Homework 47

The **due date** for this homework is **Tue 7 May 2013 12:00 AM EDT**.

Question 1

Consider the sequence $a = (a_n)$ where $a_n = \dfrac{1}{n^2+3n+2}$. Which one of the following sequences $b = (b_n)$ satisfies $\Delta b = a$?

Hint: take advantage of the fact that the multiple-choice format allows you to guess the antidifference and try differencing it.

- $b_n = \dfrac{n}{n+1}$
- $b_n = \dfrac{n-1}{n!}$
- $b_n = \dfrac{n+1}{n+2}$
- $b_n = \dfrac{1}{n+1}$
- $b_n = \dfrac{n+2}{n-3}$
- $b_n = \dfrac{n^2+1}{n+1}$

Question 2

Compute the sum $\displaystyle\sum_{n=1}^{K} \frac{1}{n^2+3n+2}$.

Hint: Use your answer to Question 1 along with the discrete version of the fundamental theorem of integral calculus.

- $\dfrac{5K}{K-2}$

- $\dfrac{K}{2(K+2)}$
- $\dfrac{K}{K+1}$
- $\dfrac{K}{K+4}$
- $\dfrac{K-1}{2(K+1)}$
- $\dfrac{K+1}{K+2}$

Question 3

Consider the sequence $s = (s_n)$ defined by

$$s_n = \pi \cdot 2^n - \frac{3}{n^2+3n+2}.$$

Which of the following sequences $r = (r_n)$ satisfies $\Delta r = s$?

Hint: it helps to use the following:

1. your answer to Question 1,
2. the fact that the sequence (2^n) forward-differentiates to itself, and
3. the fact that Δ is a *linear* operator (just like the derivative), so you know how it behaves with sums and multiples of sequences.

- $r_n = \pi \cdot 2^{n+1} - \dfrac{3}{n+1}$
- $r_n = \pi \cdot 2^n + \dfrac{6n(n+1)}{n^3+3}$
- $r_n = \pi \cdot 2^n - \dfrac{3n}{n+1}$
- $r_n = \pi \cdot 2^n - \dfrac{n!}{2^n}$
- $r_n = \pi \cdot 2^n - \dfrac{3 \cdot n!}{n+2}$
- $r_n = \pi^2 \cdot 2^n + \dfrac{3(n+2)}{n-3}$

Question 4

Use the summation by parts formula to derive an expression which equals the sum

$$\sum_{n=1}^{K} \frac{2^n}{n^2+3n+2}.$$

Hint: once again, it might be useful if you remember your answer to Question 1.

- $\frac{2^K}{K} - 1 - \sum_{n=1}^{K} \frac{2^n}{n+1}.$
- $\frac{2^K(K)}{K+1} + \sum_{n=1}^{K} \frac{2^n(n-3)}{n+2}$
- $\frac{2^K(K)}{K+1} - 1 - \sum_{n=1}^{K} \frac{2^n n}{n+1}.$
- $K! - \sum_{n=1}^{K} \frac{2^n n}{n+1}.$
- $\frac{2^{K+1}(K+1)}{K+2} - 1 - \sum_{n=1}^{K} \frac{2^n(n+1)}{n+2}.$
- $\frac{2^{K+1}(K+1)}{K+2} - 1 - \sum_{n=1}^{K} \frac{2^n(n-3)}{n+2}.$

Question 5

In the continuous case, a function $f(x)$ is said to have a *critical point* at a whenever the derivative $f'(a)$ is zero or undefined. Critical points can be local maxima, minima or saddles depending on the outcome of the second derivative test. In this Question, we will try to understand the relationship between the forward difference operator Δ and local maxima and minima of sequences.

A *local maximum* of the sequence $a = (a_n)$ occurs at the position k if the following two inequalities hold: $a_{k-1} \leq a_k \geq a_{k+1}$. Look at a sequence b which starts like $1, 3, 4, 2, \ldots$ That 4 in the third position is larger than the 3 and 2 that come before and after it, so we say that this sequence achieves a local maximum value of 4 at the third position. However, neither $(\Delta b)_2$ nor $(\Delta b)_3$ equal zero.

If some sequence $a = (a_n)$ attains a local maximum at the k-th position. Which of the following must be a true statement about its forward difference Δa near k?

Hint: for each statement, try to find a counterexample. If you cannot find one, try to figure out why it might be true. There is no need to give a proof, but this problem will give you a bit of practice at thinking like a mathematician.

- $(\Delta a)_{k-1} \leq 0$ and $(\Delta a)_k \geq 0$.
- Both $(\Delta a)_{k-1}$ and $(\Delta a)_k$ equal 0.
- None of these statements has to be true.
- $(\Delta a)_{k-1} \geq 0$ and $(\Delta a)_k \leq 0$.
- $(\Delta a)_{k-1} \geq 0$ and $(\Delta a)_k = 0$.
- $(\Delta a)_{k-1} \leq 0$ and $(\Delta a)_k = 0$.

Question 6

Assume that the sequence $a = (a_n)$ satisfies the second-order recurrence relation

$$a_{n+2} = 2a_n - a_{n+1} \quad ; \quad a_0 = 0 \quad a_1 = 1$$

Find a closed-form formula for a_n if the first two terms are $a_0 = 0$ and $a_1 = 1$.

Hint: follow the steps we took for solving the Fibonacci sequence. First, rewrite the recurrence relation in terms of operators E and I. Then factor this equation and find two basic solutions. The general solution will be some linear combination of these two basic solutions.

- $a_n = \dfrac{2^n}{3} - \dfrac{(-1)^n}{3}$
- $a_n = \dfrac{5^n}{9} - \dfrac{(-4)^n}{9}$

- $a_n = \frac{(2+\sqrt{5})^n}{4} - \frac{(2-\sqrt{5})^n}{4}$
- $a_n = \frac{5^n}{6} - \frac{(-1)^n}{6}$
- $a_n = \frac{1}{3} - \frac{1}{3}(-2)^n$
- $a_n = 2 - (-2)^n$

Question 7

Start with two sequences $f = (f_n)$ and $g = (g_n)$. We can define a new sequence which we call $f * g$ as follows:

$$(f * g)_n = \sum_{j=0}^{n} f_j g_{n-j} = f_0 g_n + f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_{n-2} g_2 + f_{n-1} g_1 + f_n g_0$$

This sequence $f * g$ is called the *discrete convolution* of f and g, and it is of fundamental importance in digital signal processing, image analysis, and much, much more.

Let us do a simple example. Let δ^T be the sequence

$$\delta^T = (0, 0, 0, \ldots, 0, 0, 1, 0, 0, 0, \ldots)$$

that has a single 1 in the T^{th} position (if you're into engineering, this sequence represents a ping or blip at time T). Given another sequence $a = (a_n)$, what is the convolution $a * \delta^T$?

Hint: as always, try out a few examples. See what you get and look for a pattern. That is how mathematics works!

- $a * \delta^T = \Delta^T a$
- $a * \delta^T = a$
- $(a * \delta^T)_n = (a)_T$
- $a * \delta^T = E^T a$

- $(a * \delta^T)_n = (a)_{nT}$
- $a * \delta^T = E^{-T} a$

In accordance with the Honor Code, I certify that my answers here are my own work.

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