10 ALGEBRAIC METHODS OF SYNTHESIS USING DISPLACEMENT EQUATIONS

10-1 DISPLACEMENT EQUATION OF THE FOUR-BAR LINKAGE

Consider a planar four-bar linkage $O_A ABO_B$ (Fig. 10-1). This linkage is characterized by having four revolutes with parallel axes, the distances between successive axes being the parameters a_1, a_2, a_3, a_4. The synthesis of four-bar linkages, or the determination of the four parameters that will yield an approximation to a desired function between the input (crank) and output (follower) angles, has been approached in the last chapters by geometric methods. In this chapter, algebraic methods for the synthesis of four-bar linkages as well as other planar mechanisms will be considered. Such methods of synthesis are based on displacement equations, i.e., equations relating the input and output variables of a mechanism in terms of its fixed parameters.

The displacement equation of the four-bar linkage may be obtained by considering a rectangular-coordinate system $O_A xy$ (Fig. 10-1) with respect to which the coordinates of A and B may be written as follows:

For A:
$$x_2 = a_1 \cos \phi$$
$$y_2 = a_1 \sin \phi$$
For B:
$$x_3 = -a_4 + a_3 \cos \psi$$
$$y_3 = a_3 \sin \psi$$

Since the distance AB is fixed and equal to a_2, application of Pythagoras' theorem yields

$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = a_2^2$$

or
$$(a_1 \cos \phi + a_4 - a_3 \cos \psi)^2 - (a_1 \sin \phi - a_3 \sin \psi)^2 = a_2^2$$

After trigonometric simplifications this may be written

$$A \sin \psi + B \cos \psi = C \tag{10-1}$$

where[1]

$$A = \sin \phi \qquad B = \frac{a_4}{a_1} + \cos \phi \qquad C = \frac{a_4}{a_3} \cos \phi + \frac{a_1^2 - a_2^2 + a_3^2 + a_4^2}{2a_1a_3}$$

Equation (10-1) may be solved for a displacement analysis of the four-bar linkage; that is, ψ is found explicitly as a function of ϕ and the parameters a_1, a_2, a_3, a_4. Such a solution is obtained by expressing $\sin \psi$ and $\cos \psi$ in terms of $\tan (\psi/2)$,

$$\sin \psi = \frac{2 \tan (\psi/2)}{1 + \tan^2 (\psi/2)} \qquad \cos \psi = \frac{1 - \tan^2 (\psi/2)}{1 + \tan^2 (\psi/2)}$$

and substituting those values in Eq. (10-1) to get

$$2A \tan \frac{\psi}{2} + B\left(1 - \tan^2 \frac{\psi}{2}\right) = C\left(1 + \tan^2 \frac{\psi}{2}\right)$$

or
$$(B + C) \tan^2 \frac{\psi}{2} - 2A \tan \frac{\psi}{2} - B + C = 0$$

from which
$$\tan \frac{\psi}{2} = \frac{A \pm \sqrt{A^2 + B^2 - C^2}}{B + C}$$

For each value of ϕ the quantities A, B, C may be obtained and

[1] Do not confuse the quantities A, B, and C with the points A and B (Fig. 10-1).

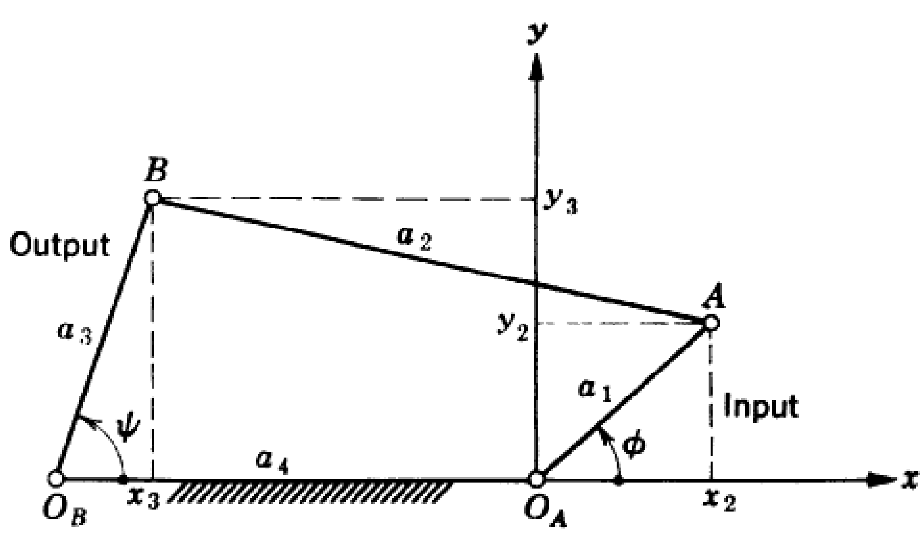

FIGURE 10-1 Planar four-bar linkage; coordinates of A and B.

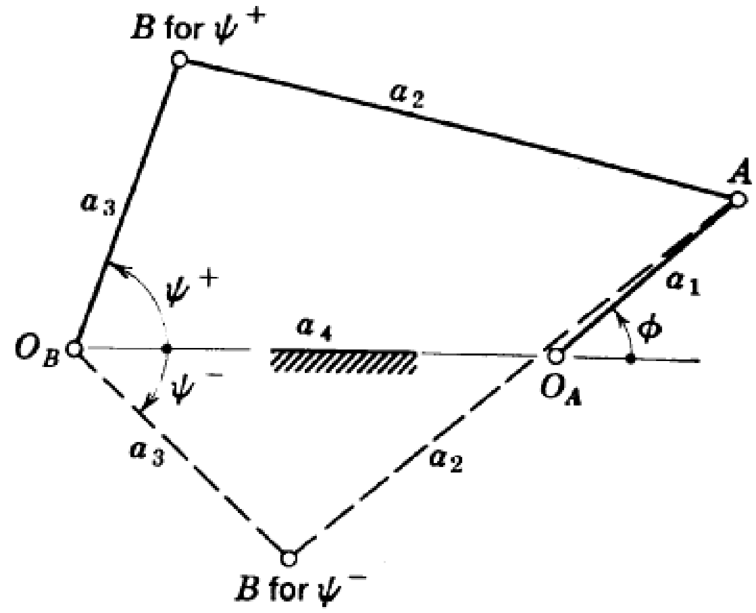

FIGURE 10-2 Two solutions of the equation of the four-bar linkage.

two distinct values of ψ found as

$$\psi^+ = 2 \arctan \frac{A + \sqrt{A^2 + B^2 - C^2}}{B + C}$$
$$\psi^- = 2 \arctan \frac{A - \sqrt{A^2 + B^2 - C^2}}{B + C} \qquad (10\text{-}2)$$

These two values correspond to the two ways in which a four-bar linkage may be closed (Fig. 10-2).

10-2 CRANK AND FOLLOWER SYNTHESIS: THREE ACCURACY POINTS

Consider the problem of designing a planar four-bar linkage such that to three given positions of the crank, defined by angles ϕ_1, ϕ_2, and ϕ_3, there correspond three prescribed positions of the follower, ψ_1, ψ_2, and ψ_3. The solution consists in finding the proper values of a_1, a_2, a_3, and a_4 for three related pairs (ϕ_1, ψ_1), (ϕ_2, ψ_2), and (ϕ_3, ψ_3). The procedure is based on the displacement equation[1]

$$K_1 \cos \phi - K_2 \cos \psi + K_3 = \cos (\phi - \psi) \qquad (10\text{-}3)$$

with $\qquad K_1 = \dfrac{a_4}{a_3} \qquad K_2 = \dfrac{a_4}{a_1} \qquad K_3 = \dfrac{a_1^2 - a_2^2 + a_3^2 + a_4^2}{2a_1a_3}$

This equation was deduced from Eq. (10-1) by rearranging the terms. When written for three pairs of values, (ϕ_1, ψ_1), (ϕ_2, ψ_2), (ϕ_3, ψ_3), this equation yields a system of three equations linear with respect to K_1, K_2, K_3,

$$K_1 \cos \phi_1 - K_2 \cos \psi_1 + K_3 = \cos (\phi_1 - \psi_1)$$
$$K_1 \cos \phi_2 - K_2 \cos \psi_2 + K_3 = \cos (\phi_2 - \psi_2)$$
$$K_1 \cos \phi_3 - K_2 \cos \psi_3 + K_3 = \cos (\phi_3 - \psi_3)$$

[1] This is also known as the Freudenstein equation (see first reference in Bibliography at the end of this chapter).

Tedious third-order determinants may be avoided by first subtracting the second and third equations from the first, thus eliminating K_3,

$$K_1(\cos\phi_1 - \cos\phi_2) - K_2(\cos\psi_1 - \cos\psi_2) = \cos(\phi_1 - \psi_1) - \cos(\phi_2 - \psi_2)$$
$$K_1(\cos\phi_1 - \cos\phi_3) - K_2(\cos\psi_1 - \cos\psi_3) = \cos(\phi_1 - \psi_1) - \cos(\phi_3 - \psi_3)$$

and solving the resulting system of two equations with two unknowns; thus,

$$K_1 = \frac{w_2w_6 - w_3w_5}{w_2w_4 - w_1w_5} \qquad K_2 = \frac{w_1w_6 - w_3w_4}{w_2w_4 - w_1w_5}$$

in which

$$\begin{aligned}
w_1 &= \cos\phi_1 - \cos\phi_2 \\
w_2 &= \cos\psi_1 - \cos\psi_2 \\
w_3 &= \cos(\phi_1 - \psi_1) - \cos(\phi_2 - \psi_2) \\
w_4 &= \cos\phi_1 - \cos\phi_3 \\
w_5 &= \cos\psi_1 - \cos\psi_3 \\
w_6 &= \cos(\phi_1 - \psi_1) - \cos(\phi_3 - \psi_3)
\end{aligned}$$

Substituting values of K_1 and K_2 into one of the three original equations yields K_3 as

$$K_3 = \cos(\phi_i - \psi_i) - K_1\cos\phi_i + K_2\cos\psi_i \qquad i = 1, 2, \text{ or } 3$$

With the values of K_1, K_2, and K_3 known, the parameters of the linkage may be found from the relations

$$a_1 = \frac{a_4}{K_2} \qquad a_3 = \frac{a_4}{K_1} \qquad a_2 = \sqrt{a_1^2 + a_3^2 + a_4^2 - 2a_1a_3K_3}$$

The parameter a_4 may be given a positive but arbitrary value, usually taken as unity. This parameter merely determines the size of the linkage and has no effect on the angular relationships.

10-3 EXAMPLES: FOUR-BAR FUNCTION GENERATORS WITH THREE ACCURACY POINTS

The design of four-bar function generators, already carried out by geometric methods in Secs. 8-4 and 8-5, is reconsidered here as an application of the three-accuracy-point synthesis developed in the last section.

