Hedging and Pricing OTC Options under UVM

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Introduction

The pricing and the hedging of the OTC Binary and Vanilla Call options under Uncertain Volatility Model is the main topic of discussion in the paper.

In Brief, the paper examines the ways to find the optimal spread for the bid/ask prices of OTC derivative contracts and the optimal static hedge ratios for effective pricing contracts.

Uncertain Volatility Model, used to price derivatives under uncertain parameters is applied in the context of uncertain volatility to price OTC Binary & Vanilla options and construct hedges to achieve narrower bid/ask spreads. Consequently, the use of static hedging accomplishes smaller spreads in pricing OTC contracts and diminishes the arbitrage opportunities.

VBA code algorithms are enclosed as a reference to the main models used in the software application.

Theory & Methodology

This section discusses the hypothesis of the implemented models, design and architecture of the software application.

Uncertain Volatility Model (UVM)

$$\frac{\partial V^-}{\partial t} + \frac{1}{2}\,\sigma^2 S^2 \frac{\partial^2 V^-}{\partial S^2} + rS\frac{\partial V^-}{\partial S} - rV^- \;=\; 0$$

$$\Gamma \;=\; \frac{\partial^2 V^2}{\partial S^2}$$

$$\sigma(\Gamma) = \begin{cases} \sigma^-, & if \quad \Gamma > 0 \\ \sigma^+, & if \quad \Gamma < 0 \end{cases}$$

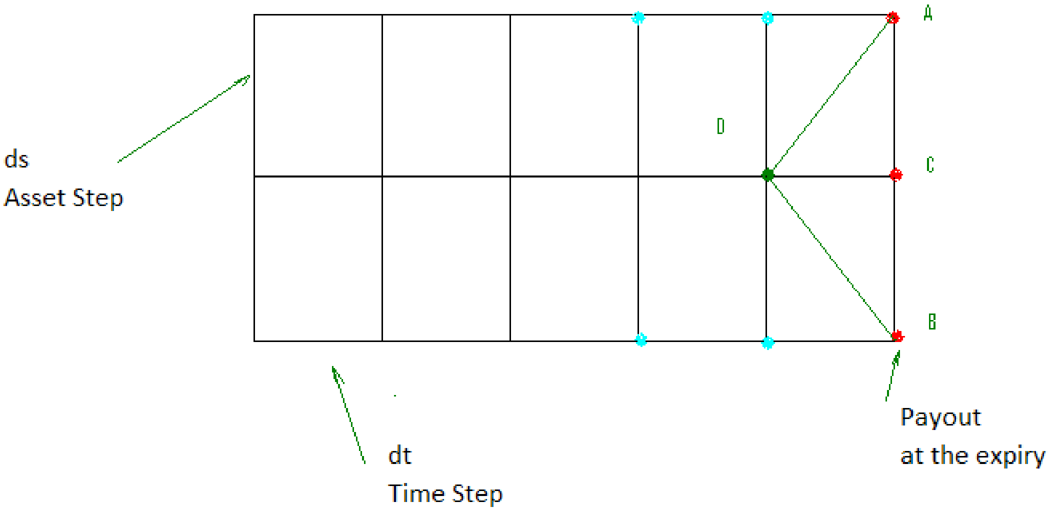
The above model is a Black-Scholes PDE equation, a version of Uncertain Parameters Model in the context of uncertain volatility, where V^+ and V^- are the best and the worst option values respectively for the volatility values too.

The sign of the gamma is the main factor in finding the best/worst case volatility in this non-linear model. This is particularly helpful in identifying the worst case scenario for options that change gamma sign during its lifetime especially the path–dependent OTC options or for portfolio of options where the net-gamma sign changes during portfolio revaluation.

For options such as Vanillas which don't change the their gamma sign, the UVM model's result will match Black-Scholes. However, we should carefully input the worst case volatility parameter as the input to the Black-Scholes PDE to achieve this result.

Explicit Finite Difference (EFD)

The Explicit Finite Difference (EFD) method has been selected to price the UVM. The diagram below illustrates a sample FD grid with data points marked as A, C, B and D.



The value of data point A, C and B are calculated as payouts at **expiry**. So, they are Max(S-E, 0) for vanilla-call, Max(E-S, 0) for vanilla-put options and, H(S-E) for binary-call and H(E-S) for binary-put options.

Approximations:

Boundary Conditions: The "Blue" points in the above diagram are the boundary conditions of the asset price steps for the respective time steps. They are approximated as:

*Value of beginning Asset Step at time step n = Value (0, n -1) * (1 – INTRate * dt)*
*Value of end Asset Step at time step n = 2 * Value (End-1, n) -Value (End-2, n)*

Delta:

$$Delta = \frac{(V_{i+1}^{n} - V_{i-1}^{n})}{2ds}$$

The above central difference approximation for Delta has an error of O (dS^2)

Gamma:

$$Delta = \frac{(V_{i+1}^{n} + V_{i-1}^{n} - 2V_{i}^{n})}{ds^2}$$

The above approximation for Gamma also has an error of O (dS^2)

Theta:

$$\frac{\partial V^-}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^-}{\partial S^2} + rS\frac{\partial V^-}{\partial S} - rV^- = 0$$

$$\Gamma = \frac{\partial^2 V^2}{\partial S^2}$$

$$\sigma(\Gamma) = \begin{cases} \sigma^-, & if \quad \Gamma > 0 \\ \sigma^+, & if \quad \Gamma < 0 \end{cases}$$

The Theta is approximated by rewriting the UVM PDE to yield us:

Option Value:

The option value at point “D” (in the above diagram) is computed using point “G” and the Delta, Theta approximations as discussed above.

