Appendix A

The Evolution of Stochastic Mathematics that Changed the Financial World

A.1. Introductory thoughts

Let us think of the first act that the human race made, which could be attributed to the area of finance. This is easy to trace as the first day that a group of humans decided to swap an asset with another group of humans and not kill or steal to get it. Since then, million of years have gone by before the area of finance reached the point where it was thought to be a scientific discipline.

Let us now consider a rather fuzzy random process $\{X_t\}_{t\geq 0}$ which expresses the scientific progress achieved in Mathematics and its applications in the time interval $[t, t+dt)$. Then intuitively one would feel that, a good model for $\{X_t\}_{t\geq 0}$ would be a diffusion process and more specifically a mean reverting one but with jumps. In what follows, we will try to pin-point the jumps in the random evolution of stochastic mathematics that led to the fascinating and important scientific discipline now days known as Mathematical Finance or Theory of Finance or Stochastic Finance. Naturally, this is not an easy task and there is a great danger that some important moments might be omitted in this small space. However the jumps that we will mention are ones that surely had a great impact in the evolution which led to today's dense research and applications of Stochastic Finance.

We will discuss what is thought to be the genesis of stochastic finance, i.e. the Bachelier thesis at Sorbone. Interesting details of the life and work of Bachelier are presented taking into account the limited time and space. Then we discuss the decisive steps in the progress of stochastic mathematics that, led to the nowadays enormous flow of research on Mathematical Finance. These are measure theory, martingale theory, stochastic integration, Girsanov's theorem and the Black, Merton and Scholes partial differential equation. In this path we also refer briefly to their interrelation with financial problems. We also mention the tragic story of Wolfgang Doeblin who discovered stochastic integration in the barracks of the second World War and this has been a secret over 60 years. The basic sources of historical accounts for the present chapter have been the works of Jarrow (1999), Bru and Yor (2002), Taqqu (2001), and Jarrow and Protter (2004).

A.2. Genesis

The modeling of risky asset prices begin with Brownian motion, so let us begin there too. The first thing is to define Brownian motion. We assume given some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

DEFINITION A.1. A real valued stochastic process $\{B_t\}_{t\geq 0}$ is a Brownian motion if it has the properties : (i) the map $t \longrightarrow B_t(\omega)$ is a continuous function of $t \in \mathbb{R}^+$ for all ω; (ii) for every $t, h \geq 0$, $B_{t+h} - B_t$ is independent of $\{B_s : 0 \leq u \leq t\}$, and has a Gaussian distribution with mean 0 and variance h.

Brownian motion is a rich and beautiful object in its own right (Rogers and Williams (2000)). Brownian motion is a martingale, a Gaussian process, a diffusion, a Levy process, a Markov process etc.; Brownian motion is sufficiently concrete that one can do explicit calculations, which are impossible for more general objects; Brownian motion can be used as a building block for other processes.

The earliest attempts to model Brownian motion mathematically (Jarrow and Protter (2004)) can be traced to three sources, each of which knew nothing about the others: the first was that of T.N. Thiele of Copenhagen, who effectively created a model of Brownian motion while studying time series in 1880, (Thiele (1880)); the second was that of L. Bachelier, who created a model of Brownian motion while deriving the evolution of the Paris asset prices, in 1900, (Bachelier (1900a,b)); and the third was that of Einstein, who proposed a model of the motion of small particles suspended in a liquid, in an attempt to convince other physicists of the molecular nature of matter, in 1905, (Einstein (1905)).

The date March 29, 1900, should be considered as the birth date of Mathematical Finance. On that day, a French postgraduate student, Louis Bachelier, successfully defended at the Sorbone his thesis *Théorie de la Spéculation*. This work together with his subsequent was for many years neglected by the economic community but not by the probabilists such as Kolmogorov. In the present day as a testimony of his great contribution, the international Finance Society is named after him. At this point we go into a little detail about what happened to Bachelier and have a glance at the environment into which his discoveries took place.

