Article Review : Efficient Monte Carlo methods for value-at-risk

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1 Introduction

The calculation of value at risk (VaR) for large portfolios consisting of complex financial instruments presents a trade-off between accuracy and speed. The accuracy of the model for calculating the VaR of a portfolio depends on the assumptions made about the changes in the underlying risk factors and how these changes in risk factors impacts the portfolio change. Simplified assumptions, for example using the "variance-covariance" method to determine VaR, assumes that the portfolio change is linearly or quadratically related to the risk factor changes. This method speeds up the computing time but leads to less accurate results. More complex models such as full Monte Carlo simulation are more accurate since it is applicable in almost any model of changes in risk factors but comes at a cost that it is computationally more expensive. The paper under review, "Efficient Monte Carlo Methods" Glasserman *et al.* (2000*a*), proposes a theoretical methodology for reducing the computing time of the Monte Carlo method for full revaluation in each market scenario and empirically tests the proposed method.

When using Monte Carlo simulation to revalue the portfolio, reduction in computing time can be achieved either through reducing the number of simulations, or through reducing the number of revaluations of the portfolio in each scenario. Reducing the number of simulations leads to less accurate results. Glasserman *et al.* (2000*a*), proposes using the delta-gamma approximation of the change in a portfolio value in conjunction with strategic sampling of important scenarios to limit the number of scenarios that need to be revalued. Importance sampling and stratified sampling are the proposed methods to reduce the variance of the losses simulated and therefore the number of simulations required to be revalued in order to get a accurate estimate.

This paper will review the article by Glasserman *et al.* (2000*a*). We will look at how well the article explains the relevant work and how it is presented to the reader. An overall evaluation will be done for the theoretical and practical implementation by performing similar empirical tests on different data as well discussing the results obtained.

2 Concepts and mathematical background

To sufficiently review the article some important concepts will be explained and derived at a high level. A more mathematical rigorous explanation can be found in the article "Variance reduction techniques for estimating value-at-risk" , Glasserman *et al.* (2000*b*) for the methods used in the article. All results and derivations are the same as in the paper although some notation might to differ. Some of the derivations have been used as in the class notes. The following notation will be used to facilitate the discussion :

- S = vector of risk factors (say consisting of m risk factors)
- δt = VaR horizon.
- δS = change in risk factors over δt.
- $V(S_1, ..., S_n)$ = value of the porfolio as a function of the risk factors.
- L = loss in portfolio value resulting from from changes , δS, over δt
- $\Sigma = =$ covariance matrix of returns.
- N = number of simulations.

2.1 Monte Carlo Simulation

Monte Carlo simulation in the context of VaR involves simulating the changes in the risk factors and then revaluing the portfolio based on theses changes for the number of simulations that were then used. The distribution of profit and losses can then easily obtained and the necessary VaR can then be calculated. The procedure is as follows:

1. Simulate N scenarios for the different changes in risk factors $\delta S(1), ..., \delta S(N)$ over a horizon δS.
2. Revalue portfolio at end of horizon δt in scenarios $S + \delta S(1), ..., S + \delta S(N)$;the losses $L(1), ..., L(N)$ are calculated as $V(S_t) - V(S_t + \delta S)$.
3. The distribution of the losses are then known and therefore the loss probabilities can be calculated as $x : \frac{1}{N} \sum_{i=1}^{N} I(L(i) > x)$ for a given threshold x.

To estimate VaR, the quantiles at different loss probabilities can then be found. To get the an approximate distribution of VaR one can interpolated between the different estimated loss probabilities correpsonding to the quantiles calculated at the loss probabilities.

2.2 VaR Delta-Gamma Approximation

When calculating the VaR of a portfolio at the tail distribution there are two main problems: estimating the loss probability, $P(L > X)$, given a threshold; and the inverse problem of estimating the quantile associated with a given loss threshold. Calculating loss probabilities is a prerequisite to finding the quantiles so we will focus on the first problem.

The change in the portfolio from time t to $t + \delta$ can be written as :

$$\delta V(S(t) = V(S(t) + \delta S(t)) - V(S(t)$$

.