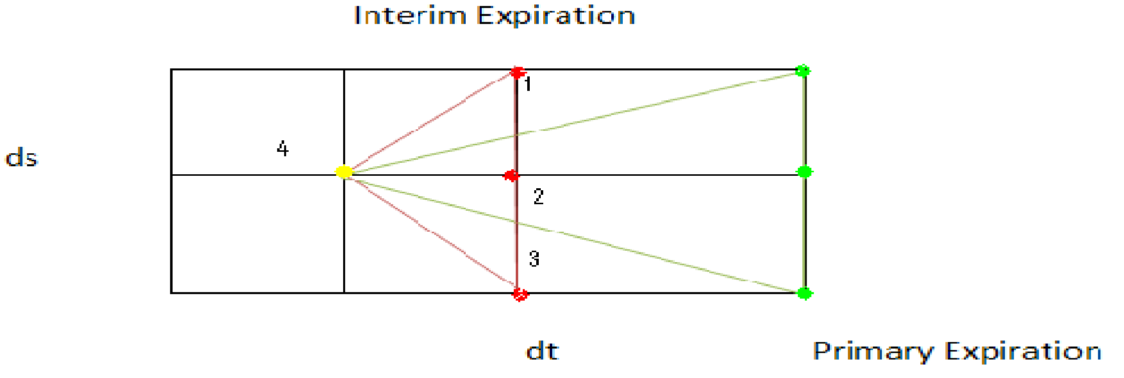
*Value (Option at Point “D”) = Value (Option at Point “C”) – Theta * Delta*

Code Snippet on approximation & value computation (VBA):

```
For n = 1 To NTS
    St = Smin
    For i = 1 To (NAS - 1)
        St = St + dS
        tmp = St / dS
        gamma = 0
        If i >= 1 Then
            gamma = (v(i + 2, n) - 2 * v(i + 1, n) + v(i, n)) / (dS * dS) 'using previous time
        End If
        If gamma < 0 Then
            Vol = VolMin
        ElseIf gamma > 0 Then
            Vol = VolMax
        Else
            Vol = (VolMin + VolMax) / 2 'For safety
        End If
        gammaVec(i) = gamma
        A = 0.5 * (Vol ^ 2 * tmp ^ 2 - (r - d) * tmp) * dT
        b = -(Vol ^ 2 * tmp ^ 2 + r) * dT
        C = 0.5 * (Vol ^ 2 * tmp ^ 2 + (r - d) * tmp) * dT
        v(i + 1, n + 1) = A * v(i, n) + (1 + b) * v(i + 1, n) + C * v(i + 2, n)
    Next
Next
```

Explicit Finite Difference with Jump Volatility

The targeted options and their hedge options do not always have the same expiry within the portfolio, which is one of the main concerns in pricing them under a non-linear model like UVM.



The above diagram depicts the interim expiration of the option at line 1,2&3 points, which has not been modelled with the vanilla options in the previously shown models.

Those intermediate expirations are regarded as their own EFD grids and discounting the values back to previous expiry on the main EFD. At the above diagram, point 4 represents a situation where we would have more than one payoff: one from the main EFD grid (which is pricing the primary expiration portfolio) and the other from the intermediate expiring option.

The gap between green and red/blue dots is “dt”, but the gap between red and blue dots is *not* “dt”. This is one of the situations where the diffusion models have been integrated within finite difference modelling with jumps.

The chosen model has a few limitations, which should be taken under consideration:

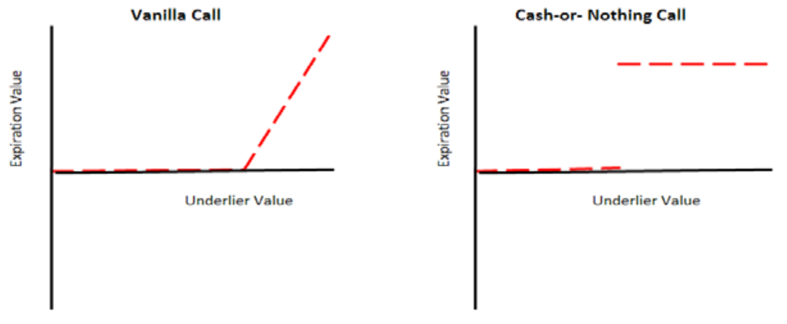
- The jumps are modelled using only one time step between the red dots and the blue/yellow dots. Although, this makes the implementation slightly easier, it does reduce the accuracy/convergence of the option value.
- Gamma sign change is not factored in during the jump.

