Preparing for the Calculus: Single Variable Final Exam

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Typical Topics

Here is a summary list of the core topics you are expected to have mastered for the Final Exam. This is only meant to be a guide for how to allocate your studying. Not everything listed below will be on the exam, and there may be problems that combine several things creatively into one problem, to test how well you integrate your knowledge.

- Taylor series: definitions, computations, convergence
- Limits of functions and l'Hopital's rule
- Asymptotic growth of functions
- Big-O notation
- Derivatives: definitions and interpretations
- Linearization of functions and linear approximation
- Logarithmic differentiation
- Indefinite and definite integrals
- The Fundamental Theorem of Integral Calculus
- Techniques of integration (lots of topics here!!!)
- Improper integrals
- Separable ODEs and their applications
- Stable and unstable equilibria of ODEs
- Area between curves in the plane and in polar coordinates
- Volumes and volumes of revolution
- Arc length and surface area
- Work
- Averages
- Centroids, mass, and centers of mass

- Probability density functions; probability; expectation
- Sequences
- Forward-differencing
- Infinite series
- Convergence tests for series (lots of tests here!!!)
- Absolute versus conditional convergence
- Power series and radius of convergence
- Error estimation for series

Things that are **not** on the Final Exam include:

- Newton's method : the arithmetic quickly gets too involved.
- Formula for integrating factors for linear ODEs: just know how to do separable ODEs and you will be fine.
- Computer algebra and tables for integration.
- High-dimensional spaces.
- Variance and standard deviation: but be sure you know the expectation!
- Moments of inertia.
- Present and future value of income streams.
- Numerical methods (for ODEs and integrals).

1 General Test Common Sense Tips

1. Answer every question. If you are not sure about a problem, make an educated guess by eliminating answers you know to be wrong.
2. Read the question carefully. Don't do more work than necessary. For example, if we ask for radius of convergence of a power series, you need not check the endpoints of your interval for convergence.
3. Simplify your answers. If you are not getting your answer to match the multiple choice answers, double check your work, and make sure you have simplified your answer.

   This includes things like rationalizing denominators (example: $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$), reducing fractions, and log simplifications: $\ln(a) - \ln(b) = \ln(a/b)$, and $\ln(a) + \ln(b) = \ln(ab)$.

2 Big O

Remember, when using terms like $O(x)$, make sure you know whether $x \to 0$ or $x \to \infty$. In either case, the definition is the same: for example, $f \in O(x^n) \iff \|f(x)\| < x^n$, as x goes to...(you fill in). Remember the directionality implied by the $<$ sign: x^2 is in $O(e^x)$, but e^x is NOT in $O(x^2)$. Recall the hierarchy of orders:

$$\text{CONSTANT} < \text{LOGARITHMIC} < \text{POLYNOMIAL} < \text{EXPONENTIAL} < \text{FACTORIAL}$$

3 Area, Volume, Arc length, Surface area

Our main recommendation is to do some practice problems to get these fresh in your mind. Also, draw yourself a small labeled cartoon to remind yourself of each of the elements and what the R, r, h, etc. are in each cartoon:

1. Area element (between curves $f(x)$ and $g(x)$): $dA = (f(x) - g(x))dx$. For polar, it is different! What is it?
2. Volume element (perpendicular to axis of rotation): $dV = \pi(R_o^2 - R_i^2)\,dx$ (or dy, depending on orientation). Don't just blindly write this down! Make sure you are using it correctly by drawing a picture.
3. Volume element (parallel to axis of rotation): $dV = 2\pi rh\,dx$ (or dy, depending on orientation). Same as above.
4. Arc length element: $dL = \sqrt{1 + (f'(x))^2}dx$ (or with all the x's replaced with y's if f is a function of y). Remember how the algebra usually goes here.

   Example:

$$\sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2} = \sqrt{1 + (x^2)^2 - \frac{1}{2} + \left(\frac{1}{4x^2}\right)^2} = \sqrt{x^4 + \frac{1}{2} + \left(\frac{1}{4x^2}\right)^2} = \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2} = x^2 + \frac{1}{4x^2}.$$

5. Surface area of revolution element: $dS = 2\pi r dL = 2\pi r\sqrt{1 + (f'(x))^2}dx$ if f is function of x and you are revolving around the x-axis. Replace the x's for y's if going about the y-axis.

