JOURNAL OF MULTIVARIATE ANALYSIS 24, 218–236 (1988)

White Noise Approach to Stochastic Integration

HUI-HSIUNG KUO* AND ANDRZEJ RUSSEK

Louisiana State University

Communicated by P. R. Krishnaiah

1. INTRODUCTION

The well-known Ito integral

$$\int_a^b \phi(t, \omega)\, dB(t, \omega)$$

with respect to a Brownian motion $B(t, \omega)$ is defined for $\phi(t, \omega)$ satisfying the following conditions:

(i) ϕ is nonanticipating.

(ii) $\int_a^b |\phi(t\ \omega)|^2\, dt < \infty$, ω-almost surely.

The purpose of this paper is to extend the Ito integral by using the white noise calculus. This approach is motivated by the following three considerations:

(A) Extension of the Ito integral to not necessarily nonanticipating integrand $\phi(t, \omega)$. This was done first by Ito [9] using the stochastic integrals whith respect to a quasi-martingale. It has also been studied by Skorohod [16], Huang and Cambanis [7], and Berger and Mizel [2].

(B) An open problem on the series expansion of the Ito integral. Suppose $\{e_k(t);\ k = 1, 2, ...\}$ is an orthonormal basis for $L^2(a, b)$. Then by condition (ii) on ϕ, we can expand ϕ by

$$\phi(t,\ \omega) = \sum_{k=1}^{\infty} a_k(\omega)\, e_k(t),$$

where $a_k(\omega) = \int_a^b \phi(t, \omega)\, \overline{e_k(t)}\, dt$. It has been an open problem as to whether

Received February 13, 1986.

AMS 1980 subject classifications: 60H05, 60J65.

Key words and phrases: white noise calculus, stochastic integration, Ito integral.

* Research supported by NSF Grant DMS-8501775.

0047-259X/88 $3.00

in R^n of radius one. For a sequence C_i of nonempty subsets of R^n, the *topological limit inferior* is given by

$$\liminf C_i = \{x\colon x = \lim x_i, \text{ with } x_i \text{ in } C_i \text{ for all } i\},$$

and the *topological limit superior* is given by

$$\limsup C_i = \{x\colon x = \lim\ x_{i_j}, \text{ with } x_{i_j} \text{ in } C_{i_j}, \text{ a subsequence of } C_i\}.$$

We say that C_i *lower semiconverges* to C if $\liminf C_i$ includes C; we say that C_i *upper semiconverges* to C if $\limsup C_i$ is included in the closure of C. (These semiconvergences are sometimes called lower and upper convergences.) We say that C_i *converges* to C if it is both lower and upper semiconvergent to C; then we write $C = \lim C_i$. Note that the limit does not distinguish between a set and its closure.

The convergence of sets is metrizable, or rather semi–metrizable since we allow non–closed sets. The induced topology is the convergence in the Hausdorff distance for a one-point compactification of R^n. We seldom need an explicit metric; when needed, we adopt the stereographic Hausdorff distance (see [10, p. 25]), denoted by $\text{haus}^s(\cdot, \cdot)$. The Hausdorff distance between two bounded sets is denoted by $\text{haus}(\cdot, \cdot)$.

Let Ω be a complete separable metric space, with metric $d(\cdot, \cdot)$. A multifunction Γ is a mapping that assigns to each ω in Ω a subset $\Gamma(\omega)$ of R^n. The multifunction is upper semicontinuous (respectively lower semicontinuous, or continuous) at ω_0 if $\omega_i \to \omega_0$ in Ω implies that $\Gamma(\omega_i)$ upper semiconverge (respectively lower semiconverge, or converge) to $\Gamma(\omega_0)$.

Consider now the space Ω with its Borel σ-field Σ. A multifunction Γ is measurable if for every open set G in R^n the set $\Gamma^-(G) = \{\omega\colon \Gamma(\omega) \cap G \neq \varnothing\}$ is in Σ.

For a set D in R^n we denote by int D its interior, by cl D its closure, and by co D its convex hull (namely the set of convex combinations of elements in D, which may not be closed even if D is closed). We set $\|D\| = \sup\{|x|\colon x \text{ in } D\}$ and leas $D = \inf\{|x|\colon x \text{ in } D\}$. If Γ is a multifunction we write co Γ for the multifunction $(\text{co}\, \Gamma)(\omega) = \text{co}(\Gamma(\omega))$; the multifunctions cl Γ and leas Γ are defined similarly. We say that Γ is bounded if $\|\Gamma(\omega)\|$ is a bounded real function.

The support function of a set D, denoted by $s(v, D)$ and defined for v in R^n, is given by

$$s(v, D) = \sup\{v \cdot x\colon x \text{ in } D\}.$$

The Minkowski sum $C + D$ of two subsets in R^n is $\{x + y\colon x \text{ in } C, y \text{ in } D\}$. A generalization of that is the integral of a multifunction Γ with respect to a probability measure P, introduced by Aumann [2]. We denote

Moreover, it will be shown that the series in Eq. (1) converges not to the Ito integral, but to the Stratonovich integral

$$\int_a^b \phi(t, \omega) \circ dB(t) = \frac{1}{2}\left[\int_a^b \phi(t, \omega)\, dB(t+, \omega) + \int_a^b \phi(t, \omega)\, dB(t-, \omega)\right].$$

2. The White Noise Calculus

Let $\mathscr{S}$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$. The dual space $\mathscr{S}^*$ of $\mathscr{S}$ consists of the tempered distributions and carries a Gaussian measure μ with the characteristic functional given by

$$\int_{\mathscr{S}^*} e^{i\langle\omega, \xi\rangle}\, d\mu(\omega) = \exp[-\|\xi\|^2/2], \qquad \xi \in \mathscr{S},$$

where $\|\cdot\|$ denotes the $L^2(\mathbb{R})$-norm. Note that $\mathscr{S} \subset L^2(\mathbb{R}) \subset \mathscr{S}^*$. Define

$$B(t, \omega) = \begin{cases} \langle \omega, 1_{(0, t]}\rangle, & t \geqslant 0, \quad \omega \in \mathscr{S}^* \\ -\langle \omega, 1_{(t, 0]}\rangle, & t < 0, \quad \omega \in \mathscr{S}^*. \end{cases}$$

Then $B(t, \omega)$, $t \in \mathbb{R}, \omega \in \mathscr{S}^*$, is a Brownian motion and $\dot{B} = \omega$. Thus $(\mathscr{S}^*, \mu)$ can be regarded as a realization of white noise.

