Partial differential equations and operators

Fundamental solutions and semigroups Part I

Jan A. Van Casteren

Jan A. Van Casteren

Partial differential equations and operators

Fundamental solutions and semigroups

Part I

Partial differential equations and operators: Fundamental solutions and semigroups: Part I
2nd edition

978-87-403-0999-7

2010 *Mathematics Subject Classification.* 35A08, 42A38, 46F05, 46F10, 47D06, 47D07, 47D08

Key words and phrases. Distribution, Fourier transform, fundamental solutions, semigroup theory, Markov processes, Feynman-Kac semigroups

The author is obliged to the University of Antwerp for its material support. He also is grateful to the Department of Mathematics of the Catholic University of Leuven (KUL) for inviting him to deliver some lectures based on the subject of this work (1979). The author is thankful for comments and logistic support by numerous students who have taken courses based on preliminary versions of this text. In particular, Evert De Nolf, master student at the University of Antwerp, did quite a bit of Latex work, and suggested several corrections. Last but not least, the author also acknowledges that he learned several concepts from Walter Rudin's book [**113**]. Some of the present material is taken from Rudin's book. In particular this is true for parts of Chapter 4. The author is also grateful to the people of Bookboon who assisted him in the final stages of the preparation of the book. In particular this is true for Karin Hamilton Jakobsen and for Ahmed Zsolt Dakroub.

Contents

Preface

The contents of this book came into existence during the time the author has been teaching a course on partial differential equations and on operator theory for about twenty years. Let us discuss the contents somewhat more closely. Basically, the book contains topics from distribution theory, from operator theory, and from operator semigroups. These lecture notes follow to some extent the book by Rudin [**113**]. The subject of distribution theory is covered to a large extent: distributions and their derivatives, compactly supported distributions, tempered distributions and their Fourier transforms, convolution products of distributions, existence of fundamental solutions of partial differential operators with constant coefficients, several concrete examples, multiplicative distributions. Most of these topics can be found in Chapter 1. The Chapters 2 and 3 contain several concrete examples of fundamental solution (for the heat equation, free Schrödinger equation, for the Laplace equation, and for the wave equation. Some theory on Banach algebras and (unbounded) operators in Hilbert space can be found in Chapter 5. Chapter 6 is devoted to some aspects of operator semigroup theory. The chapter on operators in Hilbert space contains some basic results on Banach algebras, like spectral theory, symbolic calculus, and Gelfand transforms. It also contains a discussion on commutative C^*-algebras, including the Gelfand-Naimark theorem, symbolic calculus, taking square roots. Unbounded closed, and closable operators are treated as well. Again the spectral mapping theorem for normal operators is included, and also the polar decomposition. The chapter on operator semigroups discusses some basic generation results, like the Hille-Yosida theorem, and the Lumer-Phillips theorem. The maximum principle, and its connection with dissipative operators is explained. It also contains a result which shows the close connection with initial value problems. In addition, relations with Markov processes are described. In Chapter 6 some attention is paid to Feynman-Kac semigroups, and it concludes with a discussion on the KMS-function. In Chapter 1 the fundamental ideas for the theoretical concepts of distribution theory, or generalize function theory, are explained. Many (continuity) results about distributions are proved in 4. Section 7 contains, among other things, a solution to Cauchy's equation, *i.e.* $g(x+y) = g(x) + g(y)$, for (almost all) $x, y \in \mathbb{R}^n$, where the function g is Borel measurable. The final Chapter 8 contains some results on Functional analysis, which are used in the other chapters of the book. Most, if not all, functional analytic aspects of the course are explained. The final chapter serves as sort of a reference frame.

Readership. From the description of the contents it is clear that the text is designed for students at the graduate or master level. The author believes that also Ph.D. students, and even researchers, might benefit from these notes. The reader is introduced in the following topics: generalized function or distribution theory, operators in Hilbert space (including rudiments of C^*-algebras) and their spectral decompositions, and operator semigroup theory.

CHAPTER 1

Distributions, differential operators and examples

Introduction

The main purpose of these notes is to give explicit formulas for fundamental solutions of some important partial differential operators from mathematical physics. We shall do this for the heat equation, the Laplace equation, the wave equation and for the free non-relativistic Schrödinger equation. We shall also pay attention to the Cauchy-Riemann operator. The study of partial differential operators fits nicely into the framework of distribution theory. That is the reason why we begin with a quick survey of the theory of distributions. An important auxiliary tool will be the Fourier transformation. We shall devote a chapter to this topic and derive explicit formulas for Fourier transforms of some specific distributions. In addition, we will pay some attention to operator semigroups.

All the above mentioned partial differential operators are of the form $P(D)$, where P is a complex polynomial in n variables $x_1, \dots, x_n$ and $D = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right)$, *i.e.*

$$P(D)u = \sum_{|\alpha| \leqslant m} c_\alpha D^\alpha u.$$

for a suitable complex-valued function u on $\mathbb{R}^n$. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is in $\mathbb{N}^n$ and $|\alpha| = \sum_{j=1}^n \alpha_j$. The polynomial P has coefficients $c_\alpha \in \mathbb{C}$ and is of degree m. We also used the abbreviation

$$D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \circ \cdots \circ \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

More generally, the coefficients c_α can be complex-valued functions on $\mathbb{R}^n$ and u may be a distribution.

Now, let f be a suitable complex-valued function on $\mathbb{R}^n$. The theorem of Malgrange and Ehrenpreis says that there always exists a distribution u for which $P(D)u = f$. Here we must assume that the coefficients c_α are constants. For the above mentioned partial differential operators we shall derive explicit formulas for their distributional solutions. A precise formulation and proof of this theorem can be found in [**113**]. The latter proof is also supplied in the present book: see Theorem 4.20 in Chapter 4.

Topics to be treated in this book. The following problems will be studied in the context of distribution theory. Let the distribution, or generalized function f be given. The problem is to find a distribution u with the property that $P(D)u = f$. Under the assumption that the differential operator $P(D)$ has constant coefficients, the following "heuristics" will be mathematically justified in these lecture notes.

A first solution is based on fundamental solutions together with the Theorem of Malgrange and Ehrenpreis. Let $\delta = \delta_0$ be the Dirac distribution at the origin: $\delta(\varphi) = \varphi(0)$, when φ is a test function, i.e. φ is a C^∞-function with compact support. Suppose that a distribution E satisfies $P(D)E = \delta$. (The theorem of Malgrange and Ehrenpreis yields the existence of such a distribution E.) Such a distribution u is called a *fundamental solution*. Put $u = E * f$, provided we can make sense out of a convolution product of distributions. Of course, if E and f were L^1-functions or bounded complex Borel measures, then taking convolution products poses no problem. In the context of distributions this is a somewhat delicate issue. Anyway, if $u := E * f$ is defined, then $P(D)u = P(D)(E * f) = (P(D)u) * f = \delta * f = f$. It follows that $u = E * f$ satisfies $P(D)u = f$. Here we used the fact differentiating a convolution product can be done by differentiating one of the factors. We also used the fact that the Dirac measure δ is the identity for the convolution product.

A second method is based on the use of Fourier transforms. Again, let the distributions u and f be such that $P(D)u = f$. By taking Fourier transforms we obtain

$$P(\xi)\widehat{u}(\xi) = \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n. \tag{1.1}$$

Here $\widehat{u} = \mathcal{F}u$ and $\widehat{f} = \mathcal{F}f$ are Fourier transforms. If u is an L^1-function, then its Fourier transform $\widehat{u}$ is given by

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}u(x)\,dx, \quad \xi \in \mathbb{R}^n.$$

The inverse Fourier formula reads as follows:

$$\mathcal{F}^{-1}v(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}v(\xi)\,d\xi, \quad x \in \mathbb{R}^n.$$

If u is an appropriate function, the equality in (1.1) is obtained by employing integration by parts. From (1.1) we obtain

$$\widehat{u}(\xi) = \frac{1}{P(\xi)}\widehat{f}(\xi), \quad P(\xi) \neq 0,$$

and hence,

$$u(x) = \left[\mathcal{F}^{-1}\frac{1}{P}\mathcal{F}f\right](x), \quad x \in \mathbb{R}^n.$$

Here it is required that the singularities of the function $\xi \mapsto \dfrac{1}{P(\xi)}$, that is, the zeros of the polynomial $\xi \mapsto P(\xi)$, are not too "severe". So this method works provided we can make these heuristics rigorous. The corresponding techniques for distributions will be developed in this book. Operators of the form $\mathcal{F}^{-1}p\mathcal{F}$, where the "symbol"

p is an appropriate function, are called pseudo-differential operators. Since, for polynomials $P(\xi)$, the equality $\mathcal{F}^{-1}P\mathcal{F} = P(D)$ holds, differential operators are pseudo-differential operators.

In these notes we also discuss the spectral decomposition E_T of self-adjoint and normal operators T. Formally the normal (or self-adjoint) operator T, which need not be bounded, can be written in the form $T = \int_{\sigma(T)} \lambda \, dE_T(\lambda)$. The mapping $E_T : \mathcal{B}_{\mathbb{R}} \to \mathcal{L}(H)$ is also called a resolution of the identity. For every B belonging to the Borel field $\mathbb{B}_{\mathbb{R}}$ the operator $E_T(B)$ is an orthogonal projection that commutes with T. For more details the reader is referred to Definition 5.27, and to the Theorems 5.37, 5.28, 5.31, 5.33.

A third topic which we address in this book is the theory of operator semigroups. Let L be an appropriate differential operator. For instance L is $\frac{1}{2}$ times the Laplace operator:

$$Lu(x) = \frac{1}{2}\Delta u(x) = \frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}u(x),$$

or the generator

$$Lu(x) = \frac{1}{2}\Delta u(x) - x\cdot\nabla u(x) = \frac{1}{2}e^{\frac{1}{2}|x|^2}\Delta\left(e^{-\frac{1}{2}|x|^2}u(x)\right) + \left(n - |x|^2\right)u(x)$$

of the so-called Ornstein-Uhlenbeck process $(X(t))_{t\geqslant 0}$. The Laplace operator generates standard Brownian motion or Wiener process $(W(t))_{t\geqslant 0}$. Problems we want to solve with this kind of operators take the form

$$\frac{\partial u}{\partial t}(t,x) = Lu(t,\cdot)(x), \quad u(0,x) = f(x),$$

where f is an appropriate initial condition. Formally a solution is given by $u(t,x) = e^{tL}f(x)$. However, there are several problems. As usually we have to formulate the problem in the right context. This means, we have to choose the right (Banach) space X in which we treat this initial problem. Then we have to give a meaning to the operators e^{tL}. If L is a bounded operator in the Banach space $(X,\|\cdot\|)$, then this definition does not pose a serious problem. Things are different when the operator L is not bounded. In such a case we have to build a theory. In case we deal with the Laplace operator, we may choose $X = C_0(\mathbb{R}^n)$, *i.e.*, the space of all bounded continuous complex functions which vanish at ∞. The latter means that a continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ belongs to $C_0(\mathbb{R}^n)$, if for every $\alpha > 0$ the set $\{|f| \geqslant \alpha\}$ is a compact subset of $\mathbb{R}^n$. Put, for $(t,x,y) \in (0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$,

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{2t}\right), \quad \text{and,}$$

$$p_{\mathrm{OU}}(t,x,y) = \frac{1}{\left(1-e^{-2t}\right)^{n/2}}\exp\left(-\frac{e^{-2t}|x|^2 + e^{-2t}|y|^2 - 2e^{-t}\langle x,y\rangle}{1-e^{-2t}}\right).$$

If $L = \frac{1}{2}\Delta$, then e^{tL} is given by

$$e^{tL}f(x) = \int_{\mathbb{R}^n} f(y)p(t,x,y)\,dy = \mathbb{E}_x\left[f\left(W(t)\right)\right], \quad f \in C_0(\mathbb{R}^n),$$

and if $L = \frac{1}{2}\Delta - x\cdot\nabla$, then e^{tL} is given by

$$e^{tL}f(x) = \int_{\mathbb{R}^n} f(y)p_{\mathrm{OU}}(t,x,y)\,dy = \mathbb{E}_x\left[f\left(X(t)\right)\right], \quad f \in C_0(\mathbb{R}^n).$$

Here we have also given a probabilistic interpretation. These semigroups can be considered as the one-dimensional distributions of the Wiener process and of the Ornstein-Uhlenbeck process respectively. These processes can be considered as Markov processes. This means that with certain initial value problems we may associate a certain Markov process, which gives a solution via a probabilistic interpretation. By applying stopping time arguments, certain boundary value problems may be solved using (generators of) Markov processes.

Partition of unity. This subsection is devoted to a short introduction to (locally finite) partitions of unity subordinate to open covers of an open subset Ω of $\mathbb{R}^n$. In fact, very similar results hold for n-dimensional C^∞-manifolds. For proofs and further details the reader is referred to, *e.g.*, [**151**]. We begin with some definitions.

1.1. Definition. If $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ is a cover of Ω, then a subset of $\mathcal{U}$ which is also a cover is called a subcover.

1.2. Definition. If $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ is a cover of Ω, then an open cover $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$ is a refinement of $\mathcal{U}$ if for all $\gamma \in \Gamma$ there exists an $\alpha \in \mathcal{A}$ such that $V_\gamma \subset U_\alpha$.

1.3. Definition. A collection of subsets of Ω, $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$, is called locally finite, if for every $x \in \Omega$ there exists a neighborhood O of x such that $U_\alpha \cap O \neq \varnothing$ for only finitely many $\alpha \in \mathcal{A}$.

Let $\varphi : \Omega \to \mathbb{C}$ be a continuous function. Then, by definition, the support of φ is the closure in Ω of the subset $\{\varphi \neq 0\}$. The support of φ is denoted by $\operatorname{supp}(\varphi)$.

1.4. Definition. A partition of unity on Ω is a collection of smooth functions ($= C^\infty$-functions) $\{\varphi_i : \Omega \to \mathbb{R} : i \in I\}$ with the following properties:

(1) the collection $\{\operatorname{supp}(\varphi_i) : i \in I\}$ is locally finite;
(2) $0 \leqslant \varphi_i(x) \leqslant 1$ for all $x \in \Omega$ and for all $i \in I$;
(3) $\sum_{i \in I} \varphi_i(x) = 1$ for all $x \in \Omega$.

Notice that, since the partition in Definition 1.4 is locally finite, the sum in (3) represents a finite sum for every $x \in \Omega$.

1.5. Definition. A partition of unity $\{\varphi_i : \Omega \to \mathbb{R} : i \in I\}$ is subordinate to the open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ if for all $i \in I$ there exists an $\alpha \in \mathcal{A}$ such that the support of φ_i is contained in U_α.

1.6. Lemma. *Let Ω be an open subset of $\mathbb{R}^n$. Then there exists a sequence of open subsets $(O_i)_{i\in\mathbb{N}}$ which covers Ω, and is such that, for every $i \in I$, the closure $\overline{O}_i$ of O_i is a compact subset of O_{i+1}.*

1.7. Proposition. *Suppose that $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ is a basis for the topology on the open subset Ω of $\mathbb{R}^n$ with the property that the closure of every U_α is a compact subset of Ω. Let $\mathcal{W} = \{W_\beta : \beta \in \mathcal{B}\}$ be an arbitrary open cover of Ω. Then there exists a countable locally finite cover $\mathcal{V} = \{V_i : i \in I\}$ which is a refinement of $\mathcal{W}$ such that V_i belongs to $\mathcal{U}$ for all $i \in I$.*

Proof. Without loss of generality we assume that Ω is connected; otherwise we apply the procedure below for each component separately. For a connected open subset Ω we choose a sequence of open subsets $(O_i)_{i\in\mathbb{N}}$ as in Lemma 1.7. Put $O_0 = \varnothing$. Then, for $i = 1, 2, \ldots$, the subset $\overline{O}_{i+1}\backslash O_i$ is a compact subset of the open subset $O_{i+2}\backslash\overline{O}_{i-1}$. Notice that $\mathcal{O} := \{O_{i+2}\backslash\overline{O}_{i-1} : i = 1, 2, \ldots\}$ is an open cover of Ω. Then the refinement $\mathcal{V}$ is obtained in two steps. Let $\mathcal{U}'$ be the collection of all those $U_\alpha \in \mathcal{U}$ such that $U_\alpha \subset W_\beta$ for some $\beta \in \mathcal{B}$, and such that $U_\alpha \subset O_{i+2}\backslash\overline{O}_{i-1}$ for some $i \geqslant 1$. Then $\mathcal{U}'$ is a basis for the topology of Ω; *i.e.* for every $x \in \Omega$ $\mathcal{U}'$ contains an open neighborhood basis of x. In words, $\mathcal{U}'$ inherits this property from $\mathcal{U}$. Then choose, for every $i \geqslant 1$, a finite subcollection $\mathcal{U}'_i$ of $\mathcal{U}'$ with the following properties: every $U_\alpha \in \mathcal{U}'_i$ is contained in $O_{i+2}\backslash\overline{O}_{i-1}$, and $\mathcal{U}'_i$ covers the compact subset $\overline{O}_{i+1}\backslash O_i$. Put $\mathcal{V} = \bigcup_{i=1}^{\infty}\mathcal{U}'_i$. Then $\mathcal{V}$ is a countable open cover of Ω. It is subordinate to $\mathcal{W}$, because $\mathcal{U}'$ has this property. It is locally finite, because an open subset in $\mathcal{U}'_i$, $i \geqslant 1$, can only meet members from $\bigcup_{j=i-1}^{i+2}\mathcal{U}'_j$.

This completes the proof of Proposition 1.7. $\square$

1.8. Lemma. *Let U be an open subset of the open subset Ω of $\mathbb{R}^n$. Then there exists a C^∞-function φ_U on U which is strictly positive on U and vanishes on $\Omega\backslash U$: $\varphi_U(x) > 0$, $x \in U$, $\varphi_U(x) = 0$, $x \in \Omega\backslash U$. If the closure of U is a compact subset of Ω, then the function φ_U belongs to $\mathcal{D}_{\overline{U}}$.*

Let K be a compact subset of $\mathbb{R}^n$. Then a function φ belongs to $\mathcal{D}_K$ provided it is a C^∞-function $\mathbb{R}^n$ (or it is defined on K and extends to a C^∞-function on $\mathbb{R}^n$) with support in K.

PROOF. Let $\{K_\ell : \ell \in \mathbb{N}\}$ be a sequence of compact subsets of U with the following properties: $U = \bigcup_{\ell=1}^{\infty} K_\ell$, $K_\ell \subset \overset{\circ}{K}_{\ell+1}$, $K_0 = \varnothing$. Here $\overset{\circ}{K}$ is the interior of the subset K. The interiors O_ℓ of the sets K_ℓ have the properties, relative to U, as described in Lemma 1.6. Choose functions $\varphi_\ell \in \mathcal{D}_\Omega$, $\ell \in \mathbb{N}$, such that $\mathbf{1}_{K_\ell} \leqslant \varphi_\ell \leqslant \mathbf{1}_{K_{\ell+1}}$. Define the constants c_ℓ, $\ell \in \mathbb{N}$, by

$$c_\ell = \max_{|\alpha| \leqslant \ell} \sup_{x \in \Omega} |D^\alpha \varphi_\ell(x)|,$$

and define the function φ_U by

$$\varphi_U(x) = \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} \frac{\varphi_\ell(x)}{c_\ell}. \tag{1.2}$$

Then the function φ_U possesses the properties as described in Lemma 1.8 □

The main result we need is the following one. A similar result is true for an n-dimensional C^∞-manifold.

1.9. THEOREM. *If Ω is an open subset of $\mathbb{R}^n$ and $\mathcal{W} = (\Omega_j)_{j \in \Gamma}$ is any open cover of Ω, then Ω admits a countable partition $(\varphi_\ell)_{\ell \in \mathbb{N}}$ of unity subordinate to the cover $\mathcal{W}$ with the support of each function φ_ℓ a compact subset of Ω.*

PROOF. Let B be the open unit ball in $\mathbb{R}^n$, and let $d(x, \mathbb{R}^n \backslash \Omega)$ be the Euclidean distance between x and $\mathbb{R}^n \backslash \Omega$; if $\Omega = \mathbb{R}^n$, then this distance is taken to be ∞. Applying Proposition 1.7 with $\mathcal{U} = \{U_{x,r} = x + rB : x \in U, 0 < r < d(x, \mathbb{R}^n \backslash \Omega)\}$ as a basis for the topology on the open subset Ω yields the existence of a locally finite countable cover of Ω, $\mathcal{V} = \{U_{x_i,r_i} : i \in \mathbb{N}\}$, which is subordinate to $\mathcal{W}$. For every $i \in \mathbb{N}$ there exists a function $\psi_i \in \mathcal{D}(\Omega)$ with the property that $\{\psi_i > 0\} = U_{x_i,r_i}$ and $\psi_i(x) = 0$ for $x \in \Omega \backslash U_{x_i,r_i}$. The function $\psi_\Omega(x) = \sum_{i=1}^{\infty} \psi_i(x)$, $x \in \Omega$, is well defined and C^∞ on Ω. Finally, put $\varphi_i(x) = \dfrac{\psi_i(x)}{\psi_\Omega(x)}$. Then the sequence $\{\varphi_i : i \in \mathbb{N}\}$ possesses the properties of Theorem 1.9. □

1. Test functions and distributions

1.1. Convergence of test functions. Let Ω be an open subset of $\mathbb{R}^n$. A function $\varphi \colon \Omega \to \mathbb{C}$ is said to belong to $\mathcal{D}(\Omega)$ if it is infinitely many times differentiable and if its *support*, denoted by $\operatorname{supp}(\varphi)$, is a compact subset of Ω. Here $\operatorname{supp}(\varphi)$ is the closure in $\mathbb{R}^n$ of the set $\{x \in \mathbb{R}^n \colon \varphi(x) \neq 0\}$. Next, let $(\varphi_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega)$. It is called $\mathcal{D}(\Omega)$-*convergent* if there exists a function φ in $\mathcal{D}(\Omega)$ such that the following two conditions are fulfilled:

(a) There exists a compact subset K of Ω such that $\operatorname{supp}(\varphi_k) \subseteq K$ for all $k \in \mathbb{N}$.

(b) For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$ the sequence $(D^\alpha \varphi_k)_{k\in\mathbb{N}}$ converges uniformly to $D^\alpha \varphi$, *i.e.*

$$\lim_{k\to\infty} \sup_{x\in\Omega} \left| D^\alpha \varphi_k(x) - D^\alpha \varphi(x) \right| = 0.$$

The symbol D^α stands for

$$D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \circ \cdots \circ \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}. \tag{1.3}$$

Instead of $\dfrac{1}{i}\dfrac{\partial}{\partial x_k}$ we will often write D_k. If $(\varphi_k)_{k\in\mathbb{N}}$ is a $\mathcal{D}(\Omega)$-convergent sequence, then its limit φ is denoted by

$$\varphi = \mathcal{D}(\Omega)\text{-}\lim_{k\to\infty} \varphi_k.$$

It is possible to prove that there exists a topology $\mathcal{T}_\Omega$ on $\mathcal{D}(\Omega)$ which turns $\mathcal{D}(\Omega)$ into a locally convex topological vector space with the property that a sequence in $\mathcal{D}(\Omega)$ is $\mathcal{T}_\Omega$-convergent if and only if it is $\mathcal{D}(\Omega)$-convergent. The elements of $\mathcal{D}(\Omega)$ are often called *test functions*. We mention a few examples of test functions. With $|x|$ we mean the Euclidian norm of $x \in \mathbb{R}^n$, *i.e.* $|x|^2 = x_1^2 + \cdots + x_n^2$. The space $\mathcal{D}(\Omega)$ is an example of a barreled space which is not a Fréchet space. A barreled space is a locally convex space in which every barrel is a neighborhood of the origin. A *barrel* is an absolutely convex, closed, and absorbing subset. A Fréchet space is a locally convex topological vector space which is complete metrizable. However, the space $(\mathcal{D}(\Omega), \mathcal{T}_\Omega)$ is not metrizable. Nevertheless, it is barreled. For more details see Example 8.23 in Chapter 8. In this connection the following result, which is taken from Rudin [**113**], Theorem 1.24, is appropriate.

1.10. Theorem. *If X is a Hausdorff topological vector space with a countable local base, then there exists a metric d on X such that*

(a) *d is compatible with the topology of X,*
(b) *the open balls centered at the origin are balanced, and*
(c) *d is invariant: $d(x+z, y+z) = d(x,y)$ for all x, y, $z \in X$.*

If, in addition, X is locally convex, then d can be chosen as to satisfy (a), (b), (c), and also

(d) *all open balls are convex.*

1.11. Example. Let r be a positive real number and define the function φ_r by

$$\varphi_r(x) = \begin{cases} \exp\left(-\dfrac{1}{r^2 - |x|^2}\right) & |x| < r, \\ 0 & |x| \geqslant r. \end{cases}$$

Then the function φ_r belongs to $\mathcal{D}(\mathbb{R}^n)$ and its support is the closed ball of radius r which is centered at the origin.

1.12. Example. Let a and r be positive real numbers with $a > r$. Let φ_r be the function from the previous example. Put

$$c_r = \int_{\mathbb{R}^n} \varphi_r(y)\, dy,$$

and define the function $\psi_{a,r}$ by

$$\psi_{a,r}(x) = c_r^{-1}(\varphi_r * \chi_{\{|y| \leqslant a\}})(x) = c_r^{-1} \int_{|y| \leqslant a} \varphi_r(x - y)\, dy, \ x \in \mathbb{R}^n.$$

Then $\psi_{a,r}$ belongs to $\mathcal{D}(\mathbb{R}^n)$. Its support coincides with the closed ball of radius $a + r$ which is centered at the origin. Moreover $\psi_{a,r}(x) = 1$ for $|x| \leqslant a - r$.

1.13. Example. Let K be a compact subset of $\mathbb{R}^n$ and let ε be a positive real number. There exists a function ψ in $\mathcal{D}(\mathbb{R}^n)$ with $0 \leqslant \psi \leqslant 1$, such that $\psi = 1$ on K and such that its support is contained in the set $K + \{x \in \mathbb{R}^n : |x| \leqslant \varepsilon\}$, *i.e.*

$$\operatorname{supp} \psi \subseteq K + \varepsilon B_1,$$

where B_1 is the closed unit ball around the origin in $\mathbb{R}^n$.

1.2. Space of test functions. The inequality in (1.6) below is a consequence of the following arguments. Let K be a compact subset of $\mathbb{R}^n$ with non-empty interior. Since $\mathbb{R}^n$ is locally compact such compact subsets exist in abundance. By definition, the space $\mathcal{D}_K$ consists of the functions $\varphi \in C^\infty(\mathbb{R}^n)$ with support contained in K. The locally convex topology $\mathcal{T}_K$ on $\mathcal{D}_K$ is determined by the (semi-)norms $p_{m,K}$, $m \in \mathbb{N}$, given by

$$p_{m,K}(\varphi) = \max_{\alpha\in\mathbb{N}^n,\, |\alpha|\leqslant m} \sup_{x\in K} |D^\alpha \varphi(x)|, \quad \varphi \in \mathcal{D}_K.$$

So a basis of $\mathcal{T}$-neighborhoods of the origin is given by the collection of open convex subsets:

$$\left[\left\{\varphi \in \mathcal{D}_K : p_{m,K}(\varphi) < 2^{-\ell}\right\} : m,\, \ell \in \mathbb{N}\right].$$

Let $((K_\ell)_{\ell\in\mathbb{N}})$ be a sequence of compact subsets of Ω with the following properties:

(i) $K_0 = \varnothing$ and the subset K_ℓ is contained in the interior of $K_{\ell+1}$;
(ii) the sequence $((K_\ell)_{\ell\in\mathbb{N}})$ covers Ω: $\Omega = \bigcup_\ell K_\ell$.

Then $\mathcal{D}(\Omega) = \bigcup_\ell \mathcal{D}_{K_\ell}$, and the topology $\mathcal{T}_\Omega$ on $\mathcal{D}(\Omega)$ determined by calling a subset $W \subset \mathcal{D}(\Omega)$ $\mathcal{T}_{K_\ell}$-open in $\mathcal{D}(\Omega)$ if its intersection with $\mathcal{D}_{K_\ell}$ is open in $\mathcal{D}_{K_\ell}$ for all $\ell \in \mathbb{N}$. A $\mathcal{T}_\Omega$-neighborhood basis of the zero-function consists of those subsets $W \subset \mathcal{D}(\Omega)$ with the property that $W \cap \mathcal{D}_{K_\ell}$ is a $\mathcal{T}_{K_\ell}$-neighborhood of zero in $\mathcal{D}_{K_\ell}$ for all $\ell \in \mathbb{N}$. The (locally convex) topological space $(\mathcal{D}_\Omega, \mathcal{T}_\Omega)$ obtained in this way is called the (strict) inductive limit of the the sequence $(\mathcal{D}_{K_\ell})_{\ell\in\mathbb{N}}$. It is also called the final topology corresponding to the latter sequence. Sometimes the notation $(\mathcal{D}(\Omega), \mathcal{T}_\Omega) = \varinjlim (\mathcal{D}_{K_\ell}, \mathcal{T}_{K_\ell})$ is used. In fact $\mathcal{T}_\Omega$ is the strongest, or finest, topology on $\mathcal{D}(\Omega)$ which makes all the inclusions $\mathcal{D}_{K_\ell} \hookrightarrow \mathcal{D}(\Omega)$, $\ell \in \mathbb{N}$, continuous. The locally convex topology $\mathcal{T}_\Omega$ with which $\mathcal{D}(\Omega)$ is generated by the semi-norms of the form

$$p_{(m_\ell,c_\ell)}(\varphi) = \sum_{\ell=1}^{\infty} \max_{|\alpha|\leqslant m_\ell} c_\ell \sup_{x\in K_\ell\setminus K_{\ell-1}} |D^\alpha \varphi(x)|, \tag{1.4}$$

where $K_0 = \varnothing$, and where $(m_\ell)_{\ell\geqslant 1}$ and $(c_\ell)_{\ell\geqslant 1}$ are arbitrary non-decreasing sequences of non-negative integers and positive real numbers, respectively.