4 Integration Techniques

A few pointers for each of the different techniques:

1. u-substitution. Remember: either change the bounds of integration, or substitute back in later.
2. Integration by parts. When choosing what to make your u, the helpful mnemonic for precedence is LIPET: **L**ogarithm, **I**nversefunctions, **P**olynomial, **E**xponential, **T**rigonometric. For example:

$$\int x \sin(x)\,dx \qquad \text{(the } x \text{ is polynomial which comes before trig, so } u = x)$$

$$\int x^3 \ln(x)\,dx \qquad (\ln(x) \text{ is logarithm which comes before polynomial, so } u = \ln(x))$$

$$\int e^x \sin(x)\,dx \qquad \text{(exponential before trig so } u = e^x\text{, though here it's a tossup)}$$

3. Trigonometric substitutions. Do some practice problems, and try to remember:
   - $1 - x^2$, substitute $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$.
   - $x^2 - 1$, substitute $x = \sec\theta$ so that $dx = \sec\theta\tan\theta\,d\theta$.
   - $x^2 + 1$, substitute $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$.

   Do not forget to replace the dx**!** Also, remind yourself how to deal with other constants. For instance: $4 - 9x^2$, substitute $x = \frac{2}{3}\sin\theta$ (so that the constants cancel/factor nicely). Also, you should probably have your Pythagorean identities on hand:

- $\sin^2 x + \cos^2 x = 1$.
- $\tan^2 x + 1 = \sec^2 x$.

4. Trigonometric integrals. Remember some of the most famous/nice ones:

   - $\int \sin^m(x)\cos^n(x)\,dx$ works well when (at least one of) m or n is odd. Example:

$$\begin{aligned}
\int \sin^3 x \cos^3 x\,dx &= \int \sin^2 x \cos^2 x \sin x\,dx \\
&= \int (1-\cos^2 x)\cos^2 x \sin x\,dx && \text{(now sub } u = \cos x\text{)} \\
&= -\int (1-u^2)u^2\,du && \text{(the } - \text{ comes from } du = -\sin x\,dx\text{)} \\
&= -\frac{u^3}{3} + \frac{u^5}{5} + C \\
&= \frac{-\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.
\end{aligned}$$

   If both m and n are even, then you may need to use one or both of the power reduction identities:

$$\begin{aligned}
\sin^2 x &= \frac{1-\cos(2x)}{2} \\
\cos^2 x &= \frac{1+\cos(2x)}{2}.
\end{aligned}$$

   - $\int \sec^m(x)\tan^n(x)\,dx$ works well when both m and n are odd ($u = \sec x$ works), or if m is even ($u = \tan x$ works).
   - For other types of trig integrals, try to convert all functions to sin and cos and hope there is a good u substitution you can make.

5. Partial fractions. There are some technical cases which we did not talk about in the lectures, but focus on the basics. Two such examples:

   - $\int \frac{5x-10}{x^2-3x-4}\,dx$. Factor denominator $x^2-3x-4 = (x+1)(x-4)$. Express as

$$\frac{5x-10}{x^2-3x-4} = \frac{A}{x+1} + \frac{B}{x-4}.$$

   Now solve for A and B. Multiply through gives

$$A(x-4) + B(x+1) = 5x-10.$$

   Then choosing $x = 4$ gives $B = 2$. And $x = -1$ gives $A = 3$. So we get

$$\int \frac{3}{x+1}\,dx + \int \frac{2}{x-4}\,dx = 3\ln|x+1| + 2\ln|x-4| + C.$$

   - $\int \frac{4x^2-11x+12}{(x-2)^2(x+1)}\,dx$. Express as

$$\frac{4x^2-11x+12}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}.$$

   Multiply out:

$$A(x-2)(x+1) + B(x+1) + C(x-2)^2 = 4x^2-11x+12.$$

Plugging in a few nice values:

$$\begin{aligned} x &= 2 & 3B &= 16 - 22 + 12 = 6 \\ x &= -1 & 9C &= 4 + 11 + 12 = 27. \end{aligned}$$

So $B = 2$ and $C = 3$. Now pick some other value of x (I'll do $x = 0$) to find

$$-2A + B + 4C = 12.$$

Using the values of B and C just found we see that $-2A + 2 + 12 = 12$, so $A = 1$. Thus,

$$\int \frac{4x^2 - 11x + 12}{(x-2)^2(x+1)}\,dx = \int \left(\frac{1}{x-2} + \frac{2}{(x-2)^2} + \frac{3}{x+1} \right) dx = \ln|x-2| - 2(x-2)^{-1} + 3\ln|x+1| + C.$$

5 ODEs

There is not too much to say here. Do some practice with separable ODEs. You need to do some algebra to get the variables separated, then integrate both sides. Use the initial condition to find your integration constant. Always know the solution to the linear ODE $x' = \lambda x$.

