(1) $a_n=\left(1+\frac{i}{2}\right)^{-n}$

$\left|1+\frac{i}{2}\right|=\frac{\sqrt{5}}{2}$

$\therefore \lim_{n\to\infty}\left(1+\frac{i}{2}\right)^{-n}=0$，收敛于0

(4). $a_n=e^{-n\pi i/2}$.

$=i\sin\left(-\frac{n\pi}{2}\right)+\cos\left(\frac{n}{2}\pi\right)$

$\therefore \{a_n\}$ 发散.

(5) $a_n=\frac{1}{n}e^{-n\pi\frac{i}{2}}$

$=\frac{1}{n}\left(\cos\frac{n\pi}{2}-i\sin\frac{\pi}{2}\right)=0.$

$\therefore \{a_n\}$ 收敛于 0

3. (2). $\sum\frac{i^n}{\ln n}=\frac{1}{\ln n}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^n.$

$=\frac{1}{\ln n}\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right).$

$\sum\frac{1}{\ln n}\cos\frac{n\pi}{2}=\sum\frac{1}{\ln 2k}(-1)^k.$

$\sum\frac{1}{\ln n}\sin\frac{n\pi}{2}=\sum\frac{1}{\ln(2k+1)}(-1)^k$

两项均为收敛的交错级数

$\therefore$ 收敛.

(3). $\left|\frac{6+5i}{8}\right|=\frac{\sqrt{61}}{8}<1$

$\therefore \sum\left(\frac{6+5i}{8}\right)^n$ 绝对收敛.

6. (1) $\sum\frac{z^n}{n^p}$.

$\frac{n^p}{(n+1)^p}=\left(\frac{n}{n+1}\right)^p<0.$

$\therefore R=1$

(3). $\left|\frac{c_{n+1}}{c_n}\right|=|1+i|=\sqrt{2}$. $R=\frac{\sqrt{2}}{2}$.

(5). $\operatorname{ch}\left(\frac{i}{n}\right)=\frac{e^{\frac{i}{n}}+e^{-\frac{i}{n}}}{2}$

$=\frac{\cos\frac{1}{n}+i\sin\frac{1}{n}+\cos\left(-\frac{1}{n}\right)+i\sin\left(-\frac{1}{n}\right)}{2}$

$=\cos\frac{1}{n}$

$$\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|=\lim_{n\to\infty}\left|\frac{\cos\frac{1}{n+1}}{\cos\frac{1}{n}}\right|=1$$

∴收敛半径为 1

8. 设 $\lim\limits_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|=m$,

则 $R_1=\frac{1}{m}$

$$\lim_{n\to\infty}\left|\frac{\frac{C_{n+1}}{n+2}}{\frac{C_n}{n+1}}\right|=\left|\frac{n+1}{n+2}\cdot\frac{C_{n+1}}{C_n}\right|=m$$

$R_2=\frac{1}{m}$

$$\lim_{n\to\infty}\left|\frac{(n+1)C_{n+1}}{nC_n}\right|=\lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{C_{n+1}}{C_n}\right|=m$$

$R_3=\frac{1}{m}$ ∴ $R_1=R_2=R_3$.

11. (1). $\frac{1}{z+1}=1-z+z^2-\cdots+(-1)^n z^n$.

$\frac{1}{z^3+1}=1-z^3+z^6-\cdots+(-1)^n z^{3n}$

(2) $\frac{1}{1+z^2}=\sum(-1)^n z^{2n}$

$\left(\frac{1}{1+z^2}\right)'=\frac{-2z}{(1+z^2)^2}=\sum(-1)^n 2n\cdot z^{2n-1}$

$\frac{1}{(1+z^2)^2}=\frac{1}{2}\sum(-1)^{1+n}\cdot z^{2n-2}$

(3) $\cos z^2$

$\cos z=\sum\frac{(-1)^{n+1}}{2n!}z^{2n}$

$\cos z^2=\sum\frac{(-1)^{n+1}}{2n!}z^{4n}$.