By the Wiener–Ito theorem, $L^2(\mathscr{S}^*)$ has the following orthogonal decomposition

$$L^2(\mathscr{S}^*) = \sum_{n=0}^{\infty} \oplus K_n,$$

where $K_0 = \mathbb{R}$ and K_n consists of multiple Wiener integrals ϕ of order n, i.e.,

$$\phi(\omega) = \int_{\mathbb{R}^n} f(t_1, ..., t_n)\, dB(t_1, \omega) \cdots dB(t_n, \omega), \qquad \omega \in \mathscr{S}^* \tag{4}$$

with $f \in \hat{L}^2(\mathbb{R}^n)$, the symmetric L^2-functions on $\mathbb{R}^n$.

The S-transform on $L^2(\mathscr{S}^*)$ is defined by

$$S\phi(\xi) = \int_{\mathscr{S}^*} \phi(\omega + \xi)\, d\mu(\omega), \qquad \xi \in \mathscr{S}, \phi \in L^2(\mathscr{S}^*).$$

It can be checked easily that if ϕ is the multiple Wiener integral of f as in Eq. (4), then its S-transform is given by

$$(S\phi)(\xi) = \int_{\mathbb{R}^n} f(t_1, ..., t_n)\, \xi(t_1) \cdots \xi(t_n)\, dt_1 \cdots dt_n. \tag{5}$$

Moreover, if ϕ and ψ in K_n are multiple Wiener integrals of f and g, respectively, then

$$(\phi, \psi) = n!\,(f, g). \tag{6}$$

Let $H^{\alpha}(\mathbb{R}^n)$ denote the Sobolev space of order α, $\alpha \geqslant 0$, of functions defined on $\mathbb{R}^n$. Let $\hat{H}^{\alpha}(\mathbb{R}^n) = H^{\alpha}(\mathbb{R}^n) \cap \hat{L}^2(\mathbb{R}^n)$ and $\hat{H}^{-\alpha}(\mathbb{R}^n)$ be the dual space of $\hat{H}^{\alpha}(\mathbb{R}^n)$. Then we have the continuous inclusions $\hat{H}^{\alpha}(\mathbb{R}^n) \subset \hat{L}^2(\mathbb{R}^n) \subset \hat{H}^{-\alpha}(\mathbb{R}^n)$.

Let $\{\alpha_n; n \geqslant 1\}$ be a sequence of nonnegative numbers. For each $n \geqslant 1$, let $K_n^{(\alpha_n)}$ consist of multiple Wiener integrals of functions in $\hat{H}^{\alpha_n}(\mathbb{R}^n)$. The dual space $K_n^{(-\alpha_n)}$ of $K_n^{(\alpha_n)}$ consists of the generalized multiple Wiener integrals [5] of functions in $\hat{H}^{-\alpha_n}(\mathbb{R}^n)$. Define

$$(L^2)^+_{\{\alpha_n\}} = \sum_{n=0}^{\infty} \oplus K_n^{(\alpha_n)},$$

$$(L^2)^-_{\{\alpha_n\}} = \sum_{n=0}^{\infty} \oplus K_n^{(-\alpha_n)},$$

where $K_0^{(\alpha_0)} = K_0^{(-\alpha_0)} = \mathbb{R}$. Obviously, we have

$$(L^2)^+_{\{\alpha_n\}} \subset L^2(\mathscr{S}^*) \subset (L^2)^-_{\{\alpha_n\}}$$

for any choice of the sequence $\{\alpha_n\}$. It can be shown that for each $t \in \mathbb{R}$, $\dot{B}(t) \in K_1^{(-1)}$. The white noise calculus is the analysis of the space $(L^2)^-_{\{\alpha_n\}}$ of generalized Brownian functionals with $\{\dot{B}(t); t \in \mathbb{R}\}$ as a coordinate system. The coordinate differentiation $\partial_t \equiv \partial/\partial\dot{B}(t)$ is defined as follows. Let U be the S-transform of ϕ in $L^2(\mathscr{S}^*)$. Suppose the first variation δU of U is given by

$$(\delta U)_{\xi}(\eta) = \int_{\mathbb{R}} U'_{\xi}(u)\, \eta(u)\, du, \qquad \xi, \eta \in \mathscr{S},$$

and that $U'_{(\cdot)}(t)$ is the S-transform of $\psi(t)$ in $L^2(\mathscr{S}^*)$. Then we define $\partial_t \phi = \psi(t)$. If $\phi \in K_n^{(\alpha_n)}$, $\alpha_n > \frac{1}{2}$, is given by Eq. (4), then for every t, the trace $f(t, u_1, ..., u_{n-1})$ exists and $f(t, \cdot) \in L^2(\mathbb{R}^{n-1})$ [1, p. 216; or 15, p. 41]. Moreover, it can be checked from Eq. (5) that

$$\partial_t \phi = n \int_{\mathbb{R}^{n-1}} f(t, u_1, ..., u_{n-1})\, dB(u_1) \cdots dB(u_{n-1}). \tag{7}$$

The adjoint operator ∂_t^* of ∂_t is defined by duality:

$$\langle \partial_t^* \phi, \psi \rangle = \langle \phi, \partial_t \phi \rangle, \qquad \phi \in (L^2)^-_{\{\alpha_n\}}, \psi \in (L^2)^+_{\{\alpha_n\}}.$$

It can be shown that if $\phi \in K_n$ is given by Eq. (4), then

$$\partial_t^* \phi = \int_{\mathbb{R}^{n+1}} \delta_t \hat{\otimes} f(u_1, ..., u_{n+1})\, dB(u_1) \cdots dB(u_{n+1}),$$

where $\hat{\otimes}$ denotes the symmetric tensor product. It follows from this equation that if $\phi(t, \omega)$ is the multiple Wiener integral of $f(t; u_1, ..., u_n)$, $a \leqslant t \leqslant b$, then

$$\int_a^b \partial_t^* \phi(t)\, dt = \int_{\mathbb{R}^{n+1}} \hat{h}(u_1, ..., u_{n+1})\, dB(u_1) \cdots dB(u_{n+1}), \tag{8}$$

where $\hat{h}$ is the symmetrization of

$$h(u_1, ..., u_{n+1}) = 1_{(a,b]}(u_1) f(u_1; u_2, ..., u_{n+1}).$$

In the later sections, we need a special case of the product formula [11]. Suppose ϕ is the multiple Wiener integral of f, $f \in \hat{L}^2(\mathbb{R}^n)$ and ψ is the Wiener integral of g, $g \in L^2(\mathbb{R})$. Then

$$\phi\psi = \int_{\mathbb{R}^{n+1}} f \hat{\otimes} g(u_1, ..., u_{n+1})\, dB(u_1) \cdots dB(u_{n+1})$$
$$+ \int_{\mathbb{R}^{n-1}} \left(n \int_{\mathbb{R}} f(u_1, u_2, ..., u_n)\, g(u_1)\, du_1 \right) dB(u_2) \cdots dB(u_n). \tag{9}$$

