一、单项选择题（每小题 4 分，共 16 分）CADC

二、填空

5. $x+3y+5z-11=0$　　6. 4　　7. $e^{\sin xy}\cos xy(y\mathrm{d}x+x\mathrm{d}y)$

8. $\int_0^1\mathrm{d}y\int_y^{\sqrt{2-y^2}}f(x,y)\mathrm{d}x$　　9. -1　　10. $(\frac{1}{e},e)$

三、(1)解　由于所求平面与平面$2x+y+2z=20$平行，于是可设所求平面为

$$\pi: 2x+y+2z+D=0.$$

因为平面π与球面$x^2+y^2+z^2=4$相切，所以球心$(0,0,0)$到平面π的距离为

$$\frac{|0+0+0+D|}{\sqrt{2^2+1^2+2^2}}=2,$$

求得$D=\pm6$，故所求平面为$2x+y+2z+6=0$或$2x+y+2z-6=0$.

(2)解　$A(1,-1,2)$到点$B(2,1,3)$两点所决定的直线方程为$\frac{x-1}{1}=\frac{x+1}{2}=\frac{x-2}{1}$，对应的参数方程为

$\begin{cases}x=1+t,\\ y=-1+2t,\\ z=2+t,\end{cases}$线段$AB$对应的参数范围为$0\le t\le1$．此时

$$\mathrm{d}s=\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\mathrm{d}t=\sqrt{6}\mathrm{d}t,$$

因此$I=\int_\Gamma(x^2+2y-z^2)\mathrm{d}s=\int_0^1[(1+t)^2+2(-1+2t)-(2+t)^2]\sqrt{6}\mathrm{d}t=\sqrt{6}\int_0^1(-5+2t)\mathrm{d}t=-4\sqrt{6}$.

四、解法 1　所求体积可看成是以$z=\sqrt{5-x^2-y^2}$为曲顶柱体与以$x^2+y^2=4z$为曲顶柱体的体积之差，两曲面的交线为

$$\begin{cases}z=\sqrt{5-x^2-y^2},\\ x^2+y^2=4z,\end{cases}$$

消去z后为$x^2+y^2=4$，故

$$V=\iint\limits_D[\sqrt{5-x^2-y^2}-\frac{1}{4}(x^2+y^2)]\mathrm{d}x\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^2(\sqrt{5-\rho^2}-\frac{1}{4}\rho^2)\rho\mathrm{d}\rho$$

$$=2\pi\int_0^2(\sqrt{5-\rho^2}-\frac{1}{4}\rho^2)\rho\mathrm{d}\rho=\pi\int_0^4(\sqrt{5-t}-\frac{1}{4}t)\mathrm{d}t$$

$$=\pi\left[-\frac{2}{3}(5-t)^{\frac{3}{2}}-\frac{1}{8}t^2\right]_0^4=\frac{2\pi}{3}(5\sqrt{5}-4).$$

解法 2　设曲面$z=\sqrt{5-x^2-y^2}$与$x^2+y^2=4z$所围闭区域为Ω，两曲面的交线为

$\begin{cases}z=\sqrt{5-x^2-y^2},\\ x^2+y^2=4z,\end{cases}$消去$z$后为$x^2+y^2=4$，则所求立体的体积为

$$V=\iiint\limits_\Omega\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^{2\pi}\mathrm{d}\theta\int_0^2\mathrm{d}\rho\int_{\frac{1}{4}\rho^2}^{\sqrt{5-\rho^2}}\mathrm{d}z=2\pi\int_0^2(\sqrt{5-\rho^2}-\frac{1}{4}\rho^2)\mathrm{d}\rho=\pi\int_0^4(\sqrt{5-t}-\frac{1}{4}t)\mathrm{d}t$$

$$=\pi\left[-\frac{2}{3}(5-t)^{\frac{3}{2}}-\frac{1}{8}t^2\right]_0^4=\frac{2\pi}{3}(5\sqrt{5}-4).$$

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五、解　因为$\dfrac{\partial u}{\partial x}=f'(\dfrac{x}{y})+g(\dfrac{y}{x})-\dfrac{y}{x}g'(\dfrac{y}{x})+y$，于是

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{y}f''(\frac{x}{y})+\frac{y^2}{x^3}g''(\frac{y}{x}),\quad \frac{\partial^2 u}{\partial x\partial y}=-\frac{x}{y^2}f''(\frac{x}{y})-\frac{y}{x^2}g''(\frac{y}{x})+1.$$

因此$x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}=x[\dfrac{1}{y}f''(\dfrac{x}{y})+\dfrac{y^2}{x^3}g''(\dfrac{y}{x})]+y[-\dfrac{x}{y^2}f''(\dfrac{x}{y})-\dfrac{y}{x^2}g''(\dfrac{y}{x})+1]=y$.

