Optimal delta hedging for options[☆]

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ABSTRACT

As has been pointed out by a number of researchers, the normally calculated delta does not minimize the variance of changes in the value of a trader's position. This is because there is a non-zero correlation between movements in the price of the underlying asset and movements in the asset's volatility. The minimum variance delta takes account of both price changes and the expected change in volatility conditional on a price change. This paper determines empirically a model for the minimum variance delta. We test the model using data on options on the S&P 500 and show that it is an improvement over stochastic volatility models, even when the latter are calibrated afresh each day for each option maturity. We also present results for options on the S&P 100, the Dow Jones, individual stocks, and commodity and interest-rate ETFs.

1. Introduction

The textbook approach to managing the risk in a portfolio of options involves specifying a valuation model and then calculating partial derivatives of the option prices with respect to the underlying stochastic variables. The most popular valuation models are those based on the assumptions made by Black and Scholes (1973) and Merton (1973). When hedge parameters are calculated from these models, the usual market practice is to set each option's volatility parameter equal to its implied volatility. This is sometimes referred to as using the "practitioner Black-Scholes model." The "practitioner Black-Scholes delta" for example is the partial derivative of the option price with respect to the underlying asset price with other variables, including the implied volatility, kept constant.

Delta is by far the most important hedge parameter and fortunately it is the one that can be most easily adjusted as it only requires a trade in the underlying asset. Ever since the birth of exchange-traded options markets in 1973, delta hedging has played a major role in the management of portfolios of options. Option traders adjust delta frequently, making it close to zero, by trading the underlying asset.

Even though the Black–Scholes–Merton model assumes volatility is constant, market participants usually calculate a "practitioner Black–Scholes vega" to measure and manage their volatility exposure. This vega is the partial derivative of the option price with respect to implied volatility with all other variables, including the asset price, kept constant.[1] This approach, although not based on an internally consistent model, has the advantage of simplicity. The price of an option at any given time is, to a good approximation, a deterministic function of the underlying asset price and the implied volatility.[2] A Taylor series expansion shows that the risks being taken can be assessed by monitoring the impact of changes in these two variables.

As is well known, there is a negative relationship between an equity price and its volatility. This was first shown by Black (1976) and Christie (1982) who used physical volatility estimates.

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[1] In a portfolio of options dependent on a particular asset, the options typically have different implied volatilities. The usual practice when vega is calculated is to calculate the portfolio vega as the sum of vegas of the individual options. This is equivalent to considering the impact of a parallel shift in the volatility surface.

[2] This is exactly true if we ignore uncertainties relating to interest rates and dividends.

Other authors have shown that it is true when implied volatility estimates are used. One explanation for the negative relation is leverage. As the equity price moves up (down), leverage decreases (increases) and as a result volatility decreases (increases). In an alternative hypothesis, known as the volatility feedback effect, the causality is the other way round. When there is an increase (decrease) in volatility, the required rate of return increases (decreases) causing the stock price to decline (increase). The two competing explanations have been explored by a number of authors including French et al. (1987), Campbell and Hentschel (1992), Bekaert and Wu (2000), Bollerslev et al. (2006), Hens and Steude (2009), and Hasanhodzic and Lo (2013). On balance, the empirical evidence appears to favor the volatility feedback effect.

A number of researchers have recognized that the negative relationship between an equity price and its volatility means that the practitioner Black-Scholes delta does not give the position in the underlying equity that minimizes the variance of the hedger's position. The minimum variance (MV) delta hedge takes account of the impact of both a change in the underlying equity price and the expected change in volatility conditional on the change in the underlying equity price. Given that delta hedging is relatively straightforward, it is important that traders get as much mileage as possible from it. Switching from the practitioner Black–Scholes delta to the minimum variance delta is therefore a desirable objective. Indeed it has two advantages. First, it lowers the variance of daily changes in the value of the hedged position. Second, it lowers the residual vega exposure because part of vega exposure is handled by the position that is taken in the underlying asset.

A number of stochastic volatility models have been suggested in the literature. These include Hull and White (1987, 1988), Heston (1993), and Hagan et al. (2002). A natural assumption might be that using a stochastic volatility model automatically improves delta. In fact, this is not the case if delta is calculated in the usual way, as the partial derivative of the option price with respect to the asset price. To calculate the MV delta, it is necessary to use the model to determine the expected change in the option price arising from both the change in the underlying asset and the associated expected change in its volatility.

A number of researchers have implemented stochastic volatility models and used the models' assumptions to convert the usual delta to an MV delta. They have found that this produces an improvement in delta hedging performance, particularly for out-of-the-money options. The researchers include Bakshi et al. (1997) who implemented three different stochastic volatility models using data on call options on the S&P 500 between June 1988 and May 1991[3]; Bakshi et al. (2000), who looked at short and long-term options on the S&P 500 between September 1993 and August 1995; Alexander and Nogueira (2007), who looked at call options on the S&P 500 during a six month period in 2004; Alexander et al. (2009), who consider the hedging performance of six different models using put and call options on the S&P 500 trading in 2007; and Poulsen et al. (2009) who looked at data on S&P 500 options, Eurostoxx index options, and options on the U.S. dollar euro exchange rate during the 2004 to 2008 period. Bartlett (2006) shows how a minimum variance hedge can be used in conjunction with the SABR stochastic volatility model proposed by Hagen et al (2002).

This paper is different from the research just mentioned in that it is not based on a stochastic volatility model. It is similar in spirit to papers such as Crépey (2004), Vähämaa (2004) and Alexander et al. (2012). These authors note that the minimum variance delta is the practitioner Black–Scholes delta plus the practitioner Black–Scholes vega times the partial derivative of the expected implied volatility with respect to the asset price. Improving delta therefore requires an assumption about the partial derivative of the expected implied volatility with respect to the asset price. Crépey (2004) and Vähämaa (2004) test setting the partial derivative equal to (or close to) the (negative) slope of the volatility smile, as suggested by the local volatility model.[4] Alexander et al. (2012) build on the research of Derman (1999) and test eight different models for the partial derivative, including a number of regime-switching models.

This paper extends previous research by determining empirically a model for the partial derivative of the expected implied volatility with respect to asset price. We show that, when the underlying asset is the S&P 500, this partial derivative is to a good approximation a quadratic function of the practitioner Black-Scholes delta of the option divided by the product of the asset price and the square root of the time to maturity. This leads to a simple model where the MV delta is calculated from the practitioner Black-Scholes delta, the practitioner Black–Scholes vega, the asset price, and the time to option maturity. We show that the hedging gain from approximating the MV delta in this way is better than that obtained using a stochastic volatility model or a local volatility model. The results have practical relevance to traders, many of whom still base their decision making on output from the practitioner Black–Scholes model. The hedging gain from using our approach for options on other indices was similar to that for options on the S&P 500. The approach also led to a hedging gain for options on individual stocks and ETFs, but this was not as great as for options on indices.

The structure of the rest of the paper is as follows. We first discuss the nature of the data that we use. Second, we develop the theory that allows us to parameterize the evolution of the implied volatilities of options. The theory is then implemented and tested out-of-sample using options on the S&P 500. The results are compared with those from a stochastic volatility and a local volatility model. Based on the results for the S&P 500 we then carry out tests for options on other indices and for options on individual stocks and ETFs.