3. Stochastic Integrals by White Noise

We want to define stochastic integrals by using the white noise multiplication. For a Brownian functional $\phi(\omega)$, the $\dot{B}(t)$-multiplication is defined by $\dot{B}(t)\,\phi = (\partial_t + \partial_t^*)\,\phi$ [13, 14]. However, this is defined only t-almost everywhere. Thus when we apply $\dot{B}(t)$-multiplication to a stochastic process $\phi(t, \omega)$, $\dot{B}(t)\,\phi(t, \omega)$ may not make sense at all. The problem is caused by $\partial_t \phi(t, \omega)$. Take, for instance, the stochastic integral

$$\int_a^b B(t)\, dB(t).$$

By using $\dot{B}(t)$-multiplication, this stochastic integral can be rewritten as

$$\int_a^b B(t)\, dB(t) = \int_a^b (\partial_t + \partial_t^*)\, B(t)\, dt.$$

From Eq. (7), we have $\partial_t(B(s)) = 1_{(0,s]}(t)$. But the Brownian motion $B(t)$ can be also defined by using the interval $(0, s)$ and we would have $\partial_t(B(s)) = 1_{(0,s)}(t)$. Thus $\partial_t(B(t))$ cannot be defined in a unique way.

The above example leads us to define two kinds of white noise differentiation, ∂_{t+} and ∂_{t-}, when we deal with stochastic processes. Let $\phi(t, \omega)$ be a stochastic process, not necessarily nonanticipating. Let $U(t; \xi)$ be the S-transform of $\phi(t)$. Suppose that the first variation δU of U is given by

$$(\delta U)_\xi(\eta) = \int_{\mathbb{R}} U'_\xi(t; u)\, \eta(u)\, du, \qquad \xi, \eta \in \mathscr{S},$$

and that $U'_{(\cdot)}(t; t+)$ is the S-transform of $\psi_+(t)$. Then we define $\partial_{t+}\, \phi(t) = \psi_+(t)$. Similarly, if $U'_{(\cdot)}(t; t-)$ is the S-transform of $\psi_-(t)$, then we define $\partial_{t-}\, \phi(t) = \psi_-(t)$. We remark that when the operator ∂_t^* is applied to stochastic processes, there is no distinction between ∂_{t+}^* and ∂_{t-}^*.

DEFINITION 1. Let $\phi(t, \omega)$, $a \leqslant t \leqslant b$, $\omega \in \mathscr{S}^*$, be a stochastic process such that for each t, $\phi(t, \cdot) \in L^2(\mathscr{S}^*)$.

(a) Suppose $\partial_{t+}\phi(t)$ and $\partial_t^*\, \phi(t)$ exist and are integrable on $[a, b]$. Then we define

$$\int_a^b \phi(t)\, dB(t+) = \int_a^b (\partial_{t+}\, \phi(t) + \partial_t^* \phi(t))\, dt.$$

(b) Suppose $\partial_{t-}\, \phi(t)$ and $\partial_t^* \phi(t)$ exist and are integrable on $[a, b]$. Then we define

$$\int_a^b \phi(t)\, dB(t-) = \int_a^b (\partial_{t-}\, \phi(t) + \partial_t^* \phi(t))\, dt.$$

In order to state an existence theorem of the above stochastic integrals, we need some notations. Let

$$D_n^+ = \{(t; u_1, ..., u_n) \in (a, b) \times \mathbb{R}^n; u_1 > t\}$$
$$D_n^- = \{(t; u_1, ..., u_n) \in (a, b) \times \mathbb{R}^n; u_1 < t\}.$$

Let $H^\alpha(D_n^+)$ and $H^\alpha(D_n^-)$ denote the Sobolev spaces of order α, $\alpha \geqslant 0$, on the domains D_n^+ and D_n^-, respectively.

Let J_α^+ denote the set of stochastic processes $\phi(t, \omega)$, $a \leqslant t \leqslant b$, $\omega \in \mathscr{S}^*$, such that

$$\phi(t, \omega) = \sum_{n=0}^{\infty} \phi_n(t, \omega),$$

where $\phi_n(t, \omega)$ is the multiple Wiener integral of $f_n(t; u_1, ..., u_n)$ with $f_n \in H^\alpha(D_n^+)$, and

$$\sum_{n=0}^{\infty} (n+1)!\, \|f_n\|_{H^\alpha(D_n^+)}^2 < \infty. \tag{10}$$

Similarly, we define J_α^- by replacing D_n^+ with D_n^-.

Remark. $f_n(t; u_1, ..., u_n)$ can be obtained from $\phi(t)$ by the formula

$$f_n(t; u_1, ..., u_n) = \frac{1}{n!} E[\partial_{u_1} \cdots \partial_{u_n} \phi(t)].$$

Thus the condition for $\phi(t, \omega)$ to be in J_α^+ or J_α^- can be checked analytically without appealing to the Wiener–Ito expansion of ϕ.

THEOREM 1. (a) *The stochastic integral* $\int_a^b \phi(t)\, dB(t+)$ *exists for any* $\phi \in J_\alpha^+$ *with* $\alpha > \frac{1}{2}$.

(b) *The stochastic integral* $\int_a^b \phi(t)\, dB(t-)$ *exists for any* $\phi \in J_\alpha^-$ *with* $\alpha > \frac{1}{2}$.