Using a second order Taylor approximation this can be written as [1]:

$$\delta V \approx \frac{dV}{dt}\delta t + \sum_{i=1}^{m} \frac{dV}{dS_i}\delta S_i + \frac{1}{2}\sum_{k,l=1}^{m} \frac{d^2V}{dS_k dS_k}\delta S_k \delta S_l$$

Defining the following:

- $\Theta = \frac{dV}{dt}$: The m x 1 vector of the thetas of the underlying risk factors.
- $\delta = \frac{dV}{dSi}$: The m x 1 vector of the deltas of the underlying risk factors.
- $\gamma_{k,l} = \frac{d^2V}{dS_k dS_l}$: The m x m matrix of the gammas of the underlying risk factors.

The equation can then be written as :

$$\delta V(S(t)) \approx \Theta'\delta S + \frac{1}{2}\delta S'\Gamma\delta S$$

This equation is known as the delta-gamma approximation for the change in portfolio value. The equation simplifies the revaluation of complex securities or financial instruments since each instrument price does not have to be calculated again. The change in price of the security can be approximated if the theta,delta and gamma are known as well as the change in the underlying risk factor. If the time horizon for the change in the portfolio is relatively small (say 1-day) the theta of the portfolio will be negligible and we can therefore further approximate it as[2] :

$$\delta V(S(t)) \approx \Theta'\delta S + \frac{1}{2}\delta S'\Gamma\delta S$$

We define the loss in the portfolio as:

$$L \approx -\Theta'\delta S - \frac{1}{2}\delta S'\Gamma\delta S$$

By finding the Choleski decomposition of the covariance matrix we can write $\Sigma = CC'$, so that $\delta S = CZ$ [3], where Z is a column vector of standard normal variables. Then δS has a $N(0,\Sigma)$ distribution. This simplifies the simulation since we can simulate from a $N(0,I)$ distribution where the changes in risk factors are uncorrelated. The eigenvalues can be found of the matrix $-\frac{1}{2}\delta S'\Gamma\delta S$, which simplifies the equation so that the gamma part of the loss equation can be written as $Z'\Lambda Z$, where Λ is a matrix of the eigenvalues.

[1] The portfolio as a function of time has been dropped in order to simplify the equations. to indicate has been dropped in order

[2] In the paper they include it but for our purposes we will leave it out to simplify things

[3] It is important the matrix C is a lower triangular matrix

Using these change of variables, we can rewrite this approximation in the form :

$$L \approx b'Z + Z'\Lambda Z = Q \ (1)$$

And in discrete notation as:

$$L \approx \sum_i b_i Z_i + \sum_i \Lambda Z_i^2$$

The moment generating function of the loss equation can then be found and subsequently the characteristic function by using equation (1). The derivation that follows is used from the class notes and adapted to suite the paper.

$$E[e^{\theta L}] = \prod_{i=1}^{n} E[e^{\theta(b_i Z_i + \lambda_i Z_i^2)}]$$

$$= \prod_{i=1}^{n} E[e^{\theta(\lambda_i (Z_i + \frac{b_i}{2\lambda})^2 - \frac{b_i^2}{4\lambda_i})}] \ (2)$$

$$= \prod_{i=1}^{n} E[e^{\psi(\theta)}]$$

Using the identity:

$$E[e^{u(z+c)^2}] = (1-2u)^{-\frac{1}{2}} e \frac{uc^2}{1-2u}$$

We can write the expected value in equation (2) as :

$$E = (1-2\theta\lambda_i)^{-\frac{1}{2}} e^{\frac{\theta\lambda(\frac{b}{2\lambda})^2}{1-2\theta\lambda}} e^{\frac{b_i^2}{4}}$$

$$E[e^{\psi(\theta)}] = (1-2\theta\lambda_i)^{-\frac{1}{2}} exp(\frac{\theta^2 b_i^2}{2(1-2\lambda\theta)})$$