Example 1 The function $y = \log x$ is to be generated in the interval $1 \le x \le 2$ by means of a four-bar linkage O_AABO_B (Fig. 10-3). The basic elements of the problem are here the same as in Sec. 8-4. The variables x and y are represented, respectively, by the crank and follower

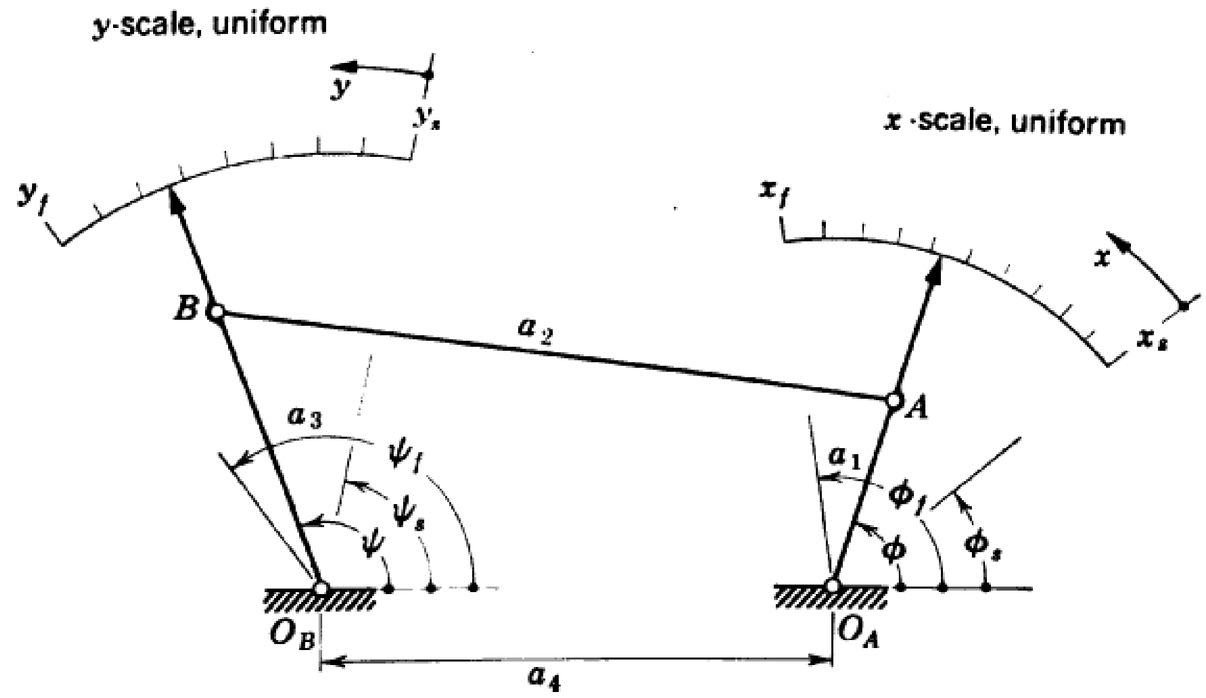

FIGURE 10-3 Principle of four-bar-linkage function generator.

angles ϕ and ψ through the relations

$$\frac{\phi - \phi_s}{\Delta\phi} = \frac{x - x_s}{\Delta x}$$
$$\frac{\psi - \psi_s}{\Delta\psi} = \frac{y - y_s}{\Delta y}$$

The reader is referred to Sec. 8-4 for the details of the formulation of the problem and the definitions of the symbols used. Three accuracy points are taken in the interval $1 \leq x \leq 2$ with Chebyshev spacing, whence the corresponding values of the variables x and y are

$$x_1 = 1.067 \qquad y_1 = 0.0282$$
$$x_2 = 1.5 \qquad y_2 = 0.1761$$
$$x_3 = 1.933 \qquad y_3 = 0.2862$$

The ranges of variation of ϕ and ψ must be selected. They are chosen as $\Delta\phi = \Delta\psi = 60°$. The rotations of the crank and follower from the position corresponding to the first accuracy point to the positions corresponding to the other two are, with the computation carried to $\frac{1}{10}°$,

$$\phi_2 - \phi_1 = \frac{x_2 - x_1}{x_f - x_s}\Delta\phi = 26.0° \qquad \psi_2 - \psi_1 = \frac{y_2 - y_1}{y_f - y_s}\Delta\psi = 29.4°$$
$$\phi_3 - \phi_1 = \frac{x_3 - x_1}{x_f - x_s}\Delta\phi = 52.0° \qquad \psi_3 - \psi_1 = \frac{y_3 - y_1}{y_f - y_s}\Delta\psi = 51.4°$$

With the present method, the angles ϕ_1 and ψ_1, crank and follower positions corresponding to the first accuracy point, must also be selected at the start. Choosing $\phi_1 = 0$ and $\psi_1 = 0$ yields

$$\phi_2 = 26.0° \qquad \psi_2 = 29.4°$$
$$\phi_3 = 52.0° \qquad \psi_3 = 51.4°$$

from which $w_1 = 0.1011$ $\quad w_2 = 0.1285$ $\quad w_3 = 0.0017$

$w_4 = 0.3839$ $\quad w_5 = 0.3766$ $\quad w_6 = 0$

giving $K_1 = -0.05777$ $\quad K_2 = -0.05900$ $\quad K_3 = 0.99877$

With the frame $a_4 = 1$ unit of length, the other three parameters of the linkage are found as

$$a_1 = -16.95 \qquad a_2 = 1.36 \qquad a_3 = -17.31$$

This linkage, with two long links (crank $a_1 = 16.95$, follower $a_3 = 17.31$) and two relatively short links (frame $a_4 = 1.00$, coupler $a_2 = 1.36$), has poor force-transmission qualities and is not an acceptable solution. In point of fact, application of the Grashof criterion shows this linkage to have change points; on second thought, this was inevitable, since both crank and follower had starting angles of 0°. The unsatisfactory solution was compounded from unhappy choices of arbitrary values—starting angles and ranges of motion.

If the idea of the spread associated with the 60° ranges seems desirable and this feature is to be retained, only one alternative exists, viz., different starting positions for ϕ and ψ.

A second attempt, in which $\phi_1 = 45°$ ($\phi_2 = 71°$, $\phi_3 = 97°$) and $\psi_1 = 0°$ ($\psi_2 = 29.5°$, $\psi_3 = 51.4°$) were assumed, with $a_4 = 1.0$, yielded $a_1 = -1.031$, $a_2 = 2.682$, $a_3 = -2.310$. These linkage proportions are favorable to force transmission, and the design may be considered as acceptable, if it is recognized that it is a double rocker. The new linkage, drawn in position 1, is shown in Fig. 10-4. The negative signs for a_1 and

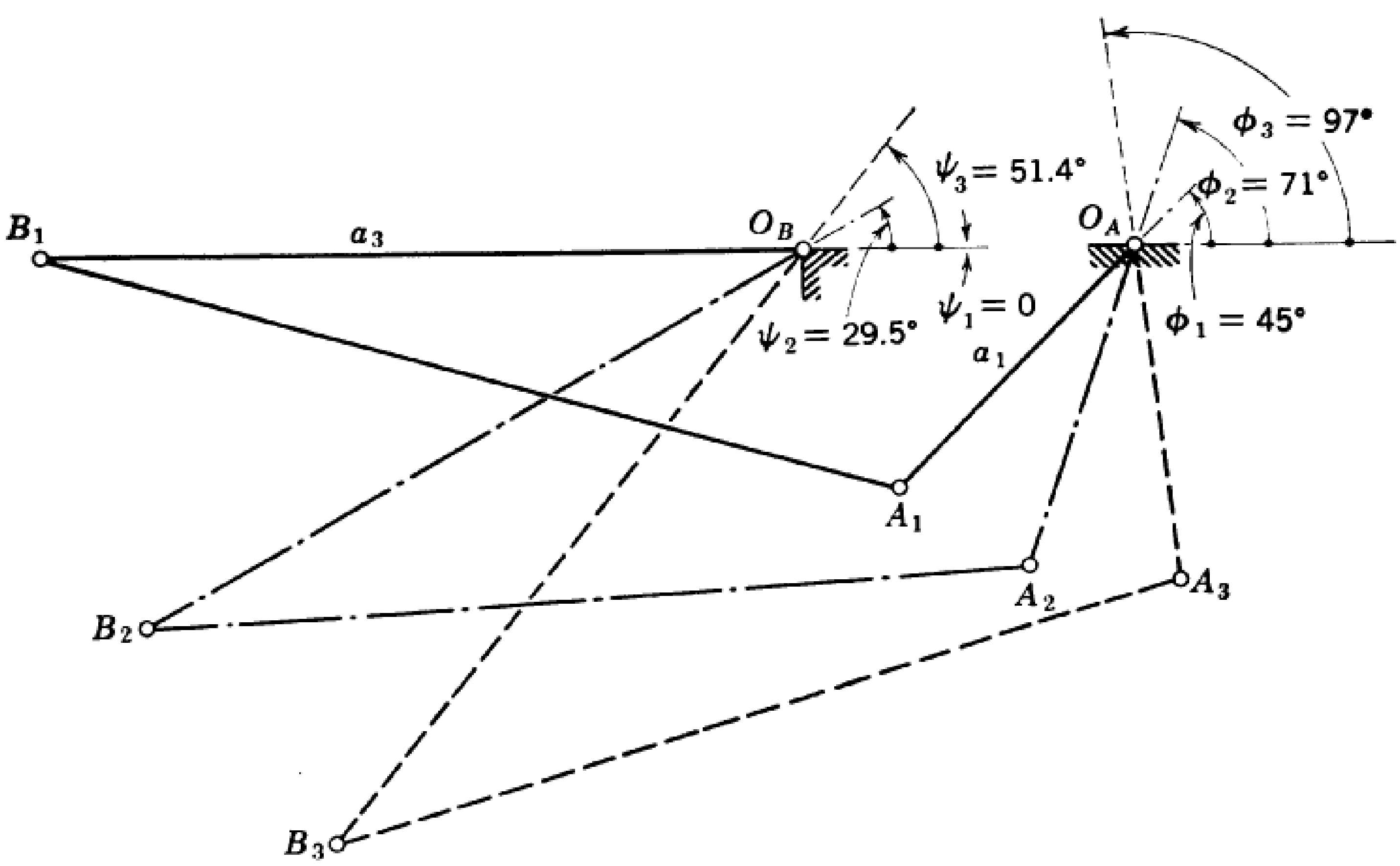

FIGURE 10-4 Example 1, function generator $y = \log x$, $1 \leq x \leq 2$, with three accuracy points, second attempt.

a_3 are interpreted by considering O_AA and O_BB as vectors: the angles ϕ and ψ define their direction; the parameters a_1 and a_3 define their magnitudes and the sense in which they are to be laid off. A graphical check of this linkage for the three accuracy points shows that no large error is present. To determine the structural error accurately, an analysis must be carried out by using Eqs. (10-2) developed in Sec. 10-1. The results of this analysis for values of ϕ in the interval $\phi_s \leq \phi \leq \phi_f$ at 6° intervals are summarized in Table 10-1.