Proof. Suppose $\phi \in J_\alpha^+$, $\alpha > \frac{1}{2}$. Let

$$\phi(t, \omega) = \sum_{n=0}^{\infty} \phi_n(t, \omega),$$

where $\phi_n(t, \omega)$ is the multiple Wiener integral of $f_n(t; u_1, ..., u_n)$. By assumption $f_n \in H^\alpha(D_n^+)$ Therefore, by the trace theorem [1, p. 216; or 15, p. 41], the trace $f_n(t; t+, u_1, ..., u_{n-1}) \in L^2((a, b) \times \mathbb{R}^{n-1})$. Hence $\partial_{t+} \phi_n(t)$ exists and is given by

$$\partial_{t+} \phi_n(t) = n \int_{\mathbb{R}^{n-1}} f_n(t; t+, u_1, ..., u_{n-1})\, dB(u_1) \cdots dB(u_{n-1}),$$

i.e., $\partial_{t+} \phi_n(t)$ is the multiple Wiener integral of $nf_n(t; t+, \cdot)$. Moreover,

$$\int_a^b \int_{\mathbb{R}^{n-1}} |f_n(t; t+, u)|^2\, du dt \leqslant c\, \|f_n\|^2_{H^\alpha(D_n^+)}, \tag{11}$$

where c is some constant independent of n. Note that by Eq. (6),

$$\begin{aligned} E\, |\partial_{t+}\, \phi_n(t)|^2 &= (n-1)!\, \|nf_n(t; t+, \cdot)\|^2_{L^2(\mathbb{R}^{n-1})} \\ &= n!\, n\ \ f_n(t; t+, \cdot)\|^2_{L^2(\mathbb{R}^{n-1})}. \end{aligned}$$

Hence, by Eq. (11),

$$\begin{aligned} \int_a^b E\, |\partial_{t+} \phi_n(t)|^2\, dt &\leqslant cn!\, n\, \|f_n\|^2_{H^\alpha(D_n^+)} \\ &\leqslant c(n+1)!\, \|f_n\|^2_{H^\alpha(D_n^+)}. \end{aligned}$$

Thus, by (10), $\int_a^b \partial_{t+} \phi(t)\, dt$ exists.

On the other hand, by Eq. (8), we have

$$\int_a^b \partial_t^* \phi(t)\, dt = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \hat{g}_n(u_1, ..., u_n)\, dB(u_1) \cdots dB(u_n),$$

where $\hat{g}_n$ is the symmetrization of g_n given by

$$g_n(u_1, ..., u_n) = 1_{(a,b]}(u_1) f_{n-1}(u_1; u_2, ..., u_n).$$

Note that

$$\begin{aligned}\sum_{n=1}^{\infty} n! \, \|\hat{g}_n\|^2 &\leqslant \sum_{n=1}^{\infty} n! \, \|g_n\|^2 \\ &= \sum_{n=1}^{\infty} n! \, \|f_{n-1}\|^2_{L^2((a,b)\times \mathbb{R}^{n-1})} \\ &\leqslant \sum_{n=1}^{\infty} n! \, \|f_{n-1}\|^2_{H^\alpha(D_n^+)}.\end{aligned}$$

Therefore, $\int_a^b \partial_t^* \phi(t)\, dt$ exists as a Brownian functional in $L^2(\mathscr{S}^*)$. This finishes the proof of the assertion (a). The proof for part (b) is similar.

EXAMPLE 1. $B(1)$ is not nonanticipating with respect to $\{B(t); 0 \leqslant t < \infty\}$. However, $B(1)$ is nonanticipating with respect to $\{\mathscr{F}_t; 0 \leqslant t < \infty\}$, where $\mathscr{F}_t = \sigma\{B(1), B(s), s \leqslant t\}$. Moreover, $\{B(t); 0 \leqslant t < \infty\}$ is a quasimartingale with respect to $\mathscr{F}_t$. With this observation, Ito [9] has shown that

$$\int_a^b B(1)\, dB(t) = B(1)(B(b) - B(a)).$$

In the white noise approach, we do not need to construct the new σ-fields. In fact, we have

$$\int_a^b B(1)\, dB(t+) = \int_a^b B(1)\, dB(t-) = B(1)(B(b) - B(a)).$$

This can be checked easily as follows. Note that $B(1) = \langle \omega, 1_{(0,1]} \rangle$ and, so $\partial_{t+} B(1) = 1_{(0,1]}(t+) = 1_{[0,1)}(t)$ and $\partial_{t-} B(1) = 1_{(0,1]}(t-) = 1_{(0,1]}(t)$. Hence

$$\int_a^b \partial_{t+} B(1)\, dt = \int_a^b \partial_{t-} B(1)\, dt = \int_a^b 1_{(0,1]}(t)\, dt.$$

On the other hand, by Eq. (8), we have

$$\begin{aligned}\int_a^b \partial_t^* B(1)\, dt &= \int_{\mathbb{R}^2} 1_{(a,b]}(s)\, 1_{(0,1]}(t)\, dB(s)\, dB(t) \\ &= B(1)(B(b) - B(a)) - \int_a^b 1_{(0,1]}(t)\, dt.\end{aligned}$$

Therefore,

$$\int_a^b B(1)\,dB(t+) = \int_a^b (\partial_{t+} B(1) + \partial_t^* B(1))\,dt$$
$$= B(1)(B(b) - B(a)).$$

Similarly,

$$\int_a^b B(1)\,dB(t-) = B(1)(B(b) - B(a)).$$

EXAMPLE 2. With the similar computation as in Example 1, we can show that

$$\int_0^1 B(1-t)\,dB(t+) = \int_0^1 B(1-t)\,dB(t-)$$
$$= B\left(\frac{1}{2}\right)^2 + 2\int_{1/2}^1 B(1-t)\,dB(t),$$

where the last integral is an Ito integral.

EXAMPLE 3. As mentioned in the beginning of this section, $\partial_t(B(s)) = 1_{(0,s]}(t)$. Thus $\partial_{t+}(B(t)) = 0$ and $\partial_{t-}(B(t)) = 1$. Therefore, we have

$$\int_a^b \partial_{t+} B(t)\,dt = 0$$

$$\int_a^b \partial_{t-} B(t)\,dt = b - a.$$

Moreover, it follows from Eq. (8) that

$$\int_a^b \partial_t^* B(t)\,dt = \frac{1}{2}\left[B(b)^2 - B(a)^2 - (b-a)\right]$$
$$= \int_a^b B(t)\,dB(t),$$

where the last integral is an Ito integral. Therefore,

$$\int_a^b B(t)\,dB(t+) = \frac{1}{2}\left[B(b)^2 - B(a)^2 - (b-a)\right]$$

$$\int_a^b B(t)\,dB(t-) = \frac{1}{2}\left[B(b)^2 - B(a)^2 + (b-a)\right].$$

4. Some Theorems

Theorem 2. *Suppose $\phi(t, \omega)$ is a nonanticipating process such that $E\int_a^b |\phi(t, \omega)|^2\, dt < \infty$. Then $\partial_{t+}\phi(t, \omega) = 0$ a.e. on $[a, b] \times \mathscr{S}^*$.*

Remark. It follows from this theorem that

$$\int_a^b \phi(t)\, dB(t+) = \int_a^b \partial_t^* \phi(t)\, dt.$$

Thus, by Eq. (3), we have

$$\int_a^b \phi(t)\, dB(t+) = \int_a^b \phi(t)\, dB(t),$$

where the Last integral is an Ito integral. However, $\int_a^b \phi(t)\, dB(t-)$ is different from Ito's integral in view of Example 3.