Bachelier was born in Le Havre to a well-to-do family on March 11, 1870, (see Taqqu (2001)). His father, Alphonse Bachelier, was a wine dealer at Le Harve and his mother Cécile Fort-Meu, was a banker's daughter. But he lost his parents in 1889 and was forced to abandon his studies in order to earn a livelihood. It is known however that he registered in Sorbonne in 1892. The Paris Stock Exchange, had become by 1850, the world market for the *rentes*, which are perpetual government bonds. It all began with "the emigrants' billion" (le milliard des émigrés). During the French revolution, the nobility left and their holdings were sold as national property. When they returned in 1815, it was necessary to make restitution. Through the bonds the French state took a loan of a billion francs at the time, which was a considerable sum. The securities had a nominal value of 100 francs, but once a bond was issued, its price fluctuated. The sums that went through Paris were enormous. The French state paid always the interest but never paid the capital. When finally default appeared considerable fortunes were made and lost. These extreme fluctuations were not addressed by Bachelier in his thesis, he was merely concerned with the ordinary day-by-day fluctuations. Bachelier's subject of thesis was out of the ordinary. The "appropriate" thesis of the era for Sorbone were theses on the theory of functions (Borel, Baire, Lebesgue). Therefore, it was not an acceptable thesis topic. We must not forget that Probability as a mathematical discipline dates from after 1925, see the special invited paper by Cramer Harald (1976) in the Annales of Probability. As usual the thesis went to Poincaré, where all the thesis that at first glance did not seemed interesting, ended. The beginning of the report is as follows:

> *The subject chosen by Mr. Bachelier is somewhat removed from those which are normally dealt with our applicants. His thesis is entitled "Theory of Speculation" and focuses on the application of probability to the stock market. First, one may fear that the author had exaggerated the applicability of probability as is often done. Fortunately, this is not the case. In his introduction and further in the paragraph entitled "Probability in Stock Exchange Operations", he strives to set limits within which one can legitimately apply this type of reasoning. He does not exaggerate the range of his results, and I do not think he is deceived by his formulas.*

It must be said that, Poincaré was after the Dreyfus Affair, very doubtful that probability could be applied to anything in real life. He took a different view in 1906 after the articles of Emile Borel. Bachelier did not take the highest possible grade in his thesis and that influenced badly his academic career. The other factor of Bachelier's misfortune was the wrong estimate by Paul Lévy on one of his research findings. Later in life Lévy apologized for that but it was rather late for Bachelier. However, it was Bachelier (1906) and its extension to the multidimensional case Bachelier (1910), that prompted Kolmogorov toward the end of the 1920s, to develop his theory, the analytical theory of Markov processes, Kolmogorov (1931 and 1991).

A.3. The decisive steps

Measure theory started with Lebesgue's thesis in 1902, (see Doob (1996)), which extended the definition of volume in $\mathbb{R}^N$ to the Borel sets. Radon (1913) made the further step to general measures of Borel sets of $\mathbb{R}^N$ (finite on compact sets). In 1913 Daniell's approach to measure theory appeared, and it was these ideas, combined with Fourier series, that N. Wiener used in 1923 to construct Brownian motion. Indeed, Wiener used the ideas of measure theory to construct a measure on the path space of continuous functions, giving the canonical path projection process the distribution of what we now know as Brownian motion.

It must be said however (Williams (1991)) that measure theory, that most arid of subjects when done for its own sake, becomes amazingly more alive when used in probability, not only because it is then applied, but also because it is immensely enriched. In Finance we need a way to mathematically model the information on which future decisions can be based. There is no other model than the appropriate σ-algebra.

You cannot avoid measure theory: Think! An *event* in probability is a measurable set, a *random variable* is a measurable function on the sample space, the expectation of a random variable is its integral with respect to the probability measure and so on. Stochastic Finance really enriches and enlivens things in the sense that we deal with lots of different σ-algebras, not just the one σ-algebra which is the concern of measure theory. Of course, intuition in the use of measure theory is much more important than the actual knowledge of technical results.

Wiener and others proved many properties of the paths of Brownian motion, an activity that continuous to this day. Two key properties are that

(1) The paths of Brownian motion have a non-zero finite quadratic variation, such that on an interval (s, t), the quadratic variations is equal with $(t - s)$ and

(2) The paths of Brownian motion have infinite variation on compact time intervals, almost surely.

In recognition of his work, his construction of Brownian motion is often referred to as the *Wiener process.* It might worth noting that, the original terminology suggested by Feller (1957) in his famous treatise *An Introduction to Probability Theory and its Applications* was the *Wiener-Bachelier* process.