Remember, an equilibrium for $x' = f(x)$ is simply a root $x_\star$ of f, at which $x' = 0$. Such an equilibrium is **stable** if $f'(x_\star) < 0$ and **unstable** if $f'(x_\star) > 0$. Stable equilibria are important, because they tell you what the limit $\lim_{t\to\infty} x(t)$ is for initial conditions not too far away from the equilibrium.

6 Sequences

Since you have recently covered this material, it should not consume too much of your study-time. Our suggestions are:

1. Do some practice.
2. Make sure you know your rates of growth and asymptotics, so you can quickly find what dominates in the numerator and denominator.
3. Use l'Hopital's rule if necessary (obviously this will not work if you have an $n!$ somewhere).
4. Also, remember how to deal with forms like $f(n)^{g(n)}$ (taking logs) and the form $f(n) \cdot g(n)$ when $f(n) \to 0$ and $g(n) \to \infty$ (flip into a form for l'Hopital, say $\frac{f(n)}{1/g(n)}$).

7 Series Convergence Tips

Go over all the convergence tests, their hypotheses, when they succeed, and when they fail! When testing for series convergence, here are some tips:

1. Do you see factorials, or things which are similar to factorials (e.g. $1 \cdot 3 \cdot 5 \cdots (2n-1)$)? If so, use Ratio Test.
2. Is everything raised to the nth power (or multiples-of-n power)? If so, use Root Test.
3. Is it alternating? If so, use Alternating Series Test. If convergence, check for absolute convergence (depending on what the question is asking).

4. Any time you see a polynomial (whether it is in the numerator, denominator, under a square root sign, etc.), drop all its lower order terms. This will often help you simplify your series and find a good candidate series $\sum b_n$ to do the Limit or Comparison test on. Examples:

$$\sum \frac{n^2+3n+1}{n^4-n^2-5} \approx \sum \frac{n^2}{n^4} = \sum \frac{1}{n^2} \text{ (converges by p-series).}$$

$$\sum \frac{\sqrt[3]{n-5}}{\sqrt{n^3-5}} \approx \sum \frac{\sqrt[3]{n}}{\sqrt{n^3}} = \sum \frac{1}{n^{3/2-1/3}} = \sum \frac{1}{n^{7/6}} \text{ (converges by p-series)}$$

$$\sum \frac{n^2+1}{n^3-2n^2+5} \approx \sum \frac{n^2}{n^3} = \sum \frac{1}{n} \text{ (diverges, harmonic).}$$

5. Remember the relative sizes of growth

$$\ln(n) < n < e^n < n! < n^n. \tag{1}$$

Here, n could be any polynomial function (this includes things like n^3, $\sqrt{n} = n^{1/2}$, or even really small powers like $n^{.001}$) or polynomials of n (but again, only the highest power matters). e^n includes other exponentials like 2^n, 1.1^n, or any other constant bigger than 1 raised to the nth power.

When dealing with sums or differences of functions which grow differently, you can isolate the leading order term, that which grows the fastest. This will hopefully give you something which you can quickly evaluate as convergent or divergent, and the Limit Test (or maybe Comparison Test) will make it official. Examples:

$$\sum \frac{e^n - n^2 + \ln(n)}{n^3 + e^{2n}} \approx \sum \frac{e^n}{e^{2n}} = \sum \frac{1}{e^n} \text{ (geometric, converges)}$$

$$\sum \frac{\sqrt{n} + (2n)!}{n! + e^{2n}} \approx \sum \frac{(2n)!}{n!} \text{ (diverges by } n\text{th term test).}$$

When dealing with functions of different rates which are multiplied together, you have to be more careful. For instance:

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)} \text{ diverges by integral test,}$$

whereas

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2} \text{ converges by integral test.}$$

The point is, if we tried to ignore the factors of $\ln(n)$ in the denominator, it would not give us the correct answer in the latter example. In such situations, the ratio test will probably fail, which leads me to the next point.