Proof. By the Wiener–Ito decomposition theorem, it is sufficient to prove the assertion for $\phi(t, \cdot) \in K_n, n \geqslant 1$. Let $\phi(t, \cdot)$ be the multiple Wiener integral of $f(t; u_1, ..., u_n)$, i.e.,

$$\begin{aligned}\phi(t) &= \int_{\mathbb{R}^n} f(t; u_1, ..., u_n)\, dB(u_1) \cdots dB(u_n) \\ &= n! \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{u_1} \cdots \left[\int_{-\infty}^{u_{n-1}} f(t; u_1, ..., u_n)\, dB(u_n) \right] \cdots dB(u_2) \right\} dB(u_1),\end{aligned}$$

where the last integral is an iterated Ito integral. But $\phi(t, \omega)$ is nonanticipating by assumption, so

$$\begin{aligned}\phi(t) &= n! \int_{-\infty}^{t} \left\{ \int_{-\infty}^{u_1} \cdots \left[\int_{-\infty}^{u_{n-1}} f(t; u_1, ..., u_n)\, dB(u_n) \right] \cdots dB(u_2) \right\} dB(u_1) \\ &= \int_{(-\infty, t]^n} f(t; u_1, ..., u_n)\, dB(u_1) \cdots dB(u_n).\end{aligned}$$

By taking the S-transform of ϕ as in Eq. (5) and applying the definition of ∂_{t+}, we see easily that $\partial_{t+}\phi(t) = 0$.

Theorem 3. *Suppose $\phi(t, \omega)$ is a stochastic process such that $E\int_a^b |\phi(t, \omega)|^2\, dt < \infty$. Then*

$$E\left| \int_a^b \partial_t^* \phi(t)\, dt \right|^2 = E\int_a^b |\phi(t)|^2\, dt + \int_a^b \int_a^b E(\partial_s \phi_t)(\overline{\partial_t \phi_s})\, dt\, ds.$$

Remark. When $\phi(t)$ is nonanticipating, the first integral is Ito's integral in view of Eq. (3). Obviously, the double integral vanishes. Thus the above equality reduces to the well-known identity

$$E\left|\int_a^b \phi(t)\, dB(t)\right|^2 = E\int_a^b |\phi(t)|^2\, dt.$$

Proof. By the Wiener–Ito decomposition theorem, it is sufficient to prove the theorem for the case when ϕ is a multiple Wiener integral. Let $\phi(t)$ be the multiple Wiener integral of $f(t; u_1, ..., u_n)$. By Eq. (8), we have

$$\int_a^b \partial_t^* \phi(t)\, dt = \int_{\mathbb{R}^{n+1}} \hat{g}(u_1, ..., u_{n+1})\, dB(u_1) \cdots dB(u_{n+1}),$$

where $\hat{g}$ is the symmetrization of the function

$$g(u_1, ..., u_{n+1}) = 1_{(a,b]}(u_1) f(u_1; u_2, ..., u_{n+1}).$$

Therefore,

$$E\left|\int_a^b \partial_t^* \phi(t)\, dt\right|^2 = (n+1)! \int_{\mathbb{R}^{n+1}} |\hat{g}(u_1, ..., u_{n+1})|^2\, du_1 \cdots du_{n+1}. \quad (12)$$

Now,

$$\hat{g}(u_1, ..., u_{n+1}) = \frac{1}{(n+1)!} \sum_\sigma 1_{(a,b]}(u_{\sigma(1)})\, f(u_{\sigma(1)}; u_{\sigma(2)}, ..., u_{\sigma(n+1)}),$$

where the summation is over all permutations σ of the set $\{1, 2, ..., n+1\}$. Note that $f(u_1; u_2, ..., u_{n+1})$ is symmetric in the variables $u_2, ..., u_{n+1}$. Therefore,

$$\begin{aligned}\hat{g}(u_1, ..., u_{n+1}) &= \frac{1}{(n+1)!} \cdot n! \sum_{i=1}^{n+1} 1_{(a,b]}(u_i) f(u_i; \tilde{u}_i) \\ &= \frac{1}{n+1} \sum_{i=1}^{n+1} 1_{(a,b]}(u_i) f(u_i; \tilde{u}_i),\end{aligned}$$

where $\tilde{u}_i$ denotes the variables $u_1, ..., u_{n+1}$ with u_i being deleted. Hence, we have

$$\begin{aligned}|\hat{g}|^2 = \frac{1}{(n+1)^2} \Bigg\{ &\sum_{i=1}^{n+1} 1_{(a,b]}(u_i) |f(u_i; \tilde{u}_i)|^2 \\ &+ \sum_{i \neq j} 1_{(a,b]}(u_i) f(u_i; \tilde{u}_i)\ 1_{(a,b]}(u_j) \overline{f(u_j; \tilde{u}_j)} \Bigg\}. \quad (13)\end{aligned}$$

Note that for any $i=1, ..., n+1$,

$$\int_{\mathbb{R}^{n+1}} 1_{(a,b]}(u_i)\,|f(u_i;\ \tilde{u}_i)|^2\,du_1 \cdots du_{n+1}$$

$$= \int_a^b \int_{\mathbb{R}^n} |f(t; u_1, ..., u_n)|^2\,du_1 \cdots du_n\,dt$$

$$= \int_a^b \frac{1}{n!} E\,|\phi(t)|^2\,dt; \tag{14}$$

and for any $i \neq j$,

$$\int_{\mathbb{R}^{n+1}} 1_{(a,b]}(u_i) f(u_i; \tilde{u}_i)\,1_{(a,b]}(u_j)\overline{f(u_j; \tilde{u}_j)}\,du_1 \cdots du_{n+1}$$

$$= \int_a^b \int_a^b \int_{\mathbb{R}^{n-1}} f(t;\ s, u_2, ..., u_n)\overline{f(s; t, u_2, ..., u_n)}\,du_2 \cdots du_n\,dt\,ds$$

$$= \frac{1}{n^2} \int_a^b \int_a^b \int_{\mathbb{R}^{n-1}} nf(t; s, u_2, ..., u_n)\,\overline{nf(s; t, u_2, ..., du_n)}\,du_2 \cdots du_n\,dt\,ds$$

$$= \frac{1}{n^2} \int_a^b \int_a^b \frac{1}{(n-1)!} E\left[(\partial_s \phi_t)\overline{(\partial_t \phi_s)}\right] dt\,ds. \tag{15}$$

The assertion of the theorem follows easily from Eqs. (12)–(15).