The previous observations prove part of the following theorem.

1.14. Theorem. *Let $(\mathcal{D}_\Omega, \mathcal{T}_\Omega) = \varinjlim (\mathcal{D}_{K_\ell}, \mathcal{T}_{K_\ell})$ be the locally convex space as described above. Then the topology $\mathcal{T}_\Omega$ on $\mathcal{D}(\Omega)$ is independent of the chosen sequence of compact subsets $(K_\ell)_\ell$ as long as (i) and (ii) are satisfied. Then the relative topology of $\mathcal{T}_\Omega$ with respect to $\mathcal{D}_K$, with K a compact subset of Ω, coincides with $\mathcal{T}_K$. Let $\mathcal{D}'(\Omega)$ denote the space of complex continuous linear functionals on $\mathcal{D}(\Omega)$. Let $\mathcal{B} \subset \mathcal{D}(\Omega)$. The following assertions are equivalent:*

(1) *the subset $\mathcal{B}$ is $\mathcal{T}_\Omega$-bounded in the sense that for every zero-neighborhood V there exists a finite number $t > 0$ such that $\mathcal{B} \subset tV$;*
(2) *for every seminorm of the form (1.4) the expression $\sup_{\varphi\in\mathcal{B}} p_{(m_\ell,c_\ell)}(\varphi)$ is finite;*

(3) *for every* $u \in \mathcal{D}'(\Omega)$ *the expression* $\sup_{\varphi\in\mathcal{B}} |u(\varphi)|$ *is finite;*

(4) *there exists a compact subset* K *of* Ω *such that* $\mathcal{B} \subset \mathcal{D}_K$*, and such that* $\mathcal{B}$ *is* $\mathcal{T}_K$*-bounded, i.e.,* $\sup_{\varphi\in\mathcal{B}} p_{m,K}(\varphi)$ *is finite for all* $m \in \mathbb{N}$.

The functionals in $\mathcal{D}'(\Omega)$ are called *distributions* or *generalized functions*.

Proof. The assertions prior to those related to the boundedness properties of the subset $\mathcal{B}$ are left to the reader.

The equivalence of the assertions (1) and (2) follows from the fact that every $\mathcal{T}_\Omega$-open zero-neighborhood V contains an $\mathcal{T}_\Omega$-open zero-neighborhood of the form $\left\{p_{(m_\ell,c_\ell)} < \delta\right\}$, for some $\delta > 0$, where $p_{(m_\ell,c_\ell)}$ is as in (1.4).

The equivalence of the assertions (2) and (3) is (standard) result in functional analysis: see, e.g., Theorem 8.10 in Chapter 8.

In order to show the equivalence of the assertions (2) and (4) it suffices to prove that $\mathcal{B}$ is contained in the space $\mathcal{D}_{K_\ell}$ for some ℓ, because the topology $\mathcal{T}_\Omega$ confined $\mathcal{D}_{K_\ell}$ coincides with $\mathcal{T}_{K_\ell}$. The assumption that for no $\ell \in \mathbb{N}$ the subset $\mathcal{B}$ is contained in $\mathcal{D}_{K_\ell}$ implies that there exists a strictly increasing sequence of positive integers $(\ell_k)_k$ and a sequence $(x_k)_k \subset \Omega$ together with a sequence $(\varphi_k)_k \subset \mathcal{B}$ with the following properties: $x_k \in K_{\ell_k} \backslash K_{\ell_k - 1}$, and $\varphi_k(x_k) \neq 0$. Put

$$c_\ell = \frac{\ell_k}{|\varphi_k(x_k)|}, \quad \ell_{k-1} + 1 \leqslant \ell \leqslant \ell_k, \ k \in \mathbb{N},$$

and $m_\ell = 0$, $\ell \geqslant 1$. Then

$$\sup_{\varphi\in\mathcal{B}} p_{(m_\ell,c_\ell)}(\varphi) \geqslant \sup_{k\in\mathbb{N}} c_{\ell_k} |\varphi_k(x_k)| \geqslant \sup_{k\in\mathbb{N}} \frac{\ell_k}{|\varphi_k(x_k)|} |\varphi_k(x_k)| = \sup_{k\in\mathbb{N}} \ell_k = \infty,$$

and hence, $\mathcal{B}$ is not $\mathcal{T}_\Omega$-bounded. Consequently, $\mathcal{B}$ is a ($\mathcal{T}_{K_\ell}$-bounded) subset of $\mathcal{D}_{K_\ell}$ for some $\ell \in \mathbb{N}$, provided it is $\mathcal{T}_\Omega$-bounded.

This completes the proof of Theorem 1.14. □

A sequence $(\varphi_k)_k \subset \mathcal{D}(\Omega)$ is called a $\mathcal{T}_\Omega$-Cauchy sequence provided for every zero-neighborhood V there exists an integer m such that $\varphi_{k_2} - \varphi_{k_1} \in V$ for all k_1, $k_2 \geqslant m$. This notion is equivalent to saying that

$$\lim_{k_1,k_2\to\infty} p_{(m_\ell,c_\ell)}(\varphi_{k_2} - \varphi_{k_1}) = 0 \tag{1.5}$$

for all non-decreasing $(m_\ell)_{\ell\geqslant 1} \subset \mathbb{N}$, and for all non-decreasing sequences $(c_\ell)_{\ell\geqslant 1} \subset [0,\infty)$. From (1.5) it follows that such a $\mathcal{T}_\Omega$-Cauchy sequence is $\mathcal{T}_\Omega$-bounded.

1.15. Corollary. *The space* $(\mathcal{D}(\Omega), \mathcal{T}_\Omega)$ *is sequentially complete. Moreover, if* $(\varphi_k)_{k\in\mathbb{N}}$ *is a* $\mathcal{T}_\Omega$*-Cauchy sequence in* $\mathcal{D}(\Omega)$*, then there exists a compact subset* $K \subset \Omega$ *such that this sequence is* $\mathcal{T}_K$*-Cauchy sequence in* $\mathcal{D}_K$.

Proof. Let $(\varphi_k)_k \subset \mathcal{D}(\Omega)$ be a $\mathcal{T}_\Omega$-Cauchy sequence. From (1.5) it follows that such a $\mathcal{T}_\Omega$-Cauchy sequence is $\mathcal{T}_\Omega$-bounded. Consequently, from assertion (4) it follows that the $\mathcal{T}_\Omega$-Cauchy sequence $(\varphi_k)_k$ is contained in $\mathcal{D}_{K_\ell}$, and is a Cauchy-sequence in the space $(\mathcal{D}_K, \mathcal{T}_K)$ for some compact subset K of Ω. Since spaces of the form $(\mathcal{D}_K, \mathcal{T}_K)$ are complete metrizable, the space $(\mathcal{D}(\Omega), \mathcal{T}_\Omega)$ is sequentially complete.

The proof of Corollary 1.15 is complete now. $\square$

1.3. Distributions. Again Ω denotes an open subset of $\mathbb{R}^n$. A *distribution* on Ω is a linear map u from $\mathcal{D}(\Omega)$ to $\mathbb{C}$ with the following continuity property. For any sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ with $\lim_{k\to\infty} \varphi_k = 0$, the limit $\lim_{k\to\infty} u(\varphi_k)$ exists and equals zero, *i.e.*

$$\varphi_k \xrightarrow{\mathcal{D}(\Omega)} 0 \implies u(\varphi_k) \to 0.$$

Using the remarks from the previous section, it is possible to show that a distribution on Ω is in fact a linear functional on $\mathcal{D}(\Omega)$ which is continuous with respect to the above mentioned topology $\mathcal{T}_\Omega$ on $\mathcal{D}(\Omega)$. See Remark 1.21 below as well. The space of distributions, or generalized functions, on Ω is denoted by $\mathcal{D}'(\Omega)$. We mention some examples.

1.16. Example. Locally integrable functions are distributions. A Borel measurable function f from Ω to $\mathbb{C}$ is called *locally integrable* if for each compact subset K of Ω the Lebesgue integral $\int_K |f(x)|\,dx$ is finite. The set of all such functions is denoted by $L^1_{\text{loc}}(\Omega)$. A function f in $L^1_{\text{loc}}(\Omega)$ gives rise to the distribution u_f defined by

$$u_f(\varphi) = \int_\Omega f(x)\varphi(x)\,dx,\ \varphi \in \mathcal{D}(\Omega).$$

1.17. Example. Let μ be a Radon measure on Ω, *i.e.* an inner regular and locally finite measure. The corresponding distribution u_μ is given by

$$u_\mu(\varphi) = \int_\Omega \varphi\,d\mu,\ \varphi \in \mathcal{D}(\Omega).$$

An example of a Radon measure is given by $\mu(B) = \int_B f(x)\,dx$ where f is a function which belongs to $L^p_{\text{loc}}(\Omega)$ for some $p \geqslant 1$. In general, the restriction of a Radon measure μ (on an open subset Ω) to a compact subset $K \subset \Omega$ is a genuine complex-valued measure on (the Borel field of) K. Its variation $|\mu|(B)$ on a Borel subset B of Ω is defined by

$$|\mu|(B) = \sup\left\{\sum_{j=1}^m |\mu(K_j)| : K_j \subset B,\ K_{j_1} \cap K_{j_2} = \varnothing,\ 1 \leqslant j,\ j_1,\ j_2 \leqslant m\right\},$$

where every K_j, $1 \leqslant j \leqslant m$, is compact.

1.18. Example. Let p be a point in Ω and let α be a multi-index. Maps of the form

$$\varphi \mapsto D^\alpha\varphi(p),\ \varphi \in \mathcal{D}(\Omega),$$

are distributions too.

1.19. Remark. For a point p in Ω the distribution $\varphi \mapsto \varphi(p)$, $\varphi \in \mathcal{D}(\Omega)$, is called the *Dirac distribution at* p and is denoted by δ_p. If $p = 0$, then δ_p is usually written as δ.

1.20. Example. The function $x \mapsto \exp(1/x)$, $x \in \mathbb{R}\backslash\{0\}$, is a distribution on $(0,\infty)$, but not on $\mathbb{R}$.

1.21. Remark. It is fairly easy to show that a linear functional u from $\mathcal{D}(\Omega)$ to $\mathbb{C}$ is a distribution if and only if for each compact subset K of Ω there exists a constant $C = C_K$ and a positive integer $N = N_K$ such that

$$|u(\varphi)| \leqslant C \max_{|\alpha|\leqslant N} \sup_{x\in K} \left|D^\alpha\varphi(x)\right| \tag{1.6}$$

for each φ in $\mathcal{D}(\Omega)$ with $\text{supp}(\varphi) \subseteq K$. If it is possible to choose N independent of K, then u is said to have *finite order*. The *order* of u is the smallest non-negative integer N with the property that, for each compact subset K of Ω, the above inequality holds for some constant $C > 0$ (depending on K) and all φ in $\mathcal{D}(\Omega)$ with $\text{supp}(\varphi) \subseteq K$. Then it is clear that a linear functional $u : \mathcal{D}(\Omega) \to \mathbb{C}$ belongs to $\mathcal{D}'(\Omega)$ if and only if all restrictions $u\restriction_{\mathcal{D}_{K_\ell}}$, $\ell \in \mathbb{N}$, are continuous. But $u\restriction_{\mathcal{D}_{K_\ell}}$ is $\mathcal{T}_{K_\ell}$-continuous if and only there exist $m \in \mathbb{N}$ and $\delta > 0$ with the property that $p_{m,K_\ell}(\varphi) < \delta$, $\varphi \in \mathcal{D}_{K_\ell}$, implies $|u(\varphi)| \leqslant 1$. Next, suppose that the restriction $u\restriction_{\mathcal{D}_{K_\ell}}$ possesses the latter property. If $\varphi \in \mathcal{D}_{K_\ell}$ is arbitrary, then ψ_η defined by

$$\psi_\eta = \frac{\delta\varphi}{p_{m,K_\ell}(\varphi) + \eta}, \quad \eta > 0,$$

satisfies $p_{m,K_\ell}(\psi_\eta) < \delta$, and hence $|u(\psi_\eta)| \leqslant 1$. It follows that

$$|u(\varphi)| \leqslant \frac{p_{m,K_\ell}(\varphi) + \eta}{\delta}. \tag{1.7}$$

In (1.7) we let $\eta \downarrow 0$ to obtain: $|u(\varphi)| \leqslant C p_{m,K_\ell}(\varphi)$, $\varphi \in \mathcal{D}_{K_\ell}$, where $C = \delta^{-1}$. It follows that a linear functional $u : \mathcal{D}(\Omega) \to \mathbb{C}$ belongs to $\mathcal{D}'(\Omega)$ if and only if (1.6) is satisfied.

The following theorem which characterizes those linear functionals, defined on $\mathcal{D}(\Omega)$, which are distributions, partly follows from the observations in Remark 1.21.

1.22. THEOREM. *Let $u : \mathcal{D}(\Omega) \to \mathbb{C}$ be a linear functional. Then the following assertions are equivalent:*

(1) *the functional u belongs to $\mathcal{D}'(\Omega)$;*
(2) *the restrictions $u\restriction_{\mathcal{D}_K}$ of u to $\mathcal{D}_K$ are $\mathcal{T}_K$-continuous for all compact subsets K of Ω;*
(3) $\lim_{k\to\infty} u(\varphi_k) = 0$ *whenever the sequence $(\varphi_k)_{k\in\mathbb{N}}$ converges to 0 in the space $(\mathcal{D}(\Omega), \mathcal{T}_\Omega)$.*

PROOF. If $u \in \mathcal{D}'(\Omega)$, the there exists a seminorm of the form $p_{(m_\ell,c_\ell)}$ with the property that

$$|u(\varphi)| \leqslant p_{(m_\ell,c_\ell)}(\varphi) \tag{1.8}$$

for all $\varphi \in \mathcal{D}(\Omega)$. It then easily follows that, for every compact subset K of Ω, there exists a constant C_K and $m \in \mathbb{N}$ such that

$$|u(\varphi)| \leqslant C_K p_{m,K}(\varphi), \quad \varphi \in \mathcal{D}_K. \tag{1.9}$$

Whence, (1) implies (2). Conversely, if for every compact subset K of Ω the restriction of u to $\mathcal{D}_K$ satisfies an inequality of the form (1.9), then it is not so difficult to prove that the functional u itself satisfies an inequality of the form (1.8). In fact, let $(\varphi_j)_{j\in\mathbb{N}}$ be a locally finite partition of unity subordinate to the cover $\left\{\mathring{K}_{\ell+2}\backslash K_{\ell-1}\right\}_{\ell\geqslant 1}$, $K_0 = \varnothing$: compare with the proof of Proposition 1.7 with $K_\ell = \overline{O}_\ell$. Choose, for $j \in \mathbb{N}$, $\ell_j \in \mathbb{N}$, $\ell_j \geqslant 2$, in such a way that $\operatorname{supp}(\varphi_j) \subset \mathring{K}_{\ell_j+1}\backslash K_{\ell_j-2}$. Then, by assertion (2),

$$|u(\varphi)| \leqslant \sum_{j=1}^{\infty} |u(\varphi_j\varphi)| \leqslant \sum_{j=1}^{\infty} C_{\ell_j+1} \max_{|\alpha|\leqslant m_{\ell_j+1}} \sup_{x\in K_{\ell_j+1}\backslash K_{\ell_j-2}} |D^\alpha(\varphi_j\varphi)(x)|$$

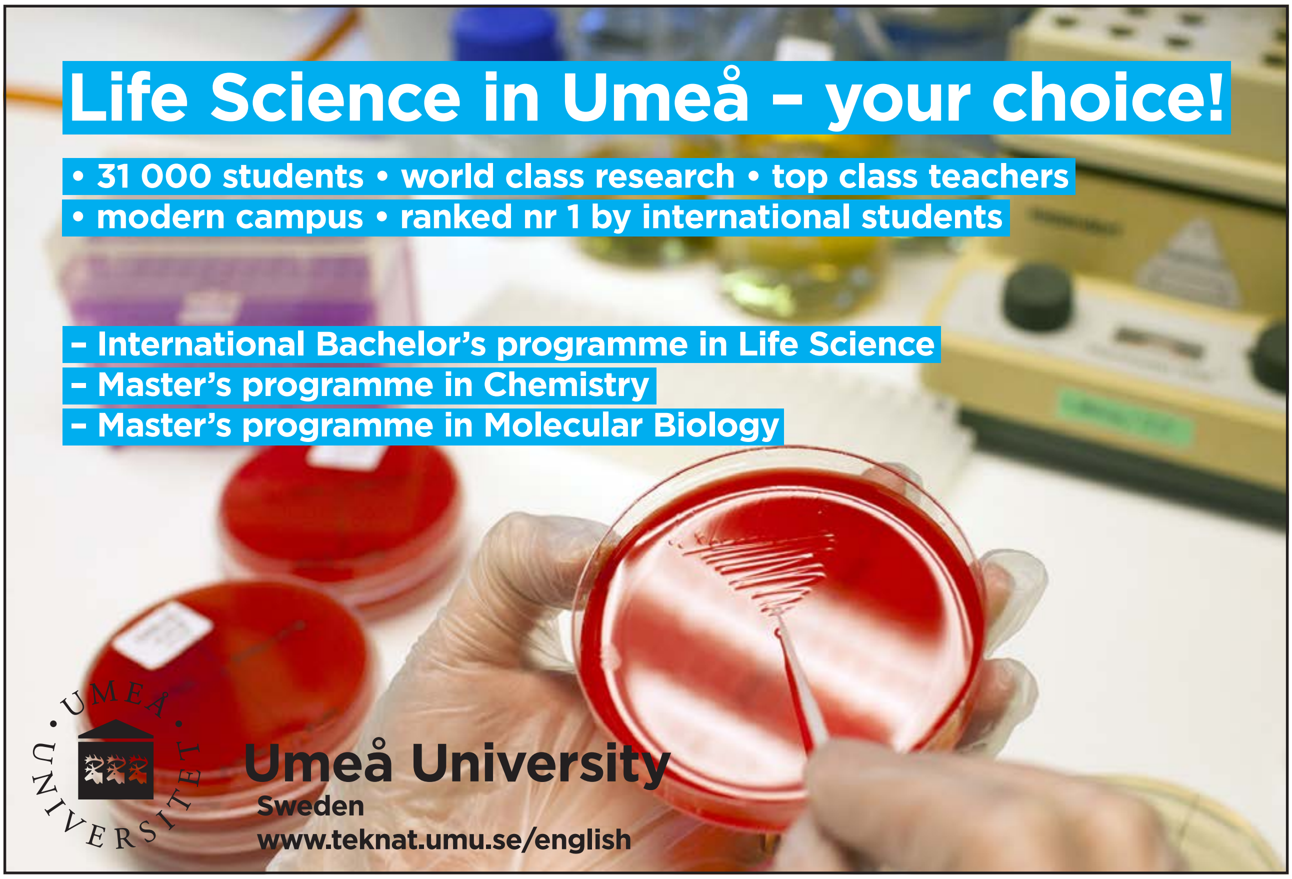
(employ Leibniz' product rule for differentiation)

$$
\begin{aligned}
&\leqslant \sum_{j=1}^{\infty} C_{\ell_j+1} \max_{|\alpha|\leqslant m_{\ell_j+1}} \sup_{x\in K_{\ell_j+1}\setminus K_{\ell_j-2}} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \left|D^{\alpha-\beta}\varphi_j(x) D^{\beta}\varphi(x)\right| \\
&\leqslant \sum_{j=1}^{\infty} C_{\ell_j+1} \max_{|\alpha|\leqslant m_{\ell_j+1}} \sup_{x\in K_{\ell_j+1}\setminus K_{\ell_j-2}} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \left|D^{\alpha-\beta}\varphi_j(x)\right| \cdot \left|D^{\beta}\varphi(x)\right| \\
&\leqslant \sum_{\ell=3}^{\infty} C_{\ell} \sum_{j=1,\ell_j+1=\ell}^{\infty} \max_{|\alpha|\leqslant m_{\ell}} \sup_{x\in K_{\ell}\setminus K_{\ell-3}} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \left|D^{\alpha-\beta}\varphi_j(x)\right| \cdot \left|D^{\beta}\varphi(x)\right| \\
&\leqslant \sum_{\ell=3}^{\infty} \widetilde{C}_{\ell} \max_{|\beta|\leqslant m_{\ell}} \sup_{x\in K_{\ell}\setminus K_{\ell-3}} \left|D^{\beta}\varphi(x)\right| \leqslant \sum_{\ell=1}^{\infty} c_{\ell} \max_{|\beta|\leqslant m_{\ell}} \sup_{x\in K_{\ell}\setminus K_{\ell-1}} \left|D^{\beta}\varphi(x)\right| \\
&= p_{(m_\ell, c_\ell)}(\varphi)\,. \qquad (1.10)
\end{aligned}
$$

(For notation and Leibniz' rule see the beginning of Subsection 1.6.) Here, $c_\ell = \widetilde{C}_\ell + \widetilde{C}_{\ell+1} + \widetilde{C}_{\ell+2}$, $\ell \geqslant 3$, $c_2 = \widetilde{C}_3 + \widetilde{C}_4$, $c_1 = \widetilde{C}_3$ and

$$
\widetilde{C}_\ell = C_\ell \sup_{x\in K_\ell\setminus K_{\ell-3}} \sum_{j=1,\ell_j+1=\ell}^{\infty} \max_{|\alpha|\leqslant m_\ell} \sup_{x\in K_\ell\setminus K_{\ell-3}} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \left|D^{\alpha-\beta}\varphi_j(x)\right|, \quad \ell \geqslant 3. \qquad (1.11)
$$

Consequently, (2) implies (1). Here we employed the fact that the sequence m_ℓ is non-decreasing, and that, because of local finiteness the sum in (1.11) is finite.



Observe that a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converges to 0 relative to $\mathcal{T}_\Omega$ if and only if there exists a compact subset K of Ω such that $(\varphi_k)_{k\in\mathbb{N}}$ is contained in $\mathcal{D}_K$ and converges to 0 relative to $\mathcal{T}_K$. Since the spaces $(\mathcal{D}_K, \mathcal{T}_K)$ are metrizable, continuity of linear functionals on $\mathcal{D}_K$ can be described by its convergence on convergent sequences, it follows that assertion (3) implies assertion (2).

The implication (1) $\Longrightarrow$ (3) being clear completes the proof of Theorem 1.22. □

Also observe that the locally convex space $(\mathcal{D}_K, \mathcal{T}_K)$ is complete metrizable, and therefore, a Frechet space. In fact the metric $d : \mathcal{D}_K \times \mathcal{D}_K \to [0,1]$, defined by

$$d(\varphi, \psi) = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \frac{p_{m,K}(\varphi - \psi)}{1 + p_{m.K}(\varphi - \psi)},$$

is a $\mathcal{T}_K$-compatible metric that turns $(\mathcal{D}_K, \mathcal{T}_K)$ into a complete metric space.

The space $\mathcal{D}'(\Omega)$ is supplied with the topology of pointwise convergence. So a sequence $(u_k)_{k\in\mathbb{N}}$ converges in $\mathcal{D}'(\Omega)$ if and only if the limit $\lim_{k\to\infty} u_k(\varphi)$ exists for each φ in $\mathcal{D}(\Omega)$. The linear functional u defined by

$$u(\varphi) = \lim_{k\to\infty} u_k(\varphi), \ \varphi \in \mathcal{D}(\Omega),$$

is again a distribution. In other words $\mathcal{D}'(\Omega)$ is sequentially complete.

1.23. Theorem. *The space $\mathcal{D}'(\Omega)$ is sequentially complete.*

Proof. The proof is based on the Banach-Steinhaus theorem in Fréchet spaces: see Corollary 8.22. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}'(\Omega)$ with the property that $u(\varphi) := \lim_{k\to\infty} u_k(\varphi)$ exists for every $\varphi \in \mathcal{D}(\Omega)$. We will prove that $\restriction_{\mathcal{D}_K}$ is continuous for all compact subsets K of Ω. Since the space $(\mathcal{D}, \mathcal{T}_K)$ is complete metrizable, and the sequence $(u_k(\varphi))_{k\in\mathbb{C}}$ is bounded in $\mathbb{C}$ for all $\varphi \in \mathcal{D}_K$, from the Banach-Steinhaus theorem it follows that the sequence $(u_k\restriction_{\mathcal{D}_K})_{k\in\mathbb{N}}$ is equi-continuous. This means that there exists a closed neighborhood U of the origin in $\mathcal{D}_K$ such that $u_k(U) \subset \{z \in \mathbb{C} : |z| \leqslant 1\}$. Since $u(\varphi) = \lim_{k\to\infty} u_k(\varphi)$ we see that $\varphi \in U$ implies $|u(\varphi)| \leqslant 1$. In other words $u\restriction\mathcal{D}_K$ is continuous. Since K is an arbitrary compact subset of Ω it follows that u belongs to $\mathcal{D}'(\Omega)$.

This completes the proof of Theorem 1.23. □

1.4. Differentiation of distributions. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index and if u is in $\mathcal{D}'(\Omega)$, then $D^\alpha u$ defined by the formula

$$D^\alpha u(\varphi) = (-1)^{|\alpha|} u(D^\alpha \varphi), \ \varphi \in \mathcal{D}(\Omega),$$

with $|\alpha| = \sum_{j=1}^n \alpha_j$, is again a distribution on Ω. Notice that

$$D^{\alpha+\beta} u = D^\alpha(D^\beta u) = D^\beta(D^\alpha u)$$

for all multi-indices α and β, since $D^{\alpha+\beta}\varphi = D^{\alpha}(D^{\beta}\varphi) = D^{\beta}(D^{\alpha}\varphi)$ and $|\alpha+\beta| = |\alpha| + |\beta|$. Next, let u be a distribution which is given by the formula

$$u(\varphi) = \int_{\Omega} f(x)\varphi(x)\,dx,\ \varphi \in \mathcal{D}(\Omega),$$

where f is a complex-valued function on Ω with the property that $D^{\alpha}f$ exists and is continuous for each multi-index α with $|\alpha| \leqslant N$ for a certain $N \in \mathbb{N}$. The derivations $D^{\alpha}u$, $|\alpha| \leqslant N$, are then given by

$$D^{\alpha}u(\varphi) = \int_{\Omega} D^{\alpha}f(x)\varphi(x)\,dx,\ \varphi \in \mathcal{D}(\Omega).$$

This fact can be proved by a simple integration by parts. However, and this is important, the almost everywhere derivative of a function f does not necessarily correspond to the derivative of the corresponding distribution u. An example will illustrate this fact.

1.24. Example. Take $\Omega = \mathbb{R}$ and let u be the distribution on $\mathbb{R}$ defined by

$$u(\varphi) = \int_{0}^{\infty} \varphi(x)\,dx,\ \varphi \in \mathcal{D}(\mathbb{R}).$$

In other words, the distribution u is associated to the so-called *Heaviside function* H which is defined by

$$H(x) = \begin{cases} 1 & x \geqslant 0, \\ 0 & x < 0. \end{cases}$$

Since for all φ in $\mathcal{D}(\mathbb{R})$ the equalities

$$\frac{du}{dx}(\varphi) = -u(\varphi') = -\int_{0}^{\infty} \varphi'(x)\,dx = \varphi(0) = \delta(\varphi)$$

are valid, the distributional derivative $\frac{du}{dx}$ equals δ. However, the almost everywhere derivative $\frac{dH}{dx}$ equals zero almost everywhere. It follows that the distribution $\frac{du}{dx}$ is not associated to the function $\frac{dH}{dx}$.

1.5. The space $C^{\infty}(\Omega)$. The space $C^{\infty}(\Omega)$ consists of all infinitely many times differentiable functions on Ω. This space will be equipped with the topology of uniform convergence of all derivatives on all compact subsets. More precisely, let $(K_m)_{m\in\mathbb{N}}$ be a sequence of compact subsets of Ω such that K_m is contained in $\overset{\circ}{K}_{m+1}$, the interior of K_{m+1}, and such that $\Omega = \bigcup_{m=1}^{\infty} K_m$. For each $m \in \mathbb{N}$ the semi-norm p_m on $C^{\infty}(\Omega)$ is defined by

$$p_m(f) = \max_{|\alpha|\leqslant m} \sup_{x\in K_m} \left|D^{\alpha}f(x)\right|,\ f \in C^{\infty}(\Omega).$$

This sequence of semi-norms turns $C^\infty(\Omega)$ into a Fréchet space, *i.e.* a completely metrizable locally convex space. The corresponding metric can be given by

$$d(f,g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(f-g)}{1+p_m(f-g)}, \quad f, g \in C^\infty(\Omega).$$

A sequence $(f_k)_{k\in\mathbb{N}}$ in $C^\infty(\Omega)$ converges with respect to this metric if and only if there exists a function f in $C^\infty(\Omega)$ such that for each compact subset K of Ω and each positive integer m, the limit

$$\lim_{k\to\infty} \max_{|\alpha|\leqslant m} \sup_{x\in K} |D^\alpha f_k(x) - D^\alpha f(x)|$$

exists and equals zero. Notice that $\mathcal{D}(\Omega)$ is a dense subspace of $C^\infty(\Omega)$.