So that we have the log of the moment generating function:

$$\psi(\theta) = \sum_{i=1}^{m} \frac{\theta^2 b_i^2}{2(1-2\theta\lambda_i)} - \frac{1}{2} log(1-2\theta\lambda_i)$$

The derivative of this function is :

$$\psi'(\theta) = \sum_{i}^{m} \frac{(\theta b_i^2)(1-\theta\lambda_i)}{(1-2\theta\lambda_i)^2} + \frac{\lambda_i}{1-2\theta\lambda_i)}$$

The characteristic function is then:

$$E[e^{i\omega Q}] = exp^{\psi(i\omega)}$$

Transform inversion can then be used to calculates value of the distribution $P(Q < x)$.

2.3 Importance Sampling

The estimation at the tail of the distributions to determine $P(L > x)$ is often inaccurate since there are few samples at the important region where $L \approx x$. Importance sampling can be used to change the distribution of the underlying risk factors simulated from a standard normal distribution to a normal distribution with mean vector and covariance matrix Σ that will generate samples of the loss function closer to the quantile when x is large. If we approximate $L \approx Q$, we can generate large samples that are close to Q with a high probability, in fact the expected value of Q will be equal to x.
The following methodology is used:

$$P(L > x) = \int_x^{\infty} 2\pi^{-\frac{n}{2}} e^{-\frac{1}{2}Z'Z} dx$$

$$= \int_x^{\infty} \frac{2\pi^{-\frac{n}{2}} e^{-\frac{1}{2}Z'Z}}{2\pi^{-\frac{n}{2}} |\Sigma^{-\frac{1}{2}}| e^{-\frac{1}{2}(Z-\mu)'\Sigma^{-1}(Z-\mu)}} \pi^{-\frac{n}{2}} |\Sigma^{-\frac{1}{2}}| e^{-\frac{1}{2}(Z-\mu)'\Sigma^{-1}(Z-\mu)} dx$$

$$= E_{\mu,\Sigma}[l(x)I(L > x)]$$

Where the likelihood ratio is

$$l(x) = |\Sigma|^{\frac{1}{2}} exp[-\frac{1}{2}\mu'\Sigma^{-1}\mu] exp[\frac{1}{2}(Z'(I - \Sigma^{-1})Z - 2\mu'\Sigma - Z]$$

The expectation in the equation shows that it is the expectation with regards to the new distribution with mean vector μ and covariance matrix Σ. We now need to find choices of μ and Σ that will produce large losses. The likelihood ratio however does not give a clear indication of how this can be achieved. A discussion is given in Appendix A of how this can be achieved and we will just state the results here. We choose the mean and standard deviation as:

$$\Sigma(\theta) = (I - 2\theta)^{-1}, \mu = \theta\Sigma(\mu\theta)b$$

which we can write as:

$$\mu_i(\theta) = \frac{\theta b_i}{1 - 2\theta\lambda_i}, \sigma_i^2(\theta) = \frac{1}{1 - 2\theta\lambda_i}$$

.

The theta in these equations is called the twisting parameter. By restricting the IS so that it corresponds to "exponential twisting" of the quadratic form of Q, with θ the twisting parameter, the IS distribution will produces losses around the quantile. Exponential twisting can be used when working with rare events with IS procedures (Sadowsky and Bucklew (1990) Heidelberger (1995)).

The likelihood ratio then becomes :

$$l(Z) = e^{-\theta(b'Z+Z\lambda Z')+\psi(\theta)}$$

$$= e^{-Q+\psi(\theta)}$$

And the we finally have :

$$P(L > x) = E_{\mu,\Sigma}[l(x)I(L > x)]$$
$$= E_{\mu,\Sigma}[e^{-Q+\psi(\theta)}I(L > x)]$$

To find the value of theta we see that:

$$\frac{d}{d\theta}\phi(\theta) = E_{\mu,\Sigma}[Q]$$

For a specified x we have that :

$$\frac{d}{d\theta}\psi(\theta_x) = E_{\mu,\Sigma}[Q] = x \ (3)$$

Using this equation we can find the value for θ for a specified x. Using different values of x and estimating the loss probabilities, interpolation can be used to calculate the full distribution of $P(L > x)$.

