# Homework 55

The **due date** for this homework is **Tue 7 May 2013 12:00 AM EDT -0400**.

## Question 1

For which values of $x$ does the Taylor series about $x = 0$ of $\ln(3 + x)$ converge?

- $-1 \leq x \leq 1$
- $-3 \leq x \leq 3$
- $-1 < x < 1$
- $-3 < x \leq 3$
- $-3 < x < 3$
- $-1 < x \leq 1$

## Question 2

For which values of $x$ does the Taylor series about $0$ of $\arctan 2x$ converge?

- $-\frac{1}{2} < x \leq \frac{1}{2}$
- $-1 < x \leq 1$
- $-\frac{1}{2} \leq x \leq \frac{1}{2}$
- $-1 < x < 1$
- $-\frac{1}{2} < x < \frac{1}{2}$
- $-1 \leq x \leq 1$

## Question 3

The *error function* (that already made an appearance in Homework 44) is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{t=0}^{x} e^{-t^2}\, dt$$

Which one of the following is its Taylor series expansion about $x = 0$?

**Hint:** recall that the Taylor series about $x = 0$ of the exponential function converges on the whole real line, and so you can integrate it term by term.

- $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

- $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}$
- $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$
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- $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$
- $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n! \cdot 2n}$

## Question 4

For which of the following functions is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (4n) x^{4n-1}}{(2n)!}$ its Taylor series expansion about $x = 0$ ?

- $\displaystyle\int_{t=0}^{x} \sin 2t \, dt$
- $\dfrac{d}{dx} \cos(x^2)$
- $\displaystyle\int_{t=0}^{x} \cos(t^4) \, dt$
- $\dfrac{d}{dx} \cos 2x$
- $\displaystyle\int_{t=0}^{x} \sin(t^2) \, dt$
- $\dfrac{d}{dx} \sin(x^4)$

## Question 5

The *complete elliptic integral of the first kind* is defined as

$$K(k) = \int_{\theta=0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_{t=0}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}$$

It is a function of the parameter $k$, and although it does not have an expression in terms of elementary functions (polynomials, exponentials, sines and cosines, etc.), it is a real-analytic function at $k = 0$. Its Taylor series expansion about that point is

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[ \frac{(2n)!}{2^{2n}(n!)^2} \right]^2 k^{2n} = \frac{\pi}{2} \left[ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots + \left( \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \right)^2 k^{2n} + \cdots \right]$$

But what is the radius of convergence, $R$, of this Taylor series?

**For the curious:** elliptic integrals are extremely useful in a number of problems. In mathematics, they show up when computing the arc-lengths of conic curves (ellipses, hyperbolas and parabolas), or the surface area of an ellipsoid. In physics, they appear in problems as diverse as pendulums making complete revolutions, the Friedmann model of the universe, or Euler's three-body problem.

- $R = \frac{2}{\pi}$
- $R = 0$
- $R = \frac{1}{k}$
- $R = \frac{\pi}{2}$
- $R = 1$
- $R = +\infty$

## Question 6

**For the brave:** There is an enormous and enormously important class of functions called the *hypergeometric functions*. They come in many types and have an *intimidating* notation. For example, consider the so-called *ordinary* hypergeometric function

$$ {}_2F_1(a, b; c; x) = 1 + \frac{ab}{1!c}\,x + \frac{a(a+1)b(b+1)}{2!c(c+1)}\,x^2 + \cdots + \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}\,x^n + \cdots $$

where $a$, $b$ and $c$ are real parameters, and $c$ is not a negative integer. What is the radius of convergence, $R$, of this Taylor series?

- $R = \frac{c}{ab}$
- $R = \frac{a+b}{c}$
- $R = +\infty$
- $R = 0$
- $R = \frac{ab}{c}$
- $R = 1$

☐ In accordance with the Honor Code, I certify that my answers here are my own work.

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