1.6. Convergence properties of distributions. If u is in $\mathcal{D}'(\Omega)$ and if f belongs to $C^\infty(\Omega)$, the distribution fu is defined by

$$(fu)(\varphi) = u(f\varphi), \quad \varphi \in \mathcal{D}(\Omega).$$

Its derivatives $D^\alpha(fu)$ for $\alpha \in \mathbb{N}^n$ can be computed by Leibniz' formula:

$$D^\alpha(fu) = \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \left(D^{\alpha-\beta} f\right) \left(D^\beta u\right).$$

Here $\beta = (\beta_1, \ldots, \beta_n) \leqslant \alpha = (\alpha_1, \ldots, \alpha_n)$ means $\beta_k \leqslant \alpha_k$ for all $k = 1, \ldots, n$ and

$$\binom{\alpha}{\beta} = \prod_{k=1}^{n} \binom{\alpha_k}{\beta_k} = \prod_{k=1}^{n} \frac{\alpha_k!}{\beta_k!(\alpha_k - \beta_k)!}.$$

Next, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}'(\Omega)$ which converges to u in $\mathcal{D}'(\Omega)$ and let α be a multi-index in $\mathbb{N}^n$. Then

$$\mathcal{D}'(\Omega)\text{-}\lim_{k\to\infty} D^\alpha u_k = D^\alpha u.$$

If, in addition, $(f_k)_{k\in\mathbb{N}}$ is a sequence in $C^\infty(\Omega)$ which converges in the topology of $C^\infty(\Omega)$ to a function f, then

$$\mathcal{D}'(\Omega)\text{-}\lim_{k\to\infty} f_k u_k = fu.$$

1.7. Supports of distributions. Let Ω be an open subset of $\mathbb{R}^n$ and let u_1 and u_2 be distributions on Ω. Furthermore let V be an open subset of Ω. Then $\mathcal{D}(V)$ can be naturally embedded into $\mathcal{D}(\Omega)$. We say that u_1 is equal to u_2 on V if $u_1(\varphi) = u_2(\varphi)$ for all φ in $\mathcal{D}(V)$. Next, let $(\Omega_j)_{j\in J}$ be an open cover of Ω and suppose that for each $j \in J$ there is a distribution u_j on Ω_j. Assume that for all j_1 and j_2 in J, $u_{j_1} = u_{j_2}$ on $\Omega_{j_1} \cap \Omega_{j_2}$. Using a partition of unity subordinate to the open cover $(\Omega_j)_{j\in J}$, a distribution u can be constructed on Ω in such a way that $u = u_j$ on Ω_j.

1.25. Theorem. *Let $\{\Omega_j : j \in J\}$ be an open cover by open subsets of Ω. Suppose that for every $j \in J$ there exists $u_j \in \mathcal{D}'(\Omega_j)$ in such a way that $u_{j_1} = u_{j_2}$ on $\Omega_{j_1} \cap \Omega_{j_2}$ whenever $\Omega_{j_1} \cap \Omega_{j_2} \neq \varnothing$. Then these distribution can be patched together in the sense that there exists $u \in \mathcal{D}'(\Omega)$ such that $u = u_j$ on Ω_j.*

Proof. Let $(\varphi_k)_{k\in\mathbb{N}}$ be a locally finite partition of unity subordinate to the open cover $(\Omega_j)_{j\in J}$. For every $k \in \mathbb{N}$ choose $j_k \in J$ such that $\operatorname{supp}(\varphi_k) \subset \Omega_{j_k}$. Put $u(\varphi) = \sum_{k=1}^{\infty} u_{j_k}(\varphi_k\varphi)$, $\varphi \in \mathcal{D}(\Omega)$. Then u is well-defined and belongs to $\mathcal{D}'(\Omega)$. It is not difficult to see that u is a distribution; here we use the fact that the partition of unity $(\varphi_k)_{k\in\mathbb{N}}$ is locally finite. That it is well-defined can be seen as follows. Let $(\psi_\ell)_{\ell\in\mathbb{N}}$ be another locally finite partition of unity subordinate to the open cover $(\Omega_j)_{j\in J}$, and choose, for every $\ell \in \mathbb{N}$, $j'_\ell \in J$ such that $\operatorname{supp}(\psi_\ell) \subset \Omega_{j'_\ell}$. Then, by consistency, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} u_{j_k}(\varphi_k\varphi) &= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} u_{j_k}(\varphi_k\psi_\ell\varphi) \\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} u_{j'_\ell}(\varphi_k\psi_\ell\varphi) = \sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty} u_{j'_\ell}(\varphi_k\psi_\ell\varphi) = \sum_{\ell=1}^{\infty} u_{j'_\ell}(\psi_\ell\varphi), \qquad (1.12)
\end{aligned}$$

which shows that $u(\varphi)$ does not depend on the particular choice of the locally finite partition of unity subordinate to the cover $\{\Omega_j : j \in J\}$. For details on partition of unity see Subsection 1. For more extensive information consult, e.g., [**151**]. The previous arguments show Theorem 1.25. $\square$

Next, pick u in $\mathcal{D}'(\Omega)$ and let W be the union of those open subsets V of Ω where u vanishes. Then u vanishes on W. Here u is said to *vanish on* V if $u = 0$ on V, *i.e.* if $u(\varphi) = 0$ for all φ in $\mathcal{D}(V)$. The complement of W is called the *support* of u. It is denoted by $\operatorname{supp}(u)$. If the distribution u happens to come from a continuous function f, then the support of u is the ordinary support of f. In the same fashion, one can consider the smallest closed subset S of Ω outside of which a distribution u is a C^∞-function. The set S is called the *singular support* of u. For example, let the distribution u correspond to the function $f(x) = |x|^{-1}$, $x \in \mathbb{R}^n \backslash \{0\}$. Then the support of u coincides with the support of f, both being equal to $\mathbb{R}^n$. The singular support of u is $\{0\}$.

1.8. Distributions with compact support. Distributions with compact support are of finite order. Since $\mathcal{D}(\Omega)$ is dense in $C^\infty(\Omega)$, they can be uniquely extended to a continuous linear functionals on $C^\infty(\Omega)$. Conversely, the restriction to $\mathcal{D}(\Omega)$ of a continuous linear functional on $C^\infty(\Omega)$ defines a distribution with compact support. The proof of these statements goes as follows. First, let u in $\mathcal{D}'(\Omega)$ have compact support S, *i.e.* $u(\varphi) = 0$ for all φ in $\mathcal{D}(W)$ with $W = \Omega \backslash S$. There exists a function ψ in $\mathcal{D}(\Omega)$ which is identically 1 on an open neighbourhood V of S. Now define the functional $\Lambda \colon C^\infty(\Omega) \to \mathbb{C}$ by

$$\Lambda(f) = u(\psi f),\ f \in C^\infty(\Omega).$$

Then Λ is a well defined, continuous linear functional on $C^\infty(\Omega)$ that extends u. Conversely, let $\Lambda \colon C^\infty(\Omega) \to \mathbb{C}$ be a continuous linear functional. Then there exists a compact subset K of Ω and a positive integer N such that

$$|\Lambda(f)| \leqslant C \max_{|\alpha| \leqslant N} \sup_{x \in K} \left| D^\alpha f(x) \right|,\ f \in C^\infty(\Omega),$$

for some constant $C > 0$. It follows that the restriction of Λ to $\mathcal{D}(\Omega)$ has its support contained in K.

1.26. Remark. We could have started as follows. If $u \in \mathcal{D}'(\Omega)$ has compact support, then there exists a fine constant C and a non-negative integer m such that

$$|u(\varphi)| \leqslant C \max_{|\alpha| \leqslant m} \sup_{x \in \Omega} |D^\alpha \varphi(x)|, \quad \varphi \in \mathcal{D}_{K+rB}, \tag{1.13}$$

where $r > 0$ is small enough and B is the closed unit ball in $\mathbb{R}^n$. By the Hahn-Banach theorem the functional u extends to a continuous linear functional on $C^\infty(\Omega)$ in such a way that the inequality in (1.13) holds for all $\varphi \in C^\infty(\Omega)$.

Suppose that the distribution u with compact support satisfies (1.13). Then it has order $\leqslant m$. Let $\varphi \in \mathcal{D}(\Omega)$ be arbitrary. Then choose $\psi \in \mathcal{D}(\Omega)$ in such a way that $\mathbf{1}_{K+\frac{1}{2}rB} \leqslant \mathbf{1}_{K+rB}$. Then $u(\varphi) = u(\psi\varphi)$, and so $u(\varphi)$ satisfies an estimate of the form

$$|u(\varphi)| = |u(\psi\varphi)| \leqslant C \max_{|\alpha| \leqslant m} \sup_{x \in \Omega} |D^\alpha (\psi\varphi)(x)| \leqslant C' \max_{|\alpha| \leqslant m} \sup_{x \in K+rB} |D^\alpha \varphi(x)|,$$

where we employed Leibniz' rule.

We include and prove some theorems on distributions with compact support. For more details we refer to Rudin [**112**] Chapter 6 and to Schwartz [**121**] Chapter 3.

1.27. Theorem. *Let p be a point in $\mathbb{R}^n$. A distribution u has support $\{p\}$ if and only if there are constants c_α, $|\alpha| \leqslant m$, not all zero, such that*

$$u = \sum_{|\alpha|\leqslant m} c_\alpha D^\alpha \delta_p.$$

Here δ_p is the Dirac distribution at p, i.e. $\delta_p(\varphi) = \varphi(p)$.

Proof. Let $u \in \mathcal{D}'(\Omega)$ be such that its support is the singleton p. Then it has finite order m. Let $\varphi \in \mathcal{D}(\Omega)$ be such that $D^\alpha\varphi(p) = 0$ for all multi-indices α with $|\alpha| \leqslant m$. From Theorem 1.29 it follows that $u(\varphi) = 0$. Put $u_\alpha = D^\alpha\delta$. From the previous arguments it follows that $\bigcap_{|\alpha|\leqslant m} \operatorname{Ker}(u_\alpha) \subset \operatorname{Ker}(u)$. Consequently, there constants c_α, $|\alpha| \leqslant m$, such that $u = \sum_{|\alpha|\leqslant m} c_\alpha D^\alpha \delta_p$. This proves Theorem 1.27 $\square$

Our proof of Theorem 1.29 requires the following lemma.

1.28. LEMMA. *Let K be a compact subset of the open subset Ω of $\mathbb{R}^n$. Let $\varphi \in C^\infty(\Omega)$ be such that $D^\alpha\varphi(x) = 0$ for all $x \in K$ and for all $\alpha \in \mathbb{N}^n$ for which $|\alpha| \leqslant N$ for some $N \in \mathbb{N}$. Then there exists a family of functions $\varphi_r \subset \mathcal{D}(\Omega)$, $0 < r \leqslant r_0$, $r_0 > 0$ small enough with the following properties:*

(1) $\operatorname{supp}(\varphi_r) \subset K + rB$, $0 < r \leqslant r_0$;
(2) $\mathbf{1}_{K+\frac{1}{2}rB}(x) \leqslant \varphi_r(x) \leqslant \mathbf{1}_{K+rB}(x)$, $x \in \Omega$, $0 < r \leqslant r_0$;
(3) $\lim\limits_{r\downarrow 0} \max\limits_{|\alpha|\leqslant N} \sup\limits_{x\in\Omega} |D^\alpha(\varphi_r\varphi)(x)| = 0$.

PROOF OF LEMMA 1.28. Let B be the closed unit ball in $\mathbb{R}^N$, and choose a function $\psi_1 \in \mathcal{D}(\mathbb{R}^n)$, $\psi_1 \geqslant 0$, such that $\operatorname{supp}(\psi_1) \subset B$, and such that $\int \psi_1(y)\,dy = 1$. Let φ be as in Lemma 1.28. Put $\psi_r(x) = \dfrac{1}{r^n}\psi_1\left(\dfrac{x}{r}\right)$, and

$$\varphi_r(x) = \mathbf{1}_{K+\frac{3}{4}rB} * \psi_{\frac{1}{4}r}(x) = \int_{K+\frac{3}{4}rB} \left(\frac{4}{r}\right)^n \psi_1\left(\frac{4}{r}(x-y)\right) dy.$$

Then the functions φ_r, $0 < r \leqslant r_0$, belong to $\mathcal{D}(\Omega)$, and satisfy (1) and (2). The fact that they also satisfy (3) can be proved as follows. For $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leqslant N$ we have

$$\begin{aligned}
D^\alpha(\varphi_r\varphi)(x) &= \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \int_{K+\frac{3}{4}rB} \left(\frac{4}{r}\right)^{n+|\alpha-\beta|} D^{\alpha-\beta}\psi_1\left(\frac{4}{r}(x-y)\right) dy\, D^\beta\varphi(x) \\
&= \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \int_{K+\frac{3}{4}rB} \left(\frac{4}{r}\right)^{n+|\alpha-\beta|} D^{\alpha-\beta}\psi_1\left(\frac{4}{r}(x-y)\right) dy\, D^\beta\varphi(x) \\
&= \int_{K+\frac{3}{4}rB} \left(\frac{4}{r}\right)^{n} \psi_1\left(\frac{4}{r}(x-y)\right) dy\, D^\alpha\varphi(x) \qquad (1.14) \\
&\quad + \sum_{\beta\leqslant\alpha,\beta\neq\alpha} \binom{\alpha}{\beta} \int_{K+\frac{3}{4}rB} \left(\frac{4}{r}\right)^{n+|\alpha-\beta|} D^{\alpha-\beta}\psi_1\left(\frac{4}{r}(x-y)\right) dy\, D^\beta\varphi(x).
\end{aligned}$$

In addition, for $\beta \leqslant \alpha$, $\beta \neq \alpha$, integration by parts reveals the identity:

$$\begin{aligned}
&D^\beta\varphi(x) - \sum_{j=0}^{|\alpha-\beta|-1} \frac{1}{j!} \sum_{|\gamma|=j} (i(x-z))^\gamma D^{\gamma+\beta}\varphi(z) \\
&= \int_0^1 \frac{(1-\rho)^{|\alpha-\beta|-1}}{(|\alpha-\beta|-1)!} \sum_{|\gamma|=|\alpha-\beta|} (i(x-z))^\gamma D^{\gamma+\beta}\varphi((1-\rho)z+\rho x)\, d\rho. \qquad (1.15)
\end{aligned}$$

Recall that, see (1.3),

$$D^\gamma = \left(\frac{\partial}{i\partial x_1}\right)^{\gamma_1} \circ \cdots \circ \left(\frac{\partial}{i\partial x_n}\right)^{\gamma_n},$$

and that is why we write the extra "i's" in the formula in (1.15). Notice that $|\alpha| \leqslant N$. Since z belongs to K, and since $D^{\gamma}\varphi(z) = 0$ for $\gamma \leqslant \alpha$, the equality in (1.15) implies:

$$D^{\beta}\varphi(x) = \int_0^1 \frac{(1-\rho)^{|\alpha-\beta|-1}}{(|\alpha-\beta|-1)!} \sum_{|\gamma|=|\alpha-\beta|} \left(i\,(x-z)\right)^{\gamma} D^{\gamma+\beta}\varphi\left((1-\rho)\,z+\rho x\right)\,d\rho. \quad (1.16)$$

Inserting (1.16) into (1.14) yields

$$\begin{aligned}
D^{\alpha}\left(\varphi_r\varphi\right)(x) &= \int_{K+\frac{3}{4}rB} \left(\frac{4}{r}\right)^{n} \psi_1\left(\frac{4}{r}\,(x-y)\right)\,dy\, D^{\alpha}\varphi(x) \\
&\quad + \sum_{\beta\leqslant\alpha,\beta\neq\alpha} \binom{\alpha}{\beta} \int_{K+\frac{3}{4}rB} \left(\frac{4}{r}\right)^{n+|\alpha-\beta|} D^{\alpha-\beta}\psi_1\left(\frac{4}{r}\,(x-y)\right)\,dy \\
&\quad \times \int_0^1 \frac{(1-\rho)^{|\alpha-\beta|-1}}{(|\alpha-\beta|-1)!} \sum_{|\gamma|=|\alpha-\beta|} \left(i\,(x-z)\right)^{\gamma} D^{\gamma+\beta}\varphi\left((1-\rho)\,z+\rho x\right)\,d\rho \\
&= \int_{4r^{-1}K+3B} \psi_1\,(y)\,dy\, D^{\alpha}\varphi(x) \qquad (1.17) \\
&\quad + \sum_{\beta\leqslant\alpha,\beta\neq\alpha} \binom{\alpha}{\beta} \int_{4r^{-1}K+3B} D^{\alpha-\beta}\psi_1\,(y)\,dy \\
&\quad \times \int_0^1 \frac{(1-\rho)^{|\alpha-\beta|-1}}{(|\alpha-\beta|-1)!} \sum_{|\gamma|=|\alpha-\beta|} \left(\frac{4i}{r}\,(x-z)\right)^{\gamma} D^{\gamma+\beta}\varphi\left((1-\rho)\,z+\rho x\right)\,d\rho.
\end{aligned}$$

Since $z \in K$, and $|z-x| \leqslant r$, from (1.17) we derive:

$$\begin{aligned}
&\left|D^{\alpha}\left(\varphi_r\varphi\right)(x)\right| \\
&\leqslant \int_{\mathbb{R}^n} \psi_1\,(y)\,dy\,\left|D^{\alpha}\varphi(x)\right| \\
&\quad + \sum_{\beta\leqslant\alpha,\beta\neq\alpha} 4^{|\alpha-\beta|} \binom{\alpha}{\beta} \int_{\mathbb{R}^n} \left|D^{\alpha-\beta}\psi_1\,(y)\right|\,dy \sum_{|\gamma|=|\alpha-\beta|} \prod_{j=1}^{n} \frac{|x_j-z_j|^{\gamma_j}}{r^{\gamma_j}} \\
&\quad \times \int_0^1 \frac{(1-\rho)^{|\alpha-\beta|-1}}{(|\alpha-\beta|-1)!} \left|D^{\gamma+\beta}\varphi\left((1-\rho)\,z+\rho x\right)\right|\,d\rho \\
&\leqslant \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \frac{4^{|\alpha-\beta|}}{|\alpha-\beta|!} \int_{\mathbb{R}^n} \left|D^{\alpha-\beta}\psi_1\,(y)\right|\,dy \sup_{0\leqslant\rho\leqslant1} \sum_{|\gamma|=|\alpha-\beta|} \left|D^{\gamma+\beta}\varphi\left((1-\rho)\,z+\rho x\right)\right| \\
&\leqslant \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \frac{4^{|\beta|}}{|\beta|!} \int_{\mathbb{R}^n} \left|D^{\beta}\psi_1\,(y)\right|\,dy \sup_{0\leqslant\rho\leqslant1} \sum_{|\gamma|=|\alpha|} \left|D^{\gamma}\varphi\left((1-\rho)\,z+\rho x\right)\right| \\
&\leqslant C\,(\alpha) \sup_{z\in K+rB} \sum_{|\gamma|=|\alpha|} \left|D^{\gamma}\varphi\,(z)\right|, \qquad (1.18)
\end{aligned}$$

where

$$C(\alpha) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \frac{4^{|\beta|}}{|\beta|!} \int_{\mathbb{R}^n} \left| D^{\beta} \psi_1 (y) \right| dy.$$

From (1.18) we obtain:

$$\sup_{x \in \Omega} \left| D^{\alpha} \left(\varphi_r \varphi \right) (x) \right| \leqslant C(\alpha) \sup_{z \in K + rB} \sum_{|\gamma| = |\alpha|} \left| D^{\gamma} \varphi \left(z \right) \right|. \tag{1.19}$$

The continuity of the functions $D^{\alpha}\varphi$, $|\alpha| \leqslant N$, and the inequality in (1.19) then imply assertion (3). Altogether this completes the proof of Lemma 1.28. □

1.29. THEOREM. *Let K be a compact subset of Ω and let u be a distribution on Ω with support K and order N. If $D^{\alpha}\varphi(x) = 0$ for all $|\alpha| \leqslant N$ and $x \in K$, then $u(\varphi) = 0$.*

Proof of Theorem 1.29. Let B be the closed unit ball in $\mathbb{R}^N$, and choose a function $\psi_1 \in \mathcal{D}(\mathbb{R}^n)$, $\psi_1 \geqslant 0$, such that $\operatorname{supp}(\psi_1) \subset B$, and such that $\int \psi_1(y)\, dy = 1$. Let φ be as in Theorem 1.29 and let the functions $\{\varphi_r : 0 < r \leqslant r_0\}$ be as in (the proof of) Lemma 1.28. Then for all $0 < r \leqslant r_0$ we have $u(\varphi) = u(\varphi_r \varphi)$, and hence the absolute value of $u(\varphi)$ can be estimated as follows:

$$\begin{aligned} |u(\varphi)| = |u(\varphi_r\varphi)| &\leqslant C \max_{|\alpha|\leqslant N} \sup_{x\in\Omega} |D^\alpha(\varphi_r\varphi)(x)| \\ &\leqslant C \max_{|\alpha|\leqslant N} C(\alpha) \sup_{z\in K+rB} \sum_{|\gamma|=|\alpha|} |D^\gamma\varphi(z)|. \end{aligned} \tag{1.20}$$

Here we employed (1.19) in the proof of Lemma 1.28. Since the right-hand side of (1.20) tends to 0 when $r \downarrow 0$, the conclusion in Theorem 1.29 follows. ☐

The following theorem shows that, locally, a distribution can be written as the derivative of a continuous function.

1.30. Theorem. *Let K be a compact subset of Ω, and let $u \in \mathcal{D}'(\Omega)$. Then there exists a continuous function f on Ω and a multi-index α such that $u = D^\alpha f$ on K in the sense that*

$$u(\varphi) = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha(\varphi)(x)\, dx, \quad \varphi \in \mathcal{D}_K. \tag{1.21}$$

Proof. Fix $z = (z_1, \dots, z_n)$ and $z' = (z_1', \dots, z_n')$ in $\mathbb{R}^n$ in such a way that K is contained in $\Omega \cap \prod_{j=1}^n [z_j, z_j']$. Let $\varphi \in \mathcal{D}_K$. Define the operators T_j^k, $1 \leqslant j \leqslant n$, $k = 0, 1, \dots$ by $T_j^0\varphi(y) = \varphi(y)$, $y = (y_1, \dots, y_n) \in \Omega$,

$$T_j^k\varphi(y) = \int_{z_j}^{y_j} \frac{i^k (y_j - w_j)^{k-1}}{(k-1)!} \varphi(y_1, \dots, y_{j-1}, w_j, y_{j+1}, \dots, y_n)\, dw_j. \tag{1.22}$$

The operator T^α is defined by $T^\alpha = T_1^{\alpha_1} \circ \dots \circ T_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. Then we have, for $\varphi \in \mathcal{D}_K$ and $\alpha \in \mathbb{N}^n$,

$$T^\alpha D^\alpha \varphi = D^\alpha T^\alpha \varphi = \varphi, \quad \text{and} \tag{1.23}$$

$$\sup_{x\in K} |T^\alpha\varphi(x)| \leqslant \frac{(z'-z)^\alpha}{\alpha!} \sup_{x\in K} |\varphi(x)|. \tag{1.24}$$

Here we used the notation $(z'-z)^\alpha = \prod_{j=1}^n (z_j' - z_j)^{\alpha_j}$. The distribution u confined to $\mathcal{D}_K$ is continuous, and so there exist $m \in \mathbb{N}$ and $C \in (0, \infty)$ such that

$$|u(\varphi)| \leqslant C \max_{|\alpha|\leqslant m} \sup_{x\in K} |D^\alpha\varphi(x)|, \quad \varphi \in \mathcal{D}_K. \tag{1.25}$$

Put $\mathcal{D}_K^{(m)} = T_1^m \circ \dots \circ T_n^m \mathcal{D}_K$. Then by (1.23) $\mathcal{D}_K$ is a subspace of $\mathcal{D}_K^{(m)}$. By the Hahn-Banach extension theorem (see Theorem 8.2) there exists a functional $u_m : \mathcal{D}_K^{(m)} \to \mathbb{C}$ with the following properties: $u(\varphi) = u_m(\varphi)$, $\varphi \in \mathcal{D}_K$, and

$$|u_m(\varphi)| \leqslant C \max_{|\alpha|\leqslant m} \sup_{x\in K} |D^\alpha\varphi(x)|, \quad \varphi \in \mathcal{D}_K^{(m)}. \tag{1.26}$$

Fix $\varphi \in \mathcal{D}_K$. The following inequalities follow from (1.23), (1.24) and (1.26):

$$\begin{aligned}\left|u_m\left(T_1^m \circ \cdots \circ T_n^m \varphi\right)\right| &\leqslant C \max_{|\alpha| \leqslant m} \sup_{x \in K} \left|\left(D^\alpha T_1^m \circ \cdots \circ T_n^m \varphi\right)(x)\right| \\ &\leqslant C \max_{|\alpha| \leqslant m} \sup_{x \in K} \left|\left(T_1^{m-\alpha_1} \circ \cdots \circ T_n^{m-\alpha_n} \varphi\right)(x)\right| \\ &\leqslant C \max_{|\alpha| \leqslant m} \prod_{j=1}^n \frac{\left(z_j' - z_j\right)^{m-\alpha_j}}{(m-\alpha_j)!} \sup_{x \in K} |\varphi(x)| \\ &= C_m \sup_{x \in K} |\varphi(x)|, \end{aligned} \tag{1.27}$$

where

$$C_m = C \max_{|\alpha| \leqslant m} \prod_{j=1}^n \frac{\left(z_j' - z_j\right)^{m-\alpha_j}}{(m-\alpha_j)!}$$

Upon considering the functional $\varphi \to u_m\left(T_1^m \circ \cdots \circ T_n^m \varphi\right)$, $\varphi \in \mathcal{D}_K$, and employing (1.27), the Riesz representation theorem yields the existence of complex measure on $\mathbb{R}^n$ with the property that:

$$u_m\left(T_1^m \circ \cdots \circ T_n^m \varphi\right) = \int \varphi(x)\, d\mu(x), \quad \varphi \in \mathcal{D}_K. \tag{1.28}$$

Let D_j, $1 \leqslant j \leqslant n$, be the operator $D_j = \dfrac{\partial}{i\partial x_j}$. From (1.23) and from (1.28) we deduce

$$u(\varphi) = u_m\left(T_1^m \circ \cdots \circ T_n^m \circ D_1^m \circ \cdots \circ D_n^m \varphi\right) = \int D_1^m \circ \cdots \circ D_n^m \varphi(x)\, d\mu(x). \tag{1.29}$$

Define the continuous function f on $\mathbb{R}^n$ by

$$f(x) = (-1)^{nm} \int_{z_1}^{x_1} \cdots \int_{z_n}^{x_n} \prod_{j=1}^n \left(x_j - w_j\right)\, d\mu(w). \tag{1.30}$$

Employing integration by parts and using (1.30), it is not so difficult to see that

$$u(\varphi) = (-1)^{nm} \int f(x) D_1^{m+2} \circ \cdots \circ D_n^{m+2} \varphi(x)\, dx, \quad \varphi \in \mathcal{D}_K. \tag{1.31}$$

In fact, by Fubini's theorem and integration by parts, we have for $\varphi \in \mathcal{D}_K$,

$$\begin{aligned}&(-1)^{nm} \int f(x) D_1^{m+2} \circ \cdots \circ D_n^{m+2} \varphi(x)\, dx \\ &= \int_{\mathbb{R}^n} \int_{z_1}^{x_1} \cdots \int_{z_n}^{x_n} \prod_{j=1}^n \left(x_j - w_j\right)\, d\mu(w)\, D_1^{m+2} \circ \cdots \circ D_n^{m+2} \varphi(x)\, dx \\ &= \int_{z_1}^{z_1'} \cdots \int_{z_n}^{z_n'} \int_{w_1}^{z_1'} \cdots \int_{w_n}^{z_n'} \prod_{j=1}^n \left(x_j - w_j\right)\, D_1^{m+2} \circ \cdots \circ D_n^{m+2} \varphi(x)\, dx\, d\mu(w) \\ &= \int_{z_1}^{z_1'} \cdots \int_{z_n}^{z_n'} D_1^m \circ \cdots \circ D_n^m \varphi(w)\, d\mu(w)\end{aligned}$$

$$= \int_{\mathbb{R}^n} D_1^m \circ \cdots \circ D_n^m \varphi(w)\, d\mu(w) = u(\varphi). \tag{1.32}$$

From (1.31) it follows that, on $\mathcal{D}_K$, $u = D^\alpha f$, with $\alpha_1 = \cdots = \alpha_n = m + 2$. This completes the proof of Theorem 1.30. □

1.31. Theorem. *Choose for each multi-index α, with $|\alpha| \leqslant m$, a continuous function f_α with compact support K_α in Ω. The distribution u defined by*

$$u = \sum_{|\alpha| \leqslant m} D^\alpha f_\alpha \tag{1.33}$$

has compact support. Conversely, each distribution with compact support is of this form.

Here $D^\alpha f_\alpha$ is meant to be the distributional derivative of u_{f_α}, the distribution associated to f_α. As an example we write the Dirac distribution δ on $\mathbb{R}$ in the form

$$\delta = f_0 + f_1' + f_2''.$$

Take ψ in $\mathcal{D}(\mathbb{R})$ in such a way that $\psi(0) = 1$ and put $f(x) = xH(x)$, $x \in \mathbb{R}$, where H is the Heaviside function. With $f_0 = f\psi''$, $f_1 = -2f\psi'$ and $f_2 = f\psi$, *i.e.*

$$f_k = (-1)^{2-k} \binom{2}{k} f \psi^{(2-k)}, \; k = 0, 1, 2,$$

the above equality holds since

$$f_0 + f_1' + f_2'' = f\psi'' - 2f'\psi' - 2f\psi'' + f''\psi + f\psi'' + 2f'\psi' = f''\psi$$

and $f''(x) = 2H'(x) + xH''(x)$.