Table 10-1 ERROR IN LOG-FUNCTION GENERATOR, THREE ACCURACY POINTS

x	ϕ, DEG	ψ, DEG	$\log x$	y_{mech}	$y_{\text{mech}} - \log x$
1.0	41.0	−6.1	0	−0.025	−0.025
1.1	47.0	2.8	0.041	0.042	0.001
1.2	53.0	10.4	0.079	0.080	0.001
1.3	59.0	17.3	0.114	0.115	0.001
1.4	65.0	23.5	0.146	0.146	0
1.5	71.0	29.4	0.176	0.176	0
1.6	77.0	34.9	0.204	0.203	−0.001
1.7	83.0	40.1	0.230	0.229	−0.001
1.8	89.0	45.1	0.255	0.254	−0.001
1.9	95.0	49.9	0.279	0.278	−0.001
2.0	101.0	54.5	0.301	0.302	0.001

By taking $\psi = \psi^+$, the structural error, i.e., the difference between the values of y_{mech} given by the linkage and the corresponding values of $\log x$, is shown in the last column. As expected, this structural error vanishes at the accuracy points. The maximum structural error, occurring at $x = 1.0$, is $\epsilon = -0.025$, or 8.3 percent of the range of variation of y.

With algebraic methods of synthesis and analysis, all quantities may be calculated to any desired degree of accuracy, which means the elimination of graphical error. Structural error remains, as does the mechanical error deriving from machining tolerances and deformations of links. The evaluation of this mechanical error will be considered in a later section.

Example 2 A four-bar linkage is to generate the function $y = 1/x$ over the interval $1 \leq x \leq 2$. The ranges of variation are to be $\Delta\phi = 90°$ and $\Delta\psi = 90°$ with three accuracy points having Chebyshev spacing. The accuracy points are then

$$x_1 = 1.067 \qquad x_2 = 1.5 \qquad x_3 = 1.933$$

with associated functional values of

$$y_1 = 0.9372 \qquad y_2 = 0.6666 \qquad y_3 = 0.5173$$

Taking $\phi_1 = 45°$, $\psi_1 = 0$ yields

$$\phi_2 = \phi_1 + \frac{x_2 - x_1}{x_f - x_s}\,\Delta\phi = 84.0° \qquad \psi_2 = \psi_1 + \frac{y_2 - y_1}{y_f - y_s}\,\Delta\psi = -48.7°$$

$$\phi_3 = \phi_1 + \frac{x_3 - x_1}{x_f - x_s}\,\Delta\phi = 123.0° \qquad \psi_3 = \psi_1 + \frac{y_3 - y_1}{y_f - y_s}\,\Delta\psi = -75.6°$$

from which the linkage parameters are found as

$$a_1 = 0.036 \qquad a_2 = 0.970 \qquad a_3 = 0.056 \qquad \text{with } a_4 = 1.0$$

In this linkage the crank and follower lengths are very small compared with the frame and coupler lengths, and another "try" is indicated. We hopefully choose $\phi_1 = 45°$, $\psi_1 = 90°$. The crank angles ϕ corresponding

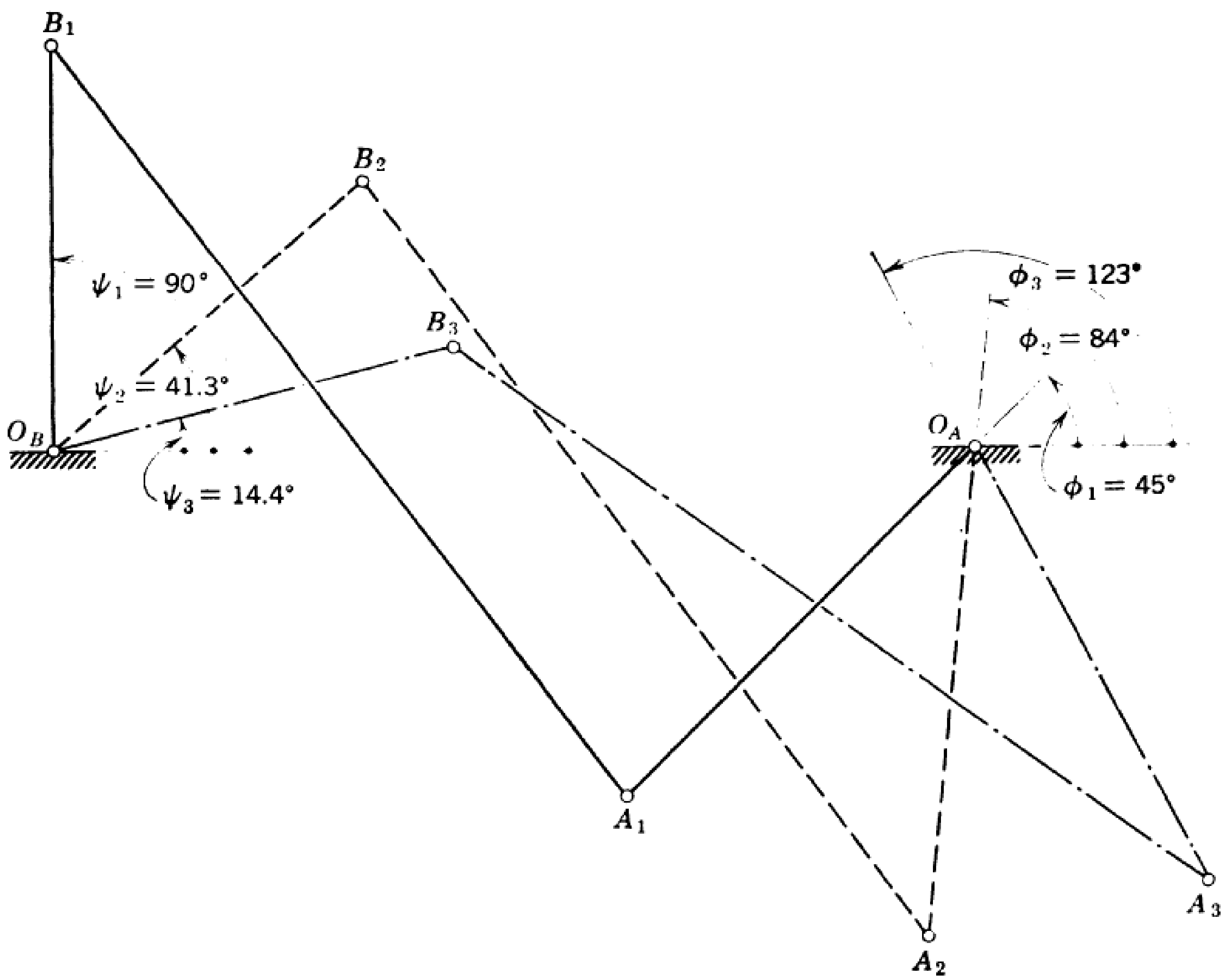

FIGURE 10-5 Example 2, function generator $y = 1/x$, $1 \le x \le 2$, with three accuracy points, second attempt.

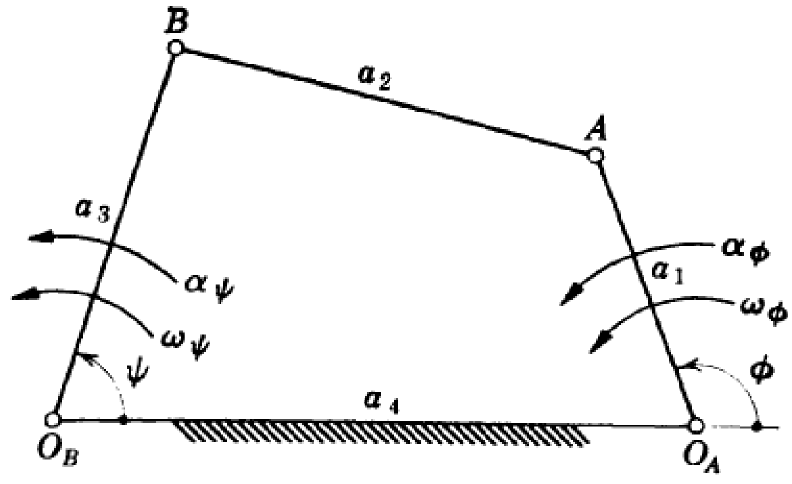

FIGURE 10-6 Notation for velocity and acceleration synthesis.

to the accuracy points remain as above, but the follower angles become

$$\psi_1 = 90° \qquad \psi_2 = 41.3° \qquad \psi_3 = 14.4°$$

With these new specifications the linkage parameters are found as

$$a_1 = -0.547 \qquad a_2 = 1.035 \qquad a_3 = 0.447 \qquad \text{when } a_4 = 1.0$$

This new linkage is shown in Fig. 10-5. A displacement analysis similar to that described in the last example shows that the maximum structural error is $\epsilon = 0.015$, that is, 3 percent of the range of variation of y.

10-4 CRANK AND FOLLOWER SYNTHESIS: ANGULAR VELOCITIES AND ACCELERATIONS

The design of a planar four-bar linkage having prescribed angular velocities and angular accelerations of the crank and follower for a desired phase, i.e., for specified values of ϕ and ψ, may be carried out in a fashion similar to the foregoing. If, in the four-bar linkage of Fig. 10-6, the crank angle is ϕ, the angular velocity and angular acceleration of the crank are defined by

$$\omega_\phi = \frac{d\phi}{dt} \qquad \text{and} \qquad \alpha_\phi = \frac{d^2\phi}{dt^2}$$

With a follower angle ψ, the angular velocity and angular acceleration of the follower become

$$\omega_\psi = \frac{d\psi}{dt} \qquad \text{and} \qquad \alpha_\psi = \frac{d^2\psi}{dt^2}$$

A linkage may be designed such that, when the crank has a specified position, angular velocity, and acceleration, the follower will also have a specified position, angular velocity, and acceleration. In other words, it is necessary to determine the parameters a_1, a_2, a_3, a_4 such that a given set of values ϕ, ω_ϕ, α_ϕ will give rise to desired values of ψ, ω_ψ, α_ψ.