2. Data

We used data from OptionMetrics. This is a convenient data source for our research. It provides daily prices for the underlying asset, closing bid and offer quotes for options, and hedge parameters based on the practitioner Black–Scholes model. We chose to consider options on the S&P 500, S&P 100, the Dow Jones Industrial Average of 30 stocks (DJIA), the individual stocks underlying the DJIA, and five ETFs. The assets underlying three of the ETFs are commodities, gold (GLD), silver (SLV) and oil (USO). The assets underlying the other two ETFs were the Barclays U.S. 20+ year Treasury Bond Index (TLT) and the Barclays U.S. 7–10 year Treasury Bond Index (IEF). The options on the S&P 500 and the DJIA are European. Both European and American options on the S&P 100 are included in our data set. Options on individual stocks and those on ETFs are American. The period covered by the data we used is January 2, 2004–August 31, 2015 except for the commodity ETFs where data was first available in 2008.[5]

Only option quotes for which the bid price, offer price, implied volatility, delta, gamma, vega, and theta were available were retained. The option data set was sorted to produce observations for the same option on two successive trading days. For every pair of

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[3] They also looked at puts on the S&P 500, but did not report the results as they were similar to calls.

[4] See for example Derman et al (1995) and Coleman et al (2001). The local volatility model was suggested by Derman and Kani (1994) and Dupire (1994).

[5] This is a much longer period than that used by other researchers except Alexander et al (2012).

observations the data was normalized so that the underlying price on the first of the two days was one. Options with remaining lives less than 14 days were removed from the data set. Call options for which the practitioner Black-Scholes delta was less than 0.05 or greater than 0.95, and put options for which the practitioner Black-Scholes delta was less than –0.95 or greater than –0.05 were removed from the data set. For options on individual stocks, in addition to the filters used for options on the indices, days on which stock splits occurred were removed.

After all the filtering there remain more than 1.3 million price quotations for both puts and calls on the S&P 500, about 0.5 million observations for the other indices and ETFs, and about 200,000 observations for options on each individual stock in the Dow Jones Industrial Index. The trading volume for puts on the S&P 500 is much greater than that for calls.[6] Puts and calls trade in approximately equal volumes for other indices. Calls trade more actively than puts for the individual stocks. Trading tends to be concentrated in close-to-the-money and out-of-the-money options. One notable feature is that the trading of close-to-the-money call options is particularly popular. The majority of trading is in options with maturities less than 91 days.

3. Background theory

In the Black-Scholes model the underlying asset price follows a diffusion process with constant volatility. Many alternatives to Black-Scholes have been developed in an attempt to explain the option prices that are observed in practice. These involve stochastic volatility, jumps in the asset price or the volatility, risk aversion, and so on. Departures from Black-Scholes tend to reduce the performance of delta hedging. For example, Sepp (2012) shows that this is so for a mixed-jump diffusion model and some of the papers referenced earlier show that this is so for stochastic volatility models. In this section we provide a theoretical result for determining the minimum variance delta from the practitioner Black-Scholes delta. The result involves the implied volatility and is exactly true in the limit for diffusion processes while being an approximation in the case of other models.

Define ΔS as a small change in an asset price and Δf as the corresponding change in the price of an option on the asset. The minimum variance delta, δ_{MV}, is the value that minimizes the variance of[7]

$$\Delta f - \delta_{MV}\Delta S \tag{1}$$

We show in Appendix A that it is approximately true that

$$\delta_{MV} = \frac{\partial f_{BS}}{\partial S} + \frac{\partial f_{BS}}{\partial \sigma_{imp}}\frac{\partial E(\sigma_{imp})}{\partial S} = \delta_{BS} + \nu_{BS}\frac{\partial E(\sigma_{imp})}{\partial S} \tag{2}$$

where f_{BS} is the Black-Scholes-Merton pricing function, σ_{imp} is the implied volatility, δ_{BS} is the practitioner Black-Scholes delta, ν_{BS} is the practitioner Black-Scholes vega, and $E(\sigma_{imp})$ is the expected value of the implied volatility as a function of S.

Other authors, in particular Alexander et al. (2012), have explored the effectiveness of various estimates $\partial E(\sigma_{imp})/\partial S$ in determining the minimum variance delta. In what follows we estimate this function empirically and then conduct out-of-sample tests of the effectiveness of the estimated function.

When presenting our results, we shall define the effectiveness of a hedge as the percentage reduction in the sum of the squared residuals resulting from the hedge. We denote the Gain from an

MV hedge as the percentage increase in the effectiveness of an MV hedge over the effectiveness of the practitioner Black–Scholes hedge. Thus:

$$\text{Gain} = 1 - \frac{SSE[\Delta f - \delta_{MV}\Delta S]}{SSE[\Delta f - \delta_{BS}\Delta S]} \tag{3}$$

where SSE denotes sum of squared errors.[8]

4. Analysis of S&P 500 options

In this section we examine the characteristics of the MV delta for options on the S&P 500 with the objective of determining the functional form of the MV delta. Once we have a candidate functional form, we will test it out of sample for both options on the S&P 500 and options on other assets.

We start with an implementation based on Eq. (1) applied to daily price changes:

$$\Delta f = \delta_{MV}\Delta S + \varepsilon \tag{4}$$

where ε is an error term. Because the mean of ΔS and Δf are both close to zero, minimizing the variance of ε in this equation, and other similar equations that we will test, is functionally equivalent to minimizing the sum of squared values. Several other variations on the model were tried such as using non-normalized data, replacing Δf with $\Delta f - \theta_{BS}\Delta t$, where θ_{BS} is the practitioner Black–Scholes theta[9] and Δt is one trading day, or including an intercept. None of the variations had a material effect on the results we present. The results that we report are for the model in (4).

We estimated Eq. (4) for options with different moneyness and time to maturity. Moneyness was measured by δ_{BS}. We created nine different moneyness buckets by rounding δ_{BS} to the nearest tenth and seven different option maturity buckets (14–30 days, 31–60 days, 61–91 days, 92–122 days, 123–182 days, 183–365 days, and more than 365 days). For each delta and each maturity bucket, the value of δ_{MV} was estimated. In all cases $\delta_{MV} - \delta_{BS}$ was negative. This result is consistent with the results of other researchers. It means that traders of S&P 500 index options should under-hedge call options and over-hedge put options relative to relative to the hedge suggested by the practitioner Black–Scholes model.[10]

The bucketed results show that $\delta_{MV} - \delta_{BS}$ is not heavily dependent on option maturity and is roughly quadratic in δ_{BS}. It is approximately true that[11]

$$\nu_{BS} = S\sqrt{T}G(\delta_{BS})$$

for some function G where T is the time to the option maturity. From this equation, Eq. (2), and the assumption of scale invariance[12] we obtain

$$\delta_{MV} = \delta_{BS} + \frac{\nu_{BS}}{S\sqrt{T}}\left(a + b\delta_{BS} + c\delta_{BS}^2\right) \tag{5}$$

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[6] The bid-offer spread for puts on the S&P 500 is smaller than that for calls except in the case of deep in-the-money options where the spreads are about the same.