Proof of Theorem 1.31. Let the compact set K be the support of u. Choose open subsets V and W with compact closure $\overline{W}$ such that $K \subset V \subset \overline{V} \subset W$. Let the distribution u have order less than or equal to m. An application of Theorem 1.30 to $\mathcal{D}_{\overline{W}}$ yields the existence of a continuous function f on $\mathbb{R}^n$ such that $u = D^\alpha f$ on $\mathcal{D}_{\overline{W}}$, where $\alpha = (m+2, \dots, m+2)$. Choose $\psi \in \mathcal{D}(\Omega)$ in such a way that $\psi(x) = 1$ for $x \in V$, and $\psi(x) = 0$, $x \in \Omega \backslash W$. Then we have, for $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned} u(\varphi) &= u(\psi\varphi) = (-1)^{nm} \int_\Omega f(x) D^\alpha (\psi\varphi)(x)\, dx \\ &= (-1)^{nm} \int_\Omega f(x) \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}\psi(x) D^\beta \varphi(x)\, dx \\ &= \sum_{\beta\leqslant\alpha} D^\beta \left\{ (-1)^{|\alpha-\beta|} \binom{\alpha}{\beta} \left(f \cdot D^{\alpha-\beta}\psi\right) \right\} (\varphi). \end{aligned} \tag{1.34}$$

It is clear that the equality in (1.34) is of the form (1.33), and so the proof of Theorem 1.31 is complete. □

1.32. Remark. From now on we adopt the convention that $u(\varphi)$ means

$$u(\varphi) = \int_\Omega u(x)\varphi(x)\, dx, \ \varphi \in \mathcal{D}(\Omega),$$

whenever the distribution u comes from a locally integrable function which we also denote by u. However, when necessary we will still use the notation u_f for the distribution associated to the function f.

1.9. Convolution of a test function and a distribution. From now on we take $\Omega = \mathbb{R}^n$, and we write $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$, $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$ and $L^1_{\text{loc}} = L^1_{\text{loc}}(\mathbb{R}^n)$. In $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ the following operations are defined:

(a) translations: $(\tau_s\varphi)(x) = \varphi(x-s)$ for all $x, s \in \mathbb{R}^n$, $\varphi \in \mathcal{D}$,
(b) reflections: $\check{\varphi}(x) = \varphi(-x)$ for all $x \in \mathbb{R}^n$, $\varphi \in \mathcal{D}$.

These operations can be defined in $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$ as well. Let u be in $\mathcal{D}'$ and let φ belong to $\mathcal{D}$:

(a) translations: $(\tau_s u)(\varphi) = u(\tau_{-s}\varphi)$ for all $s \in \mathbb{R}^n$, $\varphi \in \mathcal{D}$;
(b) reflections: $\check{u}(\varphi) = u(\check{\varphi})$.

It is readily verified that for distributions which originate from locally integrable functions, these definitions are consistent, *i.e.* $\check{u}_f = u_{\check{f}}$ and $\tau_s u_f = u_{\tau_s f}$. For f in $L^1_{\text{loc}} = L^1_{\text{loc}}(\mathbb{R}^n)$ and φ in $\mathcal{D}$, the following equality holds:

$$(f * \varphi)(x) = \int_{\mathbb{R}^n} f(y)\, \tau_x\check{\varphi}(y)\, dy, \ x \in \mathbb{R}^n,$$

since $\tau_x\check{\varphi}(y) = \check{\varphi}(y-x) = \varphi(x-y)$. So, if u belongs to $\mathcal{D}'$ and if φ is in $\mathcal{D}$, it is natural to define the convolution $u * \varphi$ by the formula

$$(u * \varphi)(x) = u(\tau_x\check{\varphi}), \ x \in \mathbb{R}^n,$$

since in this way we have $u_f * \varphi = f * \varphi$. With this definition $u * \varphi$ belongs to $C^\infty = C^\infty(\mathbb{R}^n)$. If ψ is in $\mathcal{D}$ as well, the following equalities hold:

(1) $\tau_x(u * \varphi) = (\tau_x u) * \varphi = u * (\tau_x \varphi),\ x \in \mathbb{R}^n$,
(2) $D^\alpha(u * \varphi) = (D^\alpha u) * \varphi = u * (D^\alpha \varphi),\ \alpha \in \mathbb{N}^n$,
(3) $(u * \varphi) * \psi = u * (\varphi * \psi)$.

Moreover, $\operatorname{supp}(u * \varphi) \subseteq \operatorname{supp}(u) + \operatorname{supp}(\varphi)$. For proofs and continuity properties the reader is referred to Chapter 4. In particular Theorems 4.4, 4.5, and 4.8 are of interest.

1.10. Convolution of distributions. It will be very convenient to be able to convolve two distributions. In general this cannot be done, see Schwartz [**121**] chapter 6. The convolution product $u * v$ of two distributions on $\mathbb{R}^n$ can be defined if u or v has compact support. If this is the case, the formula

$$(u * v)(\varphi) = u(\check{v} * \varphi),\ \varphi \in \mathcal{D},$$

defines the convolution of u and v. Notice that the expression $u(\check{v} * \varphi)$ makes sense if either u or v has compact support, see section 1.8. If u and v are functions in $L^1 = L^1(\mathbb{R}^n)$, then $u * v$ corresponds to the usual convolution product of functions in L^1.

If at least one of the two distributions u and v has compact support, the following assertions hold:

(1) $(u * v) * \varphi = u * (v * \varphi),\ \varphi \in \mathcal{D}$,
(2) $\operatorname{supp}(u * v) \subseteq \operatorname{supp}(u) + \operatorname{supp}(v)$,
(3) $D^\alpha(u * v) = (D^\alpha u) * v = u * D^\alpha v,\ \alpha \in \mathbb{N}^n$,
(4) $\delta * u = u$ and $D^\alpha u = (D^\alpha \delta) * u,\ \alpha \in \mathbb{N}^n$,
(5) $\delta_p * u = \tau_p u,\ p \in \mathbb{R}^n$,
(6) $u * v = v * u$,
(7) $(u * v) * w = u * (v * w)$ if w has compact support.

1.33. Remark. In general, the associativity law does not hold. For example, take $u = 1$, $v = \delta'$ and $w = H$. Then

$$u * v = 1 * \delta' = (1 * \delta)' = 1' * \delta = 0 * \delta = 0$$

and

$$v * w = \delta' * H = (\delta * H)' = H' = \delta.$$

So $(u * v) * w = 0$ and $u * (v * w) = 1 * \delta = 1$.

For proofs and continuity properties the reader is referred to Chapter 4. In particular Theorems 4.6 and 4.7, Proposition 4.10, and Theorem 4.49 are of interest.

1.11. Approximate identity. In this section $(h_j)_{j \in \mathbb{N}}$ is a sequence of distributions on $\mathbb{R}^n$ with the following properties:

(a) There exists a compact subset S of $\mathbb{R}^n$ for which $\operatorname{supp}(h_j) \subseteq S$ for all $j \in \mathbb{N}$,
(b) $\lim_{j \to \infty} h_j(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}$.

The latter property says that $\lim_{j\to\infty} h_j = \delta$ in $\mathcal{D}'$. For such a sequence the following statements hold:

(1) $\lim_{j\to\infty} h_j * \varphi = \varphi$ in $\mathcal{D}$,
(2) $\lim_{j\to\infty} h_j * u = u$ in $\mathcal{D}'$.

A sequence $(h_j)_{j\in\mathbb{N}}$ for which these properties hold is called an *approximate identity*. If each h_j is in $\mathcal{D}$, then the latter property says that distributions can be approximated by C^∞-functions (since each $h_j * u$ is in $C^\infty(\mathbb{R}^n)$).

An example of an approximate identity can be constructed as follows. Choose a function h_1 in $L^1(\mathbb{R}^n)$ with $\operatorname{supp}(h_1)$ compact and $\int_{\mathbb{R}^n} h_1(x)\,dx = 1$. Then define the functions h_j as follows:

$$h_j(x) = j^n h_1(jx) : x \in \mathbb{R}^n,\ j \in \mathbb{N}.$$

Notice that the functions h_j belong to $C^\infty(\mathbb{R}^n)$ whenever h_1 does so.

1.12. Distributions and C^∞-diffeomorphisms. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a surjective diffeomorphism. If f is a function defined on $\mathbb{R}^n$, then its transform under T is given by

$$f^T(x) = f(T^{-1}x),\ x \in \mathbb{R}^n.$$

Let J_T be the Jacobian of T, which is continuous and vanishes nowhere since T is a diffeomorphism. For f in L^1_{loc} and φ in $\mathcal{D}$, the following equality holds:

$$\int_{\mathbb{R}^n} f^T(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(y)\varphi^{T^{-1}}(y)|J_T(y)|\,dy.$$

Let u be in $\mathcal{D}'$ and suppose that T is a C^∞-diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$. In view of the latter equality, the transform u^T of the distribution u is defined by

$$u^T(\varphi) = u\big(\varphi^{T^{-1}}|J_T|\big),\ \varphi \in \mathcal{D}. \tag{1.35}$$

If $u^T = u$, then u is said to be *invariant under* T. For instance $\delta^T = \delta$ if and only if $T(0) = 0$ and $|J_T(0)| = 1$. If T is a linear map, then u^T is given by

$$u^T(\varphi) = u\big(\varphi^{T^{-1}}\big)|\det(T)|,\ \varphi \in \mathcal{D}.$$

If T is a translation over a for some $a \in \mathbb{R}^n$, *i.e.* $T(x) = x+a$, $x \in \mathbb{R}^n$, then $u^T = \tau_a u$. If T_1 and T_2 are two C^∞-diffeomorphisms from $\mathbb{R}^n$ onto $\mathbb{R}^n$, then

$$\big(u^{T_1}\big)^{T_2} = u^{T_2\circ T_1}.$$

In the following proposition we collect some elementary properties of transforms of distributions.

1.34. Proposition. *Let T be a C^∞-diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$ and let u be a distribution on $\mathbb{R}^n$. Then the following assertions hold:*

(1) *If $S = T^{-1}$ and $u^T = u$, then $u^S = u$.*
(2) *The distribution u is zero on Ω if and only if u^T is zero on $T(\Omega)$.*
(3) *If u is zero on Ω, then u is zero on*

$$\bigcup\{T^{-1}(\Omega)\colon u^T = u\}.$$

Let e_k be the k-th unit vector in $\mathbb{R}^n$. A distribution on $\mathbb{R}^n$ is said to be *independent of the k-th coordinate* if $\tau_{te_k}u = u$ for all t in $\mathbb{R}$. In the following proposition we give some alternative descriptions of this notion. For φ in $\mathcal{D}(\mathbb{R}^n)$ we define its "descent" $P_n\varphi$ in $\mathcal{D}(\mathbb{R}^{n-1})$ by

$$P_n\varphi(x_1,\dots,x_{n-1}) = \int_{-\infty}^{\infty} \varphi(x_1,\dots,x_{n-1},t)\,dt.$$

Here, of course, $(x_1,\dots,x_{n-1})$ is in $\mathbb{R}^{n-1}$.

1.35. Proposition. *Let u be in $\mathcal{D}'(\mathbb{R}^n)$. The following assertions are equivalent:*

(1) *u is independent of the n-th coordinate,* i.e. *$\tau_{te_n}u = u$, $t \in \mathbb{R}$,*
(2) *$D_n u = 0$,*
(3) *there is a distribution v in $\mathcal{D}'(\mathbb{R}^{n-1})$ such that $u(\varphi) = v(P_n\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$.*

Proof. (1) ⇒ (2) For the distributional derivative of u in $\mathcal{D}'(\mathbb{R}^n)$ with respect to the n-th coordinate we have

$$D_n u = \frac{\partial u}{\partial x_n} = \lim_{t \to 0} \frac{\tau_{-te_n} u - u}{t}.$$

Hence, from $\tau_{-te_n} u = u$ it follows that $D_n u = 0$.

(2) ⇒ (3) Choose φ_0 in $\mathcal{D}(\mathbb{R}^n)$ in such a way that $\int_{\mathbb{R}^n} \varphi_0(x)\,dx = 1$. Define v in $\mathcal{D}'(\mathbb{R}^{n-1})$ by

$$v(\chi) = u(\chi \otimes \varphi_0),\ \chi \in \mathcal{D}(\mathbb{R}^{n-1}).$$

Here $\chi \otimes \varphi_0$ is defined by

$$(\chi \otimes \varphi_0)(x_1, \ldots, x_{n-1}, x_n) = \chi(x_1, \ldots, x_{n-1})\varphi_0(x_n).$$

Let φ be in $\mathcal{D}(\mathbb{R}^n)$ and let the function $\psi\colon \mathbb{R}^n \to \mathbb{C}$ be defined by

$$\begin{aligned}\psi(x_1, \ldots, x_n) &= \int_{-\infty}^{x_n} \big(\varphi(x_1, \ldots, x_{n-1}, t) - P_n\varphi(x_1, \ldots, x_{n-1})\varphi_0(t)\big)\,dt \\ &= \int_{-\infty}^{x_n} (\varphi - P_n\varphi \otimes \varphi_0)(x_1, \ldots, x_{n-1}, t)\,dt.\end{aligned}$$

Then ψ belongs $\mathcal{D}(\mathbb{R}^n)$ as well. Moreover the equality

$$\varphi = \frac{\partial \psi}{\partial x_n} + P_n\varphi \otimes \varphi_0$$

is readily verified. Since, by (2), we have $D_n u = 0$, it follows that

$$u(\varphi) = u(P_n\varphi \otimes \varphi_0) = v(P_n\varphi).$$

(3) ⇒ (1) Let s be in $\mathbb{R}$. Then

$$(\tau_{se_n} u)(\varphi) = u(\tau_{-se_n}\varphi) = v(P_n(\tau_{-se_n}\varphi)) = v(P_n\varphi) = u(\varphi)$$

for all φ in $\mathcal{D}(\mathbb{R}^n)$. We find that $\tau_{se_n} u = u$, so u is independent of the n-th coordinate. □

1.36. Corollary. *A distribution u on $\mathbb{R}^n$ is translation invariant if and only if its gradient $\nabla u = (D_1 u, \ldots, D_n u)$ is zero, which is in turn equivalent to the existence of a constant $c \in \mathbb{R}$ such that*

$$u(\varphi) = c \int_{\mathbb{R}^n} \varphi(x)\,dx,\ \varphi \in \mathcal{D}(\mathbb{R}^n).$$

2. Tempered distributions and Fourier transforms

2.1. Rapidly decreasing functions. A very important tool in the study of differential equations is the possibility of taking Fourier transforms of certain (tempered) distributions. Pick φ in $\mathcal{D}(\mathbb{R}^n)$ and define its Fourier transform $\widehat{\varphi}$ by

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x}\varphi(x)\,dx,\ \xi \in \mathbb{R}_n.$$

1.37. Remark. Here the dual group of $\mathbb{R}^n$ is denoted by $\mathbb{R}_n$. By definition, as set the dual group of the locally compact additive group $\mathbb{R}^n$ consists of the continuous characters $\chi : \mathbb{R}^n \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Here, a function $\chi : \mathbb{R}^n \to \mathbb{T}$ is called a character, provided it is multiplicative in the sense that $\chi(x+y) = \chi(x)\chi(y)$ for all x, $y \in \mathbb{R}^n$. It is clear that the set of continuous characters, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}^n$, is a topological group. It is called the dual group. From assertion (b) in Proposition 4.53 it easily follows that this topological group can be identified with $\mathbb{R}^n$ with the usual compact open topology. The latter is a consequence of the inequalities:

$$\frac{2}{\pi} \left|(a_2 - a_1)\cdot x\right| \mathbf{1}_{[-\pi,\pi]}\left((a_2-a_1)\cdot x\right) \leqslant \left|e^{ia_2\cdot x} - e^{ia_1\cdot x}\right| \leqslant \left|(a_2-a_1)\cdot x\right|,$$

for all a_1, a_2, $x \in \mathbb{R}^n$. So henceforth $\mathbb{R}_n$ will be identified with $\mathbb{R}^n$.

It is relatively simple to prove that $\widehat{\varphi}$ is in $C^\infty(\mathbb{R}^n)$. In fact, $\widehat{\varphi}$ extends to a holomorphic function on $\mathbb{C}^n$. It is also readily verified that

$$\sup_{\xi\in\mathbb{R}^n} \left|P(\xi)\, D^\alpha \widehat{\varphi}(\xi)\right|$$

is finite for each complex polynomial P in n variables and each multi-index α in $\mathbb{N}^n$. It therefore seems plausible to study the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. A function φ belongs to $\mathcal{S}(\mathbb{R}^n)$ if φ is infinitely many times differentiable and if

$$\sup_{x\in\mathbb{R}^n} \left|P(x)\, D^\alpha \varphi(x)\right|$$

is finite for each complex polynomial P in n variables and each multi-index α in $\mathbb{N}^n$. If φ is in $\mathcal{S}$, its Fourier transform $\widehat{\varphi}$ can be defined as above. It turns out that $\widehat{\varphi}$ belongs to $\mathcal{S}(\mathbb{R}^n)$. There is a natural way to make $\mathcal{S}(\mathbb{R}^n)$ into a Féchet space. In fact, for each natural number m, the expression

$$p_m(\varphi) = \max_{|\alpha|\leqslant m} \sup_{x\in\mathbb{R}^n} \left(1+|x|^2\right)^m \left|D^\alpha\varphi(x)\right|,\ \varphi \in \mathcal{S}(\mathbb{R}^n),$$

defines a semi-norm on $\mathcal{S}(\mathbb{R}^n)$. The family of semi-norms $(p_m)_{m\in\mathbb{N}}$ can be used to describe a topology on $\mathcal{S}(\mathbb{R}^n)$. It is possible to show that the Fourier transform is an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. The inverse Fourier transformation is given by

$$\widetilde{\psi}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x}\psi(\xi)\, d\xi,\ \psi \in \mathcal{S}(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

It is clear that

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^n),$$

and that these inclusions are continuous and proper in the sense that $\mathcal{S}(\mathbb{R}^n)\backslash\mathcal{D}(\mathbb{R}^n) \neq \{0\}$ and $C^\infty(\mathbb{R}^n)\backslash\mathcal{S}(\mathbb{R}^n) \neq \{0\}$. For example, each function of the form

$$x \mapsto P(x)\exp\left(-|x|^2\right),\ x \in \mathbb{R}^n,$$

where P is a complex polynomial in n variables, belongs to $\mathcal{S}(\mathbb{R}^n)$. We also notice that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.

2.2. Tempered distributions. The space $\mathcal{S}'(\mathbb{R}^n)$ of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ is the space of so-called *tempered distributions*. So every continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$ is a tempered distribution. The restriction of a tempered distribution to $\mathcal{D}(\mathbb{R}^n)$ is a distribution in the ordinary sense. However, not every distribution in $\mathcal{D}'(\mathbb{R}^n)$ gives rise to a tempered distribution, *i.e.*

$$\mathcal{D}'(\mathbb{R}^n) \supsetneqq \mathcal{S}'(\mathbb{R}^n) \supsetneqq C^{\infty\prime}(\mathbb{R}^n) = \mathcal{E}'(\mathbb{R}^n).$$

1.38. Example. The function $\exp(x)$, $x \in \mathbb{R}$, is in $\mathcal{D}'(\mathbb{R})$ but not in $\mathcal{S}'(\mathbb{R})$. On the other hand, the function $\exp(x)\cos(\exp(x))$, $x \in \mathbb{R}$, is in $\mathcal{S}'(\mathbb{R})$. In fact,

$$\int_{\mathbb{R}} e^x \cos(e^x)\,\varphi(x)\,dx = -\int_{\mathbb{R}} \sin(e^x)\,\varphi'(x)\,dx, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

1.39. Example. Every distribution with compact support is in $\mathcal{S}'(\mathbb{R}^n)$. Functions with polynomial growth are in $\mathcal{S}'(\mathbb{R}^n)$.

1.40. Example. If $\int_{\mathbb{R}^n} |u(x)|^p\,dx$ is finite for some $p \geqslant 1$, then u is a tempered distribution.

2.3. Fourier transforms of tempered distributions. Let u be in $\mathcal{S}'(\mathbb{R}^n)$. Its Fourier transform $\widehat{u}$ is defined by

$$\widehat{u}(\varphi) = u(\widehat{\varphi}),\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The Fourier transform is an isomorphism from $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{R}^n)$. It has the following continuity property. A sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^n)$ converges pointwise in the sense that the limit

$$u(\varphi) = \lim_{k\to\infty} u_k(\varphi)$$

exists for each φ in $\mathcal{S}(\mathbb{R}^n)$, if and only if the sequence $(\widehat{u}_k)_{k\in\mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^n)$ converges pointwise in the same way. Its pointwise limit is the Fourier transform $\widehat{u}$ of u, i.e

$$\widehat{u}(\varphi) = \lim_{k\to\infty} \widehat{u}_k(\varphi),\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Next, let

$$P(D) = \sum_{|\alpha|\leqslant m} c_\alpha D^\alpha$$

be a partial differential operator with constant coefficients $c_\alpha \in \mathbb{C}$. For φ in $\mathcal{S}(\mathbb{R}^n)$ and u in $\mathcal{S}'(\mathbb{R}^n)$ the following identities hold:

(1) $(P(D)\varphi)\widehat{\ }(\xi) = P(\xi)\,\widehat{\varphi}(\xi),\ \xi \in \mathbb{R}^n$,
(2) $(P(D)u)\widehat{\ }(\varphi) = \widehat{u}(P\varphi) = (P\widehat{u})(\varphi)$,
(3) $P(-D)\,\widehat{u} = (Pu)\widehat{\ }$.

1.41. Remark. It is understood that the symbol D^α stands for

$$D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \circ \cdots \circ \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

It is also assumed that each c_α, $|\alpha| \leqslant m$, is a constant.

1.42. Remark. We will now introduce some extra notation. If ξ is in $\mathbb{R}^n$ and if α is a multi-index in $\mathbb{N}^n$, we use the abbreviation

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

If z is in $\mathbb{C}^n$ and x is in $\mathbb{R}^n$, we define $z \cdot x = \sum_{j=1}^n z_j x_j$. If z is in $\mathbb{C}^n$, the function e_z is defined by $e_z(x) = \exp(iz\cdot x)$, $x \in \mathbb{R}^n$.

In the following theorem we summarize some results on Fourier transforms of tempered distributions.

1.43. Theorem. *For Fourier transforms in (dual) Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ *the following assertions hold.*

(1) *If u is a function in $L^1(\mathbb{R}^n)$, then it belongs to $\mathcal{S}'(\mathbb{R}^n)$ and its Fourier transform $\widehat{u}$ is given by*

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\,dx,\ \xi \in \mathbb{R}^n.$$

(2) *If f is a function in $L^1(\mathbb{R}^n)$, there exists a functional u in $\mathcal{S}'(\mathbb{R}^n)$ with $\widehat{u} = f$. This tempered distribution u is given by*

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} f(\xi)\, d\xi,\ x \in \mathbb{R}^n.$$

(3) *If u is a distribution with compact support, its Fourier transform $\widehat{u}$ is given by the function*

$$\widehat{u}(\xi) = u(e_{-\xi}),\ \xi \in \mathbb{R}^n,$$

with $e_{-\xi}(x) = \exp(-i\xi \cdot x)$, $x \in \mathbb{R}^n$. Moreover, $\widehat{u}$ is the restriction of an entire function from $\mathbb{C}^n$ to $\mathbb{R}^n$.

(4) *If u is a distribution with compact support and if v is in $\mathcal{S}'(\mathbb{R}^n)$, then $u * v$ is in $\mathcal{S}'(\mathbb{R}^n)$ and*

$$(u * v)\widehat{\ } = \widehat{u} \cdot \widehat{v}.$$

Note that $\widehat{u}$ is an ordinary function because of the previous item.

(5) *(Plancherel). If u is in $L^2(\mathbb{R}^n)$, its Fourier transform $\widehat{u}$ is in $L^2(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} \left|\widehat{u}(\xi)\right|^2 d\xi = (2\pi)^n \int_{\mathbb{R}^n} |u(x)|^2\, dx.$$

(6) *If ξ is in $\mathbb{R}^n$ and if u is in $\mathcal{S}'(\mathbb{R}^n)$, then $(\tau_\xi u)\widehat{\ } = e_{-\xi}\,\widehat{u}$.*

(7) *(Theorem of Titchmarsh). If u is in $L^p(\mathbb{R}^n)$, with $1 \leqslant p \leqslant 2$, then $\widehat{u}$ is in $L^q(\mathbb{R}^n)$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and*

$$\|\widehat{u}\|_q \leqslant (2\pi)^{n/2} \|u\|_p.$$

1.44. Remark. Theorems of Paley-Wiener type characterize those entire functions on $\mathbb{C}^n$ which are the Fourier transforms of distributions with compact support.

1.45. Remark. Sometimes it is useful to consider the transformation $\Phi\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ defined by

$$\Phi(\varphi) = (2\pi)^{-n/2}\,\widehat{\varphi},\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

In some aspects Φ has better properties than the Fourier transformation itself. For example, the following assertions hold:

(1) (Theorem of Plancherel). If φ is in $\mathcal{S}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \left|\Phi(\varphi)(\xi)\right|^2 d\xi = \int_{\mathbb{R}^n} |\varphi(x)|^2\, dx.$$

(2) The eigenvalues of Φ are $\{1, i, -1, -i\}$ and the eigenfunctions are the Hermite functions.

(3) $\Phi(\varphi) = \varphi$ for $\varphi(x) = \exp\left(-\frac{1}{2}|x|^2\right),\ x \in \mathbb{R}^n$,

(4) $\Phi^2(\varphi) = \check{\varphi},\ \varphi \in \mathcal{S}(\mathbb{R}^n)$,

(5) $\Phi^4 = I$.

1.46. Remark. For more details and proofs the reader is referred to Theorem 4.11 in Chapter 4. For Titchmarsh's result see section 51 in Donoghue [**40**]. See also the article by Beckner [**10**].

2.4. Examples of Fourier transforms. In the first theorem the Fourier transforms of some simple distributions in $S'(\mathbb{R}^n)$ are given.

1.47. Theorem. *Let $P(D)$ be a partial differential operator with constant coefficients. Then*

$$\big(P(D)\,\delta\big)\widehat{\ }(\xi) = P(\xi),\ \xi \in \mathbb{R}^n.$$

In particular $\widehat{\delta} = 1$. Moreover we have

$$\widehat{P} = P(-D)\,\widehat{1} = (2\pi)^n P(-D)\,\delta.$$

In particular $\widehat{1} = (2\pi)^n\,\delta$.

In the second theorem we compute Fourier transforms of distributions which will be useful in dealing with fundamental solutions of the heat equation and the Schrödinger equation. We shall give two proofs.

1.48. Theorem. *Let a be a non-zero complex number with $\Re(a) \geqslant 0$. Define the tempered distribution u_a by*

$$u_a(x) = \big(2\sqrt{a\pi}\big)^{-n}\exp\big(-|x|^2/4a\big),\ x \in \mathbb{R}^n.$$

Its Fourier transform $\widehat{u}_a$ is given by

$$\widehat{u}_a(\xi) = \exp\big(-a|\xi|^2\big),\ \xi \in \mathbb{R}^n.$$

Proof of Theorem 1.48. If a is a positive real number, $\widehat{u}_a$ is the ordinary Fourier transform of an L^1-function. For $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned}\widehat{u}_a(\xi) &= \frac{1}{\left(2\sqrt{a\pi}\right)^n}\int_{\mathbb{R}^n} \exp\left(-i\xi\cdot x - \frac{|x|^2}{4a}\right)\,dx\\ &= \exp\left(-a|\xi|^2\right)\prod_{k=1}^n\left[\frac{1}{2\sqrt{a\pi}}\int_{-\infty}^{\infty}\exp\left(-\left(\frac{x}{2\sqrt{a}}+i\sqrt{a}\xi_k\right)^2\right)\,dx\right]\\ &= \exp\left(-a|\xi|^2\right).\end{aligned}$$

The latter equality follows from standard arguments in complex function theory. Next, consider for ξ in $\mathbb{R}^n$ fixed, the function

$$a \longmapsto \exp\left(-a|\xi|^2\right) - \frac{1}{\left(2\sqrt{a\pi}\right)^n}\int_{\mathbb{R}^n}\exp\left(-i\xi\cdot x - \frac{|x|^2}{4a}\right)\,dx$$

for a in $\mathbb{C}\backslash\{0\}$ with $\Re(a) > 0$. This function is holomorphic. Since it is zero on the positive real axis, it vanishes for each a with $\Re(a) > 0$. Taking limits yields the same equality for the case a in $\mathbb{C}\backslash\{0\}$ with $\Re(a) = 0$. $\square$

The second proof is an elaboration of exercise 6 of chapter 4 in Choquet-Bruhat [**26**].

Proof. Let a be in $\mathbb{C}\backslash\{0\}$ and suppose $\Re(a) \geqslant 0$. We are looking for a tempered distribution v for which

$$\widehat{v}(\xi) = \exp\left(-a|\xi|^2\right),\ \xi \in \mathbb{R}^n.$$

Suppose V is such a distribution. Then obviously

$$\frac{\partial\widehat{V}}{\partial\xi_k} = -2a\xi_k\widehat{V},\ k = 1,\ldots,n.$$

Since the following identities always hold (the notation is self-explanatory):

$$\frac{\partial\widehat{V}}{\partial\xi_k} = i\left(-x_kV\right)^{\widehat{}} \quad\text{and}\quad \left(\frac{\partial V}{\partial x_k}\right)^{\widehat{}} = i\xi_k\widehat{V},$$

it follows that

$$i\left(-x_kV\right)^{\widehat{}} = 2ia\left(\frac{\partial V}{\partial x_k}\right)^{\widehat{}}.$$

Since the Fourier transformation is injective, this implies

$$-x_kV = 2a\frac{\partial V}{\partial x_k},\ k = 1,\ldots,n.$$

It follows that

$$2a\frac{\partial}{\partial x_k}\left(\exp\left(|x|^2/4a\right)V\right) = \exp\left(|x|^2/4a\right)\left(x_kV + 2a\frac{\partial V}{\partial x_k}\right) = 0,$$

for $k = 1,\ldots,n$. So by corollary 1.36 it follows that V is given by

$$V(x) = C\exp\left(-|x|^2/4a\right),\ x \in \mathbb{R}^n,$$

for some constant $C \in \mathbb{R}$. The constant C is determined as follows:

$$1 = \widehat{V}(0) = C\left(\int_{-\infty}^{\infty} \exp\bigl(-|x|^2/4a\bigr)\,dx\right)^n = C\bigl(2\sqrt{a\pi}\bigr)^n.$$

This completes the second proof of Theorem 1.48. □

The example in the following theorem will be used to compute fundamental solutions of the wave equation. With $z^{-1/2}$ we always mean the principal value, *i.e.* if $z > 0$, then $z^{-1/2}$ is its positive root.

