Homework 18

The **due date** for this homework is **Tue 7 May 2013 12:00 AM EDT**.

Work on these problems. If you are stuck, turn to the appropriate discussion forum. On a few problems, you might find it helpful to use a calculator to do some of the arithmetic.

Question 1

The amount I of a radioactive substance in a given sample will decay in time according to the following equation:

$$\frac{dI}{dt} = -\lambda I$$

Nuclear engineers and scientists tend to be concerned with the *half-life* of a substance, that is, the time it takes for the amount of radioactive material to be halved.

Find the half-life of a substance in terms of its decay constant λ.

- (selected) $\frac{\ln 2}{\lambda}$
- $\frac{\lambda}{2}$
- $\frac{\lambda}{\ln 2}$
- $\frac{2}{\ln \lambda}$
- $\frac{\ln \lambda}{2}$
- $\frac{2}{\lambda}$

Question 2

After drinking a cup of coffee, the amount C of caffeine in a person's body obeys the differential equation

$$\frac{dC}{dt} = -\alpha C$$

where the constant α has an approximate value of 0.14 hours^{-1}.

How many hours will it take a human body to metabolize half of the initial amount of caffeine? Round your answer to the nearest integer.

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Question 3

In a highly viscous fluid, a falling spherical object of radius r decelerates right before reaching the bottom of the container. A simple model for this behavior is provided by the equation

$$\frac{dh}{dt} = -\frac{\alpha}{r}\,h$$

where h is the height of the object measured from the bottom, and α is a constant that depends on the viscosity of the fluid.

Find the time it would take the object to drop from $h = 6r$ to $h = 2r$ in terms of α and r.

- $\frac{\alpha}{r}\ln\frac{1}{3}$
- $\frac{\alpha}{r}\ln 3$

- $\frac{\alpha}{r} \ln 12$
- $\frac{r}{\alpha} \ln 12$
- $\frac{r}{\alpha} \ln 3$
- $\frac{r}{\alpha} \ln \frac{1}{3}$

Question 4

On a cold day you want to brew a nice hot cup of tea. You pour boiling water (at a temperature of $212^{\mathrm{o}} F$) into a mug and drop a tea bag in it. The water cools down in contact with the cold air according to Newton's law of cooling:

$$\frac{dT}{dt} = \kappa(A - T)$$

where T is the temperature of the water, $A = 32^{\mathrm{o}} F$ the ambient temperature, and $\kappa = 0.36$ min^{-1}.

The threshold for human beings to feel pain when entering in contact with something hot is around $107^{\mathrm{o}} F$. How many seconds do you have to wait until you can safely take a sip? Round your answer to the nearest integer.

Question 5

On the night of April 14, 1912, the British passenger liner RMS Titanic collided with an iceberg and sank in the North Atlantic Ocean. The ship lacked enough lifeboats to accommodate all of the passengers, and many of them died from hypothermia in the cold sea waters. Hypothermia is the condition in which the temperature of a human body drops below normal operating levels (around $36^{\circ}C$). When the core body temperature drops below $28^{\circ}C$, the hypothermia is said to have become severe: major organs shut down and eventually the heart stops.

If the water temperature that night was $-2^{\circ}C$, how long did it take for passengers of the Titanic to enter severe hypothermia? Recall from lecture that heat transfer is described by Newton's law of cooling:

$$\frac{dT}{dt} = \kappa(A - T)$$

where T is the body temperature of a passenger, A the water temperature, and $\kappa = 0.016$ min^{-1}. Give your answer in minutes and round it to the nearest integer.

Question 6

Assume the size P of a population grows following the differential equation

$$\frac{dP(t)}{dt} = bP(t)$$

The rate of growth b is the difference between the birth rate and the mortality rate. The Malthusian model supposes that this rate is constant. Of course we know that does not always happen: factors such as famines, outbreaks of disease or advances in medicine do influence these rates. When modeling some process mathematically it is important to recognize what our assumptions are and when they no longer hold. In this problem and the next we will look at one particular event that would result in a violation of Malthus' assumption that the growth rate b is constant over time: the occurrence of famines.

Experimental data suggests that food production F grows linearly over time:

$$F(t) = F_0 + st$$

We will now make two assumptions:

- Most food is perishable, so that the amount of food available at any given time is exactly the amount produced at that time. This means that we are not taking into account the effect of the possibility of preserving food for long periods of time.
- The amount of food that a person in our population eats is, on average, constant and equal to some number α. That is, the amount of food necessary to keep everybody fed is $\alpha P(t)$.

The so-called *Malthusian catastrophe* happens when there is not enough food to feed the whole population, that is, when $\alpha P(t) = F(t)$.

The Food and Agriculture Organization of the United Nations considers that a person needs around 1800 kcal/day to be considered well-fed. According to a report released in 2002 by the same organization:

- The world population in 2002 was around 6 billion.
- The population growth rate was estimated at 1.1% and expected to remain approximately constant for several decades.
- Total food production in 2002 was determined to be around $6.13 \cdot 10^{15}$ kcal, with an expected growth rate of $1.11 \cdot 10^{14}$ kcal/year.

[Source: FAO, "World Agriculture: Towards 2015/2030. Summary Report", 2002]

Estimate the world population by the year 2030.

- 8.2 billion
- 8.8 billion
- 7.8 billion
- 8.4 billion
- 8.6 billion
- 8.0 billion

Question 7

(Continued from previous question) Estimate when the Malthusian catastrophe would happen if our assumptions continue to hold.

- $0.11e^{0.011t} = 61.3 + 1.11t \Rightarrow t \approx 827$ years.
- $39.42e^{0.011t} = 61.3 + 1.11t \Rightarrow t \approx 166$ years.
- $0.11e^{1.1t} = 61.3 + 1.11t \Rightarrow t \approx 6$ years.
- $39.42e^{1.1t} = 61.3 + 1.11t \Rightarrow t \approx 0.4$ years.

In accordance with the Honor Code, I certify that my answers here are my own work.

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