[7] An early application of this type of hedging analysis to futures markets is Ederington (1979)

[8] Using standard deviations rather than SSEs would produce a similar measure but the Gain would be numerically smaller

[9] The practitioner Black-Scholes theta is the partial derivative with respect to the passage of time with the volatility set equal to the implied volatility) and time is measured in days. If the asset price and its implied volatility do not change, the option price can be expected to decline by about $\theta_{BS}\Delta t$ in one day.

[10] A call has a positive delta and the MV delta, δ_{MV}, is less positive than δ_{BS}; a put has a negative delta and δ_{MV} is more negative than δ_{BS}.

[11] For European options, $\nu_{BS} = S\sqrt{T}N'(d_1)e^{-qT}$ where $d_1 = [\ln(S/K) + (r - q + \sigma^2/2)T]/\sigma\sqrt{T}$, K is the strike price, T is the time to maturity, r is the risk-free rate, q is the dividend yield, and N is the cumulative normal distribution function. However, $\delta_{BS} = N(d_1)e^{-qT}$ so that $d_1 = N^{-1}(\delta_{BS}e^{qT})$. As a result, $\nu_{BS} = S\sqrt{T}N'(N^{-1}(\delta_{BS}e^{qT}))e^{-qT}$. If q is zero $\nu_{BS}/(S\sqrt{T})$ is dependent only on δ_{BS}. When q is small this is approximately true.

[12] A scale invariant model is one where the distribution of S_t/S_0 is independent of S_0. See for example Alexander and Nogueira (2007).
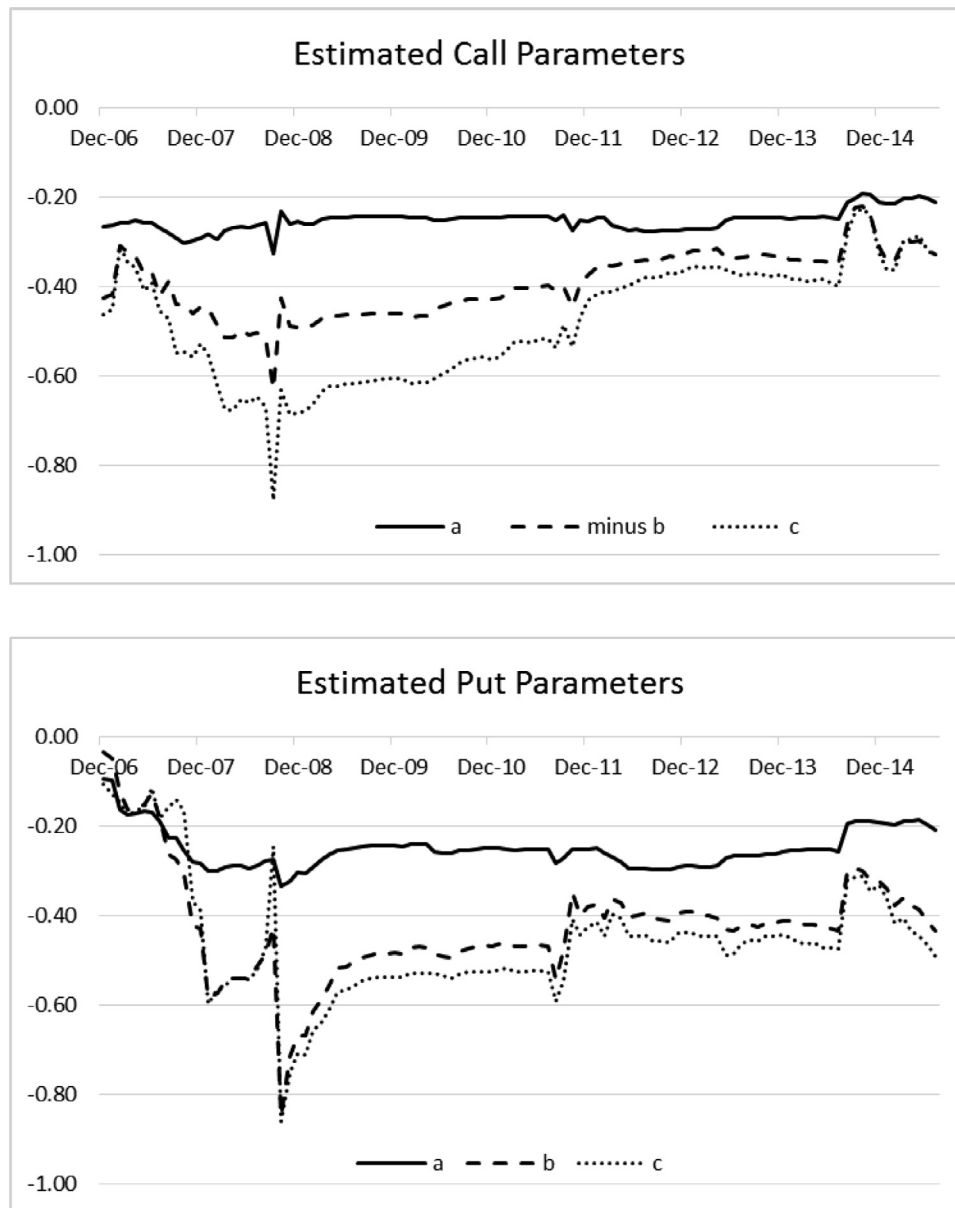
Fig. 1. The estimated parameters for the quadratic in Eq. (6) for puts and calls on the S&P 500 observed between 2004 and 2015. The estimations use overlapping 36-month periods. For call options the negative of the $\hat{b}$ parameter is plotted so that the same scale can be used for both charts.

where a, b, and c are constants. Applying (5) to Eq. (2) shows that

$$E\left(\Delta\sigma_{\text{imp}}\right) = \left(\frac{a + b\delta_{\text{BS}} + c\delta_{\text{BS}}^2}{\sqrt{T}}\right)\frac{\Delta S}{S}$$

In the balance of the paper we examine the effectiveness of the approximation in Eq. (5) for δ_{MV}.

5. Out of sample tests of S&P 500 options

To this point our work has been largely descriptive, motivated by a desire to produce a simple model of how the volatility surface for S&P 500 options evolves as a result of stock price changes. Our simple model is that for a particular moneyness and a particular stock price change, the expected size of the change in the implied volatility is inversely proportional to the square-root of the option life. For a particular option maturity and a particular percentage stock price change, the expected size of the change in the implied

volatility is a quadratic function of our measure of moneyness, δ_{BS}. The same model applies across the range of deltas considered.