The next Step was the creation of **Martingale theory**. Martingales are an important class of stochastic processes. The roots of the study of Martingales is in gambling. Their name comes from an old strategy used around 1815, where one at each stage doubles the stakes in any game until he wins for the first time. The name Martingale is due to J. Ville (1939). Martingales were extensively studied by Paul Lévy (1886–1971) and Doob (1911–2002), see Doob (1953).

In Chapter 6 it is strongly evident how important is the theory of martingales in stochastic finance. A characteristic example is the first fundamental theorem of asset pricing which states that a market is arbitrage free, if and only if, there exists an equivalent martingale measure under which the discounted asset price is a martingale.

Note that, another theorem from the theory of martingales play a most decisive role in finding a hedging strategy for a trade of derivatives. In its rather simplified form this theorem states that in a model with one asset and one Brownian motion modeling the evolution of its price the existence of a hedging strategy depends on the following Theorem:

THEOREM A.2. ***(Martingale representation, one dimension).*** *Let B_t, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{B}(t)$, be a filtration generated by this Brownian motion. Let $M(t)$, $0 \leq t \leq T$, be a martingale with respect to this filtration (i.e. for every $t, M(t)$ is $\mathcal{B}(t)-$measurable and for $0 \leq s \leq t \leq T$, $\mathbb{E}[M(t) \mid \mathcal{B}(s)] = M(s)$. Then there is is an adapted process $\Gamma(u)$, $0 \leq u \leq T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u)\, dW(u)\,, \quad 0 \leq t \leq T.$$

It is finding the adapted process $\Gamma(u)$ that creates the great mathematical difficulty for any proposed model of Brownian motion.

Another almost simultaneous big step in the groundwork was **stochastic integration.** Stochastic integration was independently discovered by Kiyosi Itô and the tragic Wolfgang Doeblin.

Kiyosi Itô attempted to establish a true *stochastic differential* to be used in the study of Markov processes and with this motivation being the primary one he studied what is known as stochastic integrals. Independently the same was studied by Doeblin before him, although of course Doeblin's work was secret, hidden away in the safe of the French Academie of Science. This is a story with many messages in its own existence and it is worth taking the time and space to mention it briefly. Thus, in what follows we will refer to the life and mathematical legacy of Wolfgang Doeblin (for more details see Bru and Yor (2002) from where the following story was taken).

The procedure of a "Pli cacheté" goes back to the very origin of the Academie des Sciences. One of the first known examples was that of the deposit by Johann Bernouli, on February 1st, 1701, of a "sealed parcel containing the problems of Isoperimetrics so that it be kept and be opened only when the solutions of the same problems by his brother, Mr. Bernouli from Basle, will appear". A "Pli cacheté", since that time, allows an author to establish a priority in the discovery of a scientific result, when he/she is momentarily unable to publish it in its entirely, in a manner which prevents anybody form exercising any control, and/or asking for some paternity, over the result. This procedure continued after the creation in 1835 of the Comptes Rendus de l'Academie des Sciences which play a comparable role (to the pli cachetés), but which, to some degree, are submitted to the judgments of peers and referees, while they do not allow in general the development of methods and proofs. This procedure is still in use today and is the subject to rules updated in 1990. These stipulate that a Pli can only be opened one hundred years after its deposit unless the author or his/her relatives explicitly demand it. Once the century has elapsed, a special commission of the Academy opens the Pli in the order of its registering and decides whether to publish it or not.

In May 2000, the sealed envelope sent in February 1940 by Wolfgang Doeblin from the front line in Lorrain to the Academy of Sciences in Paris, was finally opened. This was a long awaited event for researchers in probability, with some interest in the history of their field, and who had in the past been struck by the modernity of the ideas of Wolfgang Doeblin.

The Pli has now been published in its entirety in the *Comptes Rendus* of the Academie des Science as a Special Issue, dated December 2000, and this seems to have awakened interest in both Wolfgang Doeblin's life and work.