This problem may be solved by taking the first and second time

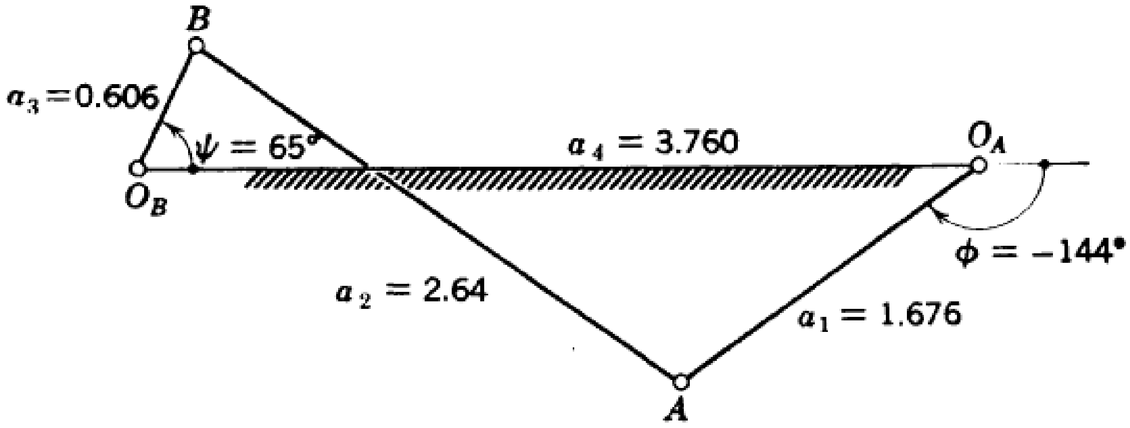

FIGURE 10-7 Example of velocity and acceleration synthesis, completed linkage.

derivatives of Eq. (10-3),

$$K_1 \cos\phi - K_2 \cos\psi + K_3 = \cos(\phi - \psi)$$
$$K_1\omega_\phi \sin\phi - K_2\omega_\psi \sin\psi = (\omega_\phi - \omega_\psi)\sin(\phi - \psi)$$
$$K_1(\alpha_\phi \sin\phi + \omega_\phi^2 \cos\phi) - K_2(\alpha_\psi \sin\psi + \omega_\psi^2 \cos\psi) = (\alpha_\phi - \alpha_\psi)\sin(\phi - \psi) + (\omega_\phi - \omega_\psi)^2 \cos(\phi - \psi)$$

Solving the last two equations of this system for K_1 and K_2 and substituting the values obtained in the first equation to get K_3 yields

$$K_1 = \frac{w_2w_6 - w_3w_5}{w_2w_4 - w_1w_5} \qquad K_2 = \frac{w_1w_6 - w_3w_4}{w_2w_4 - w_1w_5}$$

where
$$w_1 = \omega_\phi \sin\phi$$
$$w_2 = \omega_\psi \sin\psi$$
$$w_3 = (\omega_\phi - \omega_\psi)\sin(\phi - \psi)$$
$$w_4 = \alpha_\phi \sin\phi + \omega_\phi^2 \cos\phi$$
$$w_5 = \alpha_\psi \sin\psi + \omega_\psi^2 \cos\psi$$
$$w_6 = (\alpha_\phi - \alpha_\psi)\sin(\phi - \psi) + (\omega_\phi - \omega_\psi)^2 \cos(\phi - \psi)$$
and
$$K_3 = \cos(\phi - \psi) - K_1 \cos\phi + K_2 \cos\psi$$

As in the case of three-accuracy-point synthesis, the parameters of the linkage are given by

$$a_1 = \frac{a_4}{K_2} \qquad a_3 = \frac{a_4}{K_1} \qquad a_2 = \sqrt{a_1^2 + a_3^2 + a_4^2 - 2a_1a_3K_3}$$

Example Design a four-bar linkage meeting the following specifications:

$$\phi = -144° \qquad \omega_\phi = -3.0 \text{ rad/sec} \qquad \alpha_\phi = 0$$
$$\psi = 65° \qquad \omega_\psi = 8.0 \text{ rad/sec} \qquad \alpha_\psi = 0$$

Taking $a_4 = 3.760$,* the remaining parameters of the required linkage are

* This value of a_4 was chosen to allow a comparison with a solution gained from complex numbers (Sec. 11-1).

found as

$$a_1 = 1.676 \qquad a_2 = 2.640 \qquad a_3 = 0.606$$

The linkage is shown in Fig. 10-7.

10-5 GENERALIZATION OF THE SYNTHESIS METHOD BY LINEAR EQUATIONS

The foregoing methods of synthesis of the four-bar linkage in terms of linear equations only were made possible by the form of the displacement equation [Eq. (10-3)]. In this, the three coefficients K_1, K_2, and K_3 are (1) functions of three design parameters a_1/a_4, a_2/a_4, a_3/a_4 and (2) independent of the input and output variables, viz., the crank and follower angles ϕ and ψ. It is therefore possible to generalize this method to other linkages for which the displacement equation may be written in a form having essentially the same properties. Consider a general linkage in which the number of design parameters is n and for which ϕ and ψ are the input and output variables, analogous to the crank and follower angles in the case of the four-bar linkage. If the displacement equation of the linkage is written in the form

$$\sum_{i=1}^{n} K_i(n \text{ design parameters})G_i(\phi, \psi) = F(\phi, \psi)$$

then, writing this equation for n pairs of values (ϕ_j, ψ_j), $j = 1, 2, \ldots, n$, there will be n equations linear in $K_1, K_2, \ldots, K_n$,

$$\begin{aligned}
K_1G_1(\phi_1, \psi_1) + K_2G_2(\phi_1, \psi_1) + \cdots + K_nG_n(\phi_1, \psi_1) &= F(\phi_1, \psi_1)\\
K_1G_1(\phi_2, \psi_2) + K_2G_2(\phi_2, \psi_2) + \cdots + K_nG_n(\phi_2, \psi_2) &= F(\phi_2, \psi_2)\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
K_1G_1(\phi_n, \psi_n) + K_2G_2(\phi_n, \psi_n) + \cdots + K_nG_n(\phi_n, \psi_n) &= F(\phi_n, \psi_n)
\end{aligned}$$

which may be solved for $K_1, K_2, \ldots, K_n$. Since these K's are functions of n design parameters, this leads to a set of n equations from which the n design parameters may be found.

10-6 SYNTHESIS OF THE SLIDER-CRANK MECHANISM WITH THREE ACCURACY POINTS

The general approach to synthesis outlined in the last section will be applied here to the slider-crank mechanism (Fig. 10-8). The first step in the synthesis is to derive a displacement equation taking into account all possible parameters of the linkage. This may be achieved by writing the coordinates of points A and B with respect to the set of axes O_Axy as

For A:
$$x_A = a_1 \cos \phi$$
$$y_A = a_1 \sin \phi$$

For B:
$$x_B = s$$
$$y_B = a_3$$

and then expressing the distance $AB = a_2$ as

$$(AB)^2 = (x_B - x_A)^2 + (y_B - y_A)^2$$

or

$$a_2^2 = (s - a_1 \cos \phi)^2 + (a_3 - a_1 \sin \phi)^2$$

This, after manipulation and reduction by means of trigonometric identities, becomes

$$2a_1 s \cos \phi + 2a_1 a_3 \sin \phi - (a_1^2 - a_2^2 + a_3^2) = s^2 \qquad (10\text{-}4)$$

To carry out a three-point synthesis relating the crank position ϕ to the slider position s, three coefficients K_1, K_2, K_3 must be defined in terms of the three parameters a_1, a_2, a_3 of the linkage. On setting

$$K_1 = 2a_1 \qquad K_2 = 2a_1 a_3 \qquad K_3 = a_1^2 - a_2^2 + a_3^2$$

Eq. (10-4) takes the form

$$K_1 s \cos \phi + K_2 \sin \phi - K_3 = s^2 \qquad (10\text{-}5)$$

which satisfies the conditions of the last section. With the notation of the last section,

$$G_1 = s \cos \phi \qquad G_2 = \sin \phi \qquad G_3 = -1 \qquad F = s^2$$

are recognized as functions of the input and output variables ϕ and s but independent of the design parameters a. The coefficients K, on the other hand, are functions of the design parameters while independent of the

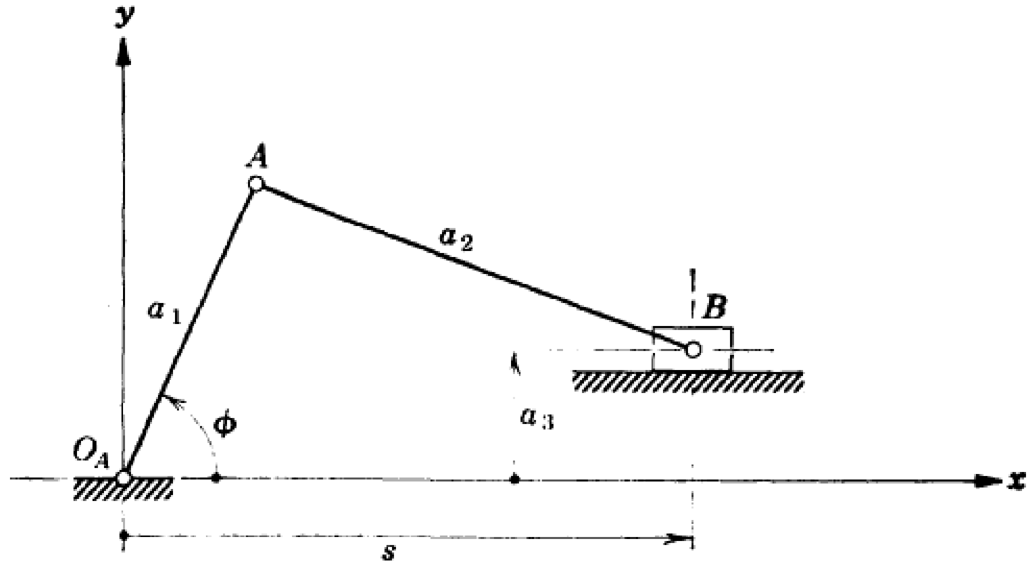

FIGURE 10-8 Slider-crank mechanism showing parameters and variables used in synthesis.

input and output variables. Writing Eq. (10-5) for three pairs of values (ϕ_1, s_1), (ϕ_2, s_2), and (ϕ_3, s_3) yields the system

$$\begin{aligned} K_1 s_1 \cos\phi_1 + K_2 \sin\phi_1 - K_3 &= s_1^2 \\ K_1 s_2 \cos\phi_2 + K_2 \sin\phi_2 - K_3 &= s_2^2 \\ K_1 s_3 \cos\phi_3 + K_2 \sin\phi_3 - K_3 &= s_3^2 \end{aligned}$$

$$K_1 = \frac{w_2 w_6 - w_3 w_5}{w_2 w_4 - w_1 w_5} \qquad K_2 = \frac{w_3 w_4 - w_1 w_6}{w_2 w_4 - w_1 w_5}$$

$$\begin{aligned} w_1 &= s_1 \cos\phi_1 - s_2 \cos\phi_2 & w_4 &= s_1 \cos\phi_1 - s_3 \cos\phi_3 \\ w_2 &= \sin\phi_1 - \sin\phi_2 & w_5 &= \sin\phi_1 - \sin\phi_3 \\ w_3 &= s_1^2 - s_2^2 & w_6 &= s_1^2 - s_3^2 \end{aligned}$$

and K_3 is conveniently given by

$$K_3 = -s_i + K_1 s_i \cos\phi_i + K_2 \sin\phi_i \qquad i = 1, 2, \text{ or } 3$$

The parameters of the linkage now follow as

$$a_1 = \frac{K_1}{2} \qquad a_3 = \frac{K_2}{2a_1} \qquad a_2 = \sqrt{a_1^2 + a_3^2 - K_3}$$

Example Design a slider-crank mechanism in which the slider displacement is proportional to the square of the crank rotation, or

$$\frac{s - s_s}{s_f - s_s} = \left(\frac{\phi - \phi_s}{\phi_f - \phi_s}\right)^2 \tag{10-6}$$

where $\Delta s = s_f - s_s$ is the total displacement of the slider corresponding to the crank rotation $\Delta\phi = \phi_f - \phi_s$. The starting and final values ϕ_s, s_s and ϕ_f, s_f are unspecified and may be chosen by the designer. The following values are assumed for the present example:

$$\begin{aligned} \phi_s &= 45^\circ & s_s &= 8 \text{ in.} \\ \phi_f &= 105^\circ & s_f &= 12 \text{ in.} \end{aligned}$$