2.4 Stratified Sampling

Further variance reduction can be accomplished through stratified sampling. This involves dividing the the loss distribution, Q, into equiprobable bins or strata and then sampling a specified number within in each bin instead of randomly drawing from the sample of Q. The first step in implementing this method is to define the strata. In order to define k equiprobable strata we need to find the points $y_i, ..., y_{k-1}$ such that

$$P(Y \leq y_i) = \frac{i}{k}, i = 1, ..., k-1$$

Once the strata has been defined, a method under which an equal fraction of the Q(i) generated needs to developed. The authors using the following iterative methodology :

Say we need n samples from the k strata then:

1. Generate a large number of independent samples Z from a $N(\mu(\theta), \Sigma(\theta))$ distribution.
2. For each Z generated evaluate Q and check which stratum it falls within.
3. If there are not n samples in the strata, we keep the generated value of Z. otherwise disregard it.
4. Repeat the procedure until all the strata have been filled

The authors use the following notation, "Let Q^{ij} denothe the j'th sample from stratum i and let Z^{ij} denote the draw from $N(\mu(\theta), \Sigma(\theta)$ that produced the sample. From Z^{ij} we get that $\delta S^{ij} = CZ^{ij}$ as before and computed the corresponding portfolio loss L^{ij}" (Glasserman *et al.* (2000*a*)). The estimator that results from this is:

$$\sum_{i=1}^{k} \frac{1}{nk} \sum_{j=1}^{n} e^{-\theta Q^{ij} + \psi(\theta)} I(L^{ij} > x)$$

By defining the the strata with probabilities $p_1, ..., p_k$ and allocating n_i samples to stratum $i, i = 1, ..., k$ the estimator is then:

$$\sum_{i=1}^{k} \frac{p_i}{n_i} \sum_{j=1}^{n_j} e^{-\theta Q^{ij} + \psi(\theta)} I(L^{ij} > x)$$

Glasserman et al. (1999), investiges various strategies for the optimal allocations of n_i and in practice 40 strata are often used.

3 Article review and results

3.1 General remarks

The article is presented well and thoroughly explains the concepts that are important for the discussion in a suitable manner. The article is however, more of a summary of the works of two of the previous papers namely: "Variance reduction techniques for estimating value at risk" Glasserman *et al.* (2000*b*); and "Portfolio value-at-risk with heavy-tailed risk factors." by the same authors. . A more mathematical rigorous explanation for all the concepts can be found in Glasserman *et al.* (2000*b*), where the authors go into great extend covering all the precise mathematics and assumptions that the results previously discussed hold for. Some new concepts are introduced in this paper however such as including volatility as a risk factor which we will look at in Appendix B.

The paper focuses mostly on the theoretical aspects for the proposed methodology and does not go into the same detail for the practical implementation. The methodology is quite complex and it remains to be seen how practical the application of these results could be used in the industry. For a large financial institution daily VaR (and even hourly) calculations need to be performed for a lot of different types of portfolios to assess the risk contained in each one. This leads to a lot of computing time therefore reducing the computing time is can be a very helpful tool. The complexity of the model also needs to be assessed in regards to how well it can be explained to users of the model and implemented. The model proposed is heavy reliant on complex mathematical concepts and can easily lead to mistakes when implementing it as it needs to be precise in order to function correctly .

There are a few key assumptions for the methodology used in the article :

1. Changes in the underlying risk factors are normally distributed.
2. The deltas and gammas of the portfolio are already calculated.
3. The gamma-delta approximation holds.

The assumption that changes in the underlying risk factors is not always accurate since a lot of the changes in the risk factors have heavy-tailed distribution. Additional methodologies could also be used to estimate the covariance matrix such as using a GARCH model. A method for evaluating these changes are explained in Glasserman *et al.* (2002). The delta-gamma approximation holds in nearly all portfolios so it can used and special cases when it does not hold are rare.

