4.3 The Black-Scholes Partial Differential Equation

Let S be the price at time t of a particular asset. After a (short) time interval of length dt, the asset price changes by dS, to $S + dS$. Rather than measuring the absolute change dS, we measure the *return* on the asset which is defined to be

$$\frac{dS}{S}.$$

Note: This return expresses the change in the asset price as a proportion of the original asset price.

One common mathematical model of the return has two components. The first is a predictable, deterministic component (similar to the return on a risk-free investment in a bank). This is

$$\mu dt.$$

The parameter μ is called the *drift*. It is a measure of the average rate of growth of the asset price.

The second contribution to the return $\frac{dS}{S}$ is

$$\sigma dX.$$

Here σ is called the *volatility* and is a measure of the standard deviation of the returns. The quantity dX is a random variable having a normal distribution with mean 0 and variance dt :

$$dX \sim N(0, (\sqrt{dt})^2).$$

This component is a random contribution to the return. For each interval dt, dX is a sample drawn from the distribution $N(0, (\sqrt{dt})^2)$ - this is multiplied by σ to produce the term σdX. The value of the parameters σ and μ may be estimated from historical data.

We obtain the following *stochastic differential equation* (stochastic analysis is the study of functions of random variables).

$$\frac{dS}{S} = \mu dt + \sigma dX. \tag{4.1}$$

Notes

1. If $\sigma = 0$ then the behaviour of the asset price is totally deterministic and we have the ordinary differential equation
$$\frac{dS}{S} = \mu dt.$$
This can be solved to give
$$S = S_0 e^{\mu t}$$
where S_0 is the asset price at time $t = 0$.

2. The equation 4.1 is an example of a *random walk*. It cannot be solved to give a deterministic path for the share price but it gives probabilistic information about the behaviour of S.

3. The equation 4.1 can be considered to be a scheme for constructing time series that may be realised by share prices.

Example 4.3.1 *The price S of a particular share today is €10. Construct a time series for the share price over three intervals if*

$$\mu = 0.4, \ \sigma = 0.2, \ dt = \frac{1}{250}.$$

(This value of dt is basically one day, assuming 250 business days in a year.)

Solution: We have

$$\frac{dS}{S} - \mu dt + \sigma dX = 0.4\left(\frac{1}{250}\right) + 0.2dX.$$

Here dX is drawn (at each step) from a normal distribution with mean 0 and standard deviation $1/\sqrt{250} \approx 0.063$.

Step 1 $S_0 = 10$. A value for dX is chosen from $N(0, 1/250)$ - choose $dX = -0.05$. Then

$$\begin{aligned}
\frac{dS}{10} &= 0.4/250 + 0.2(-0.05) \\
dS &= 10(0.4/250 + 0.2(-0.05)) \\
&= -0.084. \\
S_1 &= 10 - 0.084 \\
&= 9.916.
\end{aligned}$$

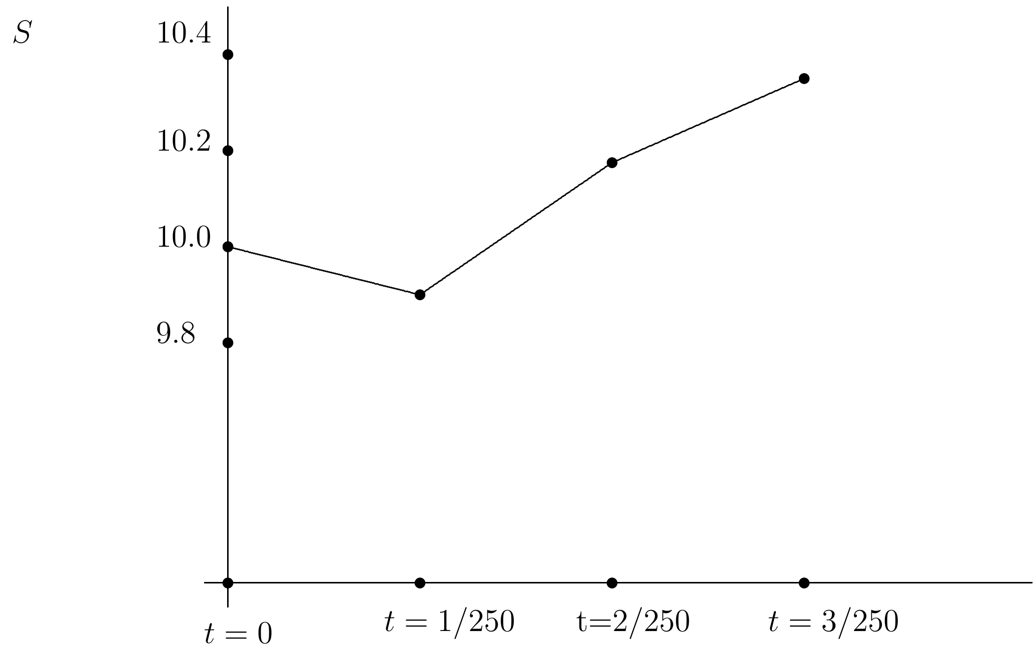
Step 2 $S_1 = 9.916$, take $dX = 0.12$.