12. (1). $f(z)=\frac{z-1}{(z-1)+2}=\sum(-1)^{n+1}\frac{1}{2^n}\times(z-1)^n.$

$R=2.$

(2). $f(z)=\frac{z}{z+1}-\frac{z}{z+2}=\frac{2}{z+2}-\frac{1}{z+1}$

$\frac{2}{z+2}=\frac{\frac{1}{4}}{1+\frac{z-2}{4}}=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{4}\right)^n$

$\frac{1}{z+1}=\frac{\frac{1}{3}}{1+\frac{z-2}{3}}=\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{3}\right)^n$

$\therefore f(z)=\sum_{n=0}(-1)^n\left(4^{-n-1}-3^{-n-1}\right)(z-2)^n$

$\therefore R=3$

(3). $\frac{1}{z}=\frac{-1}{1-(z+1)}=-\sum_{n=0}(z+1)^n.$

$\left(\frac{1}{z}\right)'=-\frac{1}{z^2}$

$\frac{1}{z^2}=\sum n(z+1)^{n-1}$

$R=1$

(5) $\mathrm{tg}\,z$

$(\mathrm{tg}\,z)'=\sec^2 z$, $(\mathrm{tg}\,z)'\big|_{\frac{\pi}{4}}=2$

$(\mathrm{tg}\,z)''=2\sec^2 z\,\mathrm{tg}\,z$, $(\mathrm{tg}\,z)''\big|_{\frac{\pi}{4}}=4.$

$\cdots$

$C_n=\frac{2^n}{n!}$

$\therefore \mathrm{tg}\,z=\sum_{n=0}^{\infty}\frac{2^n}{n!}\left(z-\frac{\pi}{4}\right)^n$

$R=\frac{\pi}{4}.$

14. $a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$

$= \frac{1}{2\pi i} \oint_C \frac{\cos(z+\frac{1}{z})}{\xi^{n+1}} d\xi.$

取 $C: e^{i\theta}$.

$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\cos(e^{i\theta}+e^{-i\theta})}{e^{i(n+1)\theta}} i e^{i\theta} d\theta.$

$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(e^{i\theta}+e^{-i\theta})}{e^{in\theta}} d\theta$

$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(2\cos\theta)}{e^{in\theta}} d\theta$

$= \frac{1}{2\pi} \int_0^{2\pi} \cos(2\cos\theta) \cdot (\cos n\theta - i\sin n\theta) d\theta$

$\cos(2\cos\theta)(-i\sin n\theta)$ 为奇函数.

QED

16. (1) $f(z) = \frac{-1-2i}{10} \cdot \frac{1}{z+i} + \frac{-1+2i}{10} \frac{1}{z-i} + \frac{1}{5} \frac{1}{z-2}$

$= \frac{-1-2i}{10} \sum \frac{(-1)^n i^n}{z^{n+1}} + \frac{-1+2i}{10} \sum \frac{i^n}{z^{n+1}} - \frac{1}{10} \sum \frac{z^n}{2^n}$

$= \frac{1}{5} \sum \frac{(-1)^{n+1}}{z^{2n+1}} + \frac{1}{5} \sum \frac{(-1)^n}{z^{2n}} - \frac{1}{10} \sum \frac{z^n}{2^n}$

(3). $f(z) = \frac{1}{z-2} - \frac{1}{z-1}$

① $0 < |z-1| < 1$

$f(z) = -\frac{1}{z-1} - \frac{1}{1-(z-1)}$

$= -\frac{1}{z-1} - \sum (z-1)^n$

$= -\sum_{n=-1}^{\infty} (z-1)^n$

② $1 < |z-2| < \infty.$

$f(z) = \sum (-1)^n \frac{1}{(z-2)^{n+2}}$

(4). $\frac{1}{1-z} = \frac{1}{z} \cdot \frac{-1}{1-\frac{1}{z}} = -\frac{1}{z} \sum (\frac{1}{z})^n$

$e^{\frac{1}{1-z}} = 1 - \frac{1}{z} - \frac{1}{2! z^2} - \frac{1}{3! z^3} - \cdots$

(9(1). $f(z) = \frac{1}{2z} \times \frac{1}{1+\frac{z}{2}}$

$= \frac{1}{2z}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \cdots\right)$