Hedging Vanilla & Exotic Derivatives

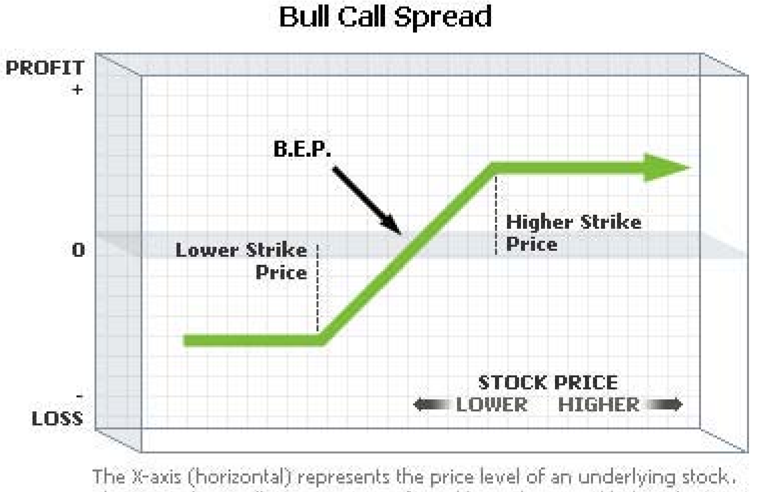
This section deals with the OTC option payoffs and the static hedging strategy for pricing Vanilla and Exotic derivatives.

The main OTC options are;
1. Binary Options (a.k.a Cash-or-Nothing or Digital)
2. Vanilla Options

The **payoff** diagram of the Vanilla call and a Binary call is shown below.

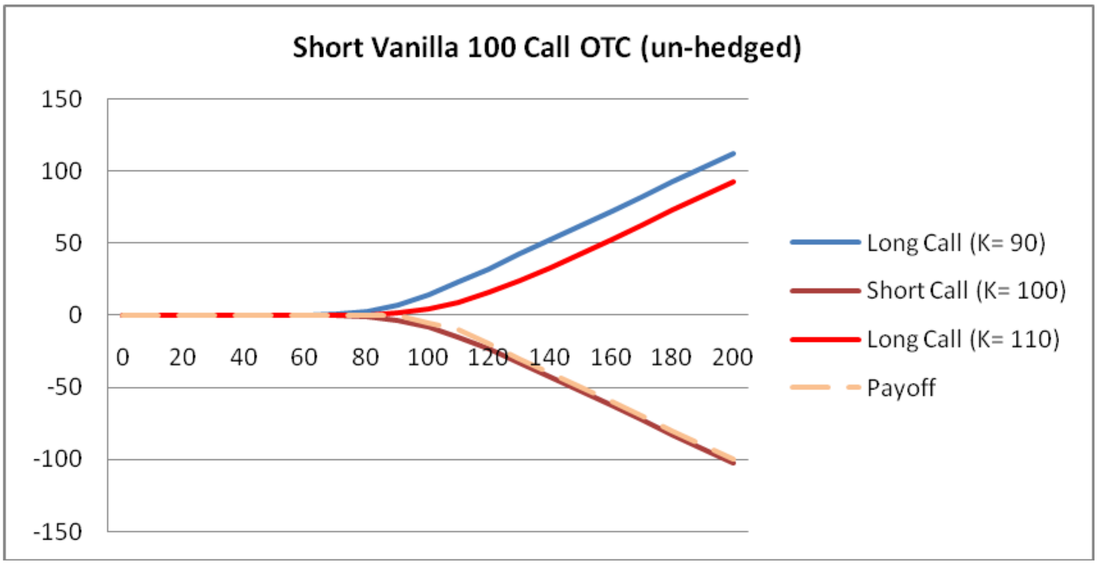


For **static hedging** OTC binary options, we use the well known Call Spread. To ensure the static hedges have the most positive impact for our portfolio, the hedge quantities need to be optimized and selectively chosen. The process of deriving the best hedge quantities is explained in detail in the Optimization section of the paper. Below is a maturity payoff diagram of a Bull Call Spread. The residual portfolio (that comes from the combination of Binary + Hedge1 + Hedge2) must be dynamically hedged to ensure no big market movements cause losses on the P&L due to net negative gamma.