The range of variation of the crank is therefore $\Delta\phi = \phi_f - \phi_s = 60^\circ$, and the corresponding displacement of the slider is $\Delta s = s_f - s_s = 4$ in. Three accuracy points are chosen with Chebyshev spacing and yield

$$\phi_1 = 49^\circ \qquad \phi_2 = 75^\circ \qquad \phi_3 = 101^\circ$$

The corresponding values of s are deduced from Eq. (10-6) as

$$s_1 = s_s + (s_f - s_s)\left(\frac{\phi_1 - \phi_s}{\phi_f - \phi_s}\right)^2 = 8.02 \text{ in.}$$

$$s_2 = 9.00 \text{ in.}$$
$$s_3 = 11.48 \text{ in.}$$

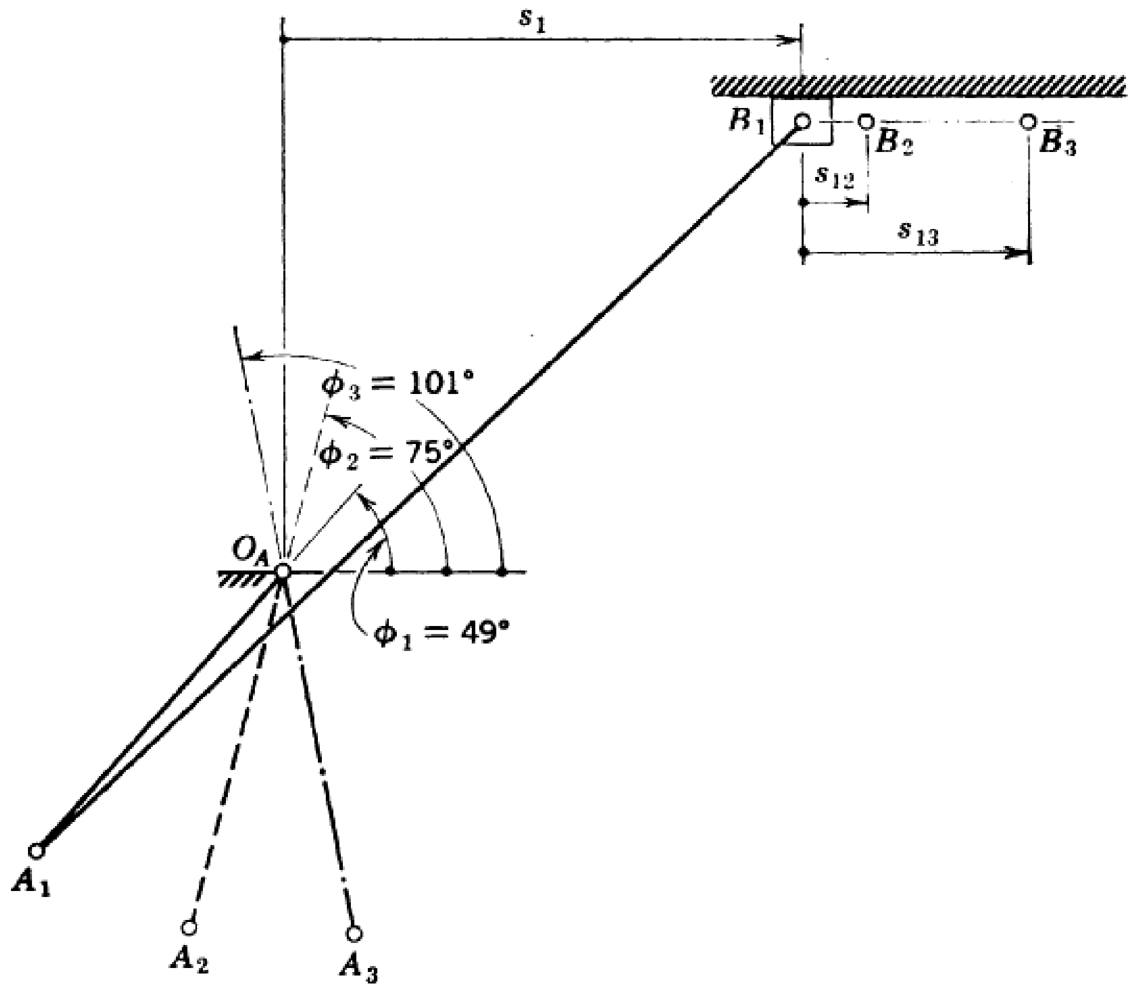

FIGURE 10-9 Example of synthesis of slider-crank mechanism, completed linkage.

The values of the coefficients w follow as

$$w_1 = 2.93 \qquad w_2 = -0.211 \qquad w_3 = -16.8$$
$$w_4 = 7.46 \qquad w_5 = -0.227 \qquad w_6 = -67.8$$

from which $K_1 = -11.7 \qquad K_2 = -84.0 \qquad K_3 = -189.1$

The dimensions of the linkage are found to be

$$a_1 = -5.85 \text{ in.} \qquad a_2 = 16.55 \text{ in.} \qquad a_3 = 7.18 \text{ in.}$$

This linkage is shown in Fig. 10-9.

10-7 SYNTHESIS OF THE SLIDER-CRANK MECHANISM WITH FOUR ACCURACY POINTS

To extend the three-accuracy-point synthesis of the slider-crank mechanism to a higher number of accuracy points, additional design parameters must be taken into account. In the case of four accuracy points, either the crank position ϕ_1 or the slider position s_1 corresponding to the first accuracy point must be included with the link lengths a_1, a_2, and a_3 as a fourth design parameter. The slider position s_1 will be taken in this section as a design parameter, leaving ϕ_1 to be chosen by the designer to meet additional requirements. The slider position corresponding to any accuracy point may now be defined as $s_j = s_1 + s_{1j}$, where s_{1j} is the slider displacement from the position corresponding to the first accuracy point to points 1, 2, 3, and 4 (s_{11} being always zero). In these terms, the displacement equation of the mechanism becomes

$$2a_1(s_1 + s_{1j}) \cos \phi_j + 2a_1a_3 \sin \phi_j - (a_1^2 - a_2^2 + a_3^2) = (s_1 + s_{1j})^2$$

or

$$2a_1s_1 \cos \phi_j + 2a_1s_{1j} \cos \phi_j + 2a_1a_3 \sin \phi_j - (a_1^2 - a_2^2 + a_3^2 + s_1^2) = 2s_1s_{1j} + s_{1j}^2$$

An attempt to apply the general approach to synthesis by linear equations to the present problem would yield four linear equations (corresponding to four accuracy points) with five unknowns,

$$K_1 \cos \phi_j + K_2s_{1j} \cos \phi_j + K_3 \sin \phi_j - K_4 = K_5s_{1j} + s_{1j}^2 \qquad j = 1, 2, 3, 4 \tag{10-7}$$

in which

$$K_1 = 2a_1s_1 \qquad K_2 = 2a_1$$
$$K_3 = 2a_1a_3 \qquad K_4 = a_1^2 - a_2^2 + a_3^2 + s_1^2 \qquad K_5 = 2s_1$$

Such a system of four equations with five unknowns is indeterminate; however, examination of the above definitions of the five K's indicates that some of these unknowns are related to one another,

$$2K_1 = K_2K_5 \qquad \text{or} \qquad 2K_1 - K_2K_5 = 0 \tag{10-8}$$

This additional relation, called the *compatibility equation*, must be added to the system of four linear equations (10-7) to yield a system of five equations for the five unknowns K_i with $i = 1, 2, 3, 4, 5$.

Since the compatibility equation (10-8) is nonlinear, the above system cannot be solved by the simple rules involving five linear equations with five unknowns. Its solution will be better understood if K_5 is denoted as λ. The compatibility equation then becomes

$$2K_1 - K_2\lambda = 0 \tag{10-9}$$

and the system of equations (10-7) is rewritten as

$$K_1 \cos \phi_j + K_2s_{1j} \cos \phi_j + K_3 \sin \phi_j - K_4 = \lambda s_{1j} + s_{1j}^2 \qquad j = 1, 2, 3, 4 \tag{10-10}$$

This system cannot be solved, since the value of λ is unknown; however, K_1, K_2, K_3, K_4 may be expressed in terms of λ. To achieve this, consider

Table 10-2 **SYNTHESIS OF SLIDER–CRANK MECHANISM, FOUR ACCURACY POINTS**

Specifications:

Crank rotations $\phi_{12}, \phi_{13}, \phi_{14}$
Slider displacements s_{12}, s_{13}, s_{14}
Crank position corresponding to first accuracy point, ϕ_1

Parameters:

a_1, a_2, a_3, s_1

Procedure:

1. Compute

$$\phi_2 = \phi_1 + \phi_{12} \qquad \phi_3 = \phi_1 + \phi_{13} \qquad \phi_4 = \phi_1 + \phi_{14}$$

2. Solve the two systems of linear equations

$$\begin{aligned}
l_1(\cos\phi_1) \qquad\qquad\qquad\quad &+ l_3(\sin\phi_1) - l_4 = 0\\
l_1(\cos\phi_2) + l_2(s_{12}\cos\phi_2) &+ l_3(\sin\phi_2) - l_4 = s_{12}\\
l_1(\cos\phi_3) + l_2(s_{13}\cos\phi_3) &+ l_3(\sin\phi_3) - l_4 = s_{13}\\
l_1(\cos\phi_4) + l_2(s_{14}\cos\phi_4) &+ l_3(\sin\phi_4) - l_4 = s_{14}
\end{aligned}$$

and

$$\begin{aligned}
m_1(\cos\phi_1) \qquad\qquad\qquad\quad &+ m_3(\sin\phi_1) - m_4 = 0\\
m_1(\cos\phi_2) + m_2(s_{12}\cos\phi_2) &+ m_3(\sin\phi_2) - m_4 = s_{12}^2\\
m_1(\cos\phi_3) + m_2(s_{13}\cos\phi_3) &+ m_3(\sin\phi_3) - m_4 = s_{13}^2\\
m_1(\cos\phi_4) + m_2(s_{14}\cos\phi_4) &+ m_3(\sin\phi_4) - m_4 = s_{14}^2
\end{aligned}$$

NOTE: The coefficients of the l's and m's are the same in both systems; only the second members differ, allowing the elimination of unknowns to be carried out in similar fashion for both systems.

3. Compute the discriminant $\Delta = (m_2 - 2l_1)^2 + 8m_1l_2$:
If $\Delta < 0$, there is no solution.
If $\Delta = 0$, $\lambda = \dfrac{2l_1 - m_2}{2l_2}$; solution is unique.
If $\Delta > 0$, $\lambda = \dfrac{2l_1 - m_2 + \sqrt{\Delta}}{2l_2}$ or $\lambda = \dfrac{2l_1 - m_2 - \sqrt{\Delta}}{2l_2}$; there are two solutions, one for each λ.