六、解　添加辅助曲面$\Sigma_1:\begin{cases}z=1,\\x^2+y^2\le 1,\end{cases}$取下侧，记$\Sigma$与$\Sigma_1$围成的闭区域立体为$\Omega$，则由高斯公式得

$$\iint_{\Sigma+\Sigma_1} z\mathrm{d}x\mathrm{d}y+(x+1)\mathrm{d}y\mathrm{d}z=-\iiint_{\Omega}(1+0+1)\mathrm{d}V=-2\int_0^{2\pi}\mathrm{d}\theta\int_0^1\rho\mathrm{d}\rho\int_{\rho^2}^1\mathrm{d}z=-\pi;$$

又$\displaystyle\iint_{\Sigma_1}(x+1)\mathrm{d}y\mathrm{d}z+z\mathrm{d}x\mathrm{d}y=0+\iint_{\Sigma_1}\mathrm{d}x\mathrm{d}y=0+\iint_{\Sigma_1}\mathrm{d}x\mathrm{d}y=-\iint_{D:x^2+y^2\le 1}\mathrm{d}x\mathrm{d}y=-\pi$.

因此$\displaystyle I=\iint_{\Sigma+\Sigma_1}z\mathrm{d}x\mathrm{d}y+(x+1)\mathrm{d}y\mathrm{d}z-\iint_{\Sigma_1}z\mathrm{d}x\mathrm{d}y+(x+1)\mathrm{d}y\mathrm{d}z=-\pi-(-\pi)=0$.

七、解　由于$\displaystyle\frac{\mathrm{e}^x-1}{x}=\frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n-1\right)=\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n!}x^n=\sum_{n=1}^{\infty}\frac{1}{n!}x^{n-1}\ (-\infty<x<+\infty,x\ne 0)$,

则
$$\frac{\mathrm{d}}{\mathrm{d}x}(\frac{\mathrm{e}^x-1}{x})=\left(\sum_{n=1}^{\infty}\frac{1}{n!}x^{n-1}\right)'=\sum_{n=2}^{\infty}\frac{n-1}{n!}x^{n-2}$$
$$=\sum_{n=1}^{\infty}\frac{n}{(n+1)!}x^{n-1}=\frac{x\mathrm{e}^x-\mathrm{e}^x+1}{x^2}\quad(-\infty<x<+\infty,x\ne 0),$$

令$x=1$，求得$\displaystyle\sum_{n=1}^{\infty}\frac{n}{(n+1)!}=1$.

八、证明　(1)设曲面上点(x,y,z)到原点的距离为r，则$r^2=x^2+y^2+z^2$.令

$$G=x^2+y^2+z^2+\lambda F(x,y,z),$$

则点M的坐标(x_0,y_0,z_0)满足下列方程组

$$\begin{cases}G_x=2x+\lambda F_x=0,\\G_y=2y+\lambda F_y=0,\\G_z=2z+\lambda F_z=0,\\G_\lambda=F(x,y,z)=0,\end{cases}$$

解之得点M的坐标为$x_0=-\dfrac{\lambda}{2}F_x$，$y_0=-\dfrac{\lambda}{2}F_y$，$z_0=-\dfrac{\lambda}{2}F_z$.所以存在常数$k=-\dfrac{\lambda}{2}$，使得点$M$的坐标为$(ka,kb,kc)$.

(2)因为曲面F在点M处的法线方程为

$$\frac{x-x_0}{F_x(M)}=\frac{x-y_0}{F_y(M)}=\frac{x-z_0}{F_z(M)},\ 即 L:\frac{x-ka}{a}=\frac{x-kb}{b}=\frac{x-kc}{c},$$

显然法线L经过原点.

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