We now consider how well our empirical hedge ratio model works in reducing uncertainty. We estimate the MV delta using historical data and then use that estimate to reduce the variance of the hedging error in the future. In carrying out this test we use a moving window where parameters are estimated over a 36-month period and then used to determine MV hedges during the following month. The first month for which MV hedges are estimated is January 2007 and the last is August 2015. We tested moving windows of length between 12- and 60-months but did not find that any one of these was materially better than the others.[13]

The only element of our simple model that is unknown is the quadratic function of moneyness in Eq. (5). We estimate the model

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[13] In all our reported results we consider one day changes in option prices and implied volatilities when estimating the MV hedge parameters. Slightly better results occur if the observation period is increased to several trading days.

parameters, a, b and c, using a regression model based on Eqs. (4) and (5).

$$\Delta f - \delta_{BS} \Delta S = \frac{v_{BS}}{\sqrt{T}} \frac{\Delta S}{S} \left(a + b \delta_{BS} + c \delta_{BS}^2 \right) + \varepsilon \qquad (6)$$

where Δf is the one-day change in the option price, ΔS is the change in the stock price, S, T is the remaining life of the option, and δ_{BS} and v_{BS} are the delta and vega calculate using the practitioner Black-Scholes model. This model is fitted to all options in each 36-month estimation period. The estimation is done separately for puts and calls. The estimated coefficients, $\hat{a}$, $\hat{b}$, and $\hat{c}$, are shown in Fig. 1. Usually, the parameters of the best fit quadratic model change slowly through time, but during the credit crisis of 2008 some extreme changes were observed.

The three coefficients estimated in a 36-month estimation period are used to determine the hedge error resulting when the estimated model is used to hedge each option on each day in the following test month. The hedging error based on this model, ε_{MV}, is

$$\varepsilon_{MV} = \Delta f - \delta_{BS} \Delta S - \frac{v_{BS}}{\sqrt{T}} \frac{\Delta S}{S} \left(\hat{a} + \hat{b} \delta_{BS} + \hat{c} \delta_{BS}^2 \right)$$

The hedging error based on standard Black-Scholes hedging, ε_{BS}, is

$$\varepsilon_{BS} = \Delta f - \delta_{BS} \Delta S$$

Once the hedging errors have been calculated for all 104 months the Gain (Eq. (3)) resulting from using our model to hedge is then calculated as

$$\text{Gain} = 1 - \frac{SSE(\varepsilon_{MV})}{SSE(\varepsilon_{BS})}$$

The Gain was calculated including the residuals for all options and then considering only the residuals from options in a particular delta bucket. This resulted in one overall Gain and 9 bucketed Gains for each test month.

When the residuals for all options are included, the average Gain is about 26% for calls and 23% for puts. The average Gain achieved for each delta bucket is shown in Table 1. This shows that for call options the Gain is largest for out-of-the-money options (a Gain of about 42% for the highest strike options) and smallest (about 17%) for in-the-money options. For put options the Gains are higher for low strike options (out-of-the-money) and lower for high strike (in-the-money) options. We confined our hedging effectiveness test to options with maturities greater than 13 days. This eliminates very short term options. Including the very short maturity options slightly worsens our results due to the large gammas of short-term options that are close to the money.

MV hedging works better for calls than puts and better for out-of-the-money options than in-the-money options. To understand why this is the case we directly estimate the relationship between implied volatility changes and stock price changes by estimating α in

$$\Delta \sigma_{imp} = \frac{\alpha}{\sqrt{T}} \frac{\Delta S}{S} + \varepsilon \qquad (7)$$

The estimation is done separately for puts and calls for every delta bucket using all options observed between 2007 and 2015. The R^2 for each delta bucket is shown in Fig. 2.

The R^2 from the estimation in Eq. (7) for calls with δ_{BS} in the 0.1 bucket is about 0.60. That is, the change in the implied volatility due to changes in the stock price explains about 60% of the total variation in implied volatilities. As δ_{BS} increases the average R^2 declines due to increased idiosyncratic noise in the implied volatility data. This explains the effectiveness of MV hedging for out-of-the money calls and the declining effectiveness of MV hedging for in-the-money calls.
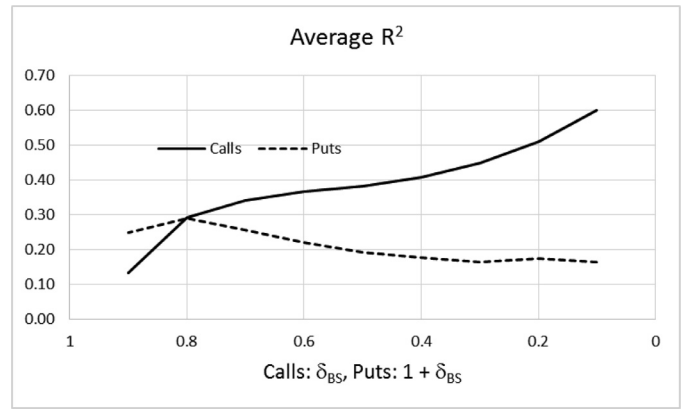

Fig. 2. The average R^2, the fraction of total variance of changes in implied volatility explained by changes in the index, for all options on the S&P 500 observed between 2004 and 2015.
The horizontal axis is ordered so that high strike prices are on the right hand end and low strike prices are on the left.

The results for put options are somewhat different. The fraction of total variance in implied volatilities explained by changes in the stock price is much smaller than that observed for call options. There is much more idiosyncratic variation in the implied volatilities of put options. As a result, MV hedging of puts is less effective than for calls. We note that the different hedging performance for puts and calls cannot be explained by the model driving prices. For any model (including jump models), put call parity shows that the hedging error for a put should in theory be the same as that for the corresponding call. The observed differences led us to carry out a test of put-call parity.

5.1. A put-call parity test

We used our quadratic model to test how well put-call parity has held over the period covered by our data. We first used the put-call parity relationship to turn all call prices in our data set into synthetic put prices. We estimated $\hat{a}$, $\hat{b}$, and $\hat{c}$ in our quadratic form using the actual put prices, and $\hat{\hat{a}}$, $\hat{\hat{b}}$, and $\hat{\hat{c}}$ using the synthetic put prices for each of our three-year calibration periods. We then calculated the root mean square error of the difference between the estimated put parameters and the put parameters calculated from the synthetic put data under the assumption that put-call parity holds:

$$RMSE = \sqrt{\frac{\left(\hat{a} - \hat{\hat{a}} \right)^2 + \left(\hat{b} - \hat{\hat{b}} \right)^2 + \left(\hat{c} - \hat{\hat{c}} \right)^2}{3}} \qquad (8)$$

The results are shown in Fig. 3. These results suggest that put-call parity was seriously violated before December 2008 but that thereafter it was approximately true. (The first observation of the post-December 2008 period is December 2011.)[14]

6. Comparison with alternative models

In the previous section we tested an empirical model to determine the minimum variance delta hedge. The results show that a reasonable improvement in hedging accuracy can be achieved in this way. However, as mentioned earlier, other researchers have calculated minimum variance deltas from stochastic volatility models and local volatility models. In this section we compare the

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[14] Some violations of put-call parity are probably created by our use of mid-market prices rather than transaction prices.