3.2 Results of article

The authors used 10 different types of portfolios consisting of between 150 and 1000 standard call and put options distributed over 10 underlying assets. The time to expiration of options varied among the portfolio from 0.1 to 0.5 years so that some of the portfolios that options with small expiration date had comparatively larger gammas. The portfolios have various traits such as portfolios that are delta hedged; or portfolios that have only positive or negative eigenvalue or mixture of the two. Different combinations of at-the-money, in-the-money and out-the-money options were also used in these portfolios. All the calculations were performed on a VaR horizon δ of 10 days.

Through the empirical analysis they found that using importance sampling can reduce the variance of the loss distribution they that found that by using importance sampling they could reduce the number of revaluations from 20 to 50 and using importance sampling in combination with stratified sampling they could reduce the number of revaluations required between 30 and 320. This means that the computation time for the calculation of VaR could be reduce by up to 320 times which is quite a significant results. The average of the time reduced is closer to 50 but is heavily dependant on the options expiry date contained in the portfolio.

An important question to answer is whether these results hold up for other type of portfolios that have other types of derivatives or financial instruments. This method is only applicable in situations where the delta and gammas of a portfolio could be calculated easily. The method proposed therefore is very useful for portfolios heavily dependant on options that already have the gammas and delta calculated for other reason relating to the portfolio.

4 Replicating the results

4.1 Data

To replicate some of the results as in the article a portfolio consisting of stock,indices and options was used where the options underlying risk factors was stock and indices that were already included in the portfolio. To keep the calculations simple, the portfolio contained only three underlying risk factors. The options consisted of call and put options and each options contract is written on 1000 of the underlying assets. The portfolio was evaluated on 1995/07/28. The following table shows the prices of the assets and the unit amount held for each.

Portfolio			
Stock and options	Amount	Price	Value
ASSD	5000	205	4100000
BARC	20000	1746	8730000
FTSE	1000	5736,1	5736100
ASSD Call	10	14218	142180
BARC Call	50	105500	5275000
FTSE Put	20	125020	2500400

The portfolio value is then 23892317.

4.1.1 Implementation of method

The first necessary steps was to calculate the gammas and the deltas of the portfolio so that delta-gamma approximation can be used. In order to get these, the Black-Scholes formula was used. Since the volatility used in the equations is unknown but the option prices are known, we can calculate the implied volatility by setting the B-S formula, will all the known variables, equal to the option price.

In our calculation of VaR we decided to use a 1-day time horizon. Since the time horizon is relatively small the theta part of the change in the portfolio in the delta-gamma approximation will be approximately zero and will therefore be left out in our calculations. The assumption was also made that the second order partial derivatives of the underlying stocks which involved cross terms were zero so that the gamma matrix only had elements on the main diagonal.

The covariance matrix of daily returns was calculated from us the previous years historical data up to the date of the portfolio. The means were also calculated from the previous year as the average of the daily returns. The next step was to find the Choleski decomposition of the co-variance matrix and to simulate the changes in the underlying risk factors. Since the co-variance matrix is of the returns of the stocks these need to be multiplied with the prices to get the nominal change in the risk factors.

All the necessary data is now ready to perform Monte Carlo simulation and use the delta-gamma approximation on these simulations in order to estimate the loss distribution which we can then use as our baseline model to in order compare the effect of importance sampling and stratified sampling on the variance of the loss distribution. For our purposes we used 10 000 simulations for the Monte Carlo method.

Figure 1 displays the loss distribution using the Monte Carlo method with the gamma-delta approximation:

