Shane Moore
April 27, 2014

Deriving and Solving the Black-Scholes Equation

Introduction

The Black-Scholes equation, named after Fischer Black and Myron Scholes, is a partial differential equation, which estimates the value of a European call option. In the European financial market, a call option gives the owner the right to purchase a share of a specified stock (or bond, commodity, etc.) at the listed strike price on a given expiration date. Black and Scholes first published their model in 1973, and in 1997 they received the Nobel Prize in Economics for their work. This paper derives the Black-Scholes equation by constructing a replicating portfolio and then solves the equation by reducing it to the diffusion equation. The Black-Scholes partial differential equation is shown below:

$$\boxed{V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + rSV_s - rV = 0 \qquad (1)}$$

Derivation

To begin the derivation of the Black-Scholes equation we define a function, which describes the stochastic characteristic of Brownian motion – the random movement of a particle colliding with microscopic moving particles:

$$dx = a(x,t)dt + b(x,t)dz \qquad (2)$$

Let *f(x,t)* be a twice differentiable function of *x* and *t*. The Taylor expansion of *df* replaced with the stochastic differential equation (2) is as follows:

$$df(x,t) = f_t dt + f_x dx + \frac{1}{2}f_{tt}(dt)^2 + \frac{1}{2}f_{xx}(dx)^2 + f_{xt}dxdt$$

$$= f_t dt + f_x(a(x,t))dt + b(x,t)dz + \frac{1}{2}f_{tt}(dt)^2 + \frac{1}{2}f_{xx}(a(x,t)dt + b(x,t)dz)^2 +$$
$$f_{xt}(a(x,t)dt + b(x,t)dz)dt$$

Since Brownian motion follows the diffusion law ($u_{zz} = u_t$), $(dz)^2$ can be replaced with (dt). Additionally, the $(dt)^2$ and ($dtdz$) terms can be ignored.

$$= f_t dt + a(x,t)f_x dt + b(x,t)f_x dz + \frac{1}{2}b^2(x,t)f_{xx}dt$$

$$df(x,t) = (f_t + a(x,t)f_x + \frac{1}{2}b^2(x,t)f_{xx})dt + b(x,t)f_x dz \qquad (3)$$

The equation (2) above is known as Ito's Lemma, which models the evolution of an option's underlying security. To continue with the derivation of the Black-Scholes equation, the following assumptions need to be made:

Assumptions:

- The stock price (*S(t)*) can be modeled by a geometric Brownian motion (μ and σ are constants):

$$dS = \mu S dt + \sigma S dz \tag{4}$$

- There is a risk-free bond (*B(t)*) that evolves with a risk-free interest rate (*r*):

$$dB = rBdt \tag{5}$$

- There are no transaction costs, taxes, or dividends during the life of the option.
- There are no risk-free arbitrage opportunities.
- The evolution of the value of the portfolio can be modeled by Ito's Lemma (3).

The Black-Scholes equation is derived by replicating a portfolio that consists of stocks and bonds. Consider a self-financing (no money is added or withdrawn) portfolio (*V*) that consists of x shares of stock and y units of the bond. The instantaneous gain in the value of the portfolio due to changes in the security prices given by (4) and (5) is as follows:

$$\begin{aligned} V &= xS + yB \\ dV &= xdS + ydB \\ &= x(\mu S dt + \sigma S dz) + y(rRdt) \\ dV &= (x\mu S + yrB)dt + x\sigma S dz \end{aligned} \tag{6}$$

Set equation (6) equal Ito's Lemma (3) so that the portfolio evolves according to a geometric Brownian motion. The equations and its corresponding coefficients should be equal; otherwise there would be an opportunity for arbitrage. Let the coefficients from Ito's Lemma be such that ($a = \mu S$) and ($b = \sigma S$), and let *f* be *V(s,t)*:

$$(x\mu S + yrB)dt + x\sigma S dz = (V_t + \mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss})dt + \sigma S V_s dz$$

$$x\sigma S = \sigma S V_s$$

$$x = V_s$$

$$x\mu S + yrB = V_t + \mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss}$$

$$V_s \mu S + yrB = V_t + \mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss}$$

$$yrB = V_t + \frac{1}{2}\sigma^2 S^2 V_{ss}$$

$$yr(\frac{V - xS}{y}) = V_t + \frac{1}{2}\sigma^2 S^2 V_{ss}$$

$$rV - rSV_s = V_t + \frac{1}{2}\sigma^2 S^2 V_{ss}$$

Rearranging the terms in this equation produces the Black-Scholes partial differential equation (1).