Wolfgang Doeblin was born on the 17th of March 1915, in Berlin. His father Alferd Doblin (1878–1957), who belonged to Jewish family, was a physician and was starting to get a name in the vanguard of German literature. He became famous in 1929 once his novel *Berlin Alexanderplatz* was published. The Doeblin family was forced into exile in March 1933 and after a short time in Zurich, the Doeblins settled in Paris. At the end of 1935, he carried out research about the theory of Markov chains under the guidance of Maurice Frechet. The young Doeblin very quickly obtained some most remarkable results. Lindvall (1993, pp. 55–56) quotes K.L. Chung's review of (Lévy 1955) in the Math. Reviews:

> *After all there can be no greater testimony of a man's work than its influence on others. Fortunately, for Doeblin, this influence has been visible and is still continuing. On limit theorems his work has been complemented by Gnedenko and other Russian authors. On Markov processes it has been carried on mostly in the United States by Doob, T.E. Harris and the reviewer. Here his mine of ideas and techniques is still being explored.*

At the age of 23 years and with only two years of active research behind him, Doeblin's performance must be considered unique, probably since Laplace (see Bru and Yor (2002)).

Wolfgang Doeblin, together with his parents and his two younger brothers Claude and Stephan, acquired French citizenship in 1936. After defending his famous thesis in Mathematics, Doeblin (1938) in Spring 1938, he was enlisted for two years military service, which had been deferred for the duration of his studies. Getting depressed by the barracks routine life, he stopped all his mathematical work for four months. After that he was trying very hard, as he wrote to Frechet, to "fight against depression. As I am not interested in alcohol, I cannot resort to getting drunk". Mathematics as a therapeutic against the blues, a nice Pascalian theme. In any case, the possibilities of intensive intellectual work were quite limited. In a letter dated November 12th, 1939, Doeblin informed Frechet that he had started work again "oh! not much, about one hour every day" during the night when the others went to sleep. Doeblin had no scientific document at hand and no place to work apart from the telephone booth.

During the first days of November 1939 in a small village of the Ardennes, he went out to buy a school exercise book of 100 pages and began to write down the development of his note "Sur l' equation de Kolmogorov". The first pages of the Pli indicate that this was a form of therapy which the author imposed upon himself. In the middle of January 1940, the dream of an early end of the war was brutally replaced by reality, with the "alert on Belgium". It may well have been in Athienville, probably around the middle of February, that Doeblin finished writing the Pli. He would then have sent it to the Academie. At the same time as the Pli a second paper was sent which was presented by Borel on March 4th, Doeblin (1940). His spirits remained high, one reason being that, at long last, he may possibly have obtained leave in the middle of March, which he may then have put to profit by going to the Institute Henri Poincare (IHP) to look for the memoirs of Hostinsky which he needed. Doeblin continued sending papers to the Comptes Rendus de la Academie de Sciences as the German offensive progressed.

During the night of June 20th to 21st, as the remains of his decimated regiment are in Vosges, completely encircled by German troops and surrender is imminent, the already decorated soldier Doeblin who, according to the opinion of his superiors, has always been a "constant model of bravery and devotion", leaves his company and tries to escape on his own. After walking all night long, he finds himself inside the German net in the village of Housseras. Wolfgang enters a farm, which belongs to the Triboulot family. There, without saying a word, he burns all his papers in the Kitchen stove. He then comes out of the farm building, enters the barn and shoots himself in the head.

Thus, if we lend the conclusion, from Bernard Bru and Mark Yor: "Wolfgang Doeblin wanted to disappear in silence. Among his burnt papers, there may have been his "research note book" in which he had always jotted down new questions to study, ideas to develop... and which has not been found. The Nazis has burnt the works of his father and had forced the family into exile. For Wolfgang Doeblin, there remained the ultimate freedom to burn his papers himself and to kill himself in order to preserve his ideal of life and the beauty of his work".

We now turn to Kiyosi Itô who's first paper on stochastic integration was published in 1944, Itô (1944). Itô has explained his motivation himself (Itô (1987)), and we let him express it: "In the papers by Kolmogorov (1931) and Feller (1936), I saw a powerful analytic method to study the transition probabilities of the process, namely Kolmogorov's parabolic equation and his extension by Feller. But I wanted to study the paths of Markov processes in the same way as Levy observed differential processes. Observing the intuitive background in which Kolmogorov derived his equation, I noticed that a Markovian particle would perform a time homogenous differential process for infinitesimal future at every instant, and arrived at the notion of a stochastic differential equation governing the paths of a Markov process that could be formulated in terms of the differentials of a single differential process".