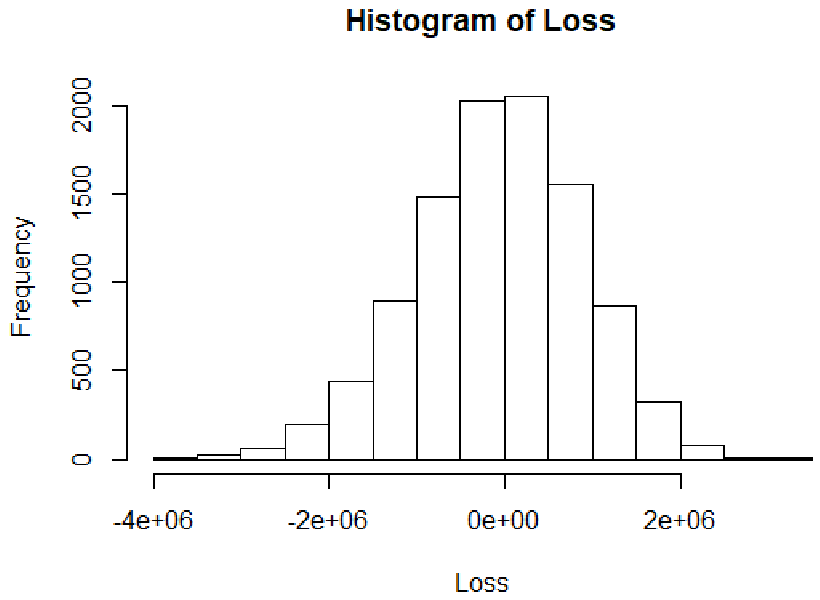

Figure 1: Histogram of losses using simple Monte Carlo with D-G approximation

The 5% 1-day VaR calculated from the distiubution is 1438534.

For the importance sampling method the following the steps are summarized below as in the paper which we used exactly :

1. Compute the Choleski decomposition by finding any matrix C such that $CC' = \Sigma$.
2. Set $b = -C\delta S$ and $-\Lambda = -\frac{1}{2}C'\Lambda C$.
3. Set $\theta = \theta_x$ and find the solution to

$$\psi'(\theta) = \sum_{i}^{m} \frac{(\theta b_i^2)(1-\theta\lambda_i)}{(1-2\theta\lambda_i)^2} + \frac{\lambda_i}{1-2\theta\lambda_i} = x$$

4. Set $\Sigma(\theta) = (I - 2\theta\lambda)^{-1}$ and $\mu(\theta) = \theta\Sigma(\theta)b$.
5. Simulate $Z^{(1)}, ..., Z^{(N)}$ from $N(\mu(\theta), \Sigma(\theta)$.
6. Set $\delta S^(i) = CZ^(i), i = 1, ..., N$
7. Calculate portfolio losses $L^{(i)}$ resulting from $\delta S^{(i)}$
8. Calculate $Q^{(i)}$ for each $Z^{(i)}$
9. Calculate the Return estimate

$$= \frac{1}{N}\sum_{i=1}^{N}[e^{-Q+\psi(\theta)}I(L > x)]$$

Mean(Q)	$P(L > x)$	$P_{IS}(L > x)$	Variance reduction
1510108	5 %	10.1 %	3.117084

Table 1: Results of simulations with delta-gamma approximation and importance sampling.

The implementation of these steps were done mostly in R excepts for step 3 which was done in excel due to R not having a function to solve the equation specified in step 3.

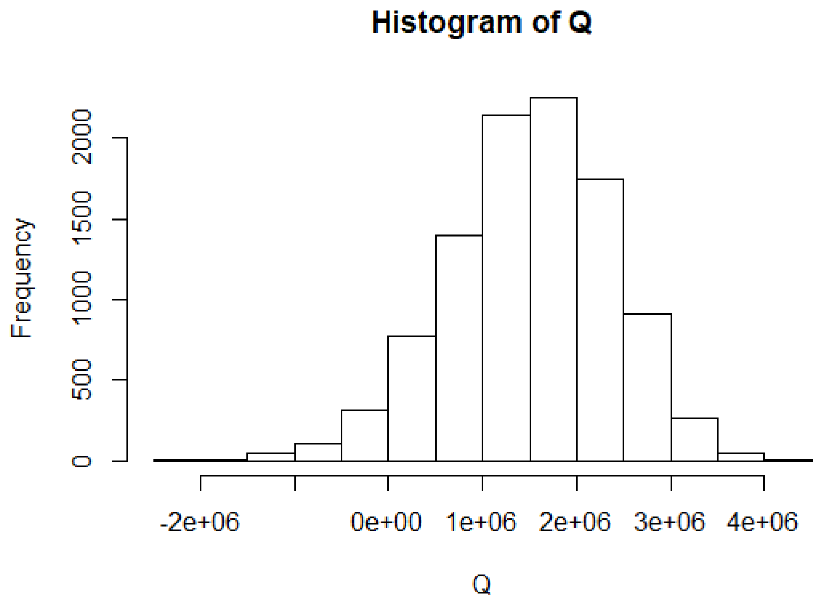

Figure 2: Histogram of losses using importance sampling Monte Carlo with D-G approximation

