Homework 35

The **due date** for this homework is **Tue 7 May 2013 12:00 AM EDT**.

Question 1

Find the arc length of the curve $y = \left(x + \frac{5}{9}\right)^{3/2}$ from $x = 0$ to $x = 3$.

- $\frac{63}{4}$
- 8
- 7
- $\frac{3}{2}$
- 1
- $\frac{21}{2}$

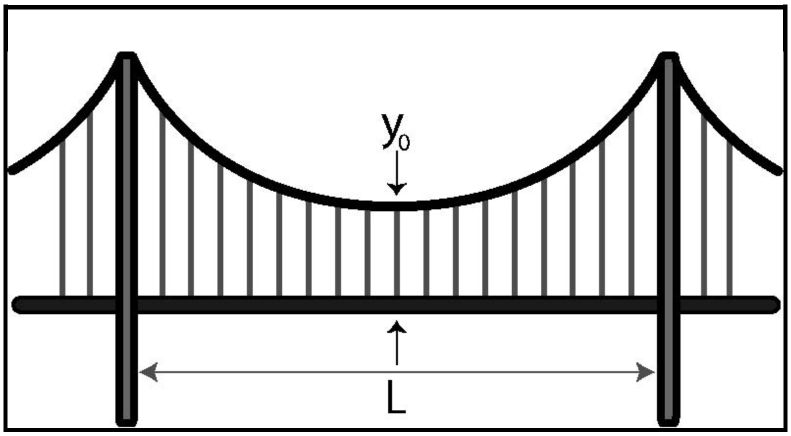
Question 2

In Lecture we saw that a catenary hangs as a hyperbolic cosine. To arrive at this conclusion, we used that the cable is uniform —that is, that the weight of an infinitesimal piece of the cable is proportional to the length element. But what happens if the cable is supporting some weight? In that case, one can show that the cable no longer takes a hyperbolic cosine shape.

For example, consider the cable supporting a suspension bridge with a horizontal roadway. If we suppose that the mass of the cable is negligible compared to the mass of the roadway below, we can show that it hangs as a parabola. Choosing coordinates as in the figure so that the bottom of the cable is located at $x = 0$ with height y_0, and the vertical suspenders stand at $x = -L/2$ and $x = L/2$, the equation describing its shape takes the form

$$y = \alpha x^2 + y_0$$

Here the constant α depends on the characteristics of the bridge —namely, the tension of the cable and the weight of the roadway. Which of the following expressions gives, in terms of α, L and y_0, the length of the cable hanging between the two vertical suspenders?



Hint: you may want to use the formula

$$\int \sqrt{1+u^2}\, du = \frac{1}{2}\left(u\sqrt{1+u^2} + \operatorname{arcsinh} u\right) + C$$

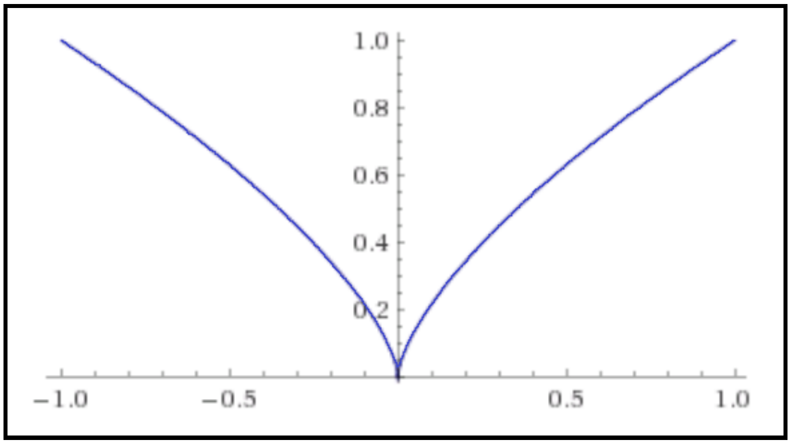
- $\frac{L}{2}\sqrt{1+\alpha^2 L^2} + \frac{1}{2\alpha}\operatorname{arcsinh}(\alpha L)$
- $\alpha L\sqrt{1+\alpha^2 L^2} + \operatorname{arcsinh}(\alpha L)$
- $\frac{L}{4}\sqrt{1+\alpha^2 L^2} + \frac{1}{4\alpha}\operatorname{arcsinh}(\alpha L)$
- $\frac{L}{2}\sqrt{1+\alpha^2 L^2}$
- $\frac{\alpha L}{2}\sqrt{1+\alpha^2 L^2} + \frac{1}{2}\operatorname{arcsinh}(\alpha L)$
- $\frac{1}{2\alpha}\operatorname{arcsinh}(\alpha L)$

Question 3

The so-called *cuspidal cubic* is given parametrically by the equations

$$x = t^3, \quad y = t^2$$

Compute the arc length of this curve as t goes from -1 to 1. Provide a numeric answer rounded to two decimal places.



Question 4

Consider the spiral given by the parametric equations

$$x = t^{-k} \cos t, \quad y = t^{-k} \sin t$$

where $k > 0$. Denote by L_k its arc length as t moves from 2π to $+\infty$. Which of the following statements are true? Select all that apply.