Let us now spent some time and space in order to understand some of the basic problems of stochastic integration and its interrelation with financial problems. We fix a positive number T and we are looking to find

$$\int_0^T \Delta(t)\, dW(t),$$

where $W(t), t \geq 0$ is a Brownian motion or a Wiener process together with a filtration $\mathcal{B}(t)$ for this Wiener process. We will let the integrand $\Delta(t)$ be an adapted stochastic process. Our reason for doing this is that, $\Delta(t)$ will eventually be the position we take in an asset at time t, and this typically depend on the price path of the asset up to time t. Requiring $\Delta(t)$ to be adapted means that we require $\Delta(t)$ to be $\mathcal{B}(t)$-measurable for each $t \geq 0$. Recall that increments of the Brownian motion after time t are independent of $\mathcal{B}(t)$ and since $\Delta(t)$ is $\mathcal{B}(t)$-measurable, it must also be independent of these future Brownian increments. Positions we take in assets may be independent of the price history of those assets, but they must be independent of the future increments of the Brownian motion that drives those prices. One of the problems we face when trying to assign meaning to the Ito integral is that Brownian motion paths cannot be differentiated with respect to time. The other basic problem is that if we consider a partition of $[0, T]$; i.e.

$$0 = t_0 \leq t_1 \leq .... \leq t_n = T,$$

and take the Riemann sum

$$\sum_{i=0}^{n-1} \Delta(t_i) \left[W(t_i + 1) - W(t_i)\right]$$

then given the σ-algebra $\mathcal{B}(t_i)$, $W(t_i+1)$ still remains a random variable and that makes the above Riemann sum a random variable. At this point to resolve the problem Ito made the logical step for a probabilist. Instead of taking the limit of the Riemann sum as the partition grows larger in number of points, which was not possible in this case, he took the convergence in mean square and thus he defined stochastic integration. Naturally, some conditions where necessary to guaranty its existence and these are given in the next theorem.

THEOREM A.3. *Let T be a positive constant and let $\Delta(t)$, $0 \leq t \leq T$, be an adapted stochastic process that satisfies the condition*

$$\mathbb{E}\left[\int_0^T \Delta^2(t)\,dt \mid \mathcal{B}(t)\right] < \infty.$$

Then

$$I(t) = \int_0^T \Delta(t)\,dW(t),$$

has the following properties:

(i) *(Continuity) As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.*

(ii) *(Adaptivity) For each t, $I(t)$ is $\mathcal{B}(t)$-measurable.*

(iii) *(Linearity) If $I(t) = \int_0^T \Delta(t)\,dW(t)$ and $J(t) = \int_0^T \Gamma(t)\,dW(t)$, then $I(t) \pm J(t) = \int_0^T [\Delta(t) \pm \Gamma(t)]\,dW(t)$; furthermore, for every constant c, $cI(t) = \int_0^T c\Delta(t)\,dW(t)$.*

(iv) *(Martingale) $I(t)$ is a martingale.*

(v) *(Isometry) $\mathbb{E}\left[I^2(t)\right] = \mathbb{E}\left[\int_0^T \Delta^2(t)\,dt\right]$.*

(vi) *(Quadratic Variation) $[I, I](t) = \int_0^T \Delta^2(t)\,dt$.*

Naturally, it is not possible to find the stochastic integral of various integrands as limits of expected mean squares. For that goal the following Ito Doeblin formula is used:

THEOREM A.4. ***(Itô-Doeblin formula for Brownian motion)****. Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined* *and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$\begin{aligned} f(T, W(T)) &= f(0, W(0)) + \int_0^T f_t(t, W(t))dt + \int_0^T f_x(t, W(t))\, dW(t) \\ &+ \frac{1}{2}\int_0^T f_{xx}(t, W(t))\, dt. \end{aligned}$$