From Figure 2 we can see that the distribution has shifted a lot from figure 1 so that it produced larger losses around the quantile x. This shows that a lot more scenarios are generated above the threshold and therefore a more accurate estimate of the loss probability can be obtained. The number of revaluations can therefore be reduced to obtain a similar accuracy then that of only using the delta-gamma approximation.

Table 2 displays the result from out simulation. The return estimate under importance sampling , 10.1%, is significantly different from the estimated loss probability without using importance sampling , which is 5%. This shows that our results are inaccurate and that there has been a calculation error. The mean of the losses under IS is 1510108 which is close to the threshold selected at 5 % which is 1438534. Since the mean is higher than for x we would expect more losses to exceed x which we increase our return estimate. The variance reduction in the estimated loss probability under IS 3.1 which means that three times less revaluations have to be done in order to get the same accuracy as without importance sampling. This variance reduction is still a lot lower than what was found in the paper.

There are a number of possible reason why our estimates are off. The possible reasons are, in order of likelihood :

1. The calculation of theta for a specified value of the loss threshold x can not be solved analytically and numerical iterative processes need to be used in order to optimize this value. R does not have such a function and therefore excel was used. The calculation of the characteristic function is very sensitive to the value of theta and and theta is a very small number. Therefore there could have been a mismatch between the solving theta in excel and the using the value in R.

2. Step 1 was adapted to suite the computation in R. The process as described in the paper is a bit more stringent than just finding the Choleksi decomposition of the covariance matrix. A orthogonal matrix V should be computed, where the columns are the eigenvectors of the matrix $-\frac{1}{2}\delta S'\Gamma\delta S$ with Λ, a diagonal matrix of associated eigenvalues. However when computing these in R, R sorts the eigenvalues in descending order so that there is a mismatch between eigenvalues and the corresponding changes in the risk factors. The eigenvalues can be manually rearranged so that they correspond to the risk factors, however we chose the simpler approach as described. Our simplified approach should get the same approximate value but there could be error which gives the wrong value of theta.

3. When calculating the nominal changes in risk factors we first simulate form a standard normal distribution and multiply by the Choleksi decomposition and then the prices of the risk factors. For the calculation of theta for a given threshold, it does not take into account the prices and therefore there could be a mismatch when the new IS distribution is sampled from and then the prices are multiplied.

4. The portfolios used on the paper consisted only of options and not of stocks. Options are in general a lot more volatile than stocks. By selecting a portfolio containing stocks the variability of the losses should be lower than for a portfolio consisting only of options. The variance reduction will likely will be the less when using importance sampling if the portfolio contains a large weight of stocks.

To solve these problems more efficient coding needs to be produced that can solve the problems. A possible suggestion is to use to MatLab instead of R so that theta can be solved more accurately. The next step is calculate the variance reduction using stratified sampling. For our analysis we only focused on importance sampling as it is the the main result of the paper. Stratified sampling however is a well known method to reduce variance and can be applied as described previously.

5 Summary

The aim of the paper was to find effective Monte Carlo methods to reduce the computing time necessary to calculate the value at risk of a portfolio. The chosen methods were strategic sampling strategies namely: importance sampling and stratified sampling. The paper successfully provided theoretical frameworks for using these method as an aid to the delta-gamma approximation and found that empirical results matched up to the proposed hypothesis. The results showed that computing time could be reduced between 20 to a 100 times. In our analysis we found that the method is quite complicated which lead to our results being accurate. The basic idea behind importance was however verified as we have shown that we could produce larger losses around the specified quantile and that indeed variance reduction is achieved by using the method.