THEOREM 4. *Let $\phi(t, \omega)$ be an $L^2(\mathscr{S}^*)$-continuous stochastic process*:

(a) *If $\phi \in J_\alpha^+$ with $\alpha > 1$, then*

$$\sum_{i=1}^{k} \phi(t_{i-1})(B(t_i) - B(t_{i-1})) \to \int_a^b \phi(t)\,dB(t+) \qquad \text{in} \quad L^2(\mathscr{S}^*)$$

as the mesh of the partition $\{t_0 = a, t_1, ..., t_k = b\}$ tends to zero.

(b) *If $\phi \in J_\alpha^-$ with $\alpha > 1$, then*

$$\sum_{i=1}^{k} \phi(t_i)(B(t_i) - B(t_{i-1})) \to \int_a^b \phi(t)\,dB(t-) \qquad \text{in} \quad L^2(\mathscr{S}^*)$$

as the mesh of the partition $\{t_0 = a, t_1, ..., t_k = b\}$ tends to zero.

Proof. We will prove only (a) since the proof for (b) is similar. By the Wiener–Ito decomposition theorem, we may assume that $\phi(t, \cdot)$ is a multiple Wiener integral, say, of $f(t; u_1, ..., u_n)$. By the product formula (9), we have

$$\phi(t_{i-1})(B(t_i)-b(t_{i-1}))$$
$$=\int_{\mathbb{R}^{n+1}}(1_{\Delta_i}\hat{\otimes} f(t_{i+1};\cdot))(u_1,...,u_{n+1})\,dB(u_1)\cdots dB(u_{n+1})$$
$$+\int_{\mathbb{R}^{n-1}}\left(n\int_{\Delta_i} f(t_{i-1};u_1,u_2,...,u_n)\,du_1\right)dB(u_2)\cdots dB(u_n). \quad (16)$$

Since $\phi(t,\ \omega)$ is $L^2(\mathscr{S}^*)$-continuous,

$$\sum_{i=1}^{k} 1_{\Delta_i}\hat{\otimes} f(t_{i-1};\cdot)\to \hat{h} \quad \text{in} \quad L^2(\mathbb{R}^{n+1}), \quad (17)$$

where $\hat{h}$ is the symmetrization of

$$h(u_1,...,u_{n+1})=1_{(a,b]}(u_1)\,f(u_1;u_2,...,u_{n+1}).$$

But, by Eq. (8), we have

$$\int_{\mathbb{R}^{n+1}}\hat{h}(u_1,...,u_{n+1})\,dB(u_1)\cdots dB(u_{n+1})$$
$$=\int_a^b \partial_t^*\,\phi(t)\,dt. \quad (18)$$

On the other hand, $f\in H^{\alpha}(D_n^+)$ with $\alpha>1$. Thus it follows from Lemma 3.F [2, p. 441] that

$$\sum_{i=1}^{k} n\int_{\Delta_i} f(t_{i-1};u_1,\cdot)\,du_1\to n\int_a^b f(t;t+,\cdot)\,dt \qquad \text{in} \quad L^2(\mathbb{R}^{n-1}). \quad (19)$$

But, by Eq. (7) for ∂_{t+}, we have

$$\int_{\mathbb{R}^{n-1}}\left(n\int_a^b f(t;t+,u_2,...,u_n)\,dt\right)dB(u_2)\cdots dB(u_n)$$
$$=\int_a^b \partial_{t+}\,\phi(t)\,dt. \quad (20)$$

From Eqs. (16)–(20), it is easy to see that

$$\sum_{i=1}^{k}\phi(t_{i-1})(B(t_i)-B(t_{i-1}))\to\int_a^b \phi(t)\,dB(t+) \qquad \text{in} \quad L^2(\mathscr{S}^*)$$

as the mesh of the partition tends to zero. This completes the proof.

Let $\phi(t, \omega)$ be a stochastic process and $0 \leqslant c \leqslant 1$. For a partition $P = \{t_0 = a, t_1, ..., t_k = b\}$ of the interval $[a, b]$, put

$$\phi_P = \sum_{i=1}^{k} (c\phi(t_{i-1}) + (1-c)\,\phi(t_i))(B(t_i) - B(t_{i-1})).$$

If the limit of ϕ_P exists in $L^2(\mathscr{S}^*)$ as mesh(P) tends to zero, then we define the limit to be the $\langle c \rangle$-Stratonovich integral of ϕ. It is denoted by $\langle c \rangle \int_a^b \phi(t) \circ dB(t)$.

THEOREM 5. *Let ϕ be an $L^2(\mathscr{S}^*)$-continuous stochastic process and $\phi \in J_\alpha^+ \cap J_\alpha^-$ with $\alpha > 1$. Then $\langle c \rangle$-Stratonovich integral of ϕ exists and is given by*

$$\langle c \rangle \int_a^b \phi(t) \circ dB(t) = c \int_a^b \phi(t)\, dB(t+) + (1-c) \int_a^b \phi(t)\, dB(t-).$$

Proof. This follows obviously from Theorem 4.

5. SERIES EXPANSION

We now consider the open problem mentioned in the first section. The answer is given in the following theorem.

THEOREM 6. *Let ϕ be an $L^2(\mathscr{S}^*)$-continuous stochastic process and $\phi \in J_\alpha^+ \cap J_\alpha^-$ with $\alpha > \frac{1}{2}$. Then, for any orthonormal basis $\{e_k; k \geqslant 1\}$ for $L^2(a, b)$, the following series converges in $L^2(\mathscr{S}^*)$ and*

$$\sum_{k=1}^{\infty} a_k(\omega) \int_a^b e_k(t)\, dB(t, \omega) = \frac{1}{2}\left[\int_a^b \phi(t)\, dB(t+) + \int_a^b \phi(t)\, dB(t-)\right],$$

where $a_k(\omega) = \int_a^b \phi(t, \omega)\, \overline{e_k(t)}\, dt$.