Table 1

The out-of-sample average hedging Gain (Eq. (3)) January 2007 to August 2015 for options on the S&P 500 from MV delta hedging. The SABR and Local Vol models are calibrated daily for each option maturity and applied to determine the hedge for the next day. The Empirical model parameters, *a*, *b* and *c* in Eq. (5) are estimated using all options traded in a 36 month window and then applied to determine the hedge on every day in the next month. Results are reported for buckets based on rounding δ_{BS} to the nearest tenth. The upper panel shows the hedging GAIN values. The lower panel shows the Newey-West adjusted *t*-statistic for the difference between the empirical model GAIN and the alternative model GAINs.

GAIN

	Call options				Put options		
δ_{BS}	empirical model	SABR model	Local vol	δ_{BS}	EMPIRICAL model	SABR model	Local vol
0.1	42.1%	39.4%	42.6%	−0.9	15.1%	11.2%	−7.4%
0.2	35.8%	33.4%	36.2%	−0.8	18.7%	19.6%	6.8%
0.3	31.1%	29.4%	30.3%	−0.7	20.3%	17.7%	9.1%
0.4	28.5%	26.3%	26.7%	−0.6	20.4%	16.7%	9.2%
0.5	27.1%	24.9%	25.5%	−0.5	22.1%	16.7%	10.8%
0.6	25.7%	25.2%	25.2%	−0.4	23.8%	17.7%	12.0%
0.7	25.4%	24.7%	25.8%	−0.3	27.1%	21.7%	16.8%
0.8	24.1%	23.5%	25.4%	−0.2	29.6%	25.8%	20.6%
0.9	16.6%	17.0%	16.9%	−0.1	27.5%	26.9%	17.7%
All	25.7%	24.6%	25.5%	All	22.5%	19.0%	10.2%

t-Statistic

	Call options			Put options	
δ_{BS}	Empirical − SABR	Empirical − local vol	δ_{BS}	Empirical − SABR	Empirical − local vol
0.1	3.94	−0.06	−0.9	9.71	27.11
0.2	3.75	0.79	−0.8	−2.34	12.34
0.3	3.22	4.31	−0.7	1.39	10.42
0.4	4.22	6.55	−0.6	3.29	10.25
0.5	4.74	6.65	−0.5	6.09	11.53
0.6	2.95	5.16	−0.4	7.22	13.89
0.7	3.24	2.82	−0.3	7.17	14.18
0.8	2.70	−0.45	−0.2	5.46	13.97
0.9	−0.01	0.44	−0.1	0.66	17.63
All	8.74	8.13	All	12.75	41.99

performance of our empirical model with these two categories of models.

6.1. Stochastic volatility model

The stochastic volatility model we use is a particular version of the SABR model discussed by Hagen et al. (2002):[15]

$$dF = \sigma F dz$$
$$d\sigma = \xi \sigma \, dw \qquad (9)$$

where F is the futures stock price when the numeraire is the zero coupon bond with maturity T. The dz and dw are Wiener processes with constant correlation ρ and ξ is a constant volatility of volatility parameter. In this model the expected change in the volatility given a particular change in the futures price is

$$E(d\sigma | dF) = \xi \rho \frac{dF}{F}$$

Hagan et al. (2002) and Rebonato et al. (2011) show that under the model defined by Eq. (9), a good analytic approximation to the implied volatility for a European option can be produced.

Define $f_{BS}(F, \sigma)$ as the value of an option given by the Black–Scholes–Merton assumptions when the futures stock price is F and the volatility is σ. An option, an estimate of the minimum variance delta given by the model is then

$$\delta_{SV} = \frac{E(\Delta f | \Delta F)}{\Delta F}$$

$$= \frac{f_{BS}\big(F_0 + \Delta F, \sigma_{\text{imp}}(F_0 + \Delta F, \sigma_0 + \xi \rho \Delta F / F)\big) - f_{BS}\big(F_0, \sigma_{\text{imp}}(F_0, \sigma_0)\big)}{\Delta F} \qquad (10)$$

The procedure for implementing this model is as follows. On each trading day the implied volatilities of all options with a particular maturity are determined.[16] The parameters for the stochastic volatility model (σ_0, ξ, and ρ) that are to be used for that particular option maturity are chosen to minimize the sum of squared differences between the market implied volatilities and the model implied volatilities.[17] Once the model parameters are determined for the particular maturity, the minimum variance delta is then determined for each option with that maturity using Eq. (10). This procedure is repeated for every option maturity observed on each trading day.

To align the tests of the stochastic volatility model with the tests of our empirical model we calibrated the model for every option maturity every day from the start of 2007 to August 2015. Puts and calls were considered separately. Since there are about 13 different maturities observed on each trading the SABR model requires about 78 model parameters to be estimated on each trading day. In total, about 29,000 optimizations are carried out and about 87,000 model parameters are estimated. The estimated parameters are reasonable and provide a good fit to the observed implied volatilities. The average initial volatility, σ_0, is about 19% which is approximately equal to the average at-the-money option implied volatility, the average volatility of the volatility, ξ, is about 1.2, and

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[15] As pointed out by Poulsen et al (2009), similar results are obtained for different stochastic volatility models. In the general SABR model $dF = \sigma F^\beta dz$. Setting $\beta = 1$ ensures scale invariance which is a reasonable property for equities and equity indices. The model we choose is equivalent to a version of the model in Hull and White (1987).

[16] In practice the SABR model is used as a model for the behavior of all options with a particular maturity. When calibrated to all options of all maturities we find that it provides poor results. This is not surprising as the model is not designed to fit the term structure of implied volatilities.