J.L. Doob realized that Itô's construction of his stochastic integral for Brownian motion did not use the full strength of the independence of the increments of Brownian motion (Jarrow and Protter (2004)). In his highly influential 1953 book he extended Ito's stochastic integral for Brownian motion first to processes with orthogonal increments (in the L^2 sense), and then to processes with conditionally orthogonal increments, that is, martingales. What he needed, however, was a martingale M such that $M^2(t) - F(t)$ is again a martingale, where the increasing process F is non-random. He established the now famous Doob decomposition theorem for submartingales:

THEOREM A.5. *If X_n is a (discrete time) submartingale, then there exists a unique decomposition $X_n = M_n + A_n$ where M is a martingale, and A is a process with non-decreasing paths, $A_0 = 0$, and with the special measurability property that A_n is $\mathcal{F}_{n-1}$ measurable.*

Since M^2 is a submartingale when M is a martingale, he needed an analogous decomposition theorem in continuous time in order to extend further his stochastic integral. As it was, however, he extended Ito's isometry relation as follows:

$$\mathbb{E}\left[\left(\int_0^T H_t dM_t\right)^2\right] = \mathbb{E}\left(\int_0^T H_t^2 dF(t)\right),$$

where F is a non-decreasing and non-random, M^2–F is again a martingale, and also the stochastic integral is also a martingale, (see Doob (1953)). Thus it became an interesting question, if only for the purpose of extending the stochastic integral to martingales in general, to see if one could extend Doob's decomposition theorem to submartingales indexed by continuous time.

The issue was resolved in two papers by the (then) young French mathematician P.A. Meyer in 1962 (Jarrow and Protter (2004)). Indeed, as if to underline the importance of probabilistic potential theory in the development of the stochastic integral, Meyer's first paper, establishing the existence of the Doob decomposition for continuous time submartingales (Meyer (1962)), is written in the language of potential theory. Meyer showed that, the theorem is false in general, but true if and only if one assumes that the submartingale has a uniform integrability property when indexed by stopping times, which he called "Class (D)", clearly in honor of Doob. Ornstein (see for example Meyer (2000)) had shown that, there were submartingales not satisfying the Class(D) property, and Johnson and Helms (1963) quickly provided an example in print, using three dimensional Brownian motion. Also in 1963, Meyer established the uniqueness of the Doob decomposition, which today is known as the Doob-Meyer decomposition theorem. In addition, in this second paper Meyer provides an analysis of the structure of L^2 martingales, which later will prove essential to the full development of the theory of stochastic integration. Two years later, in 1965, Ito and Watanabe, while studying multiplicative functionals of Markov processes, define *local martingales (1965).* This turns out to be the key object needed for Doob's original conjecture to hold. That is, any submartingale X, whether it is of Class (D) or not, has a unique decomposition

$$X_t = M_t + A_t,$$

where M is a local martingale, and A is a non-decreasing, predictable process with $A_0 = 0$.

Important parallel developments were occurring in the Soviet Union (Jarrow and Protter (2004)). The books of Dynkin on Markov processes appeared early, in 1960 and in English as Springer Verlag books in 1965. A **decisive step** was the work by Girsanov (1960) on transformation of Brownian motion which extends the much earlier work of Cameron and Martin (1949) and Maruyama (1954). It was not until Van Schuppen and Wong (1974) that these results were extended to martingales, followed by Meyer (1976) and Lengart (1977) for the current modern versions. The version which more often is applied in financial problems is the following.

THEOREM A.6. ***(Girsanov one dimension).*** *Let $W(t)$, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{B}(t)$, $0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du\right\},$$

$$\hat{W}(t) = W(t) + \int_0^t \Theta(u)\,du,$$

and assume that

$$\mathbb{E}\left[\int_0^T \Theta^2(u)\,Z^2(u)\,du < \infty\right].$$

Then

$$\mathbb{E}[Z(t)] = 1$$

and under the probability measure $\hat{\mathbb{P}}$ given by

$$\hat{\mathbb{P}}(A) = \int_A Z(\omega)\,d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F},$$

the process $\hat{W}(t)$, $0 \leq t \leq T$, is a Brownian motion.