Hint: in Lecture we studied the case $k = 1$: see the figure from the Lecture if you need help visualizing...

- [ ] L_k is finite for $k > 1$, and infinite for $k \leq 1$

- [ ] $L_k = \int_{t=2\pi}^{+\infty} \frac{\sqrt{1+k^2t^2}}{t^{k+1}}\,dt$
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- [ ] $L_k = \int_{t=2\pi}^{+\infty} \frac{\sqrt{1+t^2}}{t^{k+1}}\,dt$
- [ ] L_k is finite for $k < 1$, and infinite for $k \geq 1$
- [ ] $L_k = \int_{t=2\pi}^{+\infty} \frac{\sqrt{k^2+t^2}}{t^{k+1}}\,dt$

Question 5

Think of a roll of paper.

As the paper wraps around the roll once, the distance from the center of the roll increases by the thickness δ of the paper. This means that we can parametrize cross-sectional curve of the roll by the equations

$$x = \left(r_0 + \frac{\theta}{2\pi}\,\delta\right)\cos\theta, \quad y = \left(r_0 + \frac{\theta}{2\pi}\,\delta\right)\sin\theta$$

where r_0 is the radius of the core cylinder around which the paper is rolled.

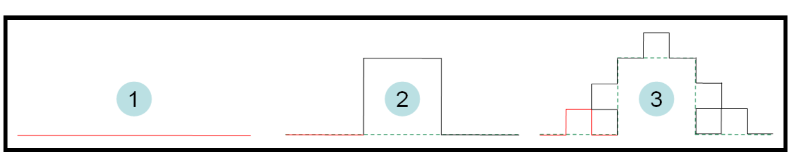
Which of the following integrals computes the length of paper used to make N full turns around the core cylinder?

[Image courtesy of Wikimedia Commons]

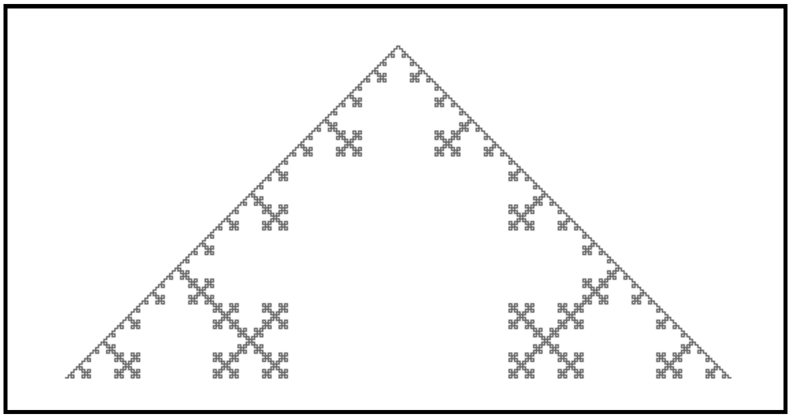
- $\frac{\delta^2}{4\pi^2}\int_{\theta=0}^{2\pi N}\sqrt{1+\left(\theta+\frac{2\pi r_0}{\delta}\right)^2}\,d\theta$
- $\frac{\delta}{2\pi}\int_{\theta=0}^{2\pi N}\sqrt{1+\left(r_0+\frac{\theta}{2\pi}\delta\right)^2}\,d\theta$
- $\frac{\delta^2}{4\pi^2}\int_{\theta=0}^{2\pi N}\sqrt{1+\left(r_0+\frac{\theta}{2\pi}\delta\right)^2}\,d\theta$
- $\frac{\delta^2}{4\pi^2}\int_{\theta=0}^{2\pi N}\sqrt{1+\left(\frac{2\pi r_0}{\delta}\right)^2+\theta^2}\,d\theta$
- $\frac{\delta}{2\pi}\int_{\theta=0}^{2\pi N}\sqrt{1+\left(\frac{2\pi r_0}{\delta}\right)^2+\theta^2}\,d\theta$
- $\frac{\delta}{2\pi}\int_{\theta=0}^{2\pi N}\sqrt{1+\left(\theta+\frac{2\pi r_0}{\delta}\right)^2}\,d\theta$

Question 6

In Lecture we saw an example of a fractal —the so-called *Koch snowflake*. A similar example is given by the following procedure. Starting with a line segment of length 1 (labelled "1" in the figure below), remove the middle third and replace it by a square hat to obtain the curve "2". Perform the same operation on each line segment in "2" to obtain "3".



Doing this *ad infinitum* yields another fractal —that is, a bounded compact curve of infinite length!



But what is the exact length of the curve obtained after a finite number n of iterations?

[Images courtesy of Wikimedia Commons]

- $\left(\frac{4}{5}\right)^n$
- $\left(\frac{3}{4}\right)^n$
- $\left(\frac{5}{3}\right)^n$
- $\left(\frac{4}{3}\right)^n$
- $\left(\frac{5}{4}\right)^n$
- $\left(\frac{3}{5}\right)^n$

In accordance with the Honor Code, I certify that my answers here are my own work.

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