be $e^{(r-\delta)T}S$, and the capital share would be priced so that

$$C = e^{-rT}(e^{(r-\delta)T}S-X) = e^{-\delta T}S-e^{-rT}X.$$

The income share would be priced so that

$$I = \int_0^T e^{-r\tau}\delta e^{(r-\delta)\tau}S\,d\tau+Xe^{-rT} = \frac{\delta_1}{\delta_1+\delta_2}S(1-e^{-(\delta_1+\delta_2)T})+Xe^{-rT}.$$

This can be rewritten as

$$I = S\left[e^{-(\delta_1+\delta_2)T}\left(1-\frac{\delta_1}{\delta_1+\delta_2}\right)+\frac{\delta_1}{\delta_1+\delta_2}\right]-c,$$

which is identical to (92).

contains testable hypotheses; the variables of the model are either readily observable or are subject to reasonably accurate estimation. However, at this time it would appear that the development of theoretical hypotheses is far ahead of the empirical work in this area. Several modifications of the basic option pricing model have been suggested, but the importance of these modifications cannot be judged in the absence of empirical work which is yet to be completed.

The development of organized secondary markets for options, such as the Chicago Board Options Exchange, the American Options Exchange, and the Philadelphia, Baltimore, Washington Options Exchange can provide valuable data for testing the implications of these models for option pricing. However, that the Black–Scholes analysis has generated substantial interest seems to derive more from its implications for a general theory of the valuation of contingent claims, than for its direct application to value simple puts and calls. Nevertheless, to date, the only example of empirical verification of this analysis which employs data other than that for put and call trading is by Ingersoll (1976). The potential benefits of empirical research in this area appear to be large.

Appendix A

The Bachelier model

Bachelier (1900) assumes that the stock price is a random variable, that price changes are independent and identically distributed, and that

$$\text{Prob}\,\{\tilde{S} \leqq S^* \mid \tilde{S} = S\} = F(S^*-S;T), \tag{A.1}$$

where F is the cumulative distribution function of the stock price changes.[37]

This describes a Wiener Process (or Arithmetic Brownian Motion). Bachelier's choice is unfortunate, for as T approaches infinity, then the Prob $\{\tilde{S}^* < S^*\}$ approaches $\frac{1}{2}$ for all S^*. Since nothing in this formulation restricts S^* to the positive numbers, there is a positive probability of negative stock prices; a violation of the property of limited liability.

Bachelier incorrectly deduces that (A.1) implies that the density function must be that of the normal,

$$F(S^*-S;T) = N\left(\frac{S^*-(S+\mu T)}{\sigma\sqrt{T}}\right), \tag{A.2}$$

where μ is the mean expected price change per time period, σ^2 the variance per time period, N the cumulative standard normal distribution.

[37] Tildes represent random variables. Eq. (A.1) says that the probability that the stock price T periods from now ($\tilde{S}^*$) is less than or equal to a given number, S^*, given that the current stock price ($\tilde{S}$) has assumed the value S, can be expressed as a function of the distance (S^*-S) and T.

Eq. (A.1) is insufficient to deduce (A.2). Any member of the stable Paretian family of distribution satisfies (A.1). It must further be assumed that the variance is finite to deduce normality.

Bachelier's next assumption suggests his specification of the process that generates the stock price is unsuitable as an equilibrium specification. He assumes that the mean expected price change per unit time (μ) equals zero. Bachelier then assumes that the call is also priced to yield a mean expected return of zero. Bachelier views the stock market as a gamble; he feels that competition will reduce the expected return to zero, which seems to deny both positive interest rates and risk aversion.

Bachelier applies the same logic to the pricing of the call option – he feels the call will be priced so that the current call price is the expected terminal call price. From (2), the terminal call price is the maximum of either the difference between the terminal stock price and the exercise price or zero, $C^* = \text{Max}\,[0, S^*-X]$, therefore Bachelier's model suggests that

$$C = E(C^*) \equiv \int_X^\infty (S^*-X)N'(S^*)\mathrm{d}S^*, \tag{A.3}$$

where $N'(S^*)$ is the normal density function for S^*.

Changing variables,

$$C = \int_{X-S/\sigma\sqrt{T}}^{\infty} (z\sigma\sqrt{T}+S-X)N'(z)\mathrm{d}z, \tag{A.4}$$

where $z \equiv (S^*-S)/\sigma\sqrt{T}$, and

$$C = S\cdot N\left\{\frac{S-X}{\sigma\sqrt{T}}\right\} - X\cdot N\left\{\frac{S-X}{\sigma\sqrt{T}}\right\} + \sigma\sqrt{T}\cdot N'\left\{\frac{X-S}{\sigma\sqrt{T}}\right\}, \tag{A.5}$$

where $N\{\cdot\}$ is the cumulative standard normal, and $N'\{\cdot\}$ the standard normal density function.

Note that as the time to expiration is increased the call price increases without bound.[38] This implication seems to violate the restriction, established in section 2.2, that the maximum value which the call price can assume is equal to the stock price. This result arises because, while Merton assumes the stock possesses limited liability, Bachelier implicitly does not.[39]

[38]The cumulative standard normal expressions in the first two terms of (A.5) go to $\frac{1}{2}$ as $T \to \infty$; therefore the first two terms go to $\frac{1}{2}[S-X]$. In the third term, the argument of the standard normal density function goes to zero, therefore that third term goes to $\lim_{t\to\infty} \sigma\sqrt{T}$ (0.3989) which is infinity.

[39]Bachelier assumes the mean future price is positive, and because he specifies a process without drift, equal to the current stock price. The process is such that if the total integral from minus to plus infinity as T is increased, the left half to minus infinity and the right, to plus infinity, in such a way as to keep the mean unchanged at S. This property, that the expected value of 'half' the distribution rises without bound as T is increased coupled with a zero discount rate, yields the anomalous result that the call price increases without bound as T is increased.

To summarize, the major objections to Bachelier's model are: (1) the assumption of Arithmetic Brownian Motion in the description of expected price movements implying both a positive probability of negative prices for the security and option prices greater than their respective security prices for large T; (2) the assumption that the mean expected price change is zero, suggesting both no time preference and risk neutrality; (3) the implicit assumption that the variance is finite, thereby ruling out other members of the stable-Paretian family except the normal.

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