[17] For a particular maturity to be included in our sample on any day we require that there be options with more than 10 different strike prices and that the root mean square error in fitting the implied volatilities be smaller than 1%.
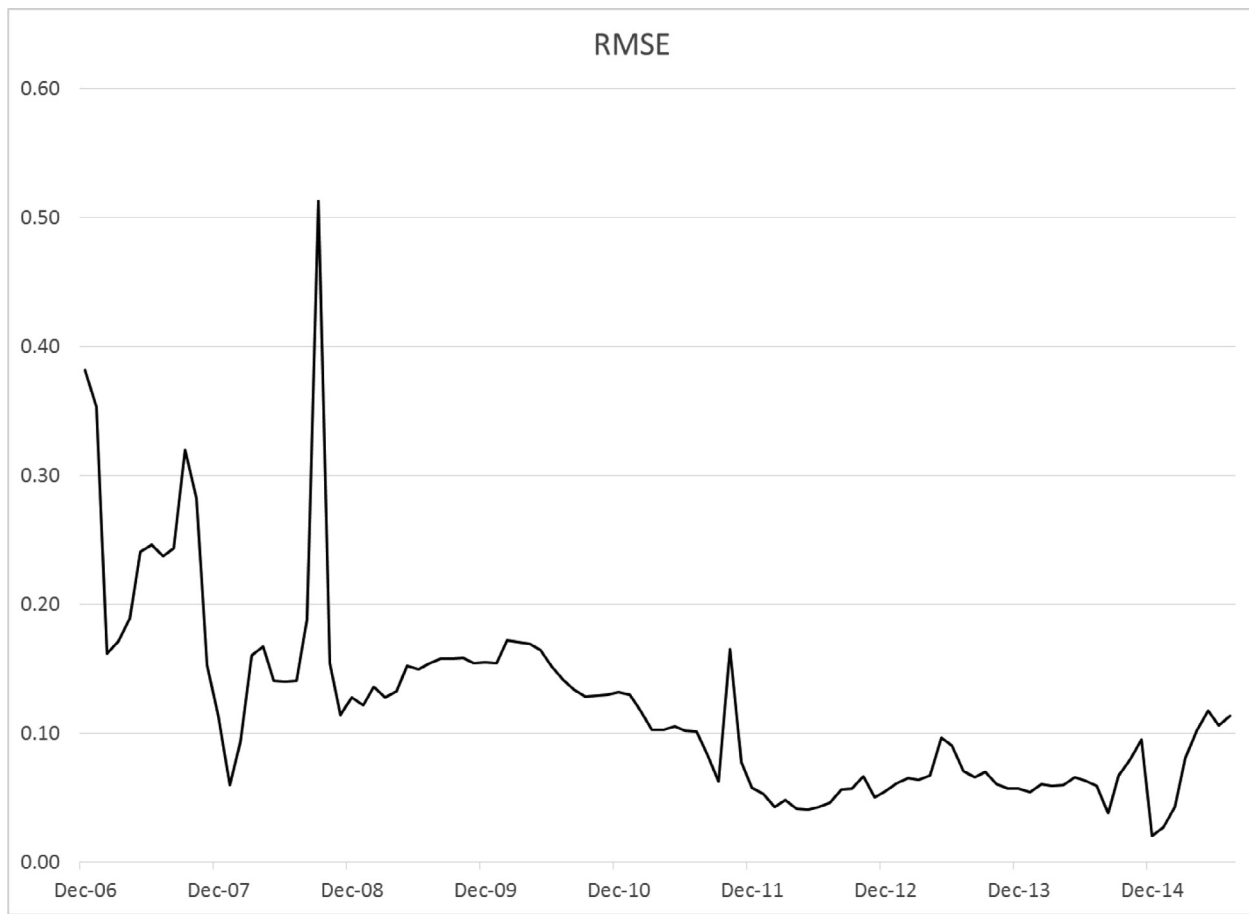
Fig. 3. The root mean squared difference between estimated parameters for the quadratic in Eq. (6) for put options, and the parameter values for put option prices that are calculated from call option prices under the assumption that put-call parity holds (Eq. (8)). The estimation uses overlapping 36-month periods based on options on the S&P 500.

the average correlation is about −0.85 while the average root mean square error in fitting the implied volatility is about 0.32%.

The upper panel of Table 1 compares the Gain from the SABR model with the Gain from the empirical model developed in this paper. The results are aggregated by practitioner Black-Scholes delta rounded to the nearest tenth. The table shows the stochastic volatility model is worse at reducing hedging variance than our empirical model. The results are particularly compelling because the SABR model utilizes many more parameters than our model. In our empirical approach we estimate only the three coefficients of the quadratic function (Eq. (6)) and update the estimates once a month. There are a total of 104 calibrations and a total of 312 parameters are estimated. This can be contrasted with the SABR model where nearly 100,000 parameters are estimated. Overall the SABR model performs less well than our empirical model. Its performance is better than the empirical model only for very-deep-in-the-model options.

In Appendix B we develop the procedure for calculating the t-statistic used to determine the statistical significance of the difference between the Gains for two different hedging procedures. Since the sample size is always greater than 100,000 this t-statistic can be considered to be a z-statistic. The lower panel of Table 1 shows the t-statistic for the difference between the Gain arising from the quadratic hedging model and the Gain from hedging with the SABR model. Considering all call options together, t is 8.24 while for all puts it is 11.52. For individual deltas it is greater than 2.5 in all cases except for deep in-the-money call and put options. For calls when $\delta = 0.9$ the difference is not statistically dif-

ferent from zero and for puts when $\delta = -0.8$ the SABR model outperforms the empirical model. However, for options which trade actively, the empirical model is clearly better than SABR.

6.2. Local volatility model

The slope of the volatility smile plays a key role determining the partial derivative of the expected implied volatility with respect to the asset price for the local volatility model. Under the local volatility model, for at-the-money options the rate of change in the implied volatility with respect to changes in the underlying price is equal to the slope of the volatility smile. This is exactly true for futures options and approximately true if the difference between the interest rate and the dividend yield are small. This has been discussed and proved by many authors (for example, Derman et al. (1995) and Coleman et al. (2001)). In this case the MV delta, Eq. (2) becomes

$$\delta_{MV} = \delta_{BS} + \nu_{BS}\frac{\partial \sigma_{imp}}{\partial K}$$

We now apply this result to the S&P 500 options we are considering. We assume this result is approximately true for options which are not at the money. This is equivalent to the assumption that (a) the volatility smile is linear and (b) the volatility smile exhibits parallel shifts. These two assumptions are approximately, but not exactly, true.

We find that a quadratic gives an excellent fit to the implied volatility smile for a particular maturity. We therefore determined the slope of smile model for each maturity on each day by fitting a

Table 2

The average out-of-sample hedging Gain (Eq. (3)) from MV delta hedging when the model parameters, a, b and c in Eq. (6) are estimated using options with all strikes and maturities observed in a 36 month window and then applied to determine the hedge in the next month. Results are reported for each delta bucket for European (XEO) and American (OEX) options on the S&P 100 and for European options on the Dow Jones Industrial Index (DJX).

|	Call options				Put options		
δ_{BS}	XEO	OEX	DJX	δ_{BS}	XEO	OEX	DJX
0.1	36.2%	35.1%	23.9%	−0.9	10.7%	32.0%	−1.3%
0.2	32.8%	26.4%	28.7%	−0.8	15.7%	28.4%	−0.7%
0.3	27.7%	22.2%	31.6%	−0.7	20.1%	27.4%	1.7%
0.4	26.0%	21.4%	29.5%	−0.6	19.4%	25.3%	5.2%
0.5	23.9%	18.5%	29.2%	−0.5	20.9%	23.8%	7.8%
0.6	24.0%	17.0%	28.3%	−0.4	20.9%	23.2%	10.9%
0.7	22.5%	17.7%	28.9%	−0.3	22.9%	24.8%	15.0%
0.8	21.2%	15.4%	24.7%	−0.2	24.9%	26.2%	17.0%
0.9	16.1%	8.2%	15.7%	−0.1	24.8%	25.9%	16.0%
All	23.0%	16.7%	26.5%	All	19.7%	27.1%	5.5%

Fig. 4. The average R^2, the fraction of total variance of changes in implied volatility explained by changes in the stock price. The results shown are averaged across the stocks underlying the DJIA.

The horizontal axis is ordered so that high strike prices are on the right hand end and low strike prices are on the left.

quadratic function to the smile and using it to determine the slope of the smile for each option. The results are shown in upper panel of Table 1 [18] and the t-statistic for the difference between the Gain arising from the quadratic hedging model and the Gain from hedging with the local volatility model are shown in the lower panel.

The performance of the local volatility model is not significantly different from that of the empirical model for deep in- and out-of-the-money calls. In all other cases the local volatility hedging is much worse than the empirical model hedging. This is particularly true for put options.