6. If the ratio test fails (i.e. $\rho = \lim \frac{|a_{n+1}|}{|a_n|} = 1$), then try to compare to a p-series. The reason this might work is that a p-series will fail the ratio test (try this and see why). If that will not work, consider the integral test.

Example:

$$\sum \frac{\ln(n)}{n} > \sum \frac{1}{n} \text{ (diverges by comparison with harmonic)}$$

This could have also been done with integral test (in general, powers of n mixed with powers of $\ln(n)$ works will with the integral test). Another example:

$$\sum \frac{\ln(n)}{n^2} < \sum \frac{n^{.5}}{n^2} = \sum \frac{1}{n^{1.5}} \text{ (converges by comparison with p-series).}$$

Again, this would have been fine with the integral test (use integration by parts with $u = \ln x$ and $dv = \frac{1}{x^2}$).

8 Power series

Find the interval of convergence for

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

A few comments:

1. Use the ratio test to find $\rho = \lim \left|\frac{a_{n+1}}{a_n}\right|$. Then the radius of convergence is $R = \frac{1}{\rho}$. Your interval of convergence is $(c-R, c+R)$ (need to check the endpoints individually).
2. Note that the interval of convergence **must be centered at c**. Example:

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n3^n}$$

must have an interval of convergence which is centered at 5. It turns out that the interval of convergence here is $[2, 8)$ (you should try this for yourself).

3. If you get something more complicated like

$$\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n},$$

then you can factor out the 2 to get

$$\sum_{n=0}^{\infty} \frac{2^n (x+1/2)^n}{n}$$

and proceed as above.

9 Taylor series

Remember your Taylor series, as well as the general form of the Taylor series for f about $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{2}$$

Remember how to do things like:

1. Find the first few terms of a series by multiplying two known series together (expand and collect like terms). Example:

$$\begin{aligned}
\sin(x)e^x &= \left(x - \frac{x^3}{3!} + O(x^5)\right)\left(1 + x + \frac{x^2}{2!} + O(x^4)\right) \\
&= x + x^2 + (1/2 - 1/6)x^3 + O(x^4) \\
&= x + x^2 + \frac{x^3}{3} + O(x^4)
\end{aligned}$$

2. Compute a Taylor series for a function directly from the definition.

3. Differentiate a function by differentiating its series (you can also integrate a function by integrating the series) within its interval of convergence. Example:

$$\frac{d}{dx}(x\sin(x)) = \frac{d}{dx}\left(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots\right) = 2x - 4\frac{x^3}{3!} + 6\frac{x^5}{5!} + \cdots$$

4. Compute a limit of a ratio by expanding as power series. Note: this will typically only work for us if the limit is taken as $x \to 0$ since most of our Taylor series are centered at $x = 0$.

10 Error bounds

Be sure you know the following:

1. Alternating series. Wherever you stop, the next term is a bound for the error.

   Example:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}.$$
   Since the next term in the series is $-\frac{1}{6}$, we know that our approximation has an error (denoted E_5), which is less than or equal to the next term, $\frac{1}{6}$. Also, our approximation is slightly larger than the true value (since we have just overshot the true value and landed to the right of it on the number line).

   Put another way, if you want to approximate an alternating series within a certain error bound, find the point at which the next term gets smaller than or equal to that bound.

   Example: How many terms are needed to approximate $(-1)^n \frac{1}{n^2}$ within .000001 (that is 10^{-6}).

   Answer: it is impractical to write out that many terms, so look at the general term and find N for which $\frac{1}{(N+1)^2} \le 10^{-6}$. This is when $(N+1)^2 \ge 10^6$ so when $N + 1 \ge 1000$, so $N = 999$ is the minimum number of terms needed to get within 10^{-6} using the alternating series bound.

2. The integral bound: assuming your a_n's are positive and decreasing and can be easily extended to an integrable function $f(x)$, the error E_N given by
$$\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{N} f(n) + E_N,$$
   is bounded as follows
$$\int_{N+1}^{\infty} f(x)\,dx \le E_N \le \int_{N}^{\infty} f(x)dx.$$
   Example: How many terms do you need to approximate
$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$
   within .01?