1.49. Theorem. *Suppose $n \geqslant 2$. Define for $a \in \mathbb{C}$ with $\Re(a) > 0$ and $\Im(a) > 0$ the distribution u_a by*

$$u_a(x) = \frac{1}{2\pi}\,\Gamma\Bigl(\frac{n-1}{2}\Bigr)\,\pi^{-\frac{1}{2}(n-1)}\,i^{n-1}\bigl(a^2 - |x|^2\bigr)^{-\frac{1}{2}(n-1)},\ x \in \mathbb{R}^n.$$

Its Fourier transform $\widehat{u}_a$ is given by the function

$$\widehat{u}_a(\xi) = \frac{\exp(ia|\xi|)}{|\xi|},\ \xi \in \mathbb{R}^n.$$

Proof. Write $|S_{n-2}|$ for the area of the $(n-2)$-dimensional unit sphere and take $|S_0| = 2$. Choose an element x in $\mathbb{R}^n$. For $a \in \mathbb{C}$ with $|a| > |x|$, $\Re(a) > 0$ and $\Im(a) > 0$, we compute the integral

$$F(x,a) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\,\frac{e^{ia|\xi|}}{|\xi|}\,d\xi = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} e^{i\xi_n|x|}\left(\int_{\mathbb{R}^{n-1}}\frac{e^{ia|\xi|}}{|\xi|}\,d\xi_1\ldots d\xi_{n-1}\right)d\xi_n$$

Transfering to polar coordinates in $\mathbb{R}^{n-1}$ gives

$$F(a,x) = \frac{2}{(2\pi)^n}\int_0^{\infty}\cos(\xi_n|x|)\left(\int_0^{\infty}\frac{\exp\bigl(ia\sqrt{p^2+\xi_n^2}\bigr)}{\sqrt{p^2+\xi_n^2}}\,p^{n-2}\,dp\right)d\xi_n\,|S_{n-2}|$$

Substituting $\xi_n = r\sin\theta$ and $p = r\cos\theta$ yields

$$F(x,a) = \frac{2}{(2\pi)^n}\int_0^{\infty} e^{iar}r^{n-2}\left(\int_0^{\pi/2}\cos(r|x|\sin\theta)(\cos\theta)^{n-2}\,d\theta\right)dr\,|S_{n-2}|$$

Expanding the cosine into a power series turns the expression for $F(x,a)$ into:

$$F(x,a) = \frac{2}{(2\pi)^n}\,|S_{n-2}|\,\sum_{k=0}^{\infty}\left[\frac{(-1)^k|x|^{2k}}{(2k)!}\int_0^{\infty} e^{iar}r^{2k+n-2}\,dr\cdot\int_0^{\pi/2}(\sin\theta)^{2k}(\cos\theta)^{n-2}\,d\theta\right]$$

Using some elementary properties of the Gamma- and the Beta-function results in

$$\begin{aligned}F(x,a) &= \frac{|S_{n-2}|}{(2\pi)^n}\,\frac{i^{n-1}}{a^{n-1}}\,\sum_{k=0}^{\infty}\left[\frac{(-1)^k|x|^{2k}}{(2k)!}\,\frac{1}{(-ia)^{2k}}\,\Gamma(2k+n-1)\,B\Bigl(k+\frac{1}{2},\frac{1}{2}(n-1)\Bigr)\right]\\ &= \frac{|S_{n-2}|}{(2\pi)^n}\,\frac{i^{n-1}}{a^{n-1}}\,\sum_{k=0}^{\infty}\left[\frac{|x|^{2k}}{a^{2k}}\,\frac{\Gamma(2k+n-1)}{\Gamma\bigl(k+\frac{1}{2}n\bigr)}\,\frac{\Gamma\bigl(k+\frac{1}{2}\bigr)}{(2k)!}\right]\Gamma\Bigl(\frac{n-1}{2}\Bigr).\end{aligned}$$

Apply the duplication formula of the Gamma-function

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma\Big(z+\frac{1}{2}\Big)$$

to

$$\Gamma(2k+n-1) = \Gamma\Big(2\Big(k+\frac{1}{2}(n-1)\Big)\Big)$$

and to

$$(2k)! = 2k\,\Gamma(2k).$$

Also use the identity

$$|S_{n-2}|\,\Gamma\Big(\frac{n-1}{2}\Big) = 2\pi^{\frac{1}{2}(n-1)},$$

to find

$$F(x,a) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\,\frac{e^{ia|\xi|}}{|\xi|}\,d\xi = \frac{1}{2\pi}\,\pi^{-\frac{1}{2}(n-1)}\,\frac{i^{n-1}}{a^{n-1}}\sum_{k=0}^{\infty}\frac{|x|^{2k}}{a^{2k}}\,\frac{\Gamma\big(k+\frac{1}{2}(n-1)\big)}{k!}.$$

For $|z| < 1$, the following equality holds:

$$\sum_{k=0}^{\infty} z^k \frac{\Gamma\big(k+\frac{1}{2}(n-1)\big)}{k!} = (1-z)^{-\frac{1}{2}(n-1)}\,\Gamma\Big(\frac{n-1}{2}\Big),$$

and hence

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\,\frac{e^{ia|\xi|}}{|\xi|}\,d\xi = \frac{1}{2\pi}\,\Gamma\Big(\frac{n-1}{2}\Big)\,\pi^{-\frac{1}{2}(n-1)}\,i^{n-1}\big(a^2-|x|^2\big)^{-\frac{1}{2}(n-1)}.$$

So for $a \in \mathbb{C}$ with $|a| > |x|$, $\Re(a) > 0$ and $\Im(a) > 0$, the equality

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\,\frac{e^{ia|\xi|}}{|\xi|}\,d\xi = \frac{1}{2\pi}\,\Gamma\Big(\frac{n-1}{2}\Big)\,\pi^{-\frac{1}{2}(n-1)}\,i^{n-1}\big(a^2-|x|^2\big)^{-\frac{1}{2}(n-1)}$$

is valid. Because of analyticity this equality also holds for $a \in \mathbb{C}$ with $\Re(a) > 0$, $\Im(a) > 0$ and $|a| \leqslant |x|$. This completes the proof of Theorem 1.49. □

The following result can be used to find a fundamental solution for the Laplace equation. In Theorem 4.14 the reader finds the same theorem with a considerably different proof. It could be that the proof below is not completely justified.

1.50. Theorem. *Let $0 < \lambda < n$ and put*

$$c_\lambda = 2^{-\lambda}\,\pi^{-\frac{1}{2}n}\,\frac{\Gamma\big(\frac{1}{2}(n-\lambda)\big)}{\Gamma\big(\frac{1}{2}\lambda\big)}.$$

The tempered distribution u_λ, given by

$$u_\lambda(x) = c_\lambda |x|^{-(n-\lambda)},\ x \in \mathbb{R}^n,$$

has a Fourier transform $\widehat{u}_\lambda$ given by

$$\widehat{u}_\lambda(\xi) = |\xi|^{-\lambda},\ \xi \in \mathbb{R}^n.$$

Proof. Let φ be in $\mathcal{S}(\mathbb{R}^n)$. Select ψ in $\mathcal{S}(\mathbb{R}^n)$ in such a way that $\widehat{\psi} = \varphi$. If $\widehat{u}_\lambda(\xi) = |\xi|^{-\lambda}$, $\xi \in \mathbb{R}^n$, then

$$\begin{aligned}
u_\lambda(\varphi) = \widehat{u}_\lambda(\psi) &= \int_{\mathbb{R}^n} \frac{1}{|\xi|^\lambda}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\varphi(x)\,dx\right)d\xi \\
&= \lim_{R\to\infty}\frac{1}{(2\pi)^n}\int_{|\xi|\leqslant R}\left(\int_{\mathbb{R}^n}\frac{e^{i\xi\cdot x}}{|\xi|^\lambda}\,\varphi(x)\,dx\right)d\xi \\
&= \lim_{R\to\infty}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{|\xi|\leqslant R}\frac{e^{i\xi\cdot x}}{|\xi|^\lambda}\,d\xi\right)\varphi(x)\,dx \\
&= \lim_{R\to\infty}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{|\xi|\leqslant R}\frac{e^{i\xi_n|x|}}{|\xi|^\lambda}\,d\xi\right)\varphi(x)\,dx \\
&= \lim_{R\to\infty}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{|\xi|\leqslant R|x|}\frac{e^{i\xi_n}}{|\xi|^\lambda}\,d\xi\right)\frac{\varphi(x)}{|x|^{n-\lambda}}\,dx.
\end{aligned}$$

Since it is possible to prove that

$$c_\lambda = \lim_{R\to\infty} \frac{1}{(2\pi)^n} \int_{|\xi|\leqslant R} \frac{e^{i\xi_n}}{|\xi|^\lambda}\, d\xi$$

exists, it follows that

$$u_\lambda(\varphi) = c_\lambda \int_{\mathbb{R}^n} \frac{\varphi(x)}{|x|^{n-\lambda}}\, dx.$$

Comparing $u_\lambda(\widehat{\varphi})$ and $\widehat{u}_\lambda(\varphi)$, with $\varphi(x) = \exp\left(-\frac{1}{2}|x|^2\right)$, yields

$$c_\lambda = 2^{-\lambda}\, \pi^{-\frac{1}{2}n}\, \frac{\Gamma\left(\frac{1}{2}(n-\lambda)\right)}{\Gamma\left(\frac{1}{2}\lambda\right)}.$$

This completes the proof of Theorem 1.50. □

The proof of the following theorem is left as an exercise to the reader.

1.51. Theorem. *Let the distribution u on $\mathbb{R}^2$ be defined by*

$$u(x,y) = (2\pi i)^{-1}(x+iy)^{-1},\ (x,y) \in \mathbb{R}^2\backslash\{0\}.$$

Its Fourier transform $\widehat{u}$ is given by

$$\widehat{u}(\xi,\eta) = -(\xi+i\eta)^{-1},\ (\xi,\eta) \in \mathbb{R}^2\backslash\{0\}.$$

Let, for $(x,\eta) \in \mathbb{R}\times\mathbb{R}$, the improper complex Riemann integral

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{e^{-\eta z}}{z}\, dz$$

be defined by

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{e^{-\eta z}}{z}\, dz = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{x-iR}^{x+iR} \frac{e^{-\eta z}}{z}\, dz = \lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} \frac{e^{-\eta(x+iy)}}{x+iy}\, dy.$$

In the proof of Theorem 1.51 the following equalities in complex analysis are relevant:

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{e^{-\eta z}}{z}\, dz = \begin{cases} 0, & x>0,\ \eta>0; \\ 1, & x>0,\ \eta<0; \\ -1, & x<0,\ \eta>0; \\ 0, & x<0,\ \eta<0. \end{cases}$$

2.5. Convergence factors. Sometimes it is necessary to introduce so-called *convergence factors*. An example will clarify this.

1.52. Example. Let H be the Heaviside function and let us compute $\widehat{H}$. Define for $\varepsilon > 0$ the function H_ε by

$$H_\varepsilon(x) = H(x)\, e^{-\varepsilon x},\ x \in \mathbb{R}.$$

Then

$$\widehat{H}_\varepsilon(\xi) = (\varepsilon + i\xi)^{-1},\ \xi \in \mathbb{R}.$$

Integration by parts yields

$$\widehat{H}_\varepsilon(\varphi) = i \int_{-\infty}^{\infty} \log(\varepsilon + i\xi)\, \varphi'(\xi)\, d\xi, \ \varphi \in \mathcal{D}(\mathbb{R}).$$

So

$$\widehat{H}(\varphi) = \lim_{\varepsilon \to 0} \widehat{H}_\varepsilon(\varphi) = \pi\varphi(0) + i \int_{-\infty}^{\infty} \log|\xi|\, \varphi'(\xi)\, d\xi.$$

Hence

$$\widehat{H} = \pi\delta - i\frac{d}{d\xi}(\log|\xi|).$$

2.6. Partial Fourier transformation. Let φ be a function in $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{m+n})$. The arguments of φ are denoted by (t, x), $t \in \mathbb{R}^m$, $x \in \mathbb{R}^n$. The Fourier transform of φ with respect to the "space variable" x is given by

$$\widetilde{\varphi}(t, \xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} \varphi(t, x)\, dx, \ \xi \in \mathbb{R}^n, \ t \in \mathbb{R}^m.$$

In the same manner we can take partial Fourier transforms of tempered distributions in $\mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n)$. If u is in $\mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n)$, its partial Fourier transform $\widetilde{u}$ is defined by

$$\widetilde{u}(\varphi) = u(\widetilde{\varphi}), \ \varphi \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n).$$

CHAPTER 2

Fundamental solutions

1. Introduction and examples

Let $P(D)$ be a linear partial with constant coefficients C_α. So $P(D) = \sum_{\alpha, |\alpha| \leqslant m} C_\alpha D^\alpha$.

Let f be a distribution in $\mathbb{R}^n$. We try to find to find a solution u for which $P(D)u = f$. Suppose that we have been able to solve this problem with $f = \delta$. This means that we have a distribution E at our disposal for which $P(D)E = \delta$. Such a distribution E is called a *fundamental solution* for $P(D)$. Let f be a distribution with compact support, and let E be a fundamental solution. With $u = E * f$ we have

$$P(D)u = P(D)\left(E * f\right) = \left(P(D)E\right) * f = \delta * f = f. \tag{2.1}$$

These equalities show the importance of fundamental solutions. The theorem of Malgrange and Ehrenpreis state that partial differential equations operators with constant coefficients always possess fundamental solutions.

Often we also want to know to what extent the differentiability properties of the distribution f carry over to u where $P(D)u = f$. This depends very strongly on the differential operator $P(D)$. We start with some examples.

2.1. Example. Wave equation in one space dimension. Here $P(D)u = \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2}$. Then we have

(a) A fundamental solution is given by

$$E(t,x) = \frac{1}{2} H\left(t - |x|\right) = \frac{1}{2} H\left(t - x\right) H\left(t + x\right), \quad (t,x) \in \mathbb{R}^2.$$

(b) If v and w are two locally integrable functions on $\mathbb{R}$, then the function u defined by $u(t,x) = v(t-x) + w(t+x)$, $(t,x) \in \mathbb{R}^2$, satisfies the equation $P(D)u = \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0$. Notice that no differentiability properties on the functions v and w are required. The function $(t,x) \mapsto v(t-x)$, $(t,x) \in \mathbb{R}^2$, can be considered as a forward traveling wave. The function $(t,x) \mapsto w(t+x)$, $(t,x) \in \mathbb{R}^2$, is then a backward traveling wave.

(c) Let $u_0 : \mathbb{R} \to \mathbb{C}$ and $u_1 : \mathbb{R} \to \mathbb{C}$ be a continuous and a continuous differentiable function respectively. A solution to the following Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = u_1'(x) \tag{2.2}$$

is given by

$$u(t,x) = \frac{1}{2}\left(u_0(x-t) - u_1(x-t)\right) + \frac{1}{2}\left(u_0(x+t) + u_1(x+t)\right), \quad (t,x) \in \mathbb{R}^2.$$

2.2. Example. The Laplace equation in $\mathbb{R}^n$. Here $P(D)u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$. Let f be a function in $C^\infty(\mathbb{R}^n)$. Every solution u of $P(D)u = f$ is coincides almost everywhere with a C^∞-function. If f vanishes identically, then u is a harmonic function, or, better, coincides with a harmonic function almost everywhere. Let the distribution E satisfy $P(D)E = \delta$. Upon taking Fourier transforms we infer $-|\xi|^2 P(\xi) = 1$, $\xi \in \mathbb{R}^n$. Using Theorem 1.50 we obtain that a solution E is given by

$$E(x) = -\frac{\Gamma\left(\frac{1}{2}n - 1\right)}{4\left(\sqrt{\pi}\right)^n |x|^{n-2}}, \quad x \in \mathbb{R}^n \backslash \{0\}, \quad n \geqslant 3.$$

Notice that the Fourier transform $\widehat{E}$ of E, and therefore E itself, is not uniquely determined. In case $n = 2$ a solution E is given by

$$E(x) = \frac{1}{2\pi} \log|x|, \quad x \in \mathbb{R}^2 \backslash \{0\}.$$

If $n = 1$, we have $E(x) = xH(x)$, $x \in \mathbb{R}$. Outside of the origin all these functions are harmonic.

2.3. Remark. The Laplace operator is the prototype of a so-called *elliptic* operator. Using Sobolev theory, refinements of Example 2.2 can be given: see Subsection 5.1.

2.4. Example. The heat equation. Here we have

$$P(D)u = \frac{\partial u}{\partial t} - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

If f belongs to $C^\infty(\mathbb{R}^n)$, then every solution u of $P(D)u = f$ is a C^∞-function. A typical fundamental solution is given by

$$E(t,x) = \frac{1}{2\left(\sqrt{\pi t}\right)^n} H(t) \exp\left(-\frac{|x|^2}{4t}\right), \quad (t,x) \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^n.$$

2.5. Remark. The heat operator is the prototype of a so-called *hypo-elliptic* operator: see subsection 1.1 as well.

2.6. Example. The Schrödinger equation. A fundamental solution E to

$$\frac{\partial E}{i\partial t} - \sum_{j=1}^{n} \frac{\partial^2 E}{\partial x_j^2} = \delta$$

is given by

$$E(t,x) = \frac{1}{2\left(\sqrt{\pi t}\right)^n} H(t) \exp\left(-i(n-2)\frac{1}{4}\pi\right) \exp\left(-\frac{|x|^2}{4it}\right), \quad (t,x) \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^n.$$

Not everywhere outside of the origin $(0,0)$ the function E is infinitely many times differentiable. In fact E is not differentiable in points of the form $(0,x)$, $x \in \mathbb{R}^n$.

2.7. Remark. Although the heat equation and the Schrödinger equation look quite similar, their solutions may behave quite differently. The following example unifies their fundamental solutions.

2.8. Example. Let $a \in \mathbb{C}\backslash\{0\}$ be such that $\Re(a) \geqslant 0$. A fundamental solution E_a to the equation

$$\frac{\partial E_a}{a\partial t} - \sum_{j=1}^{n} \frac{\partial^2 E_a}{\partial x_j^2} = \delta \tag{2.3}$$

is given by

$$E_a(t,x) = \frac{a}{2\left(\sqrt{\pi a t}\right)^n} H(t) \exp\left(-\frac{|x|^2}{4at}\right), \quad (t,x) \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^n. \tag{2.4}$$

In order to prove that the function in (2.4) is a fundamental solution, first it is verified for $\Re(a) > 0$. If $\Re(a) = 0$ the purely imaginary number a is replaced with $a + \varepsilon$, $\varepsilon > 0$, in the equality in (2.3). After that we let $\varepsilon \downarrow 0$ in (2.3). Proposition 4.16 contains a precise formulation. Its proof makes these remarks rigorous.

2.9. EXAMPLE. The wave equation in three space dimension. Here $P(D)u = \dfrac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} \dfrac{\partial^2 u}{\partial x_j^2}$. There exists a fundamental solution whose support coincides with the light cone $\{(t,x) \in \mathbb{R} \times \mathbb{R}^3 : t = |x|\}$ and which is invariant under those Lorentz transformations which leave the positive light cone invariant. For more details see Propositions 3.5, 3.7, and 3.8.

2.10. REMARK. The wave operator (equation) is a typical example of a so-called *hyperbolic* partial differential operator (equation).

1.1. Hypo-elliptic operators. Let $P(D)$ be a linear partial differential operator in an open subset Ω of $\mathbb{R}^n$. The operator $P(D)$ is called *hypo-elliptic* if for every open subset U of Ω and each function $f \in C^\infty(U)$ every distributional solution u of $P(D)u = f$ is a C^∞-function in U. For hypo-elliptic operators with constant coefficients Schwartz proved the following result: see [**136**] Chapter 1, Theorem 2.1.

2.11. THEOREM. *A partial differential operator is hypo-elliptic if and only if there exists a fundamental solution which is infinitely many times differential in* $\mathbb{R}^n \backslash \{0\}$.

PROOF OF THEOREM 2.11. Denote the differential operator in Theorem 2.11 by $P(D)$.

Sufficiency. Let f be a C^∞-function in an open subset Ω of $\mathbb{R}^n$ and pick $x_0 \in \Omega$. Let u be any solution to $P(D)u = f$ in Ω. We shall show that u is a C^∞-function in a neighborhood of x_0. Therefore we choose a function $g \in \mathcal{D}(\mathbb{R}^n)$ with the following properties:

(a) $g = 1$ in a neighborhood U of x_0. We assume that the closure $\overline{U}$ of U is compact and contained in Ω.
(b) $\operatorname{supp}(g) \subset \Omega$.

Define the distribution v by $v = P(D)(gu) - gf$. Then $x_0 \notin \operatorname{supp}(v)$. Since $\operatorname{supp}(v)$ is compact there exists a positive number ε such that

$$x_0 \notin \operatorname{supp}(v) + \{x \in \mathbb{R}^n : |x| \leqslant \varepsilon\}.$$

Next choose a function $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$, with support contained in $\{x \in \mathbb{R}^n : |x| \leqslant \varepsilon\}$, which is equal to 1 on $\left\{x \in \mathbb{R}^n : |x| \leqslant \frac{1}{2}\varepsilon\right\}$. In addition, choose a fundamental solution E for $P(D)$ which is C^∞ in $\mathbb{R}^n \backslash \{0\}$. Then

$$gu = E * (gf) + (\varphi_\varepsilon E) * v + ((1 - \varphi_\varepsilon) E) * v. \tag{2.5}$$

We consider each term in (2.5) separately. Being a convolution of a distribution and a function in $\mathcal{D}(\mathbb{R}^n)$, the distribution $E * (gf)$ is a C^∞-function. The distribution $(\varphi_\varepsilon E) * v$ is zero in a neighborhood of x_0. Since $((1 - \varphi_\varepsilon) E) * v$ is a convolution product of a C^∞-function and a distribution with compact support, it is a C^∞-function. Since $u = gu$ in a neighborhood of x_0, and x_0 is an arbitrary point in Ω, we infer that u is a C^∞-function in Ω.

Necessity. Let E be any fundamental solution. Then $P(D)E = \delta$. So outside of the origin $P(D)E = 0$. Since, by hypothesis, $P(D)$ is hypo-elliptic it follows that E is a C^∞-function on $\mathbb{R}^n\backslash\{0\}$. This completes the proof of Theorem 2.11. $\square$

1.2. Ordinary differential equations with constant coefficients. Let H be the Heaviside function, *i.e.* $H(x) = 1$, for $x \geqslant 0$, $H(x) = 0$, for $x < 0$. Then its distributional derivative $\dfrac{dH}{dx}$ equals δ, and so H is a fundamental solution for the operator $\dfrac{d}{dx}$. It is the only fundamental solution E of this operator with $\operatorname{supp}(E) \subset [0, \infty)$. Next let a be complex number. We are looking for fundamental solutions of the operator $\dfrac{d}{dx} - a$. Let E be a distribution in $\mathbb{R}$. Since $e^{ax}\dfrac{d}{dx}\left(e^{-ax}E\right) = \left(\dfrac{d}{dx} - a\right)E$, it is clear that the following statements are equivalent:

$$\text{(i)} \quad \left(\frac{d}{dx} - a\right)E = \delta, \quad \text{(ii)} \quad \frac{d}{dx}\left(e^{-ax}E\right) = \delta.$$

So the distribution E, defined by $E(x) = e^{ax}H(x)$, is the only fundamental solution of the operator $\dfrac{d}{dx} - a$ with support in the half-axis $[0, \infty)$.

This phenomenon can be generalized to systems as follows. Let A be an $m \times m$ matrix with constant entries. We try to find a matrix U with entries $u_{jk} \in \mathcal{D}'(\mathbb{R})$, $1 \leqslant j,\, k \leqslant m$, with the property that

$$e^{xA}\frac{d}{dx}\left(e^{-xA}U\right) = \left(I\frac{d}{dx} - A\right)U = \delta I. \tag{2.6}$$

Here I denotes the $m \times m$ identity matrix. It is a matter of routine to verify that the matrix U defined by $U(x) = H(x)e^{xA}$ satisfies the equation in (2.6). Next let $\overrightarrow{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ be a column vector in $(\mathcal{D}')^m$, and put $\overrightarrow{u} = U * \overrightarrow{f}$. Then $\overrightarrow{u}$ solves the equation

$$\left(I\frac{d}{dx} - A\right)\overrightarrow{u} = \overrightarrow{f},$$

provided that the matrix U satisfies $\left(I\dfrac{d}{dx} - A\right)U = \delta I$.

2.12. Remark. A square matrix U with entries $u_{jk} \in \mathcal{D}'$, $1 \leqslant j, k \leqslant m$, can be considered as a linear map from $\mathcal{D}^m$ to $\mathbb{C}^m$. In fact the formula

$$U\begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_m\end{pmatrix} = \begin{pmatrix}\sum_{k=1}^m u_{1k}\left(\varphi_k\right)\\ \vdots \\ \sum_{k=1}^m u_{mk}\left(\varphi_k\right)\end{pmatrix}, \quad \text{with} \quad \begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_m\end{pmatrix} \in \mathcal{D}^m, \tag{2.7}$$

defines this map.

If $\overrightarrow{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_k\end{pmatrix}$ belongs to $(\mathcal{D}')^m$, then the column vector $\overrightarrow{u} = U * \overrightarrow{f}$ is defined as follows. Its kth component u_k is given by

$$u_k(\varphi) = \left\langle U\left(\check{\overrightarrow{f}} * \varphi\right), e_k\right\rangle.$$

Here e_k is the kth unit vector in $\mathbb{C}^m$, and

$$\left[\check{\overrightarrow{f}} * \varphi\right](x) = \begin{pmatrix} f_1\left(\tau_{-x}\varphi\right) \\ \vdots \\ f_m\left(\tau_{-x}\varphi\right)\end{pmatrix}.$$

The reader should compare this with the definition of $u * \varphi(x) = u\left(\tau_x\check{\varphi}\right)$: see the identity in (b) in the beginning of Subsection 1.9.

To conclude this subsection we consider a differential operator of the form

$$Lu = \sum_{k=0}^{m} c_k \left(\frac{d}{dx}\right)^k u, \quad c_0, \dots, c_m \in \mathbb{C},\ c_m \neq 0. \tag{2.8}$$

Associate to the operator L the matrix Λ given by

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\dfrac{c_0}{c_m} & -\dfrac{c_1}{c_m} & -\dfrac{c_2}{c_m} & \cdots & -\dfrac{c_{m-2}}{c_m} & -\dfrac{c_{m-1}}{c_m} \end{pmatrix}.$$

Let f be a distribution in $\mathbb{R}$ with compact support, and let $\overrightarrow{f}$ be the column vector given by $\overrightarrow{f} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f c_m^{-1} \end{pmatrix}$. In addition, let e_m be the mth unit vector. Then the following result can be proven.

2.13. Proposition. *The following assertions are equivalent:*

(a) $Lu = f$,

(b) *there exists a column vector* $\overrightarrow{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in (\mathcal{D}')^m$, *with* $u = u_1$, *which satisfies* $\left(I\dfrac{d}{dx} - \Lambda\right)\overrightarrow{u} = \overrightarrow{f}$.

A solution $\overrightarrow{u}$ *to (b) is given by* $\overrightarrow{u} = U * \overrightarrow{f}$, *where* $U(x) = H(x)e^{x\Lambda}$. *If* $\overrightarrow{v}$ *denotes the vector* $\overrightarrow{v} = e^{x\Lambda}\dfrac{e_m}{c_m}$, *then, apparently,* $\overrightarrow{u} = (H\overrightarrow{v}) * f$. *It is clear that* $\overrightarrow{v}(0) = \dfrac{e_m}{c_m}$, *and that* $\left(I\dfrac{d}{dx} - \Lambda\right)\overrightarrow{v} = 0$. *A computational argument shows that the first component* v *of* $\overrightarrow{v}$ *satisfies* $Lv = 0$ *and that*

$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v^{(m-1)}(0) = \frac{1}{c_m}. \tag{2.9}$$

Summarizing, the only fundamental solution u *of* $Lu = \delta$ *with support in* $[0, \infty)$ *is of the form* $u(x) = H(x)v(x)$ *where* v *is a* C^∞*-function for which* $Lv = 0$ *and which satisfies the conditions in (2.9).*

1.3. Fundamental solutions of the Cauchy-Riemann operator. We try to find a tempered distribution E in $\mathbb{R}^2$ for which $\dfrac{\partial E}{\partial x} + i\dfrac{\partial E}{\partial y} = \delta$. Let $\widetilde{E}$ be the

Fourier transform of E relative to the second variable. Then $\widetilde{E}$ should satisfy:

$$\frac{\partial \widetilde{E}}{\partial x}(x,\eta) - \eta \widetilde{E}(x,\eta) = \delta(x)1(\eta). \tag{2.10}$$

Here the tensor product $\delta(x)1(\eta) = \delta \otimes \eta(x,\eta)$ stands for the distribution given by

$$\delta \otimes 1(\varphi) = \int_{-\infty}^{\infty} \varphi(0,\eta)\, d\eta, \quad \varphi \in \mathcal{D}\left(\mathbb{R}^2\right). \tag{2.11}$$

Let C be any function in $L^1_{\text{loc}}(\mathbb{R})$. It is clear that $\widetilde{E}$ defined by

$$\widetilde{E}(x,\eta) = (H(x) + C(\eta))\, e^{\eta x}$$

satisfies (2.10). An appropriate choice of C will turn $\widetilde{E}$ into a tempered distribution: $C(\eta) = -1$, for $\eta \geqslant 0$, $C(\eta) = 0$, $\eta < 0$, and so $C(\eta) = -H(\eta)$. For $\widetilde{E}$ we get

$$\widetilde{E}(x,\eta) = (H(x) - H(\eta))\, e^{\eta x} = \begin{cases} (H(x) - 1)\, e^{\eta x} = -H(-x)e^{\eta x}, & \eta \geqslant 0, \\ H(x)e^{\eta x}, & \eta < 0. \end{cases} \tag{2.12}$$

Since, for $x \in \mathbb{R}$ fixed, the function $\eta \mapsto \widetilde{E}(x,\eta)$, $\eta \in \mathbb{R}$, belongs to $L^1(\mathbb{R})$, the distribution E is obtained by

$$E(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{E}(x,\eta)\, e^{i\eta y}\, d\eta = \frac{1}{2\pi}\frac{1}{x+iy}. \quad (x,y) \in \mathbb{R}^2\backslash\{(0,0)\}. \tag{2.13}$$

Notice that E as given by (2.13) is locally integrable in $\mathbb{R}^2$. Define the operators $\frac{\partial}{\partial \overline{z}}$ and $\frac{\partial}{\partial z}$ respectively by

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right).$$

Define the function $E_0(x,y)$, $(x,y) \in \mathbb{R}^2\backslash\{(0,0)\}$, by

$$E_0(x,y) = \frac{1}{\pi}\frac{1}{x+iy}.$$

In Example 4.34 in Subsection 5.1 it is shown in a straightforward manner that E_0 satisfies $\frac{\partial E_0}{\partial \overline{z}} = \delta$: see the equalities in (5.87).