7. Results for other stock indices

We now return to a consideration of the empirical model and test how well it works for other stock indices. Specifically, we consider European (ticker XEO) and American (ticker OEX) options on the S&P 100, and European options on the Dow Jones Industrial Index (ticker DJX). We implement out-of-sample tests similar to those done on the S&P 500. The two contracts on the S&P 100 are the same except for exercise terms. They therefore allow us to explore the degree to which hedging differs for American options.

The out of sample test was based on estimating the three parameters of the quadratic function in Eq. (6) using options of all strikes and maturities. The model parameters were estimated using a 36-month estimation period and the three estimated parameters were then used to delta hedge for a one-month testing period. The Gain (Eq. (3)) resulting from using our model to hedge in the test periods is then calculated. The Gain achieved for puts and calls in each delta bucket is shown in Table 2.

The results for call options for all indices are essentially the same as those found for options on the S&P 500. It is tempting to think that the results for the American style (OEX) call options are the same as those for the European style options because American style call options are almost never exercised early and hence are effectively European. However, for more than 80% of the sample tested the S&P 100 dividend yield is more than 1.5% higher than the interest rate.[19] In these circumstances the probability of early exercise is high. As a result, it appears that the American feature of the option does not affect the hedging effectiveness of our rule of thumb.
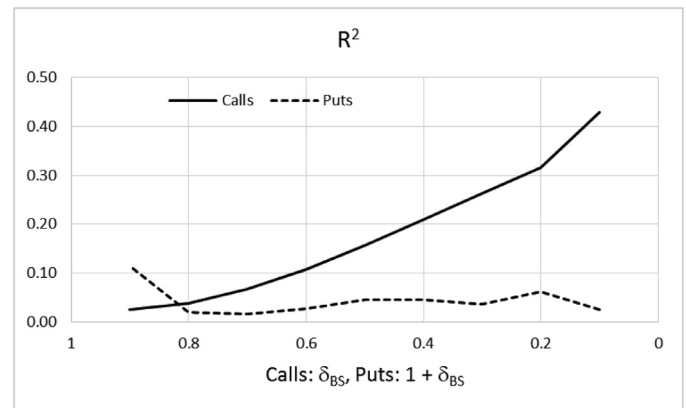
The results for put options are a bit more complicated. The results for XEO options were similar to those for options on the S&P 500 and the results for DJX options are similar but weaker. The weaker DJX results may be caused by the fact that there are only 30 stocks in the index which means that there will be more idiosyncratic variation in the implied volatilities.

Our results for in-the-money American (OEX) options are different from our results for all other assets in that the Gain for put options is large for both in- and out-of-the-money options. Overall, the conclusion that can be drawn from Table 2 is that our rule of thumb for hedging works as well for American options as for European options.

8. Results for single stocks and ETFs

We repeated the out-of-sample hedging tests based on the quadratic model in Eq. (6) for each of the thirty individual stocks underlying the DJX and each of the five ETFs. The average hedging variance reduction, averaging across the 30 stocks, the 3 commodities and the 2 interest rate products, found in these tests is reported in Table 3.

The average hedging gain for call options on single stocks are similar to but rather smaller than those for options on the Dow Jones Industrial Average. For put options the results are very poor. MV hedging contributes nothing or has a negative effect for puts. To understand why this is the case we carried out the regression in Eq. (7) for puts and calls for every delta bucket for each of the 30 stocks. The average R^2 across the thirty stocks is shown in Fig. 4.

The R^2 exhibits the same pattern observed in Fig. 2 for options on the S&P 500 but is somewhat smaller than that for the index options indicating that the idiosyncratic noise is larger for individual stocks. The increased idiosyncratic noise reduces the MV hedging effectiveness by inserting a wedge between parameters estimated in one period and the parameters that would produce the most effective MV hedge in the following period. The results for puts are quite different. The fraction of the variance of changes in the implied volatility explained by stock price changes, R^2, is essentially zero. As a result we can expect no improvement from MV hedging which is what we see.

The results for the ETFs are divided into results for options on commodities (gold, silver, and oil) and options on interest-rate products (20+ year Treasury Bonds and 7–10 year Treasury bonds). The results for options on commodities are similar to those for individual stocks while the results for interest-rate products are

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[18] We experimented with other implementations of the slope-of-smile model but did not obtain better results.

[19] The relevant interest rates were almost invariably at least 1.5% lower than the dividend yield between January 2009 and August 2015.

Table 3
The average out-of-sample hedging Gain (Eq. (3)) observed across the 30 stocks underlying the DJX, the three commodities (gold, silver and oil), and two interest-rate products (20+ year bonds and 7–10 year bonds). The model parameters, *a*, *b* and *c* in Eq. (6) are estimated using options with all strikes and maturities observed in a 36 month window and then applied to determine the hedge in the next month. Gain results are reported for each delta bucket based on rounding δ_{BS} to the nearest tenth.

	Call options				Put options		
δ_{BS}	stocks	Commodities	Int. rates	δ_{BS}	stocks	Commodities	Int. rates
0.1	32.2%	28.9%	5.2%	−0.9	0.9%	3.3%	0.0%
0.2	26.8%	20.8%	3.6%	−0.8	1.7%	5.1%	−0.6%
0.3	23.2%	15.2%	2.1%	−0.7	3.7%	5.1%	−1.1%
0.4	19.1%	12.5%	2.0%	−0.6	5.7%	5.3%	−1.1%
0.5	15.0%	9.8%	1.3%	−0.5	5.2%	4.0%	−0.5%
0.6	11.4%	7.3%	1.5%	−0.4	3.5%	1.9%	1.6%
0.7	8.3%	4.2%	1.6%	−0.3	1.8%	−0.9%	3.1%
0.8	5.4%	1.6%	0.2%	−0.2	0.7%	−3.0%	5.8%
0.9	2.5%	0.6%	−1.2%	−0.1	2.6%	−4.1%	6.2%
All	10.3%	7.7%	1.4%	All	2.5%	2.5%	0.2%

much weaker. As in most of the stock index results, MV hedging provides a much bigger Gain for call options on commodities than put options and the gain is greatest for out-of-the-money options. The negative correlation between price and implied volatility for commodities and interest rate products cannot be explained by a leverage and therefore lends support for the volatility feedback hypothesis.

9. Conclusions

Delta is by far the most important Greek letter. It plays a key role in the management of portfolios of options. Option traders take steps to ensure that they are close to delta neutral at least once a day and derivatives dealers usually specify delta limits for their traders. This paper has investigated empirically the difference between the practitioner Black–Scholes delta and the minimum variance delta. The negative relation between price and volatility for equities means that the minimum variance delta is always less than the practitioner Black–Scholes delta. Traders should underhedge equity call options and over-hedge equity put options relative to the practitioner Black–Scholes delta.

The main contribution of this paper is to show that a good estimate of the minimum variance delta can be obtained from the practitioner Black–Scholes delta and an empirical estimate of the historical relationship between implied volatilities and asset prices. We show that the expected movement in implied volatility for an option on a stock index can be approximated as a quadratic function in the option's practitioner Black–Scholes delta divided by the square root of time. This leads to a formula for converting the practitioner Black–Scholes delta to the minimum variance delta. When the formula is tested out of sample, we obtain good results for both European and American call options on stock indices. For options on the S&P 500 we find that our model gives better results that either a stochastic volatility model or a model based on the slope of the smile.