4. Compute for each λ:

$$\begin{aligned}
K_1 &= \lambda l_1 + m_1 &\quad\text{and}\quad a_1 &= \frac{K_2}{2}\\
K_2 &= \lambda l_2 + m_2 & s_1 &= \frac{\lambda}{2}\\
K_3 &= \lambda l_3 + m_3 & a_3 &= \frac{K_3}{K_2}\\
K_4 &= \lambda l_4 + m_4 & a_2 &= \sqrt{a_1^2 + a_3^2 + s_1^2 - K_4}
\end{aligned}$$

the two systems of linear equations

$$\left.\begin{aligned} l_1 \cos\phi_j + l_2 s_{1j}\cos\phi_j + l_3 \sin\phi_j - l_4 &= s_{1j} \\ \text{and}\quad m_1 \cos\phi_j + m_2 s_{1j}\cos\phi_j + m_3 \sin\phi_j - m_4 &= s_{1j}^2 \end{aligned}\right. \qquad j = 1, 2, 3, 4$$

obtained by considering one term of the second member at a time. These systems may be solved to yield l_1, l_2, l_3, l_4 and m_1, m_2, m_3, m_4. The unknowns K_1, K_2, K_3, K_4 of the system of equations (10-10) may now be expressed by superposition as

$$K_i = \lambda l_i + m_i \qquad i = 1, 2, 3, 4 \tag{10-11}$$

These values substituted into the compatibility equation (10-9) yield

$$2(\lambda l_1 + m_1) - (\lambda l_2 + m_2)\lambda = 0$$

or

$$l_2\lambda^2 + (m_2 - 2l_1)\lambda - 2m_1 = 0 \tag{10-12}$$

in which λ is the only unknown.

When the discriminant of this equation is positive, two values of λ may be found. For each of these values, K_1, K_2, K_3, K_4 may be evaluated by means of Eq. (10-11), after which the parameters a_1, a_2, a_3 and the slider position s_1 corresponding to the first accuracy point may be found. Thus, if the discriminant of Eq. (10-12) is positive, the problem has two solutions; if the discriminant is zero, there is a unique solution; and if the discriminant is negative the problem has no solution. The complete numerical procedure involved is summarized in Table 10-2.

10-8 CRANK AND FOLLOWER SYNTHESIS: FIVE ACCURACY POINTS

In the three-accuracy-point synthesis considered in Sec. 10-2, the crank and follower angles corresponding to all accuracy points were specified. If the actual angular positions of the crank and follower are left unspecified but if instead their rotations with respect to the position corresponding to the first accuracy point are given, then two additional design parameters, the crank and follower angles ϕ_1 and ψ_1, may be considered in the synthesis. Since the number of design parameters is now five, viz., the three ratios a_1/a_4, a_2/a_4, a_3/a_4 and the two angles ϕ_1 and ψ_1, a synthesis with five accuracy points may be expected. Let the crank and follower angles for the five accuracy points be ϕ_j and ψ_j ($j = 1, 2, 3, 4, 5$); then

$$\phi_j = \phi_1 + \phi_{1j} \qquad \text{with } \phi_{11} = 0$$

and

$$\psi_j = \psi_1 + \psi_{1j} \qquad \text{with } \psi_{11} = 0$$

where ϕ_{1j} and ψ_{1j} are the rotations of the crank and follower relative to the first accuracy point.

When these values are substituted into the displacement equation of the four-bar linkage (10-3), they yield a system of equations

$$\frac{a_4}{a_3}\cos(\phi_1 + \phi_{1j}) - \frac{a_4}{a_1}\cos(\psi_1 + \psi_{1j}) + \frac{a_1^2 - a_2^2 + a_3^2 + a_4^2}{2a_1a_3} = \cos(\phi_1 - \psi_1 + \phi_{1j} - \psi_{1j}) \qquad j = 1, 2, 3, 4, 5 \quad (10\text{-}13)$$

with five unknowns,

$$\frac{a_1}{a_4} \qquad \frac{a_2}{a_4} \qquad \frac{a_3}{a_4} \qquad \phi_1 \qquad \psi_1$$

As in the case of the synthesis of the slider-crank mechanism with four accuracy points (Sec. 10-7), the solution of the above system does not reduce to that of linear equations; and a compatibility equation, of third degree in this case, must be considered. The solution is lengthy, but a digital-computer program giving a complete solution of the problem has been written,[1] and this program may be used in most cases without an understanding of the details of the solution performed.

Consider the problem of generating the function $y = f(x)$ in the interval $x_s \leq x \leq x_f$ by means of a four-bar linkage O_AABO_B (Fig. 10-10), with five accuracy points such that the *structural error is minimized.* As usual, the variables x and y are represented by the crank and follower rotations through the relations

$$\frac{\phi - \phi_s}{\phi_f - \phi_s} = \frac{x - x_s}{x_f - x_s} \qquad \text{and} \qquad \frac{\psi - \psi_s}{\psi_f - \psi_s} = \frac{y - y_s}{y_f - y_s}$$

where $y_s = f(x_s)$ and $y_f = f(x_f)$. The ranges of variation $\Delta\phi = \phi_f - \phi_s$ and $\Delta\psi = \psi_f - \psi_s$ are chosen arbitrarily, and five accuracy points are

[1] Freudenstein, second reference in the Bibliography at the end of the chapter.

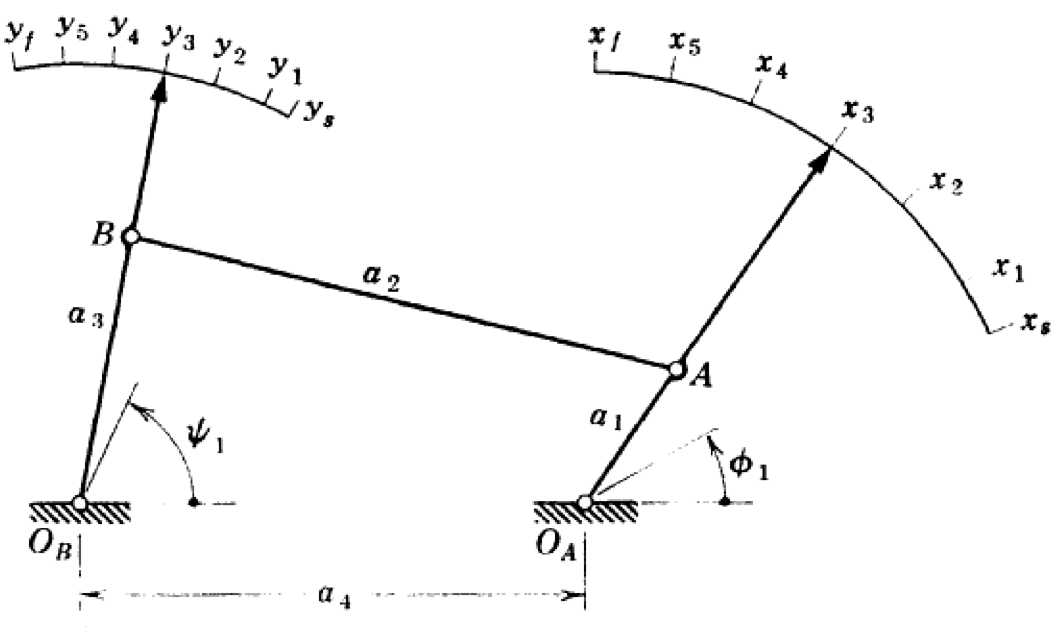

FIGURE 10-10 Four-bar function generator with five accuracy points, showing parameters and variables used in synthesis.

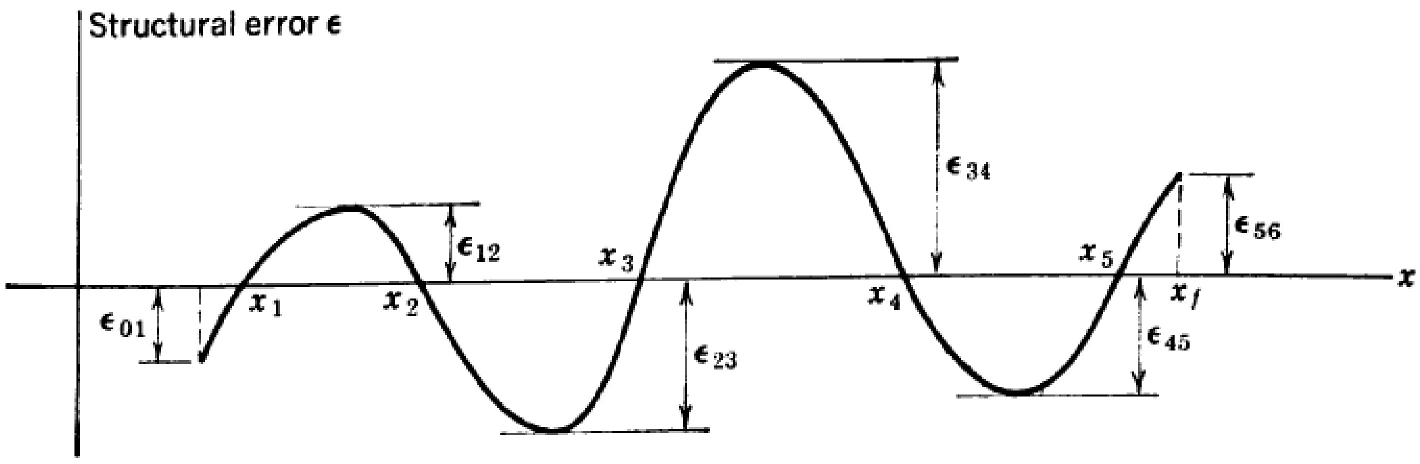

FIGURE 10-11 Structural-error curve.

selected along the curve of $y = f(x)$ for $x = x_j$ ($j = 1, 2, 3, 4, 5$) in the interval between x_s and x_f. The crank and follower rotations from the position corresponding to the first accuracy point to the positions corresponding to the other accuracy points are then

$$\phi_{1j} = \frac{x_j - x_1}{x_f - x_s}\Delta\phi \qquad \text{and} \qquad \psi_{1j} = \frac{y_j - y_1}{y_f - y_s}\Delta\psi$$

with $y_j = f(x_j)$ and $j = 1, 2, 3, 4, 5$.

A complete solution of the problem consists in solving the system of equations (10-13) to find the parameters a_1/a_4, a_2/a_4, a_3/a_4, ϕ_1, and ψ_1; analyzing the linkage to determine the structural error as a function of x; respacing the accuracy points in order to reduce the structural error; and solving Eq. (10-13) again. This process is repeated until the structural error is minimized. Since there are five accuracy points in the interval between x_s and x_f, the structural error will have the general appearance shown in Fig. 10-11. It will be zero at each accuracy point and will reach a series of maxima and minima between the accuracy points as well as at the beginning and end of the interval. For the curve shown in Fig. 10-11, the structural error ϵ_{34}, between the accuracy points x_3 and x_4, has the greatest magnitude. In respacing the accuracy points, points x_3 and x_4 must therefore be brought closer together in order to reduce ϵ_{34} at the cost of other maxima or minima. The structural error is minimized when the spacing is such that all maxima and minima are of equal magnitude.