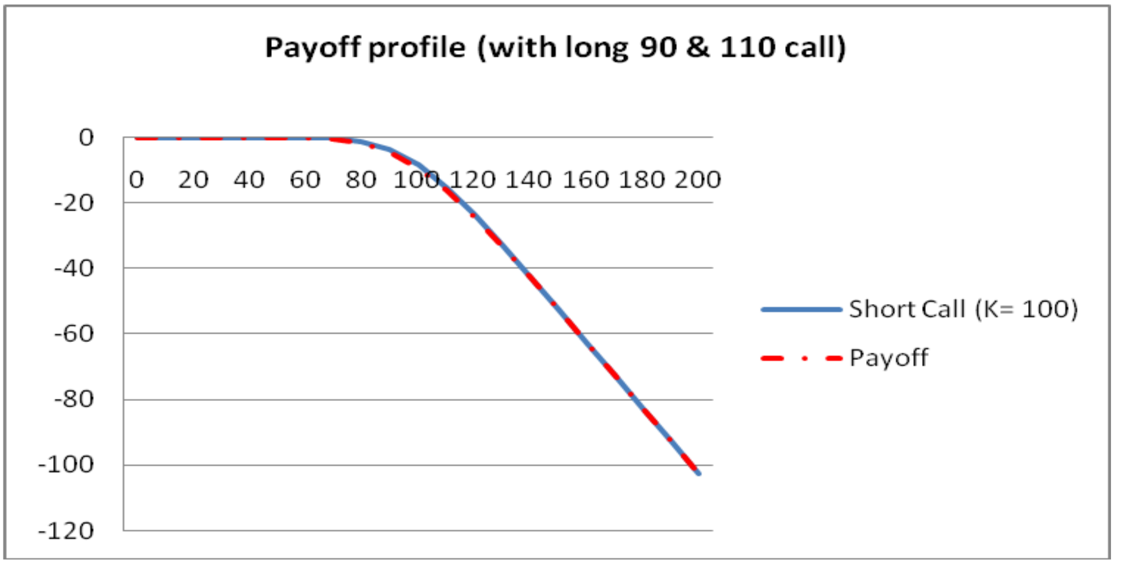


The X-axis (horizontal) represents the price level of an underlying stock. The Y-axis (vertical) represents profit and loss, above and below the X-axis intersection respectively.

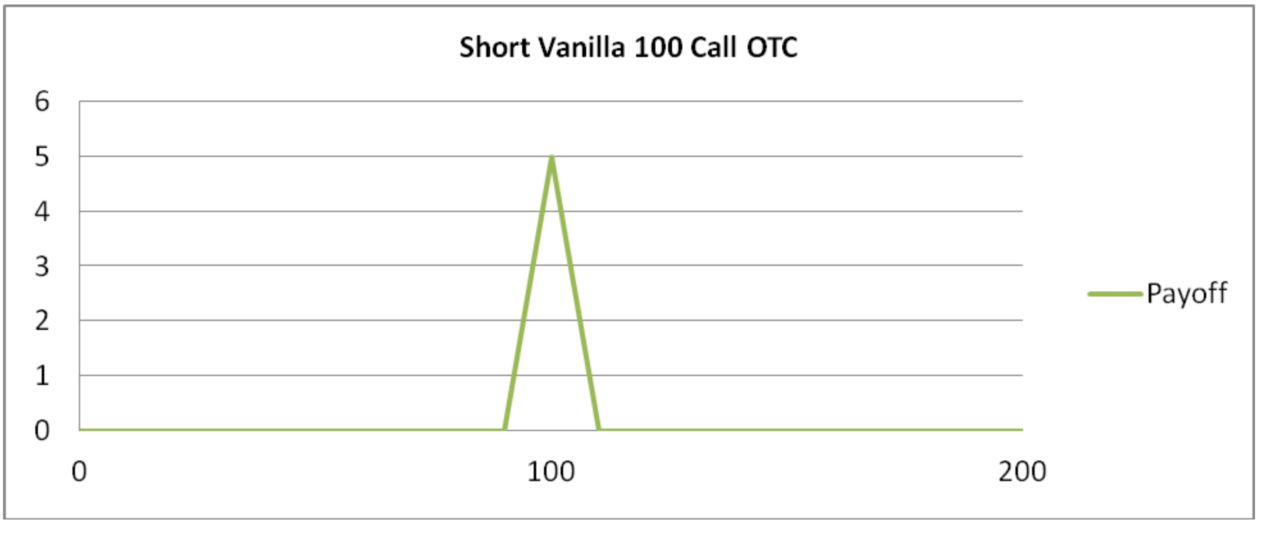
For **static hedging** OTC vanilla options, we take the opposite side of market options that is closest to the OTC strike price. That is, if our OTC option is short call with X=100. Then, we buy two call options. Assume 90 Call and 110 Call are traded, and then we would have buy *optimum quantity* of 90 Call and 110 Call.



The payoff of such a strategy is shown below.



The residual portfolio after static hedging needs dynamic hedging and the payoff of the residual in this context is shown below.



Pricing Vanilla & Exotic Derivatives

This section examines the bid/ask prices of the OTC Binary/Vanilla options and their derivation. This is accomplished using three steps.

Phase1: Hedge Ratio

The first step in pricing the OTC contracts is finding optimal hedges. Sticking to only the above mentioned hedging strategy, the first goal in finding the hedges would be to locate market vanilla options that trade around the strike price of the OTC contract.

For example, if we are to price a Short OTC Binary Call for Strike $100, then we choose $90 Vanilla & $110 Vanilla as the hedges. (Assuming 90 & 110 Vanilla are the only nearest priced options in market with sufficient liquidity).

Here is a snapshot of how the accompanying software has located the best hedges for various different OTC contracts.