$$\begin{aligned}
\frac{dS}{9.916} &= 0.4/250 + 0.2(0.12) \\
dS &= 9.916(0.4/250 + 0.2(0.12)) \\
&= 0.254. \\
S_2 &= 9.916 + 0.254 \\
&= 10.17.
\end{aligned}$$

Step 3 $S_2 = 10.17$, take $dX = 0.08$.

$$\begin{aligned}
\frac{dS}{10.17} &= 0.4/250 + 0.2(0.08) \\
dS &= 10.17(0.4/250 + 0.2(0.08)) \\
&= 0.179. \\
S_1 &= 10.17 + 0.179 \\
&= 10.35.
\end{aligned}$$

The following is a graphical representation of this time series :



In real life asset prices are quoted at discrete intervals of time, and so there is a practical lower bound for the basic time step dt of our random walk. If this time step were used in practice however, the sheer quantity of data involved would be unmanageable. One approach is to develop a *continuous* model by taking a limit as $dt \longrightarrow 0$. We finish these lecture notes now with a brief outline of such a model. We need *Itô's Lemma*, which is a version of Taylor's Theorem for functions of random variables.

Recall: Taylor Series

Let f be a function with derivatives of all orders on an interval I containing a point a. The *Taylor Series* of f at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If this series converges to f on I then

$$\begin{aligned} f(x) &= f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ \Longrightarrow f(x) - f(a) &= \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

Now replace x with $x + \Delta x$ and a with x to obtain

$$\Delta f = f(x+\Delta x) - f(x) = f'(x) + \frac{f''(x)}{2!}(\Delta x)^2 + \ldots$$

This relates the small change Δf in the function f to the small change Δx in x.

We now return to our consideration of what happens to

$$\frac{dS}{S} = \mu dt + \sigma dX$$

as $t \longrightarrow 0$. We need the following fact which we state without proof :
With probability 1 $(dX)^2 \longrightarrow dt$ *as* $dt \longrightarrow 0$.

Suppose that $f(S)$ is a function of the asset price S. If we change S by a small amount dS then by Taylor's Theorem we have

$$df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^2S}{dS^2}(dS)^2 + \dots$$

Now $dS = S(\mu dt + \sigma dS)$ and

$$(dS)^2 = S^2\left(\mu^2(dt)^2 + 2\mu\sigma dt dX + \sigma^2(dX)^2\right)$$

Now since $(dX)^2 \longrightarrow dt$ as $dt \longrightarrow 0$, the term $S^2\sigma^2(dX)^2$ dominates the above expression for $(dS)^2$ as dt becomes small. Retaining only this term we use

$$S^2\sigma^2 dt$$

as an approximation for $(dS)^2$ as $dt \longrightarrow 0$. We then have

$$\begin{aligned}
df &= \frac{df}{dS}dS + \frac{1}{2}\frac{d^2S}{dt^2}(S^2\sigma^2 dt) \\
&= \frac{df}{dS}(S\mu dt + \sigma dX) + \frac{d^2S}{dt^2}(S^2\sigma^2 dt) \\
df &= \sigma S\frac{df}{dS}dX + \left(\mu S\frac{df}{dS} + \frac{1}{2}\sigma^2 S^2\frac{d^2S}{dt^2}\right)dt.
\end{aligned}$$

This is *Itô's Lemma* relating a small change in a function of a random variable to a small change in the variable itself. There is a deterministic component dt and a random component dX.

In fact we need a version of Itô's Lemma for a function of more than one variable : if f is a function of two variables S, t we have

$$df = \sigma S\frac{\partial f}{\partial S}dX + \left(\mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt.$$

The Black-Scholes PDE

Let $V(S,t)$ be the value of an option (this is usually called $C(S,t)$ for a call and $P(S,t)$ for a put). Let r be the interest rate and let μ and σ be as above. Using Itô's Lemma we have

$$dV = \sigma S\frac{\partial V}{\partial S}dX + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt.$$

Consider a portfolio containing one option and $-\Delta$ units of the underlying stock. The value of the portfolio is

$$\Pi = V - \Delta S.$$

Thus $d\Pi = dV - \Delta dS$.

$$\begin{aligned} d\Pi &= \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt - \Delta S \mu dt - \Delta S \sigma dX \\ &= \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt \end{aligned}$$

Choose $\Delta = \dfrac{\partial V}{\partial S}$ to get

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt.$$

Now if Π was invested in riskless assets it would see a growth of $r\Pi dt$ in the interval of length dt. Then for a fair price we should have

$$\begin{aligned} r\Pi dt &= \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt \\ \Longrightarrow r\left(V - \frac{\partial V}{\partial S} S \right) &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \end{aligned}$$

Thus we obtain the *Black-Scholes PDE*.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$