The computer program referred to above considers the complete problem: the input data consist of the function $y = f(x)$; the initial and final values of x to be considered, x_s and x_f; the ranges of rotation of the crank and follower, $\Delta\phi$ and $\Delta\psi$; and the five values x_1, x_2, x_3, x_4, x_5 corresponding to the five accuracy points to be used in the first cycle of computations. With this program, the four-bar linkages shown in Table 10-3[1] have been designed to generate elementary functions within speci-

[1] This is table 1 from F. Freudenstein, Four-bar Function Generators, *Trans. Fifth Conf. Mechanisms*, Purdue University, 1958; also *Machine Design*, Nov. 27, 1958. Reprinted by courtesy of Penton Publishing Company.

Table 10-3 OPTIMUM FOUR–BAR FUNCTION GENERATORS WITH FIVE ACCURACY POINTS

FUNCTION	$\log x$	$\sin x$	$\tan x$	e^x	$\frac{1}{x}$	$x^{1.5}$	x^2	$x^{2.5}$	x^3	x^2
INTERVAL OF x	$1 \leq x \leq 2$	$0 \leq x \leq 90$ DEG	$0 \leq x \leq 45$ DEG	$0 \leq x \leq 1$	$1 \leq x \leq 2$	$0 \leq x \leq 1$	$0 \leq x \leq 1$	$0 \leq x \leq 1$	$0 \leq x \leq 1$	$-1 \leq x \leq +1$
RANGE OF ϕ, DEG	60	90	90	90	90	90	90	90	90	90
RANGE OF ψ, DEG	60	90	90	90	90	90	90	90	90	60
ACCURACY POINTS:										
x_1	1.0144678	2.7633673	1.6246052	0.02781610	1.0130385	0.011835966	0.033689272	0.022529458	0.029325922	−0.9452
x_2	1.1355580	21.988925	12.842451	0.19327052	1.1561093	0.14032685	0.24917564	0.21794875	0.25920673	−0.5259
x_3	1.3842742	48.226892	27.476764	0.49740748	1.4991776	0.40993302	0.54280174	0.54157659	0.58821332	0.0492
x_4	1.7004985	71.414168	38.542302	0.79659339	1.7984797	0.74496432	0.81636273	0.81866346	0.84128244	0.6728
x_5	1.9627009	87.549520	44.269088	0.97582616	1.9786677	0.97026835	0.97863119	0.97795758	0.98033850	0.9767
CRANK ANGLE ϕ_1, DEG AND DECIMALS	52.628390	−62.263261	269.70917	241.64463	33.804400	−5.1709711	−29.320664	−88.314667	−85.921508	−21.18
FOLLOWER ANGLE ψ_1, DEG AND DECIMALS	259.07749	75.606709	124.18966	40.422765	120.21283	211.68878	233.83630	44.492538	37.636724	−53.67
LINK PROPORTIONS:										
a_1/a_4	−3.23485	1.8343688	−2.6603189	−3.4994859	−0.38477780	0.62481545	2.5233585	−1.8009087	−1.6056207	−0.6102
a_2/a_4	0.84560215	2.2385439	7.4304440	0.87854083	1.0305418	1.3088626	3.3295928	0.90829808	0.92537009	0.5656
a_3/a_4	3.4855187	−0.69364556	8.6856194	3.3993445	0.38453884	−0.40060612	−0.55594832	1.2737203	1.1073831	0.3804
ANGULAR ERROR, DEG AND DECIMALS	0.0037	0.1900	0.038	0.0258	0.0161	0.146	0.0673	0.412	0.5095	2.34
OUTPUT ERROR, % OUTPUT TRAVEL	0.0062	0.21	0.042	0.0287	0.0179	0.162	0.0748	0.457	0.566	4.47

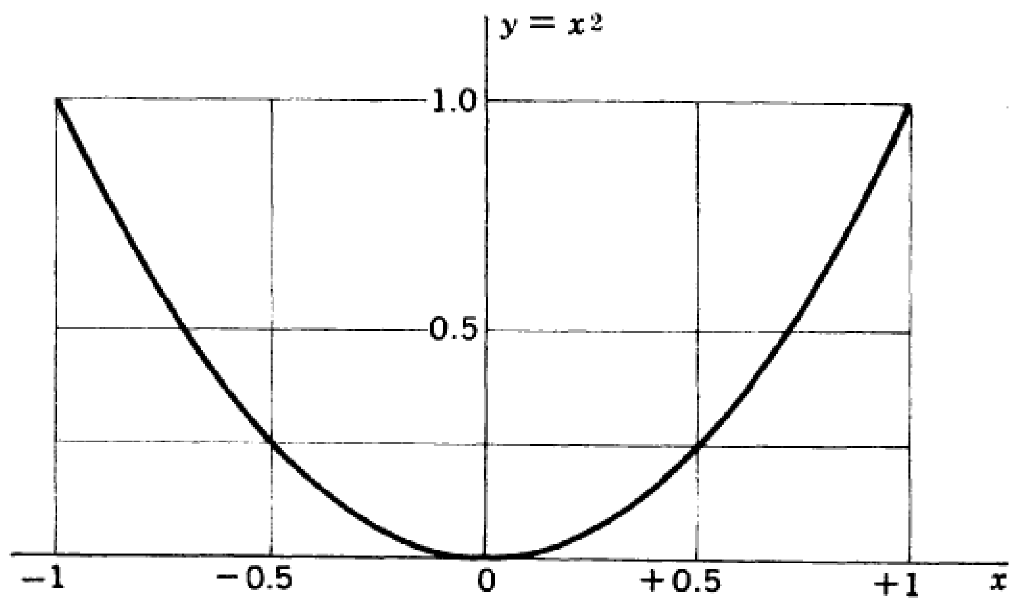

FIGURE 10-12 Symmetrical function $y = x^2$.

fied intervals. The table shows the accuracy points corresponding to optimum spacing, i.e., for minimum structural error as well as the magnitude of this error. Note that the error varies widely from function to function and depends on the interval of generation as well. For example, function $y = x^2$ in the interval $0 \leq x \leq 1$ is generated with a structural error less than 0.075 percent of the output travel. When the interval is extended to $-1 \leq x \leq +1$, the error becomes 4.47 percent, or 60 times as large. The large structural error in this case is due to the symmetry of the function $y = x^2$ in the interval $-1 \leq x \leq +1$ (Fig. 10-12). The four-bar linkage is not suited to the generation of symmetrical functions. Mechanisms better suited to this purpose will be considered in Chap. 12.

10-9 ANALYSIS OF MECHANICAL ERRORS IN LINKAGES

Consider a linkage with n constant parameters $q_1, q_2, \ldots, q_n$ transforming a motion (as shaft rotation), defined by an input variable ϕ, into another motion, defined by an output variable ψ. This linkage has been designed to generate a given function in a given interval such that, if the linkage were built to perfection, the maximum deviation between the desired function and the function generated by the linkage would not exceed ϵ_s, the mathematical or structural error. This deviation is present in all linkages designed by approximate synthesis. An additional error, ϵ_m, due to deflections of the links, play in the joints, and manufacturing tolerances, will inevitably occur in addition to ϵ_s in any actual linkage. This mechanical error ϵ_m will now be evaluated in terms of dimensional variations that may be maintained on the values of the parameters $q_1, q_2, \ldots, q_n$.

The displacement equation of the linkage, relating the constant parameters $q_1, q_2, \ldots, q_n$ to the input and output variables ϕ and ψ, may be written in general form as

$$F(q_1, q_2, \ldots, q_n, \phi, \psi) = 0 \tag{10-14}$$

The errors of the constant parameters $q_1, q_2, \ldots, q_n$ are assumed to be $\Delta q_1, \Delta q_2, \ldots, \Delta q_n$. For a given value of the input variable ϕ, the value of the output variable will be $\psi + \Delta\psi$, and we may write

$$F(q_1 + \Delta q_1, q_2 + \Delta q_2, \ldots, q_n + \Delta q_n, \phi, \psi + \Delta\psi) = 0 \quad (10\text{-}15)$$

The mechanical error manifested at the output is

$$\epsilon_m = \Delta\psi$$

For small values of the errors $\Delta q_1, \Delta q_2, \ldots, \Delta q_n$ and $\Delta\psi$, the differential of the function F may be written in terms of its partial derivatives as

$$\frac{\partial F}{\partial q_1}\Delta q_1 + \frac{\partial F}{\partial q_2}\Delta q_2 + \cdots + \frac{\partial F}{\partial q_n}\Delta q_n + \frac{\partial F}{\partial \psi}\Delta\psi = 0$$

or

$$\epsilon_m = -\frac{\dfrac{\partial F}{\partial q_1}\Delta q_1 + \dfrac{\partial F}{\partial q_2}\Delta q_2 + \cdots + \dfrac{\partial F}{\partial q_n}\Delta q_n}{\dfrac{\partial F}{\partial \psi}}$$

$$= -\sum_{k=1}^{n}\frac{\partial F/\partial q_k}{\partial F/\partial \psi}\Delta q_k$$

The total mechanical error ϵ_m in the linkage is therefore the sum of the individual errors due to each of the parameters considered separately.

10-10 MECHANICAL ERRORS IN FOUR-BAR LINKAGES

For our present purpose, the displacement equation (10-1) of the four-bar linkage is written as

$$D\sin\psi + E\cos\psi = F \quad (10\text{-}16)$$

in which

$$D = 2a_1a_3\sin\phi$$
$$E = 2a_3a_4 + 2a_1a_3\cos\phi$$
$$F = 2a_1a_4\cos\phi + a_1^2 - a_2^2 + a_3^2 + a_4^2$$

Errors Δa in the link lengths a_1, a_2, a_3, a_4 will modify the coefficients D and E and the term F by amounts ΔD, ΔE, and ΔF and will produce an error $\Delta\psi$ in the output. Each error Δa in a given link will produce separate errors in D, E, and F, whence the output error contributed by each link must be considered separately. The total mechanical error ϵ_m of the linkage will be the sum of the separate errors.