   Answer: Here $f(x) = \frac{1}{x^4}$. We want to find the smallest N such that
$$\int_{N}^{\infty} f(x)dx \le .01,$$

since this will guarantee $E_N \leq .01$. Thus

$$\int_N^\infty \frac{dx}{x} = \frac{1}{-3x^3}\bigg|_N^\infty = \frac{1}{3N^3} \leq .01.$$

So we find by algebra that $33 \leq N^3$, so $4 \leq N$. So $N = 4$ is the minimum number of terms necessary.

3. Taylor's bound. This tells you a bound for the error when you truncate the Taylor series for $f(x)$:

$$|E_N(x)| \leq \frac{C|x|^{n+1}}{(n+1)!},$$

where C is the maximum of $|f^{(n+1)}(t)|$ for all $|t| < x$.

11 Probability and averages

Know the basics:

1. PDF: $\rho(x)$ is a PDF on $a \leq x \leq b$ if $\int_a^b \rho(x)\,dx = 1$.
2. The expectation (or mean) of a PDF is $\mu = \int_a^b xp(x)\,dx$.
3. The average value of a function $f(x)$ on $a \leq x \leq b$ is

$$\frac{\int_a^b f(x)\,dx}{\int_a^b dx} = \frac{\int_a^b f(x)\,dx}{b-a}.$$

4. Centroids: know the formulae and how to use them. For example, the region between two graphs of f and g:

$$\overline{x} = \frac{\int_a^b x(f(x)-g(x))\,dx}{\int_a^b (f(x)-g(x))\,dx}$$
$$\overline{y} = \frac{\int_a^b \frac{1}{2}(f(x)^2-g(x)^2)\,dx}{\int_a^b (f(x)-g(x))\,dx}.$$

5. Mass: integrate the mass element $dM = \rho(x)dx$ (on a 1-d interval).
6. Center-of-mass: use the centroid equations, but with dM instead of dx.

12 Discrete calculus

There is a lot of fun stuff here that won't make it onto the final, since this material is more "experimental" than most. The things you should know are the forward-difference operator, Δ: what it is and how to use it on a sequence. I will not ask you to do crazy Taylor-like expansions with shift operators or solve second-order recurrence relations like what we did with the Fibonacci sequence. You do not have to memorize the formula for falling powers, but you should be comfortable with the notation and how $\Delta n^{\underline{k}} = kn^{\underline{k-1}}$.

13 Evaluating a series

If a question asks you to evaluate a series (instead of just determining whether the series converges or diverges), then it must be a (possibly disguised) Taylor series for something we know, or some sort of use of the discrete fundamental theorem (also known as a telescoping series).

First, recall all of our favorite series, and pay attention to their "signatures" (e.g. $\sin x$ has odd factorials/powers, e^x has every factorial/power, etc.):

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{3}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{4}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{5}$$

And, for $|x| < 1$ we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \tag{6}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \tag{7}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \tag{8}$$

Examples:

1. Evaluate:
$$\sum_{n=0}^{\infty} \frac{[\ln(4)]^n}{n!}.$$
Answer: compare with (3) to see that it is $e^{\ln(4)} = 4$.

2. Evaluate:
$$\sum_{k=1}^{\infty} \frac{2^{2k-1}}{5^k}.$$
Answer: there are not any factorials, which leads us to think it is a geometric series in disguise. Indeed, if we just write out a few terms, we can use the geometric series:
$$\begin{aligned}\sum_{k=1}^{\infty} \frac{2^{2k-1}}{5^k} &= \frac{2}{5} + \frac{2^3}{5^2} + \frac{2^5}{5^3} + \cdots \\ &= \frac{2}{5}\left(1 + \frac{2^2}{5} + \frac{2^4}{5^2} + \cdots\right) \\ &= \frac{2}{5} \cdot \frac{1}{1 - 4/5} \\ &= \frac{2}{5} \cdot 5 = 2.\end{aligned}$$

3. Evaluate:
$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{3^{2n}(2n+1)!}.$$
Answer: the odd factorial is one of $\sin x$'s characteristics, so we should try to transform this into something which looks like the series for $\sin x$ (for some particular x). Multiplying top and bottom by 3 (to make the exponents of π and 3 match up, we get
$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{3^{2n}(2n+1)!} &= 3\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{3^{2n+1}(2n+1)!}\\ &= 3\sum_{n=0}^{\infty}(-1)^n\frac{(\pi/3)^{2n+1}}{(2n+1)!}\\ &= 3\sin(\pi/3) = \frac{3\sqrt{3}}{2}.\end{aligned}$$