Homework 20

The **due date** for this homework is **Tue 7 May 2013 12:00 AM EDT**.

Question 1

For a differential equation

$$\frac{dx}{dt} = f(x)$$

with an equilibrium at $x = 0$, the *linearization* of the differential equation at the equilibrium is, as per our example in class,

$$\frac{dx}{dt} = f'(0)x,$$

and one *hopes* that the solution to the linearized equation provides a good approximation to the original, nonlinear equation. (It does, so long as $f'(0) \neq 0$).

The differential equation

$$\frac{dx}{dt} = (e^x - 1)(x - 1)$$

has an equilibrium at $x = 0$. Which of the following is the linearized equation at this equilibrium?

- $\frac{dx}{dt} = ex$
- $\frac{dx}{dt} = -x$
- $\frac{dx}{dt} = x$
- $\frac{dx}{dt} = ex$
- $\frac{dx}{dt} = -ex$

- $\frac{dx}{dt} = 0$

Question 2

The differential equation in Question 1 has another equilibrium at $x = 1$. Let's look at that. For a differential equation

$$\frac{dx}{dt} = f(x)$$

with an equilibrium at $x = a$, the *linearization* of the differential equation at the equilibrium is best expressed as a differential equation on a new variable, $z = x - a$. You should think of z as a small perturbation to the equilibrium. This perturbation changes at a rate given by the linearization at a:

$$\frac{dz}{dt} = f'(a)z,$$

and one *hopes* that the solution to the linearized equation provides a good approximation for the original, nonlinear equation. (It does, so long as $f'(a) \neq 0$).

The differential equation

$$\frac{dx}{dt} = (e^x - 1)(x - 1)$$

has an equilibrium at $x = 1$. What is the linearized equation at this equilibrium?

- $\frac{dz}{dt} = (e - 1)z$
- $\frac{dz}{dt} = ez$
- $\frac{dz}{dt} = -z$
- $\frac{dz}{dt} = -ez$

- $\frac{dz}{dt} = z$
- $\frac{dz}{dt} = 0$

Question 3

In our lesson, we looked at two oscillators with "sinusoidal" coupling. Other types of coupling are possible as well. Consider the system of two oscillators modeled by

$$\frac{d\theta_1}{dt} = 2 + \epsilon(e^{\theta_1 - \theta_2} - 1) \quad ; \quad \frac{d\theta_2}{dt} = 2 + \epsilon(1 - e^{\theta_1 - \theta_2})$$

Consider the phase difference $\varphi = \theta_2 - \theta_1$. Note that $\varphi = 0$ (where the oscillators are coupled) is an equilibrium. What is the linearized equation for φ about 0?

This looks intimidating, but is very straightforward. If you're not sure how to start, compute $\dot{\varphi}$. Then linearize this about $\varphi = 0$.

- $\frac{d\varphi}{dt} = \epsilon$
- $\frac{d\varphi}{dt} = \epsilon\varphi$
- $\frac{d\varphi}{dt} = -2\epsilon\varphi$
- $\frac{d\varphi}{dt} = -\epsilon\varphi$
- $\frac{d\varphi}{dt} = 2\varphi$
- $\frac{d\varphi}{dt} = 2\epsilon\varphi$

Question 4

It is easy to determine stability or instability of an equilibrium. Recall that an equilibrium $x = a$ for $\frac{dx}{dt} = f(x)$ is *stable* if $f'(a) < 0$ and is *unstable* if $f'(a) > 0$.

Recall from Lecture 18, Newton's Law of Heat Transfer, which states that

$$\frac{dT}{dt} = \kappa(A - T),$$

where $\kappa > 0$ is a thermal conductivity constant and A is the (constant) ambient temperature. Find and classify the equilibria in this system.

- $T = \kappa$ is the unique equilibrium. It is stable.
- There are no equilibria, which is why the temperature never quite stops changing.
- $T = A$ is the unique equilibrium. It is stable.
- $T = A$ is the unique equilibrium. It is unstable.
- There are equilibria at $T = 0$ (unstable) and $T = A$ (stable).
- $T = \kappa$ is the unique equilibrium. It is unstable.

Question 5

Consider the differential equation

$$\frac{dx}{dt} = f(x)$$

and assume that $f(a) = 0$ for some constant a.

Assume also that the function f has a Taylor expansion at $x = a$,

$$f(x) = b(x - a) + \mathrm{O}((x - a)^2)$$

for some constant b.

Which of the following statements about the differential equation are true? Select all that apply.

- a is a stable equilibrium if $b < 0$.
- a is an unstable equilibrium if $b > 0$.
- b is an unstable equilibrium if $a > 0$.
- b is a stable equilibrium if $a < 0$.
- a is a stable equilibrium if $b > 0$.
- a is an unstable equilibrium if $b < 0$.
- Neither a nor b is an equilibrium point.
- b is a stable equilibrium if $a > 0$.

Question 6

Consider the differential equation

$$\frac{dy}{dt} = -2y + y^2 + y^3$$

Find and classify all the equilibria.

- $y = 2$: unstable. $y = 0$ and $y = -1$: stable.
- $y = 0$: stable. $y = -1$ and $y = 2$: unstable.
- $y = 0$, $y = 1$, and $y = -2$: stable.
- $y = 1$: stable. $y = 0$ and $y = -2$: unstable.
- $y = 0$: unstable. $y = 1$ and $y = -2$: unstable.
- $y = 0$: stable. $y = 1$ and $y = -2$: unstable.

Question 7

Recall, also, from Lecture 19 how we computed the terminal velocity of a falling body with linear drag given by

$$m\frac{dv}{dt} = mg - \kappa v,$$

where, of course, m is mass, g is gravitation, v is velocity, and $\kappa > 0$ is the drag coefficient. Can you see how easily one can solve for the equilibrium $v_\infty = mg/\kappa$?

Very good. Now, let's use a more realistic model of drag that is quadratic as opposed to linear:

$$m\frac{dv}{dt} = mg - \lambda v^2,$$

where $\lambda > 0$ is a constant drag coefficient. *This* differential equation is not as easy to solve (but soon you will learn how). Is there is terminal velocity? What is it?

- $v_\infty = 0$ is an unstable equilibrium; this is the terminal velocity.
- $v_\infty = \sqrt{\dfrac{mg}{\lambda}}$ is a stable equilibrium; this is the terminal velocity.
- $v_\infty = \sqrt{\dfrac{mg}{\lambda}}$ is an unstable equilibrium; there is no terminal velocity.
- $v_\infty = \dfrac{mg}{\lambda}$ is a stable equilibrium; this is the terminal velocity.
- $v_\infty = \dfrac{mg}{\lambda}$ is an unstable equilibrium; there is no terminal velocity.
- $v_\infty = 0$ is an unstable equilibrium; there is no terminal velocity.

In accordance with the Honor Code, I certify that my answers here are my own work.

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