In order to prove this theorem, we need two lemmas on the trace of integral operators on $L^2([a, b])$.

DEFINITION 2. An operator A on a Hilbert space H is said to have a finite trace if for any orthonormal basis $\{e_k; k \geqslant 1\}$ for H, $\sum_{k=1}^{\infty} \langle Ae_k, e_k \rangle$ converges and the limit is independent of $\{e_k; k \geqslant 1\}$. The limit is denoted by tr A.

Remark. Suppose A is an operator on H such that (Ax, x) is real for all x in H and its symmetrization $\hat{A} = \frac{1}{2}(A + A^*)$ is a trace class operator. Then obviously A has a finite trace and $\operatorname{tr} A = \operatorname{Tr} \hat{A}$, where Tr denotes the trace of a trace class operator.

LEMMA 1. *Let $g \in H^{\alpha}((a,b)^2)$ with $\alpha > \frac{1}{2}$. Then the integral operator G defined on $L^2([a,b])$ by*

$$Gf(t) = \int_a^b g(t,s) f(s)\, ds, \qquad f \in L^2([a,b])$$

is a trace class operator. Moreover,

$$\operatorname{Tr} G = \int_a^b g(t,t)\, dt,$$

$$|\operatorname{Tr} G| \leqslant c \, \|g\|_{H^{\alpha}((a,b)^2)},$$

where c is a constant independent of g.

Proof. It follows from Theorem 1 [2, p. 42] that G is a trace class operator and

$$|\operatorname{Tr} G| \leqslant c_1 \, \|g\|_{H^{\alpha}((a,b)^2)},$$

where c_1 is a constant independent of g. On the other hand, from the trace theorem [1, p. 216; or 15, p. 41]

$$\left| \int_a^b g(t,t)\, dt \right| \leqslant c_2 \, \|g\|_{H^{\alpha}((a,b)^2)},$$

where c_2 is a constant independent of g. Therefore, the linear maps $g \mapsto \operatorname{Tr} G$ and $g \mapsto \int_a^b g(t,t)\, dt$ are continuous on $H^{\alpha}((a,b)^2)$. But

$$\operatorname{Tr} G = \int_a^b g(t,t)\, dt$$

holds for all g in a dense subspace $C^{\infty}([a,b]^2)$ of $H^{\alpha}((a,b)^2)$. Hence it holds for all g in $H^{\alpha}((a,b)^2)$.

LEMMA 2. *Let $Q^+ = \{(u,v) \in (a,b)^2;\ u < v\}$ and $Q^- = \{(u,v) \in (a,b)^2;\ u > v\}$. Suppose $g \in H^{\alpha}(Q^+) \cap H^{\alpha}(Q^-)$ with $\alpha > \frac{1}{2}$. Then the integral operator G given by g has a finite trace and*

$$\operatorname{tr} G = \frac{1}{2} \int_a^b (g(t,t+) + g(t,t-))\, dt.$$

Moreover,

$$|\operatorname{tr} G| \leqslant c(\|g\|_{H^\alpha(Q^+)} + \|g\|_{H^\alpha(Q^-)}),$$

where c is a constant independent of g.

Proof. By considering the real and imaginary parts of g separately, we may assume that g is a real-valued function. Thus (Gf, f) is real for all f in $L^2([a, b])$.

It follows from the assumption that $g(t, t+)$ and $g(t, t-)$ exist a.e. Define

$$\hat{g}(t, s) = \begin{cases} \frac{1}{2}(g(t, s) + g(s, t)), & t \neq s \\ \frac{1}{2}(g(t, t+) + g(t, t-)), & t = s. \end{cases}$$

Then $\hat{g}$ is the symmetrization of g. It follows from Theorem 2.2 [15, p. 13] and Theorem 8.1 [15, p. 38] that for $\alpha \leqslant 1$, $g \in H^\alpha((a, b)^2)$ implies that $\hat{g} \in H^\alpha((a, b)^2)$. Therefore, for any α, $g \in H^\alpha((a, b)^2)$ implies that $\hat{g} \in H^\beta((a, b)^2)$ with $\beta = \min(\alpha, 1)$. By Lemma 1, the integral operator $\hat{G}$ given by $\hat{g}$ is a trace class operator and

$$\begin{aligned} \operatorname{Tr} \hat{G} &= \int_a^b \hat{g}(t, t)\, dt \\ &= \frac{1}{2} \int_a^b (g(t, t+) + g(t, t-))\, dt \end{aligned}$$

and

$$\begin{aligned} |\operatorname{Tr} \hat{G}| &\leqslant c_1 \|\hat{g}\|_{H^\beta((a, b)^2)} \\ &\leqslant c_2(\|g\|_{H^\alpha(Q^+)} + \|g\|_{H^\alpha(Q^-)}). \end{aligned}$$

Therefore, by the remark following Definition 2, the integral operator G given by g has a finite trace and $\operatorname{tr} G = \operatorname{Tr} \hat{G}$. This gives the conclusion of the lemma immediately.

Proof of Theorem 6. It is sufficient to prove the case when ϕ is a multiple Wiener integral, say, of $f(t; u_1, ..., u_n)$, i.e.,

$$\phi(t, \omega) = \int_{\mathbb{R}^n} f(t; u_1, ..., u_n)\, dB(u_1, \omega) \cdots dB(u_n, \omega).$$

Then, for $k = 1, 2, ...$, we have

$$\begin{aligned} a_k(\omega) &= \int_a^b \phi(t, \omega)\, \overline{e_k(t)}\, dt \\ &= \int_{\mathbb{R}^n} f_k(u_1, ..., u_n)\, dB(u_1, \omega) \cdots dB(u_n, \omega), \end{aligned}$$

where $f_k(u_1, \ldots, u_n) = \int_a^b f(t; u_1, \ldots, u_n)\, \overline{e_k(t)}\, dt$. Therefore, by the product formula (9), we have

$$a_k(\omega) \int_a^b e_k(t)\, dB(t, \omega)$$

$$= \int_{\mathbb{R}^{n+1}} (f_k \hat{\otimes} (1_{(a,b)} e_k))(u_1, \ldots, u_{n+1})\, dB(u_1, \omega) \cdots dB(u_{n+1}, \omega)$$

$$+ \int_{\mathbb{R}^{n-1}} \left(n \int_a^b f_k(u_1, u_2, \ldots, u_n)\, e_k(u_1)\, du_1 \right) dB(u_2, \omega) \cdots dB(u_n, \omega). \tag{21}$$