2.14. Theorem. *Every fundamental solution of the Cauchy-Riemann operator* $\frac{\partial}{\partial \overline{z}}$ *is of the form* $\frac{1}{\pi z} + f(z)$, *where* $z = x + iy$, *and where* f *is holomorphic on* $\mathbb{C}$.

Proof. It remains to be shown that a distribution u for which $\frac{\partial u}{\partial \overline{z}} = 0$ is in fact a holomorphic function. So let $u \in \mathcal{D}'$ be a distribution for which $\frac{\partial u}{\partial \overline{z}} = 0$. Then we pick a function $\psi \in \mathcal{D}$ which is identically 1 in a neighborhood of the origin. Consider the equality

$$u = \delta * u = \left(\delta - \frac{\partial\left(\psi E_0\right)}{\partial \bar{z}}\right) * u + \frac{\partial\left(\psi E_0\right)}{\partial \bar{z}} * u, \tag{2.14}$$

where $E_0(x, y) = \dfrac{1}{\pi(x+iy)}$. Since

$$\frac{\partial\left(\psi E_0\right)}{\partial \bar{z}} * u = \frac{\partial}{\partial \bar{z}}\left(\left(\psi E_0\right) * u\right) = \left(\psi E_0\right) * \frac{\partial}{\partial \bar{z}} u = \left(\psi E_0\right) * 0 = 0. \tag{2.15}$$

From (2.14) and (2.15) it follows that

$$u = \left(\delta - \frac{\partial\left(\psi E_0\right)}{\partial \bar{z}}\right) * u. \tag{2.16}$$

Since, in addition, the distribution $\delta - \dfrac{\partial\left(\psi E_0\right)}{\partial \bar{z}}$ belongs to $\mathcal{D}$, the distribution u is in fact a C^∞-function. Hence, since $\dfrac{\partial u}{\partial \bar{z}} = 0$, the function u satisfies the Cauchy-Riemann conditions and so u is holomorphic on $\mathbb{C}$. This completes the proof of Theorem 2.14. □

2.15. Remark. The Cauchy-Riemann operator is an elliptic operator. It is not a "typical" elliptic operator, because it is of first order, and most elliptic operators are of even order.

1.4. Fundamental solutions of the Laplace equation in two dimensions. We formulate the result of this subsection in the form of a theorem.

2.16. THEOREM. *Every fundamental solution E of the Laplace operator*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is of the form

$$E(x,y) = \frac{1}{2\pi} \log|x+iy| + h(x,y), \quad (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\},$$

where h is a harmonic function on $\mathbb{R}^2$.

PROOF. We employ the notation of the previous Subsection 1.3. In particular E_0 is the distribution given by

$$E_0(x,y) = \frac{1}{\pi(x+iy)}, \quad (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}.$$

Let φ be any function in $\mathcal{D}(\mathbb{R}^2)$ and identify $\mathbb{C}$ and $\mathbb{R}^2$. If (x,y) is in $\mathbb{R}^2$, we have

$$\varphi(x,y) = \delta * \varphi(x,y) = \left(\frac{\partial E_0}{\partial \bar{z}} * \varphi\right)(x,y) = \left(E_0 * \frac{\partial \varphi}{\partial \bar{z}}\right)(x,y)$$

$(\zeta = \xi + i\eta)$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_0(x-\xi, y-\eta) \frac{\partial \varphi}{\partial \bar{\zeta}}(\xi,\eta)\, d\xi\, d\eta$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\xi + i\eta - x - iy} \frac{\partial \varphi}{\partial \bar{\zeta}}(\xi,\eta)\, d\xi\, d\eta$$

$(2\frac{\partial}{\partial z} \log|z| = \frac{1}{z})$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial \zeta} \log|\zeta - z| \frac{\partial \varphi}{\partial \bar{\zeta}}(\xi,\eta)\, d\xi\, d\eta$$

$$= \frac{2}{\pi} \left(\frac{\partial \log|z|}{\partial z} * \frac{\partial \varphi}{\partial \bar{z}}\right)(x,y) = \frac{2}{\pi}\left(\log|z| * \frac{\partial^2 \varphi}{\partial \bar{z} \partial z}\right)(x,y)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log|\zeta - z|\, \Delta\varphi(\xi,\eta)\ d\xi\, d\eta, \tag{2.17}$$

where $z = x + iy$. With $U(x,y) = (2\pi)^{-1} \log|x+iy|$, $(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}$, we see by (2.17)

$$\delta * \varphi = U * \Delta\varphi = \Delta U * \varphi$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$. It follows that $\delta = \Delta U$. It remains to be shown that each distribution u for which $\Delta u = 0$ is in fact a harmonic C^∞-function. This is Weyl's lemma, and can be proved in exactly the same manner as we proved the corresponding statement for the Cauchy-Riemann operator in Theorem 2.14. Altogether this completes the proof of Theorem 2.16. □

1.5. Fundamental solutions of the heat equation. We want to find a (tempered) distribution $E \in \mathcal{D}'\left(\mathbb{R}^{n+1}\right)$ for which

$$\frac{\partial E}{\partial t} - \Delta_x E = \delta. \tag{2.18}$$

Here Δ_x denotes the Laplace operator in the space variables $x = (x_1, \dots, x_n)$. Let E be a tempered distribution satisfying equation (2.18). Its Fourier transform with respect to the space variables then verifies

$$\frac{\partial \widetilde{E}}{\partial t}(t,\xi) + |\xi|^2 \, \widetilde{E}(t,\xi) = \delta(t)1(\xi) = (\delta \otimes 1)(t,\xi). \tag{2.19}$$

For the notion $\delta \otimes 1$ see, *e.g.* (2.11). A typical solution of equation (2.19) is given by

$$\widetilde{E}(t,x) = H(t)e^{-t|\xi|^2}, \quad (t,\xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{2.20}$$

Theorem 1.48 in Subsection 2.4 shows that the distribution E defined by

$$E(t,x) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} H(t) \exp\left(-\frac{|x|^2}{4t}\right), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \ t \neq 0, \tag{2.21}$$

is a candidate to satisfy the equation in (2.18). In the following theorem we will provide a direct proof of this fact.

2.17. Theorem. *Let the distribution E be given by the formula in (2.21). Then E is a fundamental solution of the heat equation,* i.e. *it satisfies*

$$\frac{\partial E}{\partial t} - \Delta_x E = \delta. \tag{2.22}$$

A distribution E is a fundamental solution to the equation in (2.22) if and only if E can be written as

$$E(t,x) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} H(t) \exp\left(-\frac{|x|^2}{4t}\right) + f(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.23}$$

where f is a C^∞-function which satisfies

$$\frac{\partial f}{\partial t}(t,x) - \Delta_x f(t,x) = 0. \tag{2.24}$$

Proof. Let $\varphi \in \mathcal{D}\left(\mathbb{R} \times \mathbb{R}^n\right)$. Then we have

$$\left(\frac{\partial E}{\partial t} - \Delta_x E\right)(\varphi) = -E\left(\frac{\partial \varphi}{\partial t} + \Delta_x \varphi\right)$$
$$= -\int_0^\infty \left(\int_{\mathbb{R}^n} E(t,x)\frac{\partial \varphi}{\partial t}(t,x)\,dx + \int_{\mathbb{R}^n} E(t,x)\Delta_x\varphi(t,x)\,dx\right) dt$$

(integration by parts in the second term relative to the space variable)

$$= -\int_0^\infty \left(\int_{\mathbb{R}^n} E(t,x)\frac{\partial \varphi}{\partial t}(t,x)\,dx + \int_{\mathbb{R}^n} \Delta_x E(t,x)\varphi(t,x)\,dx\right) dt$$

$$
\begin{aligned}
&\left(\text{employ } \frac{\partial E}{\partial t}(t,x) = \Delta_x E(t,x)\right)\\
&= -\int_0^\infty \left(\int_{\mathbb{R}^n} E(t,x)\frac{\partial \varphi}{\partial t}(t,x)\,dx + \int_{\mathbb{R}^n} \frac{\partial E}{\partial t}(t,x)\varphi(t,x)\,dx \right)\,dt\\
&= -\int_0^\infty \int_{\mathbb{R}^n} \frac{\partial\,(E\varphi)}{\partial t}(t,x)\,dx\,dt = -\int_0^\infty \frac{\partial}{\partial t}\int_{\mathbb{R}^n} E(t,x)\varphi(t,x)\,dx\,dt\\
&= -\lim_{\varepsilon\downarrow 0}\frac{1}{\left(\sqrt{\pi}\right)^n}\int_\varepsilon^\infty \frac{\partial}{\partial t}\int_{\mathbb{R}^n} e^{-|x|^2}\varphi\left(t, x\sqrt{t}\right)\,dx\,dt\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\left(\sqrt{\pi}\right)^n}\int_{\mathbb{R}^n} e^{-|x|^2}\varphi\left(\varepsilon, x\sqrt{\varepsilon}\right)\,dx = \varphi(0,0) = \delta(\varphi). \qquad (2.25)
\end{aligned}
$$

In the penultimate step we applied Lebesgue's dominated convergence theorem together with the equality $\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \left(\sqrt{\pi}\right)^n$. Next, using Theorem 2.11, we infer that a distribution f which satisfies (2.24) coincides almost everywhere with a C^∞-function, and hence every fundamental solution of the heat equation is of the form (2.23). This completes the proof of Theorem 2.17. $\square$

1.6. Fundamental solutions of the Laplace operator in several space dimensions. We summarize the results of this subsection as follows. Assertion (i) is treated in Proposition 4.15 with $\frac{1}{2}\Delta$ instead of Δ.

2.18. Theorem. *The explicit form of the fundamental solutions for the Laplace operator depends on the space dimension.*

(i) *Let the distribution E in $\mathbb{R}^n$, $n \geqslant 3$, be given by*

$$E(x) = -\frac{1}{4}\frac{1}{\left(\sqrt{\pi}\right)^n}\frac{\Gamma\left(\frac{1}{2}n-1\right)}{|x|^{n-2}}, \quad x \in \mathbb{R}^n\backslash\{0\}.$$

Then $\Delta E = \delta$. Every other fundamental solution of the Laplace operator is obtained by adding a harmonic function to E.

(ii) *If $n = 2$, the distribution E defined by*

$$E(x,y) = \frac{1}{2\pi}\log|x+iy|, \quad (x,y) \in \mathbb{R}^2\backslash\{(0,0)\},$$

satisfies $\Delta E = \delta$.

(iii) *If $n = 1$, the function $E(x) = \frac{1}{2}|x|$ satisfies $\dfrac{\partial^2 E}{\partial x^2} = \delta$.*

Proof. (i) One way of proving this assertion is appealing to Theorem 1.50 with $n \geqslant 3$ and $\lambda = 2$ in Subsection 2.4. Another way of establishing assertion (i) reads like in the proof of Proposition 4.15. More precisely, let $E(x)$ be the function in (i) of Theorem 2.18, and let $E(t,x)$ be the function in (2.20). Then $E(x) = -\int_0^\infty E(t,x)\,dt$, $x \in \mathbb{R}^n\backslash\{0\}$. Like in the proof of Proposition 4.15, for $\varphi \in \mathcal{D}\left(\mathbb{R}^n\right)$, we have

$$\begin{aligned}
\Delta E(\varphi) &= E\left(\Delta\varphi\right) = \int_{\mathbb{R}^n} E(x)\Delta\varphi(x)\,dx = -\int_{\mathbb{R}^n}\int_0^\infty E(t,x)\,dt\Delta\varphi(x)\,dx \\
&= -\int_0^\infty \left(\int_{\mathbb{R}^n} E(t,x)\Delta\varphi(x)\,dx\right)dt = -\int_0^\infty \left(\int_{\mathbb{R}^n} \Delta_x E(t,x)\varphi(x)\,dx\right)dt \\
&= -\int_0^\infty \frac{d}{dt}\left(\int_{\mathbb{R}^n} E(t,x)\Delta\varphi(x)\,dx\right)dt \\
&= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} E(\varepsilon,x)\varphi(x)\,dx = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} E(1,x)\varphi\left(\sqrt{\varepsilon}x\right)\,dx \\
&= \int_{\mathbb{R}^n} E(1,x)\varphi\left(0\right)\,dx = \delta(\varphi). \qquad (2.26)
\end{aligned}$$

The argumentation in (2.26) and in (4.91) in the proof of Proposition 4.15 are the same.

(ii) We disposed of the two-dimensional case in Subsection 1.4 Theorem 2.16.

(iii) This is an easy exercise.

This completes the proof of Theorem 2.18. □

The arguments in (2.26), and also in (4.91) follow Exercises 6.4 and 6.5 of Chapter 1 in Trèves [**136**]. The technique is related to the theory of potential and resolvent operators in semigroup theory: see *e.g.* [**117**].

1.7. The free Schrödinger equation. We state the result. The proof below could be erroneous; a rigorous proof is found in Proposition 4.16 with $\frac{1}{2}\Delta$ instead of Δ.

2.19. THEOREM. *A fundamental solution E of the (free) Schrödinger equation*

$$\frac{\partial E}{i\partial t} - \Delta_x E = \delta$$

is given by

$$E(t,x) = i\frac{1}{\left(2\sqrt{\pi t}\right)^n}\exp\left(-\frac{in\pi}{4}\right)\exp\left(-\frac{|x|^2}{4it}\right), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,\ t \neq 0. \tag{2.27}$$

In the present proof properties of Fresnel integrals are being employed. For example we have

$$\frac{1}{\sqrt{\pi i}}\int_{-\infty}^{\infty} e^{ix^2}\,dx = \frac{1}{\sqrt{\pi i}}\lim_{R\to\infty}\int_{-R}^{R} e^{ix^2}\,dx = 1. \tag{2.28}$$

PROOF. Assume that E is a tempered distribution which satisfies the equation in Theorem 2.19. Its Fourier transform with respect to the space variables $\widetilde{E}$ then satisfies:

$$\frac{\partial \widetilde{E}}{i\partial t} + |\xi|^2\,\widetilde{E} = \delta(t)1\left(\xi\right). \tag{2.29}$$

So for $\widetilde{E}$ we may choose

$$\widetilde{E}\left(t,\xi\right) = iH(t)\exp\left(-it\,|\xi|^2\right), \quad (t,\xi) \in \mathbb{R}\times\mathbb{R}^n. \tag{2.30}$$

The corresponding distribution E is given as in (2.27): see Theorem 1.48 with $a = i$. Let φ belong to $\mathcal{D}\left(\mathbb{R}\times\mathbb{R}^n\right)$. An attempt to prove Theorem 2.19 directly reads as follows. Let the distribution $E(t,x)$ be as in (2.27). Then we have

$$\left(\frac{\partial E}{i\partial t} - \Delta_x E\right)(\varphi) = -\int_0^\infty \left(\int_{\mathbb{R}^n} E(t,x)\frac{\partial\varphi}{i\partial t}(t,x)\,dx + \int_{\mathbb{R}^n} E(t,x)\Delta_x\varphi(t,x)\,dx\right)dt$$

$$= -\int_0^\infty \left(\int_{\mathbb{R}^n} E(t,x)\frac{\partial\varphi}{i\partial t}(t,x)\,dx + \int_{\mathbb{R}^n} \Delta_x E(t,x)\varphi(t,x)\,dx\right)dt$$

$$(\frac{\partial E}{i\partial t}(t,x) = \Delta_x E(t,x))$$

$$= -\int_0^\infty \left(\int_{\mathbb{R}^n} E(t,x)\frac{\partial\varphi}{i\partial t}(t,x)\,dx + \int_{\mathbb{R}^n} \frac{\partial E}{i\partial t}(t,x)\varphi(t,x)\,dx\right)dt$$

$$= -\frac{1}{i}\int_0^\infty \left(\int_{\mathbb{R}^n} \frac{\partial E(t,x)\varphi(t,x)}{\partial t}\,dx\right)dt = -\frac{1}{i}\int_0^\infty \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} E(t,x)\varphi(t,x)\,dx\right)dt$$

$$
\begin{aligned}
&= \lim_{\varepsilon \downarrow 0} \frac{1}{i} \int_{\mathbb{R}^n} i \frac{1}{\left(2\sqrt{\pi\varepsilon}\right)^n} \exp\left(-\frac{in\pi}{4}\right) \exp\left(-\frac{|x|^2}{4i\varepsilon}\right) \varphi(\varepsilon, x)\, dx \\
&= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \frac{1}{\left(2\sqrt{\pi i\varepsilon}\right)^n} \exp\left(-\frac{|x|^2}{4i\varepsilon}\right) \varphi(\varepsilon, x)\, dx \\
&= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \frac{1}{\left(\sqrt{\pi i}\right)^n} \exp\left(i\,|x|^2\right) \varphi\left(\varepsilon, 2\sqrt{\varepsilon}x\right)\, dx
\end{aligned}
$$

(formally interchange limit and integral)

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \frac{1}{\left(\sqrt{\pi i}\right)^n} \exp\left(i\,|x|^2\right) \lim_{\varepsilon \downarrow 0} \varphi\left(\varepsilon, 2\sqrt{\varepsilon}x\right)\, dx \\
&= \int_{\mathbb{R}^n} \frac{1}{\left(\sqrt{\pi i}\right)^n} \exp\left(i\,|x|^2\right)\, dx \varphi(0,0) = \varphi(0,0) = \delta(\varphi) \qquad (2.31)
\end{aligned}
$$

where in the penultimate equality we invoked the equality in (2.28). Except for the problem of interchanging limits and integrals, this proves Theorem 2.19. □

2.20. Remark. Using Cauchy's theorem from complex analysis, it can be shown that the interchange of limit and integral in (2.31) can be justified if $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ is of the form $\varphi(t,x) = \psi(t)e^{-\alpha|x|^2}p(x)$, where $\alpha > 0$, and $p(x)$ is a complex polynomial. In such case one substitutes $x = \sqrt{i}y$ and applies Cauchy's theorem.

CHAPTER 3

Fundamental solutions of the wave operator

In this chapter we will discuss fundamental solutions of the wave operator. We will see that it possesses a fundamental solution which is invariant under Lorentz transformations, and which is odd space dimensions has support in the positive light cone.

1. Fundamental solutions of the wave operator in one space dimension

The basic result in this section is the following one. Compare this result to Remark 4.24 following Proposition 4.23 in Chapter 4.

3.1. PROPOSITION. *The unique solution E_+ of the distributional differential equation*

$$\frac{\partial^2 E_+}{\partial t^2} - \frac{\partial^2 E_+}{\partial t^2} = \delta \tag{3.1}$$

with support in $\{(t,x) \in \mathbb{R}\times\mathbb{R} : t \geqslant 0\}$ is given by the formula

$$E_+(t,x) = \frac{1}{2}H\left(t-|x|\right) = \frac{1}{2}H(t)H(t-x)H(t+x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{3.2}$$

PROOF. It is easy to see that the function $E_+(t,x)$ given in (3.2) is a fundamental solution of the wave operator, *i.e.* it satisfies the equation in (3.1). The fact that it is the only one with support in the half-space $\{(t,x)\in\mathbb{R}\times\mathbb{R} : t\geqslant 0\}$ follows from Lemma 3.2 below. ☐

The following lemma should be compared with Proposition 4.23.

3.2. LEMMA. *Every distributional solution u of the equation*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{3.3}$$

is of the form

$$u(\varphi) = v\left(P^+\varphi\right) + w\left(P^-\varphi\right), \quad \varphi\in\mathcal{D}\left(\mathbb{R}^2\right), \tag{3.4}$$

where v and w belong to $\mathcal{D}(\mathbb{R})$ and where

$$P^{\pm}\varphi(t) = \int_{-\infty}^{\infty}\varphi\left(\pm\left(x-t\right),x\right)\,dx.$$

Moreover, a solution u to the equation in (3.3) is 0 if and only if its support is contained in the half-space $\{(t,x)\in\mathbb{R}\times\mathbb{R} : t\geqslant 0\}$.

The conclusion in Lemma 3.2 may be reworded by saying: A wave which will live at some in future (has lived in the past) must have lived in the past (will live in the future).

Proof. We use an approximate identity $(h_k : k \in \mathbb{N})$ consisting of C^∞-functions on $\mathbb{R}^2$ h_k such that $\operatorname{supp}(h_k) \subset \{(t,x) \in \mathbb{R}^2 : |t| \leqslant k^{-1}\}$. Then the distribution u satisfies the equation in (3.3) if and only if

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)(u * h_k) = 0, \quad k = 1, 2, \ldots \tag{3.5}$$

From (3.5) it follows that there exist C^∞-functions v_k and w_k such that

$$(u * h_k)(t,x) = v_k(x-t) + w_k(x+t), \quad k = 1, 2, \ldots. \tag{3.6}$$

We shall prove that there exists a sequence of constants $(c_k)_{k\in\mathbb{N}}$ such that together with distributions v and w in $\mathcal{D}'(\mathbb{R})$ such that

$$v = \mathcal{D}'(\mathbb{R})\text{-}\lim_{k\to\infty}(v_k - c_k), \quad \text{and} \quad w = \mathcal{D}'(\mathbb{R})\text{-}\lim_{k\to\infty}(w_k + c_k). \tag{3.7}$$

A plain computation shows

$$(u * h_k)(\varphi) = (v_k - c_k)\left(P^+\varphi\right) + (w_k + c_k)\left(P^-\varphi\right), \tag{3.8}$$

and hence,

$$u(\varphi) = v\left(P^+\varphi\right) + w\left(P^-\varphi\right),$$

provided the limits in (3.7) exist. Therefore choose $\xi_1 \in \mathcal{D}(\mathbb{R})$ in such a way that $\int_{-\infty}^{\infty} \xi_1(x)\,dx = 1$, and defined c_k by

$$c_k = v_k(\xi_1) = \int_{-\infty}^{\infty} v_k(s)\xi_1(s)\,ds.$$

Furthermore, choose $\xi_2 \in \mathcal{D}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} \xi_2(x)\,dx = 1$. Pick $\varphi_1 \in \mathcal{D}(\mathbb{R})$ arbitrarily and look at the function φ_0 defined by

$$\varphi_0(t,x) = \left(\varphi_1(x-t) - \xi_1(x-t)\int_{-\infty}^{\infty} \varphi_1(s)\,ds\right)\xi_2(x+t). \tag{3.9}$$

Then φ_0 belongs to $\mathcal{D}(\mathbb{R}^2)$, $P^-\varphi_0 = 0$, and

$$P^+\varphi_0 = \frac{1}{2}\left(\varphi_1 - \xi_1\int_{-\infty}^{\infty} \varphi_1(s)\,ds\right).$$

It follows that

$$\begin{aligned}(u * h_k)(\varphi_0) &= (v_k - c_k)\left(P^+\varphi_0\right) + (w_k + c_k)\left(P^-\varphi_0\right) \\ &= (v_k - c_k)\left(P^+\varphi_0\right) = (v_k - c_k)\left(\frac{1}{2}\varphi_1\right).\end{aligned} \tag{3.10}$$

From (3.10) we infer that, for all $\varphi_1 \in \mathcal{D}(\mathbb{R})$, the limit $\lim_{k\to\infty}(v_k - c_k)(\varphi_1)$ exists. Similarly, we see that the limit $\lim_{k\to\infty}(w_k + c_k)(\varphi_1)$ exists for $\varphi_1 \in \mathcal{D}(\mathbb{R})$. Let $v(\varphi_1)$ respectively $w(\varphi_1)$ denote these limits. Then v and w belong to $\mathcal{D}'(\mathbb{R})$ and satisfy (3.7).

Next suppose that $u \in \mathcal{D}'(\mathbb{R}^2)$ satisfies the equation in (3.3) and is such that $\operatorname{supp}(u) \subset \{(t,x) \in \mathbb{R}^2 : t \geqslant 0\}$. Let $(h_k)_{k\in\mathbb{N}}$ be an approximate identity consisting of C^∞-functions as in the beginning of the proof. Then the inclusions

$$\begin{aligned}&\{(t,x) \in \mathbb{R}\times\mathbb{R} : u * h_k \neq 0\} \subset \operatorname{supp}(u * h_k) \subset \operatorname{supp}(u) + \operatorname{supp}(h_k) \\ &\subset \left\{(t,x) \in \mathbb{R}\times\mathbb{R} : t \geqslant -\frac{1}{k}\right\}\end{aligned} \tag{3.11}$$

hold true. Let v_k and w_k be the corresponding C^∞-functions as described above. From (3.11) it is readily seen that

$$\left\{(t,x) \in \mathbb{R}\times\mathbb{R} : t < -\frac{1}{k}\right\} \subset \{(t,x) \in \mathbb{R}\times\mathbb{R} : v_k(t-x) + w_k(t+x) = 0\}. \tag{3.12}$$

From (3.12) we obtain, for $t < -k^{-1}$, and $a \in \mathbb{R}$ arbitrary,

$$v_k(2t + a) + w_k(-a) = v_k(t - (-t + a)) + w_k(t + (-t - a)) = 0.$$

Hence, $v_k(s) = -w_k(-a)$, for $s < -2k^{-1} + a$, $a \in \mathbb{R}$. It follows that $v_k = -w_k = d_k$, a constant. The equality in (3.6) implies $u * h_k = 0$. Passing to the limit yields $u = 0$. This concludes the proof of Lemma 3.2. $\square$

2. Fundamental solutions of the wave equation in several space dimensions

We try to find a tempered distribution E on $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ with the property that

$$\frac{\partial^2 E}{\partial t^2} - \Delta_x E = \delta. \tag{3.13}$$

Let E be such a distribution. Take its Fourier transform relative to the space variables $\widetilde{E}$. Then $\widetilde{E}$ satisfies

$$\frac{\partial^2 \widetilde{E}}{\partial t^2} + |\xi|^2 \widetilde{E} = \delta(t)1(\xi). \tag{3.14}$$

The unique solution $\widetilde{E}_+$ of this equation with support in the half-space

$$\{(t, \xi) \in \mathbb{R} \times \mathbb{R}^n : t \geqslant 0\}$$

is given by the function

$$\widetilde{E}_+(t, \xi) = H(t)\frac{\sin(t|\xi|)}{|\xi|}, \quad (t, \xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.15}$$

Hence, equation (3.13) possesses a unique solution E_+ with support in the half-space $\{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \geqslant 0\}$. This distribution can be written in the form

$$E_+(\varphi) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i\xi\cdot x} \varphi(t, x)\, dx \right) \frac{\sin(t|\xi|)}{|\xi|}\, d\xi\, dt, \tag{3.16}$$

where φ belongs to $\mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$. This can be proved as follows. Pick $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$. Then there exists $\psi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ such that $\widetilde{\psi} = \varphi$, and so $E_+(\varphi) = E_+\left(\widetilde{\psi}\right) = \widetilde{E}_+(\psi)$. Consequently,

$$E_+(\varphi) = \int_0^\infty \left(\int_{\mathbb{R}^n} \psi(t, \xi) \frac{\sin(t|\xi|)}{|\xi|}\, d\xi \right) dt$$

(Fourier inversion formula)

$$= \frac{1}{(2\pi)^n} \int_0^\infty \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i\xi\cdot x} \varphi(t, x)\, dx \right) \frac{\sin(t|\xi|)}{|\xi|}\, d\xi \right) dt. \tag{3.17}$$

The equalities in (3.17) show the validity of the equality in (3.16).

