Homework 49

The **due date** for this homework is **Tue 7 May 2013 12:00 AM EDT**.

Question 1

The Fundamental Theorem of Integral Calculus yields the following integral expression for π:

$$\pi = 4\int_0^1 \frac{dx}{1+x^2}$$

Using a calculator (physical or on-line), estimate the value of π by the following two methods:

1. First, the trapezoidal rule with step $h = 1/4$, and
2. Second, Simpson's rule with $h = 1/4$.

Round your answer to five decimal places and input them separated by a space. For example, if you get 3.41573 using the trapezoidal rule and 3.16331 with Simpson's rule, enter "3.41573 3.16331" (without the quotes, please).
Notice: although the amount of computations you need to make in each case is very similar, Simpson's rule provides a much better estimate. Recall that $\pi \approx 3.14159$

Question 2

For those who want to go a little deeper into the subject: in Lecture 48 we discretized first-order differential equations

$$\frac{dx}{dt} = f(x,t) \qquad (*)$$

by using a discrete version of the derivative —the forward difference—, to yield *Euler's method*:

$$x_{n+1} - x_n = hf(x_n, t_n)$$

A different approach starts by considering the integral form of our original differential equation: integrating $(*)$ from t_n to t_{n+1} and using the Fundamental Theorem of Integral Calculus, we obtain

$$x(t_{n+1}) - x(t_n) = \int_{s=t_n}^{t_{n+1}} f(x(s), s)\, ds$$

What we have on the right hand side is now an integral, so how about applying a numerical integration method to this instead?

1. The first, obvious choice is to use a left Riemann sum:

$$\int_{s=t_n}^{t_{n+1}} f(x(s), s)\, ds \approx (t_{n+1} - t_n) f(x(t_n), t_n)$$

   Now, setting $x(t_n) = x_n$, $x(t_{n+1}) = x_{n+1}$ and $h = t_{n+1} - t_n$ yields the update rule

$$x_{n+1} - x_n = hf(x_n, t_n) \tag{1}$$

   which is none other than Euler's method!
2. Using a right Riemann sum instead,

$$\int_{s=t_n}^{t_{n+1}} f(x(s), s)\, ds \approx (t_{n+1} - t_n) f(x(t_{n+1}), t_{n+1})$$

   gives

$$x_{n+1} - x_n = hf(x_{n+1}, t_{n+1}) \tag{2}$$

   This formula looks very similar to Euler's method, but it is different: it is know as the *backward Euler's method*. Notice that x_{n+1} also shows up on the right hand side, so that isolating it in terms of x_n, t_n and h might not be possible, depending on the exact shape of $f(x, t)$. For this reason, this update rule is said to be an *implicit* method (as opposed to an *explicit* one, such as Euler's method).
3. Yet a third possibility is to use the trapezoidal rule:

$$\int_{s=t_n}^{t_{n+1}} f(x(s), s)\, ds \approx (t_{n+1} - t_n)\, \frac{f(x(t_n), t_n) + f(x(t_{n+1}), t_{n+1})}{2}$$

   which gives us another implicit method.

$$x_{n+1} - x_n = h\, \frac{f(x_n, t_n) + f(x_{n+1}, t_{n+1})}{2} \tag{3}$$

Let us apply these three methods to our well-known problem: estimate the value at $t = 1$ of the solution of the differential equation

$$\frac{dx}{dt} = x$$

with initial condition $x(0) = 1$. Of course we know how to solve this problem exactly: the value we are looking for is e.

1. In Lecture 48, we saw that Euler's method —our update rule (1)— with N steps returns

$$x_N = (1+h)^N = \left(1 + \frac{1}{N}\right)^N$$

Here you have some concrete values of x_N:

N	1	2	3	4	5	6	7
x_N	2.00000	2.25000	2.37037	2.44141	2.48832	2.52163	2.54650

2. The backwards Euler's method —our update rule (2)—, gives

$$x_{n+1} - x_n = hx_{n+1} \quad \Rightarrow \quad x_{n+1} = \frac{x_n}{1-h} \quad \Rightarrow \quad x_N = \frac{1}{(1-h)^N} = \left(\frac{N}{N-1}\right)^N$$

N	2	3	4	5	6	7
x_N	4.00000	3.37500	3.16049	3.05176	2.98598	2.94190

Following the same procedure as above, find the general formula for x_N using the update rule (3) that we obtained using the trapezoidal rule.

- $x_N = \left(\frac{1-2h}{1+2h}\right)^N = \left(\frac{N-2}{N+2}\right)^N$
- $x_N = \left(\frac{1-h}{1+h}\right)^N = \left(\frac{N-1}{N+1}\right)^N$
- $x_N = \left(\frac{1+h}{1-h}\right)^N = \left(\frac{N+1}{N-1}\right)^N$
- $x_N = \left(\frac{1+2h}{1-2h}\right)^N = \left(\frac{N+2}{N-2}\right)^N$

- $x_N = \left(\frac{1+h/2}{1-h/2}\right)^N = \left(\frac{2N+1}{2N-1}\right)^N$
- $x_N = \left(\frac{1-h/2}{1+h/2}\right)^N = \left(\frac{2N-1}{2N+1}\right)^N$

In accordance with the Honor Code, I certify that my answers here are my own work.

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