Call options on individual stocks and ETFs exhibit the same general behavior as call options on stock indices, but the effectiveness of MV hedging is greatly reduced because there is more noise in the relationship between volatility changes and price changes. For nearly all the assets we considered, the results for put options are much worse than those for call options. In the case of put options on individual stocks and ETFs, the results are particularly disappointing in that virtually none of the variation in changes in implied volatility is explained by changes in stock prices. The relatively poor performance of MV hedging for put options is a puzzle because (a) in the case of the European options considered put-call parity means that puts and calls can be regarded as substitutes for

each other and (b) that in the case of American options puts are less likely to be exercised early than call options for most of our sample period. It appears that the reason for the discrepancy between calls and puts is a result of a very high level of idiosyncratic noise in the prices of put options.

The most striking result is the ubiquity of the negative relation between asset price and implied volatilities for call option prices. When asset prices rise, implied volatilities decline resulting in an MV delta that is less than the practitioner Black-Scholes delta. For options on equities and equity indices this might be explained by a leverage argument. As equity prices rise the firm becomes less levered and equity volatility declines. However, this argument does not seem to apply to commodity or bond prices. For these assets it seems likely that we have to rely on the volatility feedback effect in which an increase in volatility raises the required rate of return resulting in a stock price decline.

Appendix A

The implied volatility is defined as the volatility which when inserted into the Black-Scholes option pricing function results in a model price that equals the market price. Suppose that we observe the price of an option price, *f*, when the stock price is *S*. The implied volatility is implicitly defined by[20]

$$f = f_{BS}(S, \sigma_{imp}) \tag{A1}$$

where f_{BS} is the Black-Scholes-Merton pricing function and σ_{imp} is the implied volatility. The Black–Scholes-Merton pricing function is continuous and continuously differentiable.

First let us consider a two-factor world in which the stock price and the volatility of the stock price obey diffusion processes. In this world the option price also changes continuously which allows us to expand Eq. (A1) in a Taylor series

$$\Delta f = \frac{\partial f}{\partial S} \Delta S + \frac{\partial f}{\partial \sigma_{imp}} \Delta \sigma_{imp} + O(\Delta S^2)$$

or equivalently

$$\Delta f = \delta_{BS} \Delta S + v_{BS} \Delta \sigma_{imp} + \varepsilon \tag{A2}$$

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[20] Note that σ_{imp}, is a "catch-all" parameter capturing the difference between the Black–Scholes pricing function and how option prices are determined in the real world. It leads to an exact pricing function that is always correct in the real-world. All the changes we consider in *f*, *S*, and σ_{imp} are changes in the real world. The risk-neutral world, although used to derive any particular option pricing model, is not relevant to the derivation of Eq (2) or our empirical work in Section 4.

where δ_{BS} and v_{BS} are the delta and vega as defined by the practitioner Black–Scholes model and ε denotes the residual higher order terms in the series. Subtracting $\delta_{MV}\Delta S$ from both sides we have

$$\Delta f - \delta_{MV}\Delta S = (\delta_{BS} - \delta_{MV})\Delta S + v_{BS}\Delta\sigma_{imp} + \varepsilon \qquad (A3)$$

Conditioning on ΔS and taking expectations we obtain

$$\delta_{MV} = \delta_{BS} + v_{BS}\frac{E\left(\Delta\sigma_{imp}\right)}{\Delta S} + \frac{E(\varepsilon)}{\Delta S}$$

In the case of diffusion processes as ΔS approaches zero the last term is infinitesimal so

$$\delta_{MV} = \delta_{BS} + v_{BS}\frac{E\left(\Delta\sigma_{imp}\right)}{\Delta S} \qquad (A4)$$

which leads to Eq. (2).

Now consider the more general case in which there are many factors driving asset prices and the processes may not be diffusion processes. In this case the option price may change discontinuously and the Taylor series expansion omits the effects of other state variables. As a result, the residual term in Eq. (A2), ε, now includes the effects of the omitted state variables and, if the option price changes discontinuously, may be large. In this case Eq. (2) is only approximately true since it omits the term $E(\varepsilon)/\Delta S$.

Appendix B

This appendix describes the calculation of the t-statistic used to determine the statistical significance of the difference between the Gain values for two different hedging methods. Consider a set of n observations of one-day changes in option prices. These price changes can be hedged using the practitioner Black–Scholes delta, a minimum variance delta based on our quadratic function, or a minimum variance delta based on the SABR model. Let x_i be the squared residual from hedging the ith one-day change in option price using the quadratic function, y_i be the squared residual from hedging with the SABR model, and z_i be the squared residual from Black–Scholes hedging.

The Gain from quadratic hedging is

$$\text{Gain}_{\text{Quad}} = 1 - \sum_{i=1}^{n} x_i / \sum_{i=1}^{n} z_i$$

The Gain from SABR-hedging is

$$\text{Gain}_{\text{SABR}} = 1 - \sum_{i=1}^{n} y_i / \sum_{i=1}^{n} z_i$$

Defining $\sum_{i=1}^{n} z_i = Z$, the difference in the Gains is

$$\text{Diff} = \frac{1}{Z}\sum_{i=1}^{n}(x_i - y_i)$$

Suppose the individual difference observations, $x_i - y_i$, are draws from a distribution with mean μ and standard deviation σ. These parameters can be estimated from the observed values of the differences:

$$\hat{\mu} = \sum_{i=1}^{n}(x_i - y_i)/n$$

$$\hat{\sigma}^2 = \left[\sum_{i=1}^{n}(x_i - y_i - \hat{\mu})^2\right]\bigg/(n-1)$$

If the individual observations are independent then the estimated mean and standard deviation of Diff are $n\hat{\mu}/Z$ and $\sqrt{n}\hat{\sigma}/Z$, and the t-statistic to test whether Diff is significantly different

from zero is $\sqrt{n}\hat{\mu}/\hat{\sigma}$. If the individual observations are not independent but exhibit serial correlation, a Newey-West adjustment factor must be applied to the estimated variance. Given the way in which the data is aggregated, the procedure used to determine the adjustment factor warrants explanation.

There are approximately 54,000 unique put and call options in the S&P 500 data set. Each of these options has a unique expiry date and strike price and may fall into different aggregation groups at different times. For each of these unique options, the time series of the Diff statistic was calculated and the Newey–West adjustment factors for lags between 1 and 50 were calculated. The average adjustment factor, averaging across all the option series, was then calculated for each lag. The magnitude of the average adjustment factor plateaued at about 30 lags and so we used 30 lags to determine the adjustment factor used for each option series. (Using 20 or 40 lags did not materially change the results.) The estimate of the adjusted sample variance is

$$\hat{\sigma}^2 = \left[\sum_{i=1}^{n} w_i\left(x_i - y_i - \hat{\mu}\right)^2\right]\bigg/(n-1)$$

where w_i is the Newey–West adjustment factor for the option series on which the ith observation is based.

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