By a similar argument as in proving Eq. (17), we can check easily that the following series converges in $L^2(\mathbb{R}^{n+1})$ and

$$\sum_{k=1}^{\infty} f_k \hat{\otimes} e_k = \hat{h},$$

where $\hat{h}$ is the symmetrization of

$$h(u_1, \ldots, u_{n+1}) = 1_{(a,b]}(u_1) f(u_1; u_2, \ldots, u_{n+1}).$$

Therefore, by Eq. (18), the following series converges in $L^2(\mathscr{S}^*)$ and

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^{n+1}} (f_k \hat{\otimes} 1_{(a,b)} e_k)(u_1, \ldots, u_{n+1})\, dB(u_1, \omega) \cdots dB(u_{n+1}, \omega)$$

$$= \int_a^b \partial_t^* \phi(t)\, dt. \tag{22}$$

On the other hand, for each $(u_2, \ldots, u_n) \in \mathbb{R}^{n-1}$, let $F(u_2, \ldots, u_n)$ be the integral operator on $L^2([a, b])$ given by $f(\cdot; \cdot, u_2, \ldots, u_n)$. By the assumption on f and Lemma 2, $F(u_2, \ldots, u_n)$ has a finite trace given by

$$\operatorname{tr} F(u_2, \ldots, u_n) = \frac{1}{2} \int_a^b [f(t; t+, u_2, \ldots, u_n) + f(t; t-, u_2, \ldots, u_n)]\, dt. \tag{23}$$

But, since $\{e_k; k \geqslant 1\}$ is an orthonormal basis for $L^2([a, b])$, the trace of $F(u_2, \ldots, u_n)$ is also given by

$$\operatorname{tr} F(u_2, \ldots, u_n) = \sum_{k=1}^{\infty} \langle F(u_2, \ldots, u_n)\, e_k, e_k \rangle$$

$$= \sum_{k=1}^{\infty} \int_a^b f_k(u_1, u_2, \ldots, u_n)\, e_k(u_1)\, du_1, \tag{24}$$

where f_k is given as above. From Eqs. (23) and (24), and the assumption on f, we can check that the following series converges in $L^2(\mathbb{R}^{n-1})$ and

$$\sum_{k=1}^{\infty} n \int_a^b f_k(u_1, \cdot, \ldots, \cdot)\, e_k(u_1)\, du_1$$
$$= \frac{1}{2} \int_a^b n[f(t; t+, \cdot, \ldots, \cdot) + f(t; t-, \cdot, \ldots, \cdot)]\, dt.$$

Therefore, the following series converges in $L^2(\mathscr{S}^*)$ and

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^{n-1}} \left(n \int_a^b f_k(u_1, u_2, \ldots, u_n)\, e_k(u_1)\, du_1\right) dB(u_2, \omega) \cdots dB(u_n, \omega)$$
$$= \int_{\mathbb{R}^{n-1}} \left(\frac{1}{2} \int_a^b n[f(t; t+, u_2, \ldots, u_n) + f(t; t-, u_2, \ldots, u_n)]\, dt\right)$$
$$\times dB(u_2, \omega) \cdots dB(u_n, \omega)$$
$$= \frac{1}{2} \int_a^b (\partial_{t+} \phi(t) + \partial_{t-} \phi(t))\, dt. \tag{25}$$

From Eqs. (21), (22), and (25), it is obvious that the following series converges in $L^2(\mathscr{S}^*)$ and

$$\sum_{k=1}^{\infty} a_k(\omega) \int_a^b e_k(t)\, dB(t, \omega)$$
$$= \int_a^b \partial_t^* \phi(t)\, dt + \frac{1}{2} \int_a^b [\partial_{t+} \phi(t) + \partial_{t-} \phi(t)]\, dt$$
$$= \frac{1}{2}\left[\int_a^b \phi(t)\, dB(t+) + \int_a^b \phi(t)\, dB(t-)\right].$$

This completes the proof.

References

[1] Adams, R. A. (1975). *Sobolev Spaces*. Academic Press, New York/London.

[2] Berger, M., and Mizel, V. (1982). An extension of the stochastic integral. *Ann. Probab.* **10** 435–450.

[3] Birman, M., and Solomyak, M. (1977). Estimates of singular numbers of integral operators. *Russian Math. Surveys* **32** 15–89.

[4] Hida, T. (1975). *Analysis of Brownian Functionals*. Carleton Math. Lecture Notes No. 13. Carleton Univ., Ottawa.

[5] HIDA, T. (1978). Generalized multiple Wiener integrals. *Proc. Japan Acad. Ser. A Math. Sci.* **54** 55–58.

[6] HIDA, T. (1980). *Brownian Motion.* Application of Math. Vol. 11. Springer-Verlag, Heidelberg/Berlin/New York.

[7] HUANG, S. T., AND CAMBANIS, S. (1978). Gaussian processes: Nonlinear analysis and stochastic calculus. Lecture Notes in Math. Vol. 695, pp. 165–177, Springer-Verlag, New York/Berlin.

[8] ITO, K. (1951). Multiple Wiener integrals. *J. Math. Soc. Japan* **3** 157–169.

[9] ITO, K. (1976). Extension of stochastic integrals. *Proceedings, Internat. Sympos. on Stochastic Differential Equations, Kyoto,* pp. 95–109.

[10] KUBO, I. (1983). Ito formula for generalized Brownian functionals. Lecture Notes in Control and Information Sciences Vol. 49, pp. 156–166. Sringer-Verlag, New York/Berlin.

[11] KUBO, I., AND TAKENAKA, S. (1980). Calculus on Gaussian white noise, II. *Proc. Japan Acad. Ser. A Math. Sci.* **56** 411–416.

[12] KUBO, I. AND TAKENAKA, S. (1981). Calculus on Gaussian white noise, III. *Proc. Japan Acad. Ser. A Math. Sci.* **57** 433–437.

[13] KUO, H.-H. (1982). On Fourier transform of generalized Brownian functionals. *J. Multivariate Anal.* **12** 415–431.

[14] KUO, H.-H. (1983). Brownian functionals and applications. *Acta Appl. Math.* **1** 175–188.

[15] LIONS, J. L., AND MAGENES, E. (1972). *Non-Homogeneous Boundary Value Problems and Applications,* Vol. 1. Springer-Verlag, New York/Berlin.

[16] SKOROKHOD, A. V. (1975). On a generalization of a stochastic integral. *Theory Probab. Appl.* **20** 219–233.