Link-error Equation

In the presence of link-length errors the displacement equation (10-16) may be written as

$$(D + \Delta D) \sin (\psi + \Delta\psi) + (E + \Delta E) \cos (\psi + \Delta\psi) = F + \Delta F \tag{10-17}$$

After expansion of this equation (by use of trigonometric identities, small-angle approximations, and neglect of higher-order terms), the subtraction of Eq. (10-16), and ordering of terms, we find

$$\Delta\psi(D \cos \psi - E \sin \psi) = -\Delta D \sin \psi - \Delta E \cos \psi + \Delta F$$

from which
$$\Delta\psi = -\frac{\Delta D \sin \psi + \Delta E \cos \psi - \Delta F}{D \cos \psi - E \sin \psi} = (\epsilon_m)_{\text{link}} \tag{10-18}$$

This is the *link-error equation.*

Error Due Only to Δa_1

An error Δa_1 in link dimension a_1 produces deviations ΔD, ΔE, and ΔF in D, E, and F to yield a mechanical error $\epsilon_{m1} = \Delta\psi_1$. The deviation ΔD is found as follows:

$$\begin{aligned}\Delta D &= (D + \Delta D) - D = [2(a_1 + \Delta a_1)a_3 \sin \phi] - 2a_1a_3 \sin \phi \\ &= (2a_3 \sin \phi)\,\Delta a_1\end{aligned}$$

In like manner

$$\begin{aligned}\Delta E &= (2a_3 \cos \phi)\,\Delta a_1 \\ \Delta F &= (2a_4 \cos \phi + 2a_1)\,\Delta a_1\end{aligned}$$

Substitution of these values into Eq. (10-18) yields

$$\Delta\psi_1 = \epsilon_{m1} = 2\,\frac{a_4 \cos \phi + a_1 - a_3 \cos (\phi - \psi)}{D \cos \psi - E \sin \psi}\,\Delta a_1$$

Error Due Only to Δa_2

The deviations are

$$\Delta D = 0 \qquad \Delta E = 0 \qquad \Delta F = -(2a_2)\,\Delta a_2$$

and
$$\Delta\psi_2 = \epsilon_{m2} = \frac{-2a_2}{D \cos \psi - E \sin \psi}\,\Delta a_2$$

Error Due Only to Δa_3

Here

$$\Delta D = (2a_1 \sin \phi)\, \Delta a_3 \qquad \Delta E = (2a_4 + 2a_1 \cos \phi)\, \Delta a_3$$
$$\Delta F = 2a_3\, \Delta a_3$$

and
$$\Delta\psi_3 = \epsilon_{m3} = -\left(2\, \frac{a_4 \cos \phi + a_3 - a_1 \cos \psi}{D \cos \psi - E \sin \psi}\right) \Delta a_3$$

Error Due Only to Δa_4

Here

$$\Delta D = 0 \qquad \Delta E = 2a_3\, \Delta a_4 \qquad \Delta F = (2a_1 \cos \phi + 2a_4)\, \Delta a_4$$

and
$$\Delta\psi_4 = \epsilon_{m4} = 2\, \frac{a_1 \cos \phi + a_4 - a_3 \cos \psi}{D \cos \psi - E \sin \psi}\, \Delta a_4$$

Numerical Example Consider the four-bar linkage designed by the five-point synthesis method to generate the function log x for values of x between 1 and 2. The dimensions of this linkage, given in Table 10-3, are approximately

$$a_1 = -3.23 \text{ in.} \qquad a_2 = 0.84 \text{ in.} \qquad a_3 = 3.48 \text{ in.} \qquad a_4 = 1.0 \text{ in.}$$

with the first accuracy point at

$$\phi_1 = 53^\circ \qquad \psi_1 = 259^\circ$$

and a maximum structural error $\epsilon_s = 0.0037^\circ$. The mechanical error will now be evaluated at the first accuracy point in terms of errors Δa_1, Δa_2, Δa_3, Δa_4 that may be assumed for the link lengths a_1, a_2, a_3, a_4. At the first accuracy point,

$$D = -18.7 \qquad E = -7.04 \qquad F = 19.64$$
$$\cos \psi = -0.19 \qquad \sin \psi = -0.98$$

from which
$$\begin{aligned} D \cos \psi - E \sin \psi &= -3.38 \\ \epsilon_{m_1} &= -0.24\, \Delta a_1 \text{ rad} = -14\, \Delta a_1 \text{ deg} \\ \epsilon_{m_2} &= 0.50\, \Delta a_2 \text{ rad} = 29\, \Delta a_2 \text{ deg} \\ \epsilon_{m_3} &= -0.38\, \Delta a_3 \text{ rad} = -23\, \Delta a_3 \text{ deg} \\ \epsilon_{m_4} &= 0.20\, \Delta a_4 \text{ rad} = 12\, \Delta a_4 \text{ deg} \end{aligned}$$

Assuming $|\Delta a_1| = |\Delta a_2| = |\Delta a_3| = |\Delta a_4| = 0.001$ in.,

$$|\epsilon_m|_{\max} = |\epsilon_{m_1}| + |\epsilon_{m_2}| + |\epsilon_{m_3}| + |\epsilon_{m_4}| = 0.001(78) = 0.078^\circ$$

or
$$|\epsilon_m|_{\text{rms}} = \sqrt{\epsilon_{m_1}{}^2 + \epsilon_{m_2}{}^2 + \epsilon_{m_3}{}^2 + \epsilon_{m_4}{}^2} = 0.001(42) = 0.042^\circ$$

The maximum mechanical error with the 0.001-in. tolerance is thus more than 20 times the structural error $(0.0780/0.0038 \simeq 20)$; the ratio of the rms error is more than 11 times the structural $(0.0420/0.0037 \simeq 11)$.

10-11 GEOMETRIC INTERPRETATION OF THE ERROR DENOMINATOR

Intuition and experience indicate that the errors in a four-bar linkage are closely related to the value assumed by the angle γ (Fig. 10-13), already identified in Sec. 2-10 as the *transmission angle.* The role played by the angle γ in the mechanical errors and force transmission of the mechanism indicates that there should be a relation between this angle and the denominator, $G = D \cos \psi - E \sin \psi$, in the expressions of the mechanical errors.

The angle γ may be expressed in terms of the linkage parameters and the input variable ϕ by application of the cosine law to triangles O_AO_BA and ABO_B (Fig. 10-13),

$$(AO_B)^2 = a_2^2 + a_3^2 - 2a_2a_3 \cos \gamma = a_1^2 + a_4^2 + 2a_1a_4 \cos \phi$$

or

$$\cos \gamma = -\frac{a_1^2 - a_2^2 - a_3^2 + a_4^2 + 2a_1a_4 \cos \phi}{2a_2a_3} \tag{10-19}$$

To express G in terms of the same quantities, the angle ψ must be eliminated by using the displacement equation of the linkage [Eq. (10-1)]. Solving simultaneously the equations

$$\begin{aligned} D \sin \psi + E \cos \psi &= F \\ -E \sin \psi + D \cos \psi &= G \end{aligned}$$

yields

$$\sin \psi = \frac{\begin{vmatrix} F & E \\ G & D \end{vmatrix}}{\begin{vmatrix} D & E \\ -E & D \end{vmatrix}} \qquad \cos \psi = \frac{\begin{vmatrix} D & F \\ -E & G \end{vmatrix}}{\begin{vmatrix} D & E \\ -E & D \end{vmatrix}}$$

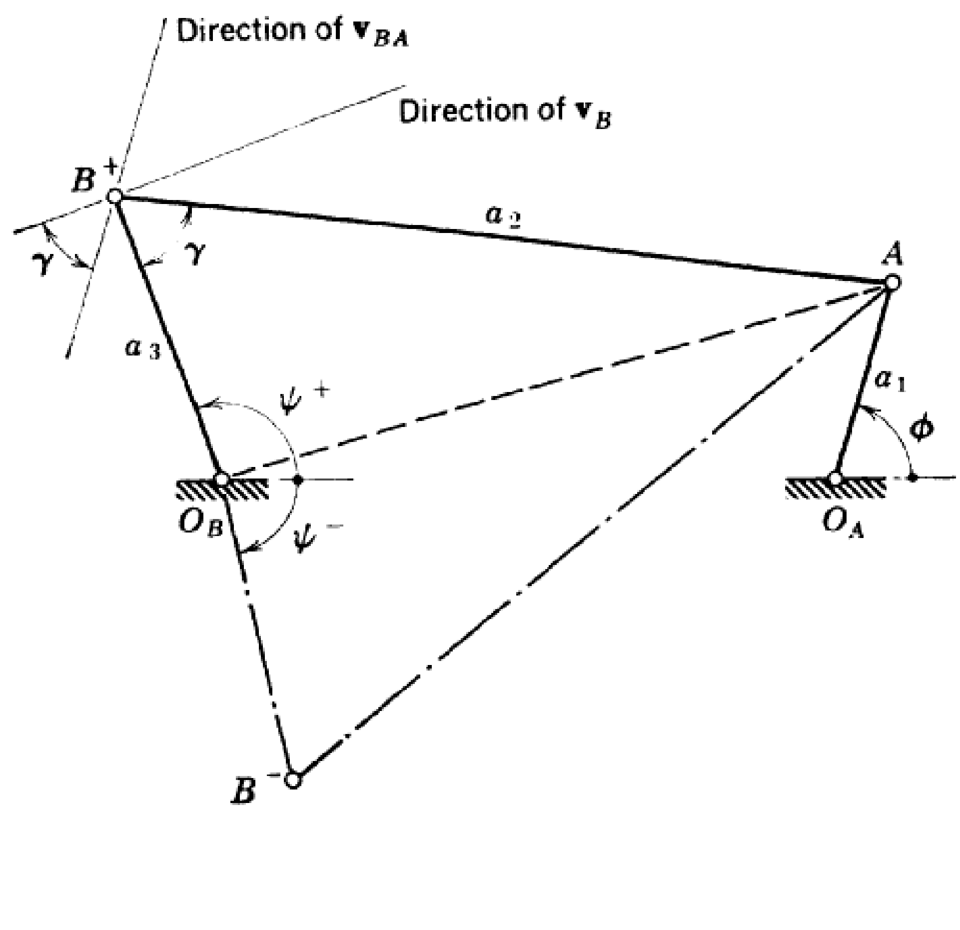

FIGURE 10-13 Determination of transmission angle γ.

which by substitution into the trigonometric identity

$$\sin^2 \psi + \cos^2 \psi = 1$$

gives
$$\begin{vmatrix} F & E \\ G & D \end{vmatrix}^2 + \begin{vmatrix} D & F \\ -E & G \end{vmatrix}^2 = \begin{vmatrix} D & E \\ -E & D \end{vmatrix}^2$$

Expanding the determinants, collecting terms, and dividing through by $D^2 + E^2$ yields

$$G^2 = D^2 + E^2 - F^2 \tag{10-20}$$

Note that, according to this relation, G is the square root which appears in the expressions of ψ^+ and ψ^- derived in Sec. 10-1. When $G = 0$, $\psi^+ = \psi^-$ and the linkage is in dead-center position, whence no torque can be transmitted from the crank to the follower.

Introducing the expressions for D, E, and F (Sec. 10-10) into Eq. (10-20) yields

$$G^2 = 4a_1^2a_3^2 \sin^2 \phi + (2a_3a_4 + 2a_1a_3 \cos \phi)^2 - (2a_1a_4 \cos \phi + a_1^2 - a_2^2 + a_3^2 + a_4^2)^2$$

By combination of terms and algebraic simplifications, this reduces to

$$G^2 = -4a_2^2a_3^2 \frac{2a_1a_4 \cos \phi + a_1^2 - a_2^2 - a_3^2 + a_4^2}{2a_2a_3} + 4a_2^2a_3^2$$

or
$$G^2 = 4a_2^2a_3^2(-\cos^2 \gamma + 1)$$

or
$$G = \pm 2a_2a_3 \sin \gamma \tag{10-21}$$

This equation shows that the quantity G and the angle γ are indeed two equivalent expressions of the same property of a four-bar linkage. It confirms the fact that a four-bar linkage having poor force transmission is also subject to large mechanical errors.

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