Static Portfolio - OTC Options

#	Category	Type	Qty Short	Strike	Expiry	Vol-	Vol+	Int	Residual Short	Residual Long	Bid No Hedge	Ask No Hedge	Bid Static Hedge	Ask Static Hedge	Bid Optimal	Ask Optimal
1	Vanilla	Call	1	100	0.5	0.2	0.3	0.05	0.6069	-1.6573	6.8887	9.6349	7.6741	8.7245	7.7711	8.6792
2	Binary	Call	1	100	0.5	0.2	0.3	0.05	-0.0333	-0.0882	0.4922	0.5288	0.8449	0.8998	0.8857	0.8903
3	Binary	Call	1	110	0.5	0.2	0.3	0.05	-0.0196	-0.0840	0.2780	0.3227	0.4266	0.4910	0.4401	0.4887
4	Binary	Call	1	120	0.5	0.2	0.3	0.05	-0.0103	-0.0605	0.1155	0.1934	0.1920	0.2422	0.1986	0.2408
Total			4			0.2	0.3	0.05			7.7744	10.6798	9.1376	10.3575	9.2956	10.2990

Static Hedges / **Optimal Hedge Ratio**

#	Category	Type	Qty Long	Strike	Expiry	Vol	Cost Long	Hedging Position	Residual	Bid	Ask	Qty Short	Cost Short	Qty Long	Cost Long
H1:1	Vanilla	Call	-0.5	90	0.5	0.25	7.2186	1	0.6069	14.4371	14.4371	0.67	9.6729	-0.62	8.9510
H1:2	Vanilla	Call	-0.5	110	0.5	0.25	2.1129	1	0.6069	4.2258	4.2258	0.55	2.3242	-0.53	2.2397
H2:1	Vanilla	Call	-0.05	90	0.5	0.25	0.7219	2	-0.0333	14.4371	14.4371	0.051	0.7363	-0.05	0.7219
H2:2	Vanilla	Call	0.05	110	0.5	0.25	0.2113	2	-0.0333	4.2258	4.2258	-0.055	0.2324	0.039	0.1648
H3:1	Vanilla	Call	-0.05	100	0.5	0.25	0.4130	3	-0.0196	8.2600	8.2600	0.05	0.4130	-0.05	0.4130
H3:2	Vanilla	Call	0.05	120	0.5	0.25	0.0976	3	-0.0196	1.9517	1.9517	-0.054	0.1054	0.044	0.0859
H4:1	Vanilla	Call	-0.05	110	0.5	0.25	0.2113	4	-0.0103	4.2258	4.2258	0.05	0.2113	-0.05	0.2113
H4:2	Vanilla	Call	0.05	130	0.5	0.25	0.0412	4	-0.0103	0.8247	0.8247	-0.055	0.0454	0.038	0.0313
Total			8		0.5	0.25	11.0277			52.5880	52.5880		13.7408		12.8188

The implementations logic to locate hedges is fairly simple. It looks for Vanilla options around the OTC strike for same expiration.

The key improvements that can be done on this algorithm include:

1. Finding cheaper hedges which are not necessarily around the strike of OTC but have very similar payoff on the residual portfolio.

2. Make markets more efficient & win more businesses by :

    a. Creating new OTC contracts to fill in the existing strike range.

    b. Provide a narrower bid/ask spread on contracts that have higher spreads.

3. Factoring in transaction costs during the location of the optimum hedges.

4. Scan for arbitrage-able prices in the market. This improvement is not just for finding hedges, but has a bigger scope in being able to create a new system for finding & exploiting arbitrage opportunities. One simple starting point on this is looking at existing market prices of various contracts & calculating if better prices can be generated for those contracts using static hedges.

5. The EFD method (with jumps) discussed earlier is can be implemented by handling pricing multi-expiration options.

Phase2: Payoff Residuals and Hedge Cost

$$\text{Hedging Cost} = \sum (\text{Cost of Hedge vanillas}) \qquad \text{(eq. C1)}$$

$$\textit{Cost of vanillas} = \textit{Qty} * \{\textit{Reference Price}\} \qquad \textit{(eq. C2)}$$

Where Reference Price is:
Market Bid for selling vanillas.
Market Ask for buying vanillas.

As a price taker during the buying/selling of the hedges, we look at the opposite side of the market book. But, the case is different with the OTC contract pricing since we are the price maker on these contracts. Thus, how we display the OTC contract prices (bid/ask) is opposite to how we use the reference prices from the market during hedging.

$$\textit{Residual Portfolio Value} = \textit{Value from EFD (under UVM)} \qquad \textit{(eq.R1)}$$

The most important thing about the residual portfolio is that it needs to be dynamically hedged as with any other actively managed portfolio. Say, for example to avoid jumps that lead to losses due to negative gammas.

In the earlier mentioned example for pricing OTC long Binary \$100 call, the residual portfolio is made up of:

- Short Binary \$100 Call (6 month expiry)
- Long Vanilla \$90 Call (50% / 6 month expiry)
- Short Vanilla \$110 Call (50% / 6 month expiry)

The payoffs being H(S-100, 0) + Max(S-110, 0) – Max(S-90, 0) The cost of hedging is : \$7.21+2.11 = \$9.32 Residual Portfolio Value is : \$ 0.61

Code Snippet on residual portfolio (VBA):