Since we took the Fourier transform of E_+ with respect to the space variables, we lost, or better, we disguised the Lorentz symmetry. To make up for this we shall take the Fourier transform of $\widetilde{E}_+$ with respect to the time variable as well. Therefore, let

ε be a strictly positive number. Then there exists a distribution $E_+^\varepsilon \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^n)$ for which

$$\widetilde{E_+^\varepsilon}(t,\xi) = H(t)e^{-\varepsilon t}\frac{\sin(t|\xi|)}{|\xi|}, \quad (t,\xi)\in\mathbb{R}\times\mathbb{R}^n. \tag{3.18}$$

As a consequence we infer

$$\widehat{E_+^\varepsilon}(\tau,\xi) = \int_0^\infty e^{-i\tau t-\varepsilon t}\frac{\sin(t|\xi|)}{|\xi|}\,dt = -\frac{1}{(\tau-i\varepsilon)^2-|\xi|^2}, \quad (\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n. \tag{3.19}$$

Next let φ be in $\mathcal{D}(\mathbb{R}\times\mathbb{R}^n)$ and suppose $\psi\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^n)$ is such that $\widehat{\psi}=\varphi$. Then

$$\psi(\tau,\xi) = \frac{1}{(2\pi)^{n+1}}\int_{-\infty}^\infty\left(\int_{\mathbb{R}^n} e^{i\tau t+i\xi\cdot x}\varphi(t,x)\,dx\right)dt, \quad (\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n. \tag{3.20}$$

Hence, we get

$$E_+^\varepsilon(\varphi) = \widehat{E_+^\varepsilon}(\psi)$$
$$= -\int_{-\infty}^\infty\left(\int_{\mathbb{R}^n}\frac{1}{(\tau-i\varepsilon)^2-|\xi|^2}\psi(\tau,\xi)\,d\xi\right)d\tau$$

(employ (3.20))

$$= -\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^\infty\int_{\mathbb{R}^n}\frac{1}{(\tau-i\varepsilon)^2-|\xi|^2}\left(\int_{-\infty}^\infty\int_{\mathbb{R}^n} e^{i\tau t+i\xi\cdot x}\varphi(t,x)\,dx\,dt\right)d\xi\,d\tau. \tag{3.21}$$

Since φ belongs to $\mathcal{D}(\mathbb{R}\times\mathbb{R}^n)$, its Fourier transform, *i.e.* the function

$$(\tau,\xi)\mapsto\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}e^{i\tau t+i\xi\cdot x}\varphi(t,x)\,dx\,dt$$

has a holomorphic extension to $\mathbb{C}\times\mathbb{C}^n$. Using Cauchy's theorem together with (3.21), we infer the following equality

$$\begin{aligned}E_+^{\varepsilon}(\varphi)=-\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{1}{(\tau-ia-i\varepsilon)^2-(\xi-ib)^2}\\ \left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}e^{i(\tau-ia)t+i(\xi-ib)\cdot x}\varphi(t,x)\,dx\,dt\right)d\xi\,d\tau\end{aligned}\tag{3.22}$$

for $a+\varepsilon>|b|$. Here $(\xi-ib)^2=\sum_{j=1}^{n}(\xi_j-ib_j)^2$, where $\xi=(\xi_1,\dots,\xi_n)$, $b=(b_1,\dots,b_n)$. Now, for $a>|b|$, it is easy to take the limit for $\varepsilon\downarrow 0$:

$$\begin{aligned}E_+(\varphi)=\lim_{\varepsilon\downarrow 0}E_+^{\varepsilon}(\varphi)=-\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{1}{(\tau-ia)^2-(\xi-ib)^2}\\ \left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}e^{i(\tau-ia)t+i(\xi-ib)\cdot x}\varphi(t,x)\,dx\,dt\right)d\xi\,d\tau.\end{aligned}\tag{3.23}$$

In order to justify the application of Cauchy's theorem in (3.22) the following lemma is required.

3.3. Lemma. *For $a\geqslant|b|$, $a\in\mathbb{R}$, $\tau\in\mathbb{R}$, $b\in\mathbb{R}^n$, and $\xi\in\mathbb{R}^n$ the following inequality is valid:*

$$\left|(\tau+ia)^2-(\xi+ib)^2\right|\geqslant a^2-|b|^2.\tag{3.24}$$

Here $|\xi|:=\sqrt{\xi_1^2+\dots+\xi_n^2}$ denotes the Euclidean norm of $\xi=(\xi_1,\dots,\xi_n)$ in $\mathbb{R}^n$. We also write

$$(\xi+i\eta)^2=\sum_{j=1}^{n}(\xi_j+i\eta_j)^2=\sum_{j=1}^{n}\left(\xi_j^2-\eta_j^2+2i\xi_j\eta_j\right).$$

Proof. We write $\left|(\tau+ia)^2-(\xi+ib)^2\right|^2$ in the form

$$\begin{aligned}&\left|(\tau+ia)^2-(\xi+ib)^2\right|^2\\ &=\left(a^2-|b|^2\right)^2+4\left(\langle\xi,b\rangle-a\tau\right)^2+\left(\tau^2-|\xi|^2\right)^2+2\left(a^2-|b|^2\right)\left(|\xi|^2-\tau^2\right).\end{aligned}\tag{3.25}$$

The expression in (3.25) is $\geqslant\left(a^2-|b|^2\right)^2$, provided $|\xi|\geqslant|\tau|$, and $a\geqslant|b|$. If $|\tau|\geqslant|\xi|$, then

$$a\,|\tau|-|\langle\xi,b\rangle|\geqslant a\,|\tau|-|\xi|\,|b|\geqslant 0,$$

and, hence

$$\begin{aligned}&4\left(\langle\xi,b\rangle-a\tau\right)^2+\left(\tau^2-|\xi|^2\right)^2+2\left(a^2-|b|^2\right)\left(|\xi|^2-\tau^2\right)\\ &\geqslant 4\left(a\,|\tau|-|\xi|\;|b|\right)^2+\left(\tau^2-|\xi|^2\right)^2+2\left(a^2-|b|^2\right)\left(|\xi|^2-\tau^2\right)\\ &=2a^2\tau^2-8a\,|\tau|\;|\xi|\,|b|+2\,|b|^2\,|\xi|^2+2a^2\,|\xi|^2+2\,|b|^2\,\tau^2\end{aligned}$$

$$= 2\left(a\left|\tau\right| - \left|b\right|\left|\xi\right|\right)^2 + 2\left(a\left|\xi\right| - \left|\tau\right|\left|b\right|\right)^2 \geqslant 0.$$

This concludes the proof of the inequality in (3.24). □

Essentially speaking these observations yield the following result.

3.4. Theorem. *The only fundamental solution E_+ of the operator $\dfrac{\partial^2}{\partial t^2} - \Delta_x$ in $\mathbb{R} \times \mathbb{R}^n$, $n \geqslant 2$, with support contained in the half-space $\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t \geqslant 0\}$ is given by the integral:*

$$E_+(\varphi) = -\frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(\tau - ia)^2 - (\xi - ib)^2} \left(\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} e^{i(\tau - ia)t + i(\xi - ib)\cdot x} \varphi(t,x)\, dx\, dt \right) d\xi\, d\tau, \quad \varphi \in \mathcal{D}\left(\mathbb{R} \times \mathbb{R}^n\right), \tag{3.26}$$

provided $a > |b|$.

It is emphasized that the condition $a > |b|$ is very important. This will also be reflected by the fact that the distribution E_+ is not invariant under all Lorentz transformations: see Subsection 2.1 below.

Proof. Using Lemma 3.3 justifies the equality in (3.22) for $a > |b|$. The equality in (3.23), which is the same as (3.26), is a consequence of (3.22).

This completes the proof of Theorem 3.4. □

2.1. Fundamental solutions which are invariant under certain Lorentz transformations. In this subsection we will discuss the matrix group of Lorentz transformations acting on the space $\mathbb{R} \times \mathbb{R}^n$, and its subgroup $\mathcal{L}_+$ of those Lorentz transformations $T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ which leave the positive light cone invariant, *i.e.* which have the property that

$$T\left\{(t,x) \in \mathbb{R} \times \mathbb{R}^n,\, t \geqslant |x|\right\} \subset \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^n,\, t \geqslant |x|\right\}.$$

It turns out that such Lorentz transformations leave the distribution E_+ invariant, *i.e.* $E_+^T = E_+$. See Definition 3.6 below. Here $E_+^T(\varphi) = E_+\left(\varphi^{T^{-1}} \left|\det(T)\right|\right)$, $\varphi \in \mathcal{D}\left(\mathbb{R} \times \mathbb{R}^n\right)$. For more details on the way transformations act on functions and distributions the reader is referred to Section 1.12. In particular see the equality in (1.35), and Proposition 1.34. The adjoint of T is written as T^*. Define the projection mapping $P_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ by

$$P_1 \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.27}$$

3.5. Proposition. *Let $T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ be a linear transformation. Then the following assertions are equivalent.*

(a) *The transformation T leaves the bilinear form*

$$\left(\begin{pmatrix} t \\ x \end{pmatrix}, \begin{pmatrix} \tau \\ y \end{pmatrix}\right) \mapsto t\tau - \sum_{j=1}^{n} x_j y_j = \left\langle (2P_1 - I)\begin{pmatrix} t \\ x \end{pmatrix}, \begin{pmatrix} \tau \\ y \end{pmatrix}\right\rangle, \quad (t,x),\ (\tau,y) \in \mathbb{R}\times\mathbb{R}^n, \tag{3.28}$$

invariant in the sense that

$$\left\langle (2P_1 - I)\, T\begin{pmatrix} t \\ x \end{pmatrix}, T\begin{pmatrix} \tau \\ y \end{pmatrix}\right\rangle = \left\langle (2P_1 - I)\begin{pmatrix} t \\ x \end{pmatrix}, \begin{pmatrix} \tau \\ y \end{pmatrix}\right\rangle, \quad (t,x),\ (\tau,y) \in \mathbb{R}\times\mathbb{R}^n. \tag{3.29}$$

(b) *The transformation T leaves the quadratic form*

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto t^2 - \sum_{j=1}^{n} x_j x_j, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,\ (\tau,y) \in \mathbb{R}\times\mathbb{R}^n, \tag{3.30}$$

invariant.

(c) *The following identity holds: $T^* (2P_1 - I)\, T = 2P_1 - I$.*

(d) *The following identity holds: $T (2P_1 - I)\, T^* = 2P_1 - I$.*

(e) *The transformation T^* leaves the bilinear form in (3.28) invariant.*

If T is a Lorentz transformation, then $\det(T) = \pm 1$.

PROOF. The equivalence of (a) and (c) is clear from (3.29). Similarly, the equivalence of (d) and (e) is clear: replace in (3.29) the transformation T with T^*. The equivalence of (a) and (b) is a consequence of the following polarization formula:

$$2t\tau - 2\sum_{j=1}^{n} x_j y_j = (t+\tau)^2 - \sum_{j=1}^{n} (x_j + y_j)^2 - \left(t^2 - \sum_{j=1}^{n} x_j^2\right) - \left(\tau^2 - \sum_{j=1}^{n} y_j^2\right). \quad (3.31)$$

Next observe that $(2P_1 - I)^2 = I$, and $\det(2P_1 - I) = (-1)^n$. From the property in (c) it follows that the transformation T is invertible. From (c) we also infer $T^*(2P_1 - I)\,T\,(2P_1 - I)\,T^* = T^*$. Consequently, $(2P_1 - I)\,T\,(2P_1 - I)\,T^* = I$. Whence, if T has property (c), then it has property (d). The proof of the converse implication is similar. It follows that all properties (a) through (e) are equivalent.

Observe that $\det(2P_1 - I) = (-1)^n$. Thus from the equality in (c) we see

$$\det(T^*)\det(T) = 1.$$

Hence, $(\det(T))^2 = 1$, and so $\det(T) = \pm 1$. This completes the proof of Proposition 3.5. □

3.6. DEFINITION. A transformation $T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ which possesses one of the equivalent properties in Proposition 3.5 is called a Lorentz transformation.

3.7. PROPOSITION. *Let $T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ be a Lorentz transformation, Denote by $\mathcal{L}$ be the group of Lorentz transformations on $\mathbb{R} \times \mathbb{R}^n$, and let $\mathcal{L}_+$ be its subgroup consisting of those Lorentz transformations which leave the positive light cone invariant. Let $T \in \mathcal{L}$. Then the following assertions are equivalent:*

(a) $T \in \mathcal{L}_+$;
(b) $T^* \in \mathcal{L}_+$;
(c) T *is such that*

$$\left\langle T\begin{pmatrix}1\\x\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\rangle > 0, \quad x \in \mathbb{R}^n; \quad (3.32)$$

(d) T^* *is such that*

$$\left\langle T^*\begin{pmatrix}1\\x\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\rangle > 0, \quad x \in \mathbb{R}^n; \quad (3.33)$$

PROOF. The equivalence of (a) and (b) follows from the equalities

$$T^*(2P_1 - I)\,T = 2P_1 - I = T\,(2P_1 - I)\,T^*.$$

The implications (a) $\Longrightarrow$ (c) and (b) $\Longrightarrow$ (d) are obvious. Suppose that (c) holds. Let $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ belong to the positive light cone, *i.e.* suppose $t > |x|$, and put $\begin{pmatrix}t'\\x'\end{pmatrix} = T\begin{pmatrix}t\\x\end{pmatrix}$. Then, since T is a Lorentz transformation $(t')^2 - |x'|^2 = t^2 - |x|^2 > 0$. Assertion (c) implies that $t' > 0$, and so $t' > |x'|$. This means that (c) implies (a). A similar argument shows the implication (d) $\Longrightarrow$ (b). This completes the proof of Proposition 3.7. □

3.8. PROPOSITION. *Suppose that the Lorentz transformation T is such that*

$$\left\langle T\begin{pmatrix}1\\x\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\rangle > 0, \quad x \in \mathbb{R}^n. \tag{3.34}$$

Then T and T^* belong to $\mathcal{L}_+$, and $E_+^T = E_+$.

From (3.34) it follows that the first coordinate of the column vector $T\begin{pmatrix}t\\x\end{pmatrix}$ is strictly positive whenever $t > 0$ and $x \in \mathbb{R}^n$.

PROOF. From assertion (d) in Proposition 3.5 it follows that the transformation T^* is a Lorentz transformation. From Proposition 3.7 and (3.34) we see that T as well as T^* belong to $\mathcal{L}_+$. In order to prove $E_+^T = E_+$ we let $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$ and proceed as is in Theorem 3.4 equality (3.26):

$$\begin{aligned}E_+^T(\varphi) &= E_+\left(\varphi^{T^{-1}}\right)\\ &= -\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{1}{(\tau - ia)^2 - (\xi - ib)^2}\\ &\qquad\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i(\tau-ia)t+i(\xi-ib)\cdot x}\varphi^{T^{-1}}(t,x)\,dx\,dt\right)d\xi\,d\tau.\end{aligned} \tag{3.35}$$

Next we make the following substitutions in (3.35):

$$\begin{pmatrix}t'\\x'\end{pmatrix} = T\begin{pmatrix}t\\x\end{pmatrix}, \quad \begin{pmatrix}\tau'\\\xi'\end{pmatrix} = \left(T^{-1}\right)^*\begin{pmatrix}\tau\\\xi\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}a'\\b'\end{pmatrix} = \left(T^{-1}\right)^*\begin{pmatrix}a\\b\end{pmatrix}. \tag{3.36}$$

By employing Lorentz invariance of the T from (3.35) and (3.36) we obtain

$$\begin{aligned}E_+^T(\varphi) &= -\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{1}{(\tau' - ia')^2 - (\xi' - ib')^2}\\ &\qquad\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i(\tau'-ia')t'+i(\xi'-ib')\cdot x'}\varphi\left(t',x'\right)\,dx'\,dt'\right)d\xi'\,d\tau'\\ &= E_+(\varphi).\end{aligned} \tag{3.37}$$

In the final step of (3.37) we used the fact that $a' > |b'|$ which is a consequence of Proposition 3.7. So the proof of Proposition 3.8 is complete now. $\square$

Next we show that there are several Lorentz transformations which leave the positive light cone invariant.

3.9. PROPOSITION. *The following equality of subsets holds:*

$$\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t < |x|\} = \bigcup_{T \in \mathcal{L}_+} T^{-1}\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t < 0\}. \tag{3.38}$$

Proof. Let H be a hyperplane in $\mathbb{R} \times \mathbb{R}^n$ whose intersection with the positive light cone $\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t \geqslant |x|\}$ reduces to the origin. Then there exists a unique $x_0 \in \mathbb{R}^n$, $|x_0| < 1$, for which the vector $(1, -x_0)$ is orthogonal to H, *i.e.*

$$H = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t - x \cdot x_0 = 0\}.$$

Define the half-space H_- by

$$H_- = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t - x \cdot x_0 < 0\}.$$

Then we shall prove that there exists a Lorentz transformation T in $\mathcal{L}_+$ with $TH_- = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t < 0\}$. Let e_1 be the first unit vector in Euclidean n-space $\mathbb{R}^n$, and choose an orthogonal map U on $\mathbb{R}^n$ for which $U^* x_0 = |x_0|\, e_1$. Then choose $\vartheta_0 \in \mathbb{R}$ for which $\tanh \vartheta_0 = |x_0|$, and let t_0 be determined by the equality

$$\begin{pmatrix} \cosh \vartheta_0 & -\sinh \vartheta_0 \\ -\sinh \vartheta_0 & \cosh \vartheta_0 \end{pmatrix} \begin{pmatrix} 1 \\ |x_0| \end{pmatrix} = \begin{pmatrix} t_0 \\ 0 \end{pmatrix}.$$

Then $t_0 > 0$ and the Lorentz transformation T defined by the product matrix

$$T = \begin{pmatrix} \cosh \vartheta_0 & \sinh \vartheta_0 & 0 & \cdots & 0 \\ \sinh \vartheta_0 & \cosh \vartheta_0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & u_{11} & \cdots & u_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & u_{n1} & \cdots & u_{nn} \end{pmatrix}$$

possesses the required properties. Here the orthogonal matrix U is given by $U = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ldots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}$. The fact is used that the hermitian matrices $\begin{pmatrix} \cosh\vartheta & \sinh\vartheta_0 \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix}$ and $\begin{pmatrix} \cosh\vartheta & -\sinh\vartheta \\ -\sinh\vartheta & \cosh\vartheta \end{pmatrix}$, $\vartheta \in \mathbb{R}$, are each other's inverses. As a consequence we obtain:

$$\{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t - x\cdot x_0 < 0\} = T^{-1}\{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t < 0\}. \tag{3.39}$$

It is elementary to verify that

$$\{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t < |x|\} = \bigcup_{x_0\in\mathbb{R}^n,\, |x_0|<1} \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t - x\cdot x_0 < 0\}. \tag{3.40}$$

The conclusion in Proposition 3.9 now follows from (3.39) and (3.40). □

Theorem 3.4 can be strengthened as follows.

3.10. Theorem. *The only fundamental solution E_+ of the operator $\dfrac{\partial^2}{\partial t^2} - \Delta_x$ in $\mathbb{R}\times\mathbb{R}^n$, $n \geqslant 2$, with support contained in the half-space $\{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t \geqslant 0\}$ is given by the integral:*

$$E_+(\varphi) = -\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \frac{1}{(\tau - ia)^2 - (\xi - ib)^2} \left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i(\tau-ia)t + i(\xi-ib)\cdot x}\varphi(t,x)\,dx\,dt\right) d\xi\,d\tau, \quad \varphi \in \mathcal{D}\left(\mathbb{R}\times\mathbb{R}^n\right), \tag{3.41}$$

provided $a > |b|$. Its support is contained in the positive light cone

$$\{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t \geqslant |x|\}.$$

The distribution E_+ is invariant under Lorentz transformations belonging to $\mathcal{L}_+$.

Proof. The result is a consequence of Theorem 3.4 which shows the representation in (3.41, of Proposition 3.8 which yields the invariance under Lorentz transformation in $\mathcal{L}_+$, and of Proposition 3.9 in conjunction with Proposition 1.34 which reduces its support to a subset of the positive light cone. □

2.2. Explicit formulas for the fundamental solutions. In this section we will use the symbol $\Box = \Box_{t,x}$ for the wave operator or d'Alembertian in $\mathbb{R}\times\mathbb{R}^n$ defined by

$$\Box = \Box_{t,x} = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} = \frac{\partial^2}{\partial t^2} - \Delta_x. \tag{3.42}$$

We formulate the results in the form of a theorem.

3.11. Theorem. *Let E_+ be the only distribution in $\mathbb{R}\times\mathbb{R}^n$ with supp $(E_+) \subset \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t \geqslant 0\}$, and which satisfies $\Box E_+ = \delta$.*

(i) *If* $n = 1$, *then* E_+ *is given by*

$$E_+ = \frac{1}{2} H(t - |x|), \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

(ii) *If* $n = 2m + 2$ *and if* $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$, *then*

$$E_+(\varphi) = \frac{2}{(4\pi)^{m+1}} \frac{1}{m!} \int_{\mathbb{R}^n} \int_{|x|}^{\infty} \frac{\square^m \varphi(t, x)}{\sqrt{t^2 - |x|^2}} \, dt \, dx.$$

(iii) *If* $n = 2m + 3$ *and if* $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$, *then*

$$E_+(\varphi) = \frac{1}{(4\pi)^{m+1}} \frac{1}{m!} \int_{\mathbb{R}^n} \frac{\square^m \varphi(|x|, x)}{|x|} \, dt \, dx.$$

PROOF. (i) The case $n = 1$ is disposed of in Section 1: see Proposition 3.1. Let $\varepsilon > 0$. There exists a distribution E_+^ε whose Fourier transform with respect to the space variables $\widetilde{E_+^\varepsilon}$ is defined by the formula:

$$\widetilde{E_+^\varepsilon} = H(t) e^{-\varepsilon|\xi|} \frac{\sin(|\xi| t)}{|\xi|}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.43}$$

Notice that the function $\widetilde{E_+^\varepsilon}$ is not the same as the one in (3.18). Then $\mathcal{S}\text{-}\lim_{\varepsilon \downarrow 0} \widetilde{E_+^\varepsilon} = \widetilde{E_+}$, *i.e.*

$$\lim_{\varepsilon \downarrow 0} \widetilde{E_+^\varepsilon}(\psi) = \widetilde{E_+}(\psi), \quad \psi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n).$$

From (4) in Theorem 4.11 we infer $\mathcal{S}\text{-}\lim_{\varepsilon \downarrow 0} E_+^\varepsilon = E_+$. Here E_+ is as in Theorem 3.11. Theorem 1.49 with $a = t + i\varepsilon$ yields

$$E_+^\varepsilon = c_n \Im \left\{ i^{n-1} \left((t + i\varepsilon)^2 - |x|^2 \right)^{-\frac{1}{2}(n-1)} \right\}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.44}$$

Here $\Im\{\cdot\}$ denotes imaginary part, and c_n is the constant:

$$c_n = \frac{1}{2\pi} \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\pi^{\frac{1}{2}(n-1)}}.$$

Now we are settled to prove (ii) and (iii).

(ii) So for n even, say $n = 2m + 2$, we have

$$E_+^\varepsilon(\varphi) = (-1)^m c_n \int_{\mathbb{R}^n} \int_0^\infty \varphi(t, x) \Re \left((t + i\varepsilon)^2 - |x|^2 \right)^{-\frac{1}{2}(n-1)} dt \, dx, \quad \varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n). \tag{3.45}$$

Since

$$\frac{c_{2m+2}}{(2m)!} = \frac{2}{(4\pi)^{m+1}} \frac{1}{m!},$$

and since we have

$$\square^m \left((t + i\varepsilon)^2 - |x|^2 \right)^{-\frac{1}{2}} = (-1)^m (2m)! \left((t + i\varepsilon)^2 - |x|^2 \right)^{-\frac{1}{2}(n-1)}$$

from (3.45) we infer, for $\varphi \in \mathcal{D}\left(\mathbb{R} \times \mathbb{R}^n\right)$,

$$E_+^{\varepsilon}(\varphi) = \frac{2}{(4\pi)^{m+1}} \frac{1}{m!} \int_{\mathbb{R}^n} \int_0^{\infty} \varphi(t,x) \Box^m \Re \left((t+i\varepsilon)^2 - |x|^2 \right)^{-\frac{1}{2}} \, dt \, dx. \tag{3.46}$$

If we integrate by parts in (3.46), then either the integration terms vanish or tend to 0 whenever $\varepsilon \downarrow 0$. Hence, for $\varphi \in \mathcal{D}\left(\mathbb{R} \times \mathbb{R}^n\right)$, we obtain

$$\begin{aligned} E_+(\varphi) &= \lim_{\varepsilon \downarrow 0} E_+^{\varepsilon}(\varphi) \\ &= \lim_{\varepsilon \downarrow 0} \frac{2}{(4\pi)^{m+1}} \frac{1}{m!} \int_{\mathbb{R}^n} \int_0^{\infty} \Box^m \varphi(t,x) \Re \left((t+i\varepsilon)^2 - |x|^2 \right)^{-\frac{1}{2}} \, dt \, dx \\ &= \lim_{\varepsilon \downarrow 0} \frac{2}{(4\pi)^{m+1}} \frac{1}{m!} \int_{\mathbb{R}^n} \int_{|x|}^{\infty} \frac{\Box^m \varphi(t,x)}{\left(t^2 - |x|^2\right)^{1/2}} \, dt \, dx. \end{aligned} \tag{3.47}$$

The equality in (3.47) proves the identity in (ii).

(iii) For n odd, say $n = 2m+3$, we obtain, for $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$, the equality

$$E_+^\varepsilon(\varphi) = (-1)^{m+1} c_n \int_{\mathbb{R}^n} \int_0^\infty \varphi(t,x) \Im \left((t+i\varepsilon)^2 - |x|^2 \right)^{-\frac{1}{2}(n-1)} \, dt\, dx, \tag{3.48}$$

for $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$. Since

$$\frac{c_{2m+3}}{4^m (m!)^2} = \frac{2}{\pi} \frac{1}{(4\pi)^{m+1}} \frac{1}{m!},$$

and since we have

$$\Box^m \left((t+i\varepsilon)^2 - |x|^2 \right)^{-1} = (-1)^m 4^m (m!)^2 \left((t+i\varepsilon)^2 - |x|^2 \right)^{-\frac{1}{2}(n-1)}$$

from (3.45) we infer, for $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$, the equality

$$E_+^\varepsilon(\varphi) = \frac{-2}{\pi} \frac{1}{(4\pi)^{m+1}} \frac{1}{m!} \int_{\mathbb{R}^n} \int_0^\infty \varphi(t,x) \Box^m \Im \left\{ \left((t+i\varepsilon)^2 - |x|^2 \right)^{-1} \right\} dt\, dx. \tag{3.49}$$

Again applying integrating by parts in (3.49) and using an argument as in the even dimensional case proves

$$\begin{aligned} E_+(\varphi) &= \lim_{\varepsilon \downarrow 0} E_+^\varepsilon(\varphi) \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \frac{1}{(4\pi)^{m+1}} \frac{1}{m!} \int_{\mathbb{R}^n} \int_0^\infty \Box^m \varphi(t,x) \Im \left\{ -2 \left((t+i\varepsilon)^2 - |x|^2 \right)^{-1} \right\} dt\, dx \\ &= \frac{1}{\pi} \frac{1}{(4\pi)^{m+1}} \frac{1}{m!} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \int_0^\infty \frac{\Box^m \varphi(t,x)}{|x|} \left\{ \frac{\varepsilon}{(t-|x|)^2 + \varepsilon^2} - \frac{\varepsilon}{(t+|x|)^2 + \varepsilon^2} \right\} dt\, dx. \end{aligned} \tag{3.50}$$

Substitution of $t = |x| + \varepsilon s$ in the first integral, and of $t = -|x| + \varepsilon s$ in the second integral in the right-hand sid of (3.50) yields

$$\begin{aligned} E_+(\varphi) &= \frac{1}{\pi} \frac{1}{(4\pi)^{m+1}} \frac{1}{m!} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \int_{-|x|/\varepsilon}^\infty \frac{\Box^m \varphi(|x| + \varepsilon s, x)}{|x|} \frac{1}{1+s^2} \, ds\, dx \\ &\quad - \frac{1}{\pi} \frac{1}{(4\pi)^{m+1}} \frac{1}{m!} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \int_{|x|/\varepsilon}^\infty \frac{\Box^m \varphi(-|x| + \varepsilon s, x)}{|x|} \frac{1}{1+s^2} \, ds\, dx \\ &= \frac{1}{(4\pi)^{m+1}} \frac{1}{m!} \int_{\mathbb{R}^n} \frac{\Box^m \varphi(|x|, x)}{|x|} \, dx. \end{aligned} \tag{3.51}$$

The equality in (3.51) proves the identity in (iii). So the proof of Theorem 3.11 is complete now. $\square$

3.12. Corollary (Huygens principle). *Let E_+ be the fundamental solution of the wave equation with support in the half-space $\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t \geqslant 0\}$. If n is odd and $n \geqslant 3$, then its support coincides with the boundary of the positive light cone. If $n = 1$ or n is even, then its support coincides with the solid positive light cone.*

There is an extensive literature on distribution theory and partial differential equations. We mention some of it in the References.

CHAPTER 4

Proofs of some main results

1. Convolution products: formulation of some results

In what follows we write $\mathcal{E} = C^{\infty}(\mathbb{R}^n)$ for the space of infinitely many times differentiable functions on $\mathbb{R}^n$, $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ for the space of rapidly decreasing functions on $\mathbb{R}^n$, and $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ for the space of C^{∞}-functions with compact support (*i.e.* the space of test functions). By $\mathcal{E}'$, $\mathcal{S}'$, and $\mathcal{D}'$ we denote their respective dual spaces.

4.1. Theorem. *Fix $m \in \mathbb{N}$. Let μ be a complex Borel measure on $\mathbb{R}^n$ with the property that*

$$\int \left(1 + |x|^2\right)^{m/2} d|\mu|(x) < \infty, \tag{4.1}$$

and let $u \in \mathcal{E}'$ be a distribution with compact support such that, for some compact subset K of $\mathbb{R}^n$ and some finite constant C,

$$|u(f)| \leqslant C \sup_{x\in K} \max_{|\alpha|\leqslant m} |D^{\alpha} f(x)|, \quad \text{for all functions } f \in \mathcal{E}. \tag{4.2}$$

Then the distribution u belongs to $\mathcal{S}'$ and its Fourier transform $\widehat{u}$ is given by the function: $x \mapsto u\left(\xi \mapsto e^{-i\langle \xi, x\rangle}\right)$. More precisely, let μ be a complex Borel measure on $\mathbb{R}^n$ with the property that

$$\int \left(1 + |x|^2\right)^{m/2} d|\mu|(x) < \infty. \tag{4.3}$$

Define $\widehat{u}(\mu)$ by $\widehat{u}(\mu) = u(\widehat{\mu})$. Then $\widehat{u}(\mu)$ is given by:

$$\widehat{u}(\mu) = \int u\left(\xi \mapsto e^{-i\langle \xi, x\rangle}\right) d\mu(x).$$

As an application, with $\mu(B) = \int_B \varphi(x)dx$, we have the following result.