It was the work of Doleans-Dade and Meyer (1970) that removed the assumption that the underlying filtration of σ-algebras was quasi left continuous or alternatively stated as saying that, the filtration had no fixed times of discontinuity thus making the theory a pure martingale theory. This can now be seen as **a key step** that led to the **fundamental papers in finance** of Harrison and Kreps (1979) and Harrison and Pliska (1981,1983). Harrison and Kreps paper has a large number of citation in the web of science and also does the paper by Harrison and Pliska. Lastly, in the same paper Doleans-Dade and Meyer coined the modern term *semimartingale,* to signify the most general process for which one knew (at that time) there existed a stochastic integral.

In 1969, Robert Merton introduced stochastic calculus into the study of Finance. Merton was motivated by the desire to understand how prices are set in financial markets, which is the classical economics question of "equilibrium", and in later papers used the machinery of stochastic calculus to begin investigation of this issue. The fact that the world had seen the emergence of a new scientific discipline, *Mathematical Finance, Stochastic Finance, or Theory of Finance* was reflected by awarding Harry Markowitz, William Sharpe, and Merton Miller the 1990 Nobel Prize in Economics. The genesis of this science has been verified by the awarding of the 1997 Nobel Prize in Economics the formal press release of which from the Royal Academy of Sciences was the following:

> *For a new method to determine the value of derivatives. Robert C. Merton and Myron S. Scholes have in collaboration with the late Fisher Black, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society.*

The 1997 Nobel prize was awarded for their papers Black and Scholes (1973) which has a very large number of citations in the web of science and Merton (1973) which also was cited too many times, followed by Merton (1974) seminal paper which introduced the theory of credit risk. The formal press release although true, is just the proverbial of the iceberg (see Jarrow (1999)). The impact of the Black-Merton-Scholes model, is greater than most people realize. Their work on option pricing has not only provided a technique for valuation, but has also created a new field within finance, known as derivative, and offered a new perspective on related areas including corporate finance, capital budgeting, and financial markets and institutions. In mathematics and computer science , the direction of study in probability theory and numerical methods has been influenced by problems arising from the use of option pricing technology. In private industry, the Black-Merton-Scholes option pricing theory has generated not just "new types of financial instruments", but also new organizational structures within corporations to help manage risks. Research in stochastic processes and numerical methods has been financed within large investment corporations, the results of which are not known since they are highly classified by them. Mathematics and Engineering departments have recently introduced masters programs specializing in derivatives and mathematical finance. In the last fifteen years mathematicians and theoretical physicists can now find alternate and high-paying demand for their skills in the financial world. Note though, that competition for these jobs is fierce and the better your skills on mathematics the better are your chances. In addition, there is no limit in the working hours per week a young researcher has to provide and as for job security the policy is hire and fire in correlation with the many turbulences of the international market.

A.4. A brief glance towards the flow of research paths

As mentioned earlier two are the basic assumptions underlying the Black-Merton-Scholes model, the constant risk-free interest rates and a constant volatility for the underlying asset. In April 1973, around the time of the publication of the Black - Merton-Scholes model, the Chicago Board Options Exchange began trading the first listed options in the United States. Since that time, the growth in exchange traded and over the counter traded options on equities, indices, foreign currencies, commodities, and interest rates has been phenomenal. In response to these new derivatives markets, new firms were created and new departments in existing firms and banks were formed to take advantage of these new trading opportunities.

From that point of time, i.e. since 1973, we have witnessed a tremendous acceleration in research efforts aimed at better comprehending, modeling and hedging all risks involved. Later through the machinery of the Theory of Martingales and Gyrsanov's theorem, martingale methods have been constructed which generalized considerably these assumptions (see Musiela and Rutkowski (2000) and Schreve (2004)). Generalizations included models in which volatility was random and models in which asset prices jumped, rather than moving smoothly. In the 1980's increased interest rate volatility occurred due to double-digit inflation. That created a new demand for interest rate derivatives for both motives insurance and speculation. For this type of problem the seminal paper is that of Heatth, Jarrow and Morton (1992). Various stochastic process models have been created which we will briefly mention in what follows. The book by Brigo and Mercurio (2006) on interest rate models is one that combines a strong mathematical background with expert knowledge of practice. This simultaneous attention is difficult to find in other available literature. Local volatility models have been introduced as straightforward analytical extensions of a geometric Brownian motion that allow skews in the implied volatility. Another excellent book in the area is that of Rebonato (1998). The more flexible models of this type, allowing for smile-shaped implied volatilities, have been proposed by Brigo, Mercurio and Sartorelli (2003) and Brigo and Mercurio (2003a). The already briefly mentioned stochastic volatility models where the volatility is assumed to follow a diffusion process have as main representatives the works of Hull and White (1987) and Heston