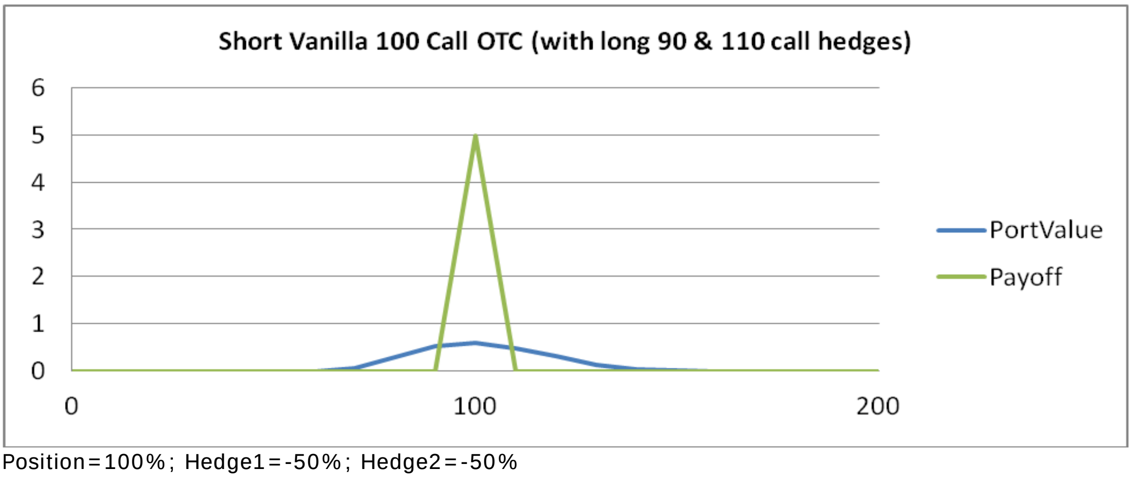
```
For j = 0 To NAS
    If Category = "Vanilla" Then
        v(j + 1, 1) = -Abs(q0) * WorksheetFunction.Max(St - K0, 0) + Abs(q1)
                         * WorksheetFunction.Max(St - K1, 0) + Abs(q2) * WorksheetFunction.Max(St - K2, 0)
    ElseIf Category = "Binary" Then

        If WorksheetFunction.Max(St - K0, 0) > 0 Then
            BinaryPayoff = 1
        Else
            BinaryPayoff = 0
        End If

        v(j + 1, 1) = -Abs(q0) * BinaryPayoff + Abs(q1) * WorksheetFunction.Max(St - K1, 0) - Abs(q2)
                                                     * WorksheetFunction.Max(St - K2, 0)

     End If

     If BuyOrSell = "s" Then
        v(j + 1, 1) = v(j + 1, 1) * -1
    End If

    St = St + dS
  Next
```

Results

This section lists the major results of the residual portfolio from our model (eq. R1). All the hedged positions below have hedge quantities that have been chosen carefully to attain minimum residual portfolio. Since the binary's payoff is set at $1 in the money, when hedging them with vanillas, the hedge percentage of vanilla calls/put is generally lower (like 5%) compared to what we observe when hedging vanillas with other vanillas (like 50%).