6 Appendices

6.1 Appendix A

Once we have found the likelihood function, the question becomes, what values of μ and Σ should we choose? The following is a summary of the discussion in the paper (Glasserman *et al.* (2000*a*)).We inspect values that will produce large losses by looking at the delta-gamma approximation. We have for the equation of losses :

$$L \approx \sum_i b_i Z_i + \sum_i \Lambda Z_i^2$$

We can see from this equation that, (Glasserman *et al.* (2000*a*)) :

1. large positive values of Z_i for those i with $b_i > 0$
2. n large negative values of Z_i for those i with $b_i < 0$
3. n large values of Z_i^2 for those i with $\lambda_i > 0$

So this can be accomplished by, (Glasserman *et al.* (2000*a*) :

1. increase the mean of Z_i for those i with $b_i > 0$;
2. n decrease the mean of Z_i for those i with $b_i < 0$;
3. n increase the variance of Z_i for those i with $\lambda_i > 0$;

We first reduce the choice of μ and Σ to the choice of a scalar parameter θ, and then specify the value of this parameter. For any $\theta > 0$:

$$\Sigma(\theta) = (I - 2\theta)^{-1}, \mu = \theta\Sigma(\mu\theta)b$$

With these parameters, Z_i becomes normal with mean and variance :

$$\mu_i(\theta) = \frac{\theta b_i}{1 - 2\theta\lambda_i}, \sigma_i^2(\theta) = \frac{1}{1 - 2\theta\lambda_i}$$

.

6.2 Appendix B

The vector S so far has been described as the market prices of risk factors. However, volatility could also be included as a risk factor. When including volatility as a risk facto, the vector S could be partitioned into asset prices and the implied volatility of those asset prices which we can write as $(\tilde{S}, \tilde{\Sigma}$, where $\tilde{\sigma_i}$ is the implied volatility of $\tilde{S_i}$. The paper makes the following asssumptions : "the correlation among asset prices, among implied volatilities, and between asset prices and implied volatilities remains unchanged over the VaR horizon".And also that "the changes $(\delta\tilde{S}\ \delta\tilde{\sigma})$ over the VAR horizon are conditionally normally distributed, given the current history of prices and implied volatilities, with a conditional mean of 0 and a known conditional covariance matrix". (Glasserman *et al.* (2000*a*)).

To include risk as a volatility factor, the partial derivative of a portfolio value with respect to the implied volatility , σ needs to be calculated. This partial derivative vector is called vega and is denoted as v_i. The authors assume that the second order partial derivatives with respect to implied volatilities is zero. The delta-gamma approximation can then be written as:

$$L \approx -(\Delta' v') \begin{pmatrix} \delta\tilde{S} \\ \delta\tilde{\sigma} \end{pmatrix} - \frac{1}{2}(\delta\tilde{S}'\ \delta\tilde{\sigma}') \begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{S} \\ \delta\tilde{\sigma} \end{pmatrix}$$

The same analysis could then be used as described throughout the paper.

References

Glasserman, P., Heidelberger, P. and Shahabuddin, P. (2000*a*). Efficient monte carlo methods for value-at-risk.

Glasserman, P., Heidelberger, P. and Shahabuddin, P. (2000*b*). Variance reduction techniques for estimating value-at-risk. *Management Science*, vol. 46, no. 10, pp. 1349–1364.

Glasserman, P., Heidelberger, P. and Shahabuddin, P. (2002). Portfolio value-at-risk with heavy-tailed risk factors. *Mathematical Finance*, vol. 12, no. 3, pp. 239–269.

Heidelberger, P. (1995). Fast simulation of rare events in queueing and reliability models. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, vol. 5, no. 1, pp. 43–85.

Sadowsky, J.S. and Bucklew, J.A. (1990). On large deviations theory and asymptotically efficient monte carlo estimation. *IEEE transactions on Information Theory*, vol. 36, no. 3, pp. 579–588.