(1993), with the related application to the LIBOR market model developed by Wu and Zhang (2002). Another class of models are the Jump-Diffusion models, which have been introduced to model discontinuities in the underlying stochastic process, namely the possibility of finite changes in the value of the related financial variable over infinitesimal time intervals. Discontinuous dynamics seem ideally suited for the interest rate market, where short-term rates can suddenly jump due to central banks interventions. The first example of Jump-Diffusion models in the financial literature is due to Merton. Jump diffusion Libor models have been developed by Glasserman and Merener (2001) and Glasserman and Kou (2003). Finally another interesting class of models are the Levy-driven models. These have been designed to allow for stochastic evolutions governed by general Levy processes. A book with the applications of Levy Processes in Finance for pricing financial derivatives is Schoutens (2003).

Another large area of the Theory of Finance is the one that deals with *default risk* (see Bielecki and Rutkowski 2004). A default risk is a possibility that a counterpart in a financial contract will not fulfill a contractual commitment to meet her/his obligations stated in the contract. If this actually happens, we say that the party defaults, or that the default event occurs. More generally, by credit risk we mean the risk associated with any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads, and the default event. There are two kinds of credit risks the *reference credit risk and the counterpart credit risk.* In the reference credit risk the two parties of the contract are default-free but some reference entity in the contract which plays an important role appears to produce a default risk. *Credit derivatives* are recently developed financial instruments that allow market participants to isolate and trade the reference credit risk. In counterpart credit risk each counterpart is exposed to the default risk of the other party. The counterpart risk emerges in a clear way in such contracts as *vulnerable claims* and *default swaps.* In both of these cases one needs to quantify the default risk of both parties in order to correctly assess the contracts value. A corporate bond is an example of a defaultable claim.

A vast majority of mathematical research devoted to the credit risk is concerned with the modeling of the random time when the default event occurs, i.e. the default time. Two competing methodologies have emerged in order to model the default/ migration times and the recovery rates: *the structural approach and the reduced-form approach.* Structural models are concerned with modeling and pricing credit risk that is specific to a particular corporate obligor. Credit events are triggered by movements of the firm's value relative to some (random or non-random) credit-event-triggering threshold (or barrier). From the long list of works devoted to structural approach, let us mention in here: Merton (1974), Black and Cox (1976), Ericson and Renedy (1998), Ericson (2000).

In the reduced-form models approach, the value of the firm's asset and its capital structure are not modeled at all, and the credit events are specified in terms of some exogenously specified jump process. We can distinguish between the reduced form models that are concerned with the modeling of the default time, and that are henceforth referred to as *intensity-based models*, and the reduced form models with migrations between credit rating classes, called the *credit migration models.* The main emphasis is put on the modeling of the random time of default as a hazard process, as well as evaluating conditional expectations under risk-neutral probability of functionals of the default time and the corresponding cash flows. Interesting works in this respect that pioneered the area are Pye (1974), Ramaswamy and Sundaresan (1986), Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Lando (1997, 1998), and Jarrow and Yu (2001).

The credit migration models assume that the credit quality of corporate debt is quantified and categorized into a finite number of disjoint credit rating classes. Each credit class is represented by an element in a finite set one of which is the default state. The assumed process for the evolution of the credit quality is referred to as the migration process. The main issue in this approach is the modeling of the transition intensities under the real world probabilities, the equivalent martingale measure and the forward measure. The next step is the evaluation of conditional expectations under the equivalent martingale measure and the forward measure of certain functionals, typically related to the default time. The most highly cited papers in the area are those of Jarrow and Turnbull (1995), and Jarrow, Lando and Turnbull (1997). References dealing with the stochastic modeling of credit migrations include Duffie and Singleton (1998), Kijima (1998), Thomas *et al*. (1998), Huge and Lando (1999), Bielecki and Rutkowski (2000), Lando (2000), Schonbucher (2000), Vasileiou and Vassiliou (2006), Guo, Jarrow, Zeng (2009).