Reduction to Diffusion Equation

The boundary value problem for the Black-Scholes equation is:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + rSV_s - rV = 0 \qquad 0 \leq S, 0 \leq t \leq T$$

$$V(S,T) = f(S), 0 \leq S \qquad f(S) = max(S - E, 0)$$

$$V(0,t) = 0, 0 \leq t \leq T$$

Where V is the value of a call option, S is the price of the underlying security, r is the risk-free interest rate, T is the time between the option's issue date and its expiration date, and E is the strike price of the option.

The following change of variables will transform this boundary value problem into a standard boundary value problem:

$$S = e^x \qquad t = T - \frac{2\tau}{\sigma^2} \qquad V(S,t) = v(x,\tau) = v(ln(S), \frac{\sigma^2}{2}(T-t))$$

The partial derivatives of V are now given by:

$$\frac{\partial V}{\partial t} = \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau}$$

$$\frac{\partial V}{\partial S} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}(\frac{1}{S}\frac{\partial v}{\partial x}) = -\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S}(\frac{\partial x}{\partial S}\frac{\partial}{\partial x})\frac{\partial v}{\partial x} = -\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}$$

Inserting these expressions into the original partial differential equation (1) results in the following:

$$-\frac{\sigma^2}{2}v_\tau + \frac{1}{2}\sigma^2 S^2(-\frac{1}{S^2}v_x + \frac{1}{S^2}v_{xx}) + rS(\frac{1}{S}v_x) - rv = 0$$

$$-\frac{\sigma^2}{2}v_\tau + \frac{1}{2}\sigma^2(-v_x + v_{xx}) + rv_x - rv = 0$$

$$\sigma^2 v_\tau + \sigma^2 v_x - \sigma^2 v_{xx} - 2rv_x + 2rv = 0$$

$$v_\tau = v_{xx} + (\frac{2r}{\sigma^2} - 1)v_x - \frac{2r}{\sigma^2}v \tag{7}$$

Let $k = 2r/\sigma^2$ and $t = \tau$ in equation (7). The boundary value problem then becomes:

$$v_t = v_{xx} + (k-1)v_x - kv, -\infty < x < \infty, 0 \le t \le \frac{\sigma^2}{2}T \tag{8}$$
$$v(x,0) = V(e^x, T) = f(e^x), -\infty < x < \infty$$

Equation (8) is similar to the diffusion equation except that it has an additional two terms on the equation's right-hand side. To eliminate these terms, another change of variables is performed and its partial derivates are computed:

$$v(x,t) = e^{Ax+Bt}u(x,t) \qquad \omega = e^{Ax+Bt}$$

$$\frac{\partial v}{\partial t} = B\omega u + \omega\frac{\partial u}{\partial t}$$
$$\frac{\partial v}{\partial x} = A\omega u + \omega\frac{\partial u}{\partial x}$$
$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}(A\omega u + \omega\frac{\partial u}{\partial x}) = A^2\omega u + 2A\omega\frac{\partial u}{\partial x} + \omega\frac{\partial^2 u}{\partial x^2}$$

Placing these partial derivatives into equation (8) results in the following:

$$B\omega u + \omega u_t = A^2\omega u + 2A\omega u_x + \omega u_{xx} + (k-1)(A\omega u + \omega u_x) - k\omega u$$
$$u_t = u_{xx} + A^2 u + 2Au_x + (k-1)(Au + u_x) - ku - Bu$$
$$u_t = u_{xx} + (A^2 + kA - A - k - B)u + (2A + k - 1)(u_x)$$

In order for this equation to turn into the heat equation, the coefficients for the u and u_x terms need to be equal to zero.

$$A^2 + kA - A - k - B = 0 \quad \rightarrow \quad B = (A-1)(A+K)$$
$$2A + k - 1 = 0$$
$$k = \frac{2r}{\sigma^2}$$

$$A = \frac{1}{2}(1-k) = \frac{\sigma^2 - 2r}{2\sigma^2}$$
$$B = -\frac{1}{4}(k+1)^2 = -(\frac{\sigma^2 + 2r}{2\sigma^2})^2 \tag{9}$$

By using the coefficients in (9) we have successfully reduced the Black-Scholes equation (1) to the following diffusion equation:

$$\begin{aligned} u_t = u_{xx}, -\infty < x < \infty, 0 \le t \le \frac{\sigma^2}{2}T \\ u(x,0) = e^{-Ax}v(x,0) = e^{-Ax}f(e^x), -\infty < x < \infty \end{aligned} \tag{10}$$

Using the fundamental solution of the heat equation (the heat kernel), the solution to equation (10) can be given by the following integral:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u(y,0) e^{-\frac{(x-y)^2}{4t}} dy$$

Works Cited

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