Case1 : Hedging Vanilla Options

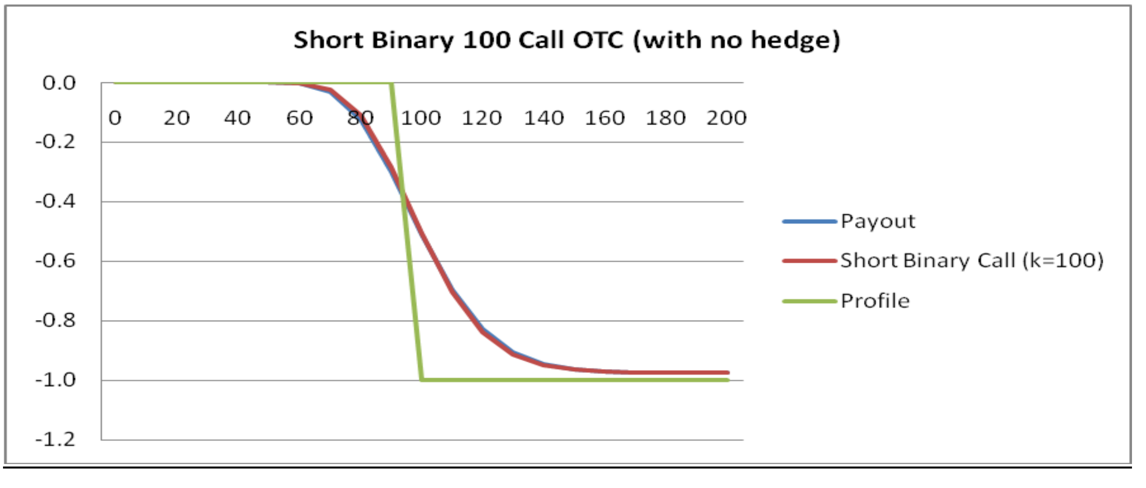
Hedged Short Vanilla Call Position



Position=100%; Hedge1=-50%; Hedge2=-50%

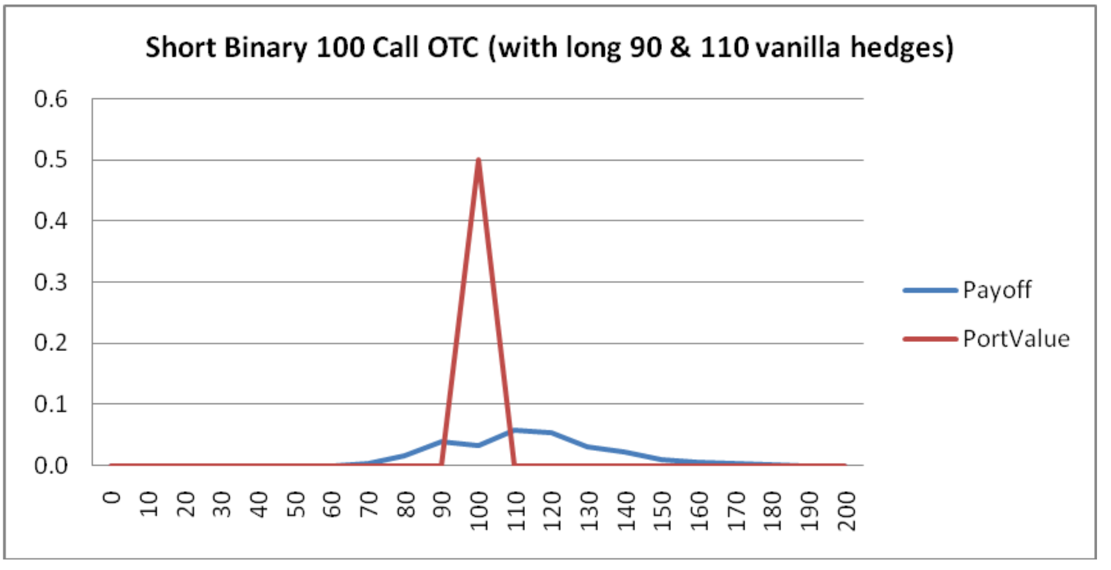
Case2 : Hedging Binary Options

Un-hedged Short Binary Call



Position=100%; No hedges

Position Hedged Short Binary Call Position



Position=100%; Hedge1=5%; Hedge2=-5%

Phase3: Hedge Quantity vs. OTC Value

Value of OTC Option = {Value of Residual Portfolio} – Cost of Hedging *(eq.V1)*

The above equation clearly highlights the importance of the hedge in the valuation. And thus, the hedge quantities of the portfolio should be optimized in such way as to get the best possible prices (a.k.a the narrowest bid/ask spread).

Static Portfolio - OTC Options

#	Category	Type	Bid (No Hedge)	Ask (No Hedge)	Bid (Static Hedge)	Ask (Static Hedge)	Bid (Optimal)	Ask (Optimal)
1	Vanilla	Call	6.8887	9.6349	7.6741	8.7245	7.7711	8.6792
2	Binary	Call	0.4922	0.5288	0.8449	0.8998	0.8857	0.8903
3	Binary	Call	0.2780	0.3227	0.4266	0.4910	0.4401	0.4887
4	Binary	Call	0.1155	0.1934	0.1920	0.2422	0.1986	0.2408

The optimal value of the hedge at the current model is derived statically. Below is a brief proposal of optimization algorithm in the future software improvements.

The key ideas behind optimization algorithm are:

1. Sequential optimizer finds the minima and maxima.

2. OTC contract derives the optimization type.
   - When buying OTC Option: Optimizer finds coordinates that gives maximum price.
   - When selling OTC Option: Optimizer finds coordinates that gives minimum price.
   - Treat the prices for buy/sell as the worst case prices that one could transact with.

3. Expect different prices to show-up based on buy/sell of OTC option. Because of:
   - UVM is a non-linear PDE and thus negative vs. positive payoffs are different even on a portfolio with fixed hedging quantities.
   - The residual portfolio's gamma net-off effect.
   - Multiple expiration options could be part of the portfolio & their effect on the overall portfolio is non-linear.

The key results from the model and the benefits of hedging & optimization are discussed in the next section.

Results

The main benefit seen with hedging & optimization achieved with this model is the reduction of Bid/Ask spreads and underlying implied volatility.

The above picture of Static Portfolio of OTC Option shows both the bid/ask (which is un-hedged) and hedged bid/ask (which is hedged and has residual portfolio as discussed in earlier section). The OTC's have been priced with volatility range 20%~30% under UVM. The hedges are call spread with 5% of both long and short vanillas.

The results are remarkable: they demonstrate that the spread for hedged bid/ask is substantially lower than the un-hedged version. It also demonstrates that the implied volatility range (backed out from BS) on the hedged bid/ask prices is in the range of 23%~27%.

This has two important implications –

• Reduced spreads indicate competitive pricing and thereby the potential for increased business from clients.

• Our ability to find & exploit wider spread option prices that can be narrowed with static hedges thus making arbitrage profits.

Application User Guide

This section is a quick summary of the features of the "OTC Option Calculator" program that implements the models & methodologies discussed in this paper. The program has three main components:

1. **OTC Option Blotter**: This is used for inputting the OTC options that user would like to price.

Input	Data
Spot Price	100
Asset Step	100
Int Rate	5.00%
Div	0.00%
Vol-	20.00%
Vol+	30.00%
Expiry(yrs)	0.5

The user entry area is marked in "Red colour" in the input blotter and, the Bid & Ask prices in the below blotter are automatically populated.