4.2. Corollary. *Let u be a distribution with compact support,* i.e. *$u \in \mathcal{E}'$, and let φ be a function in $\mathcal{S}$. Then $\widehat{u}(\varphi) = \int u\left(\xi \mapsto e^{-i\langle \xi, x\rangle}\right) \varphi(x) d(x)$.*

4.3. Theorem. *Let u be a tempered distribution, and let φ and ψ be rapidly decreasing functions in $\mathcal{S}$. Then $u * \varphi$ is a C^{∞}-function of polynomial growth, and hence it is a tempered distribution, and $(u * \varphi) * \psi = u * (\varphi * \psi)$.*

Here $u * \varphi(x)$ is defined by $u * \varphi(x) = u(\tau_x \check{\varphi})$, and $\varphi * \psi$ is the function

$$x \mapsto \int \varphi(y)\psi(x-y)\, dy = \int \varphi(y) \tau_x \check{\psi}(y)\, dy.$$

4.4. Theorem. *Let $u \in \mathcal{D}'$ be a distribution, and let φ and ψ be test functions in $\mathcal{D}$. Then $u * \varphi$ is a C^∞-function, and $(u * \varphi) * \psi = u * (\varphi * \psi)$.*

1.1. Proofs. In what follows we write $D^\alpha f(x) = \left(\frac{\partial}{i\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{i\partial x_n}\right)^{\alpha_n} f(x)$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Proof of Theorem 4.1. Let μ and $m \in \mathbb{N}$ be as in (4.1), and C as in (4.2). Fix $\varepsilon > 0$, and choose a compact subset K_ε of $\mathbb{R}^n$ in such a way that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant m$ the inequality

$$C \int_{\mathbb{R}^n \setminus K_\varepsilon} |x^\alpha| \, d|\mu|(x) \leqslant \frac{1}{2}\varepsilon \tag{4.4}$$

holds. We also choose a partition $B_{j,\varepsilon}$, $1 \leqslant j \leqslant N_\varepsilon$, of the compact set K_ε in such a way that, again for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant m$, the inequalities

$$C \sup_{\xi \in K} \sup_{y_2, y_1 \in B_{j,\varepsilon}} \left| e^{-i\langle \xi, y_2\rangle} - e^{-i\langle \xi, y_1\rangle} \right| \int_{K_\varepsilon} |x^\alpha| \, d|\mu|(x) \leqslant \frac{1}{4}\varepsilon, \quad \text{and} \tag{4.5}$$

$$C \sup_{x_2, x_1 \in B_{j,\varepsilon}} |x_2^\alpha - x_1^\alpha| \, |\mu|(B_{j,\varepsilon}) \leqslant \frac{1}{4}\varepsilon \tag{4.6}$$

hold for all $1 \leqslant j \leqslant N_\varepsilon$. The partitions $\{B_{j,\varepsilon} : 1 \leqslant j \leqslant N_\varepsilon\}$, $\varepsilon > 0$, are chosen in such a way that the partition corresponding to ε' refines the one corresponding to ε, provided $\varepsilon' < \varepsilon$. In order to achieve this the ε's should be taken of the form $\varepsilon = 2^{-k}\varepsilon_0$, $k \in \mathbb{N}$. This means that $K_{\varepsilon'} \supset K_\varepsilon$, and

$$B_{j,\varepsilon} = \bigcup_{1\leqslant k\leqslant N_{\varepsilon'}:B_{k,\varepsilon'}\cap B_{j,\varepsilon}\neq\varnothing} B_{k,\varepsilon'} \quad \text{for } 1 \leqslant j \leqslant N_\varepsilon \text{ and } 0 < \varepsilon' < \varepsilon. \tag{4.7}$$

Since $\left|e^{-i\langle\xi,y_2\rangle} - e^{-i\langle\xi,y_1\rangle}\right| \leqslant |\xi|\,|y_2 - y_1|$, the partition $B_{j,\varepsilon}$, $1 \leqslant j \leqslant N_\varepsilon$, can be chosen in such a way that inequality (4.5) holds by the hypothesis in (4.1). Choose $x_{j,\varepsilon} \in B_{j,\varepsilon}$, $1 \leqslant j \leqslant N_\varepsilon$. Put

$$g(\xi) = \int_{\mathbb{R}^n} e^{-i\langle\xi,x\rangle} d\mu(x) = \sum_{j=1}^{N_\varepsilon}\int_{B_{j,\varepsilon}} e^{-i\langle\xi,x\rangle} d\mu(x) + \int_{\mathbb{R}^n\setminus K_\varepsilon} e^{-i\langle\xi,x\rangle} d\mu(x), \quad \text{and} \tag{4.8}$$

$$g_\varepsilon(\xi) = \sum_{j=1}^{N_\varepsilon} e^{-i\langle\xi,x_{j,\varepsilon}\rangle} \mu\left(B_{j,\varepsilon}\right) = \sum_{j=1}^{N_\varepsilon}\int_{B_{j,\varepsilon}} e^{-i\langle\xi,x_{j,\varepsilon}\rangle} d\mu(x). \tag{4.9}$$

Then

$$\begin{aligned}
&D^\alpha\left(g - g_\varepsilon\right)(\xi) \\
&= (-1)^{|\alpha|}\sum_{j=1}^{N_\varepsilon}\int_{B_{j,\varepsilon}} \left(e^{-i\langle\xi,x\rangle} - e^{-i\langle\xi,x_{j,\varepsilon}\rangle}\right) x^\alpha d\mu(x) \\
&\quad + (-1)^{|\alpha|}\sum_{j=1}^{N_\varepsilon} e^{-i\langle\xi,x_{j,\varepsilon}\rangle}\int_{B_{j,\varepsilon}} \left(x^\alpha - x_{j,\varepsilon}^\alpha\right) d\mu(x) + (-1)^{|\alpha|}\int_{\mathbb{R}^n\setminus K_\varepsilon} e^{-i\langle\xi,x\rangle} x^\alpha d\mu(x),
\end{aligned}$$

and hence, by (4.4), (4.5), and (4.6), we get for all $\xi \in K$ and all $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant m$,

$$C\left|D^\alpha g_\varepsilon(\xi) - D^\alpha g(\xi)\right| \leqslant \varepsilon. \tag{4.10}$$

It follows that $\left|u\left(g_\varepsilon - g\right)\right| \leqslant \varepsilon$. By the refinement property (4.7) we also have

$$\lim_{\varepsilon\downarrow 0} u\left(g_\varepsilon\right) = \int_{\mathbb{R}^n} u\left(\xi \mapsto e^{-i\langle\xi,x\rangle}\right) d\mu(x), \tag{4.11}$$

and hence,

$$\widehat{u}\left(\mu\right) = u\left(\widehat{\mu}\right) = u\left(g\right) = \lim_{\varepsilon\downarrow 0} u\left(g_\varepsilon\right) = \int_{\mathbb{R}^n} u\left(\xi \mapsto e^{-i\langle\xi,x\rangle}\right) d\mu(x).$$

This completes the proof of Theorem 4.1. □

Proof of Theorem 4.3. Let $u \in \mathcal{S}'$ and let φ and ψ be members of $\mathcal{S}$. Then there exists a finite constant C and a non-negative integer m such that

$$\left|u\left(f\right)\right| \leqslant C p_m(f), \quad \text{for all } f \in \mathcal{S}, \tag{4.12}$$

where

$$p_m(f) = \max_{|\alpha|\leqslant m}\sup_{y\in\mathbb{R}^n}\left(1 + |y|^2\right)^{m/2}\left|D_y^\alpha\left(f\right)(y)\right|.$$

Fix $x \in \mathbb{R}^n$. Then we obtain

$$
\begin{aligned}
|u(\tau_x(\check{\varphi}))| &\leqslant C \max_{|\alpha|\leqslant m} \sup_{y\in\mathbb{R}^n} \left(1+|y|^2\right)^{m/2} \left|D_y^\alpha(\tau_x\check{\varphi})(y)\right| \\
&\leqslant C \max_{|\alpha|\leqslant m} \sup_{y\in\mathbb{R}^n} \left(1+|y-x+x|^2\right)^{m/2} |D^\alpha(\varphi)(y-x)| \\
&\leqslant C \max_{|\alpha|\leqslant m} \sup_{y\in\mathbb{R}^n} \left(1+|y+x|^2\right)^{m/2} |D^\alpha(\varphi)(y)|
\end{aligned}
$$

$(1+|y+x|^2 \leqslant 2\left(1+|x|^2\right)\left(1+|y|^2\right))$

$$
\leqslant 2^{m/2} C\left(1+|x|^2\right)^{m/2} \max_{|\alpha|\leqslant m} \sup_{y\in\mathbb{R}^n} \left(1+|y|^2\right)^{m/2} |D^\alpha(\varphi)(y)|. \tag{4.13}
$$

From (4.13) the polynomial growth of the function $u * \varphi$ follows. Let $m \in \mathbb{N}$ and the constant C be as in (4.12).

Next we prove the equality $(u * \varphi) * \psi(x) = u * (\varphi * \psi)(x)$, $u \in \mathcal{S}'$, φ, $\psi \in \mathcal{S}$, $x \in \mathbb{R}^n$. To this end, fix $\varepsilon > 0$, and fix $x \in \mathbb{R}^n$, and choose a compact subset K_ε of $\mathbb{R}^n$ in such a way that the inequality

$$
2^{m/2} C p_m(\varphi) \int_{\mathbb{R}^n\setminus K_\varepsilon} \left(1+|y|^2\right)^{m/2} |\psi(x-y)|\, dy \leqslant \frac{1}{2}\varepsilon \tag{4.14}
$$

holds. Suppose that the collection $\{U_\varepsilon : \varepsilon > 0\}$ forms a neighborhood base of the origin consisting of compact subsets. We also choose a partition $B_{j,\varepsilon}$, $1 \leqslant j \leqslant N_\varepsilon$, of the compact set K_ε in such a way that the inequalities

$$
C \sup_{y_2, y_1 \in B_{j,\varepsilon}} p_m\left(\tau_{y_2}\check{\varphi} - \tau_{y_1}\check{\varphi}\right) \int_{K_\varepsilon} |\psi(x-y)|\, dy \leqslant \frac{1}{2}\varepsilon \tag{4.15}
$$

hold for all $1 \leqslant j \leqslant N_\varepsilon$. In addition, we assume that the vector-sum $B_{j,\varepsilon} - B_{j,\varepsilon}$ is contained in the neighborhood U_ε of 0. Moreover, the partitions $\{B_{j,\varepsilon} : 1 \leqslant j \leqslant N_\varepsilon\}$, $\varepsilon > 0$, are chosen in such a way that the partition corresponding to ε' refines the one corresponding to ε, provided $\varepsilon' < \varepsilon$. As in the proof of Theorem 4.1 this means that $K_{\varepsilon'} \supset K_\varepsilon$, and

$$
B_{j,\varepsilon} = \bigcup_{1\leqslant k\leqslant N_{\varepsilon'} : B_{k,\varepsilon'}\cap B_{j,\varepsilon}\neq\varnothing} B_{k,\varepsilon'} \quad \text{for } 1 \leqslant j \leqslant N_\varepsilon \text{ and } 0 < \varepsilon' < \varepsilon. \tag{4.16}
$$

Put

$$
g(z) = \tau_x\widecheck{(\varphi * \psi)}(z) = \int_{\mathbb{R}^n} \tau_y\check{\varphi}(z)\psi(x-y)\, dy, \quad \text{and} \tag{4.17}
$$

$$
g_\varepsilon(z) = \sum_{j=1}^{N_\varepsilon} \int_{B_{j,\varepsilon}} \tau_{y_{j,\varepsilon}}\check{\varphi}(z)\psi(x-y)dy. \tag{4.18}
$$

Here $y_{j,\varepsilon}$ is chosen in $B_{j,\varepsilon}$, $1 \leqslant j \leqslant N_\varepsilon$. Then

$C p_m(g - g_\varepsilon)$

$$
\begin{aligned}
&= Cp_m \left(\sum_{j=1}^{N_\varepsilon} \int_{B_{j,\varepsilon}} \left(\tau_y \check{\varphi}(\cdot) - \tau_{y_{j,\varepsilon}} \check{\varphi}(\cdot) \right) \psi(x-y)dy + \int_{\mathbb{R}^n \setminus K_\varepsilon} \tau_y \check{\varphi}(\cdot) \psi(x-y)dy \right) \\
&\leqslant Cp_m \left(\sum_{j=1}^{N_\varepsilon} \int_{B_{j,\varepsilon}} \left(\tau_y \check{\varphi}(\cdot) - \tau_{y_{j,\varepsilon}} \check{\varphi}(\cdot) \right) \psi(x-y)dy \right) \\
&\quad + Cp_m \left(\int_{\mathbb{R}^n \setminus K_\varepsilon} \tau_y \check{\varphi}(\cdot) \psi(x-y)dy \right) \\
&\leqslant C \sum_{j=1}^{N_\varepsilon} \int_{B_{j,\varepsilon}} p_m \left(\tau_y \check{\varphi}(\cdot) - \tau_{y_{j,\varepsilon}} \check{\varphi}(\cdot) \right) |\psi(x-y)| \, dy \\
&\quad + C \int_{\mathbb{R}^n \setminus K_\varepsilon} p_m \left(\tau_y \check{\varphi}(\cdot) \right) |\psi(x-y)| \, dy \\
&\leqslant C \max_{1 \leqslant j \leqslant N_\varepsilon} \sup_{y_1, y_2 \in B_{j,\varepsilon}} p_m \left(\tau_{y_2} \check{\varphi}(\cdot) - \tau_{y_1} \check{\varphi}(\cdot) \right) \sum_{j=1}^{N_\varepsilon} \int_{B_{j,\varepsilon}} |\psi(x-y)| \, dy \\
&\quad + 2^{m/2} C p_m(\varphi) \int_{\mathbb{R}^n \setminus K_\varepsilon} \left(1 + |y|^2 \right)^{m/2} |\psi(x-y)| \, dy \leqslant \varepsilon. \qquad (4.19)
\end{aligned}
$$

The final inequality in (4.19) follows from the choice of K_ε and the partition $B_{j,\varepsilon}$, $1 \leqslant j \leqslant N_\varepsilon$: see (4.14) and (4.15). From (4.12) and (4.19) we infer $|u(g-g_\varepsilon)| \leqslant \varepsilon$, and hence

$$\begin{aligned} u*(\varphi*\psi)(x) &= u(g) = \lim_{\varepsilon\downarrow 0} u(g_\varepsilon) \\ &= \int u(\tau_y\breve{\varphi})\,\psi(x-y)\,dy = \int (u*\varphi)(y)\psi(x-y)\,dy = (u*\varphi)*\psi(x). \end{aligned} \tag{4.20}$$

In the proof of the equalities in (4.20) the refinement property in (4.16) is applied. This proves Theorem 4.3. □

Proof of Theorem 4.4. This proof is along the same lines as the previous proof of Theorem 4.3. Choose a neighborhood base of the origin consisting of decreasing compact subsets U_ε, $\varepsilon > 0$. Again we fix $x \in \mathbb{R}^n$, and we define the function g as in (4.17). The support of the function g is compact and contained in the compact subset $x - \operatorname{supp}(\varphi) - \operatorname{supp}(\psi)$. Fix $\varepsilon_0 > 0$, and put $K_{\varepsilon_0} = U_{\varepsilon_0} + x - \operatorname{supp}(\varphi) - \operatorname{supp}(\psi)$. Since u belongs to $\mathcal{D}'$, there a constant C_0 and a non-negative integer m such that

$$|u(f)| \leqslant C_0 \max_{|\alpha|\leqslant m} \sup_{x\in K_{\varepsilon_0}-K_{\varepsilon_0}} |D^\alpha f(x)|, \tag{4.21}$$

for all functions $f \in \mathcal{D}$ with support contained in $K_{\varepsilon_0} - K_{\varepsilon_0}$. The semi-norm $f \mapsto q_m(f)$, $f \in \mathcal{D}_{K_{\varepsilon_0}-K_{\varepsilon_0}} := \{f \in \mathcal{D} : \operatorname{supp}(f) \subset K_{\varepsilon_0} - K_{\varepsilon_0}\}$ is defined by

$$q_m(f) = \max_{|\alpha|\leqslant m} \sup_{x\in K_{\varepsilon_0}-K_{\varepsilon_0}} |D^\alpha f(x)|, \quad f \in \mathcal{D}_{K_{\varepsilon_0}-K_{\varepsilon_0}}. \tag{4.22}$$

For $0 < \varepsilon < \varepsilon_0$ we choose a compact subset K_ε and a partition $\{B_{j,\varepsilon}\}_{1\leqslant j\leqslant N_\varepsilon}$ of K_ε with the following properties:

(1) If $\varepsilon' < \varepsilon$, then $K_{\varepsilon'} \supset K_\varepsilon$, and in fact $\mathbb{R}^n = \cup_{0<\varepsilon<\varepsilon_0} K_\varepsilon$;
(2) For every $1 \leqslant j \leqslant N_\varepsilon$ the inclusion $B_{j,\varepsilon} - B_{j,\varepsilon} \subset U_\varepsilon$ is true, and the inequality

$$C_0 \sup_{y_2,y_1\in B_{j,\varepsilon}} q_m(\tau_{y_2}\breve{\varphi} - \tau_{y_1}\breve{\varphi}) \int_{K_\varepsilon} |\psi(x-y)|\,dy \leqslant \frac{1}{2}\varepsilon \tag{4.23}$$

holds.
(3) The inequality

$$C_0 \sup_{y'\in x-\operatorname{supp}(\psi)} q_m(\tau_{y'}\breve{\varphi}) \int_{\mathbb{R}^n\setminus K_\varepsilon} |\psi(x-y)|\,dy \leqslant \frac{1}{2}\varepsilon \tag{4.24}$$

holds for all $\varepsilon > 0$.
(4) The partition corresponding to $\varepsilon' > 0$ refines the one corresponding to $\varepsilon > 0$, provided $\varepsilon' < \varepsilon$; *i.e.* $K_{\varepsilon'} \supset K_\varepsilon$ and (4.7) holds for $1 \leqslant j \leqslant N_\varepsilon$:

$$B_{j,\varepsilon} = \bigcup_{1\leqslant k\leqslant N_{\varepsilon'}:B_{k,\varepsilon'}\cap B_{j,\varepsilon}\neq\varnothing} B_{k,\varepsilon'} \quad \text{for } 0 < \varepsilon' < \varepsilon. \tag{4.25}$$

Notice that in order to establish the refinement property we need to take ε of the form $\varepsilon_\ell = 2^{-\ell}\varepsilon_0$, $\ell \in \mathbb{N}$. The inclusion $x - \operatorname{supp}(\varphi) - \operatorname{supp}(\psi) \subset K_{\varepsilon_0} \subset K_\varepsilon$, $\varepsilon \leqslant \varepsilon_0$, is true as well.

As in (4.17) respectively (4.18) the functions g and g_ε are defined by:

$$\begin{aligned} g(z) &= \tau_x(\varphi * \psi)\check{\ }(z) = \int_{\mathbb{R}^n} \tau_y\check{\varphi}(z)\psi(x-y)\,dy \\ &= \int_{K_\varepsilon} \tau_y\check{\varphi}(z)\psi(x-y)\,dy + \int_{\mathbb{R}^n\setminus K_\varepsilon} \tau_y\check{\varphi}(z)\psi(x-y)\,dy \\ &= \sum_{j=1}^{N_\varepsilon}\int_{B_{j,\varepsilon}} \tau_y\check{\varphi}(z)\psi(x-y)\,dy + \int_{\mathbb{R}^n\setminus K_\varepsilon} \tau_y\check{\varphi}(z)\psi(x-y)\,dy, \quad \text{and} \qquad (4.26) \\ g_\varepsilon(z) &= \sum_{j=1}^{N_\varepsilon}\int_{B_{j,\varepsilon}} \tau_{y_{j,\varepsilon}}\check{\varphi}(z)\psi(x-y)dy. \qquad (4.27) \end{aligned}$$

Here, for every $1 \leqslant j \leqslant N_\varepsilon$, the point $y_{j,\varepsilon}$ is chosen in $B_{j,\varepsilon}$. Notice that

$$\operatorname{supp}(g_\varepsilon) \subset U_\varepsilon + x - \operatorname{supp}(\varphi) - \operatorname{supp}(\psi).$$

To prove this assume that $z \in \mathbb{R}^n$ is such that $g_\varepsilon(z) \neq 0$. Then there exists $1 \leqslant j \leqslant N_\varepsilon$ and $y_j \in B_{j,\varepsilon}$ such that $y_{j,\varepsilon} - z \in \operatorname{supp}(\varphi)$ and $x - y_j \in \operatorname{supp}(\psi)$. It follows that z belongs to the set

$$\begin{aligned} y_{j,\varepsilon} - y_j + y_j - \operatorname{supp}(\varphi) &\subset B_{j,\varepsilon} - B_{j,\varepsilon} + x - \operatorname{supp}(\varphi) - \operatorname{supp}(\psi) \\ &\subset U_\varepsilon + x - \operatorname{supp}(\varphi) - \operatorname{supp}(\psi). \qquad (4.28) \end{aligned}$$

In addition we have

$$\operatorname{supp}(g) \subset x - \operatorname{supp}(\varphi) - \operatorname{supp}(\psi).$$

In a similar manner as we obtained (4.19) in the proof of Theorem 4.3 we now get $\mathcal{D}\text{-}\lim_{\varepsilon\downarrow 0} g_\varepsilon = g$, and by (4.23) and (4.24) we get

$$\begin{aligned} |u(g - g_\varepsilon)| &\leqslant C_0 \sup_{y_2,y_1\in B_{j,\varepsilon}} q_m\left(\tau_{y_2}\check{\varphi} - \tau_{y_1}\check{\varphi}\right)\int_{K_\varepsilon} |\psi(x-y)|\,dy \\ &\quad + C_0 \sup_{y'\in x-\operatorname{supp}(\psi)} q_m\left(\tau_{y'}\check{\varphi}\right)\int_{\mathbb{R}^n\setminus K_\varepsilon} |\psi(x-y)|\,dy \leqslant \varepsilon. \qquad (4.29) \end{aligned}$$

The same argument which we used to show (4.20) now applies, and completes the proof of Theorem 4.4. Like in the proof of Theorem 4.3 we again use the refinement property: see (4.25). □

4.5. Theorem. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then the following assertions hold:*

(i) *$u * \varphi \in C^\infty(\mathbb{R}^n)$, and $D^\alpha(u * \varphi) = D^\alpha u * \varphi = u * D^\alpha\varphi$.*
(ii) *$u * \varphi$ has polynomial growth, hence is a tempered distribution.*
(iii) *$(u * \varphi) * \psi = u * (\varphi * \psi)$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$.*
(iv) *$\widehat{u * \varphi} = \widehat{\varphi}\widehat{u}$.*
(v) *$\widehat{u} * \widehat{\varphi} = (2\pi)^n\widehat{\varphi u}$.*

Recall that $u * \varphi$ is defined as follows: $u * \varphi(x) = u(\tau_x \check{\varphi})$, $x \in \mathbb{R}^n$, where $\tau_x \varphi(y) = \varphi(y - x)$, x, $y \in \mathbb{R}^n$.

PROOF. The assertions in (i), (ii) and (iii) follow from Theorem 4.3.

In the proof of the items (iv) and (v) we will employ some results which will be proved in Theorem 4.11 item (4).

(iv). Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then we have

$$\begin{aligned}\widehat{u * \varphi}\,(\mathcal{F}\psi) &= u * \varphi\left(\mathcal{F}^2\psi\right) = (2\pi)^n\,(u * \varphi)\left(\check{\psi}\right)\\ &= (2\pi)^n \int u\,(\tau_x\check{\varphi})\,\check{\psi}(x)\,dx\end{aligned}$$

(Fubini's theorem for tempered distributions: see Theorem 4.3)

$$\begin{aligned}&= (2\pi)^n u\left(\int \tau_x\check{\varphi}\check{\psi}(x)\,dx\right) = (2\pi)^n u\left(y \mapsto \int \tau_x\check{\varphi}(y)\check{\psi}(x)\,dx\right)\\ &= (2\pi)^n u\left(\widecheck{\varphi * \psi}\right) = u\left(\mathcal{F}^2\left(\varphi * \psi\right)\right) = \widehat{u}\left(\mathcal{F}\left(\varphi * \psi\right)\right)\\ &= \widehat{u}\left(\widehat{\varphi}\widehat{\psi}\right) = \widehat{\varphi}\widehat{u}\left(\widehat{\psi}\right). \qquad (4.30)\end{aligned}$$

From (4.30) the assertion in (iv) follows.

(v). The assertion in (v) is a consequence of the following argument. From assertion (iv) we get

$$\mathcal{F}\left(\widehat{u} * \widehat{\varphi}\right) = (2\pi)^{2n}\breve{\varphi}\breve{u} = (2\pi)^{2n}\widecheck{\varphi u} = (2\pi)^{n}\mathcal{F}^{2}\left(\varphi u\right),$$

whence

$$\widehat{u} * \widehat{\varphi} = (2\pi)^{n}\widehat{\varphi u}.$$

This completes the proof of Theorem 4.5. □

4.6. THEOREM. *Suppose that at least one of the distributions u or v has compact support. Fix $\alpha \in \mathbb{N}^n$. Then the following assertions hold:*

(i) $(u * v) * \varphi = u * (v * \varphi)$, $\varphi \in \mathcal{D}$;
(ii) $\operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v)$;
(iii) $D^{\alpha}(u * v) = D^{\alpha}u * v = u * D^{\alpha}v$;
(iv) $\delta * u = u$, $D^{\alpha}u = (D^{\alpha}\delta) * u$;
(v) $\delta_p * u = \tau_p u$, $p \in \mathbb{R}^n$;
(vi) $u * v = v * u$;
(vii) *If, in addition, w has compact support, then $(u * v) * w = u * (v * w)$.*

PROOF. (i) From the definitions it follows that

$$(u * v) * \varphi(x) = (u * v)\left(\tau_x \breve{\varphi}\right) = u\left(\breve{v} * \tau_x \breve{\varphi}\right). \tag{4.31}$$

We also have

$$\begin{aligned}\breve{v} * \tau_x \breve{\varphi}(y) &= \breve{v}\left(\tau_y\left(\widecheck{(\tau_x \breve{\varphi})}\right)\right) = \breve{v}\left(z \mapsto \tau_y\left(\widecheck{(\tau_x \breve{\varphi})}\right)(z)\right) \\ &= \breve{v}\left(z \mapsto \widecheck{(\tau_x \breve{\varphi})}(z-y)\right) = \breve{v}\left(z \mapsto \left(\tau_x \breve{\varphi}\right)(y-z)\right) \\ &= \breve{v}\left(z \mapsto \breve{\varphi}(y-z-x)\right) = v\left(z \mapsto \breve{\varphi}(y+z-x)\right) \\ &= v\left(\tau_{x-y}\breve{\varphi}\right) = \tau_x\left(\widecheck{v * \varphi}\right)(y),\end{aligned} \tag{4.32}$$

because we have

$$\tau_x\left(\widecheck{v * \varphi}\right)(y) = \widecheck{(v * \varphi)}(y-x) = v * \varphi(x-y) = v\left(\tau_{x-y}\breve{\varphi}\right). \tag{4.33}$$

Hence, from (4.31), (4.32), and (4.33) it follows that:

$$(u * v) * \varphi(x) = u\left(\tau_x \widecheck{(v * \varphi)}\right) = \left(u * (v * \varphi)\right)(x). \tag{4.34}$$

Note that (4.34) is the same as (i).

(ii) Assertion (ii) is left as an exercise.

(iii) For the proof of this equality we need the continuity theorem below: Theorem 4.7. The equality in (iii) is proved with τ_p instead of D^{α}. The equality with derivatives then follows from Theorem 4.7.

(iv) and (v). These assertions are left as an exercise for the reader.

(vi) Let φ and ψ be members of $\mathcal{D}$. Then we have:

$$(u * v) * (\varphi * \psi) = (u * v) * (\psi * \varphi) = u * (v * (\psi * \varphi))$$

$$\begin{aligned}
&= u * ((v * \psi) * \varphi) = u * (\varphi * (v * \psi)) \\
&= (u * \varphi) * (v * \psi) = (v * \psi) * (u * \varphi) \\
&= (v * u) * (\varphi * \psi) . \qquad (4.35)
\end{aligned}$$

Assertion (vi) follows from (4.35).

(vii). Let φ be a member of $\mathcal{D}$. On the one hand we have

$$u * (v * (w * \varphi)) = (u * v) * (w * \varphi) = ((u * v) * w) * \varphi. \qquad (4.36)$$

On the other hand we also have

$$u * (v * (w * \varphi)) = u * ((v * w) * \varphi) = (u * (v * w)) * \varphi. \qquad (4.37)$$

Assertion (vii) follows from (4.36) and (4.37). □

4.7. Theorem. *Let u_j be a sequence of distributions with compact support which in $\mathcal{E}'$ converges pointwise to a distribution with compact support u. In addition, let v_j be sequence of distributions which converges in $\mathcal{D}'$ to a distribution v. Then*

$$\lim_{j\to\infty} (u_j * v_j)(\varphi) = (u * v)(\varphi), \quad \varphi \in \mathcal{D}. \qquad (4.38)$$

Proof. By the theorem of Banach-Steinhaus we have:

$$\begin{aligned}
&|(u_j * v_j)