Static Portfolio - OTC Options

#	Category	Type	Qty Short	Strike	Expiry	Vol-	Vol+	Int	Residual Short	Residual Long	Bid (No Hedge)	Ask (No Hedge)	Bid (Static Hedge)	Ask (Static Hedge)	Bid (Optimal)	Ask (Optimal)
1	Vanilla	Call	1	100	0.5	0.2	0.3	0.05	0.6069	-1.6573	6.8887	9.6349	7.6741	8.7245	7.7711	8.6792
2	Binary	Call	1	100	0.5	0.2	0.3	0.05	-0.0333	-0.0882	0.4922	0.5288	0.8449	0.8998	0.8857	0.8903
3	Binary	Call	1	110	0.5	0.2	0.3	0.05	-0.0196	-0.0840	0.2780	0.3227	0.4266	0.4910	0.4401	0.4887
4	Binary	Call	1	120	0.5	0.2	0.3	0.05	-0.0103	-0.0605	0.1155	0.1934	0.1920	0.2422	0.1986	0.2408
Total			4			0.2	0.3	0.05			7.7744	10.6798	9.1376	10.3575	9.2956	10.2990

2. **Static Hedges:** This section gets automatically populated once the user populates the Input Blotter.

Static Hedges													Optimal Hedge Ratio			
#	Category	Type	Qty Long	Strike	Expiry	Vol	Cost Long	Hedging Position	Residual	Bid	Ask		Qty Short	Cost Short	Qty Long	Cost Long
H1:1	Vanilla	Call	-0.5	90	0.5	0.25	7.2186	1	0.6069	14.4371	14.4371		0.67	9.6729	-0.62	8.9510
H1:2	Vanilla	Call	-0.5	110	0.5	0.25	2.1129	1	0.6069	4.2258	4.2258		0.55	2.3242	-0.53	2.2397
H2:1	Vanilla	Call	-0.05	90	0.5	0.25	0.7219	2	-0.0333	14.4371	14.4371		0.051	0.7363	-0.05	0.7219
H2:2	Vanilla	Call	0.05	110	0.5	0.25	0.2113	2	-0.0333	4.2258	4.2258		-0.055	0.2324	0.039	0.1648
H3:1	Vanilla	Call	-0.05	100	0.5	0.25	0.4130	3	-0.0196	8.2600	8.2600		0.05	0.4130	-0.05	0.4130
H3:2	Vanilla	Call	0.05	120	0.5	0.25	0.0976	3	-0.0196	1.9517	1.9517		-0.054	0.1054	0.044	0.0859
H4:1	Vanilla	Call	-0.05	110	0.5	0.25	0.2113	4	-0.0103	4.2258	4.2258		0.05	0.2113	-0.05	0.2113
H4:2	Vanilla	Call	0.05	130	0.5	0.25	0.0412	4	-0.0103	0.8247	0.8247		-0.055	0.0454	0.038	0.0313
Total			8		0.5	0.25	11.0277			52.5880	52.5880			13.7408		12.8188

It has all the static hedge related details including the position it is trying to hedge, the market price (in bid/ask columns), the cost of specific hedge and other standard option details like Strike, Expiry, Volatility. One of the most important parameter in this section is the "Qty". This Qty is the quantity of the static hedge that needs to be either bought (if Qty <0) or sold (if Qty>0) for achieving the optimum price of the OTC Contract.

Field Description:

- The "Hi:1" and "Hi:2" are quantity % of hedge1 and hedge2 respectively to be used when hedging the OTC contract.
- "Optimal Hedge Ratio" is calculated automatically and the system finds the optimum hedge percentages around Hedge1% & Hedge2%.
- The Spot refers to the market spot price and is mandatory.
- Vol-& Vol+ indicate the volatility range's lowest & highest values
- Residual column displays the residual portfolio value.
- Int Rate column is for interest rates.
- Div is the dividend

Further Application Improvements

Many of improvements can be done to the design and the performance of the software application. They include:

- Support for other OTC path-dependent Exotic Options.
- Support for ability to choose different hedging strategies.
- Improve the hedge finding algorithm to be able to identify market options that are best suited for the given constraints in pricing the OTC.
- Improve Finite difference by using Black-Scholes valuation in time step dt for faster convergence and also migrate towards implicit finite different schemes.
- Improve on various model/implementation limitations including:

➢ Defining boundary conditions during the optimization that minimizes prices. The optimizer could potentially minimize hedge quantities to unrealistic values in an attempt to find minimum OTC prices.

➢ The four key development points on identifying hedges as stated in the "Finding Hedges" section.

➢ When an OTC option is being hedged using a shorter expiry option, then there is a need to redo a static hedge once the shorter expiry option elapses (if the gap between OTC expiry & shorter expiry is material).

➢ At this moment, there is no direct attempt made to build a residual portfolio that is as small as it could be while trying to achieve attractive bid/ask spreads. The risk of this ignorance is that if wrong hedge % is chosen, the residual portfolio could be fairly large even though we have attained good bid/ask prices. Such large residual portfolio's demand higher maintenance/dynamic hedging and higher transaction costs, which at the end of day could lead to negative P&L. One approach to resolve this problem could be to build features that would allow portfolio optimization on the residual portfolio based on the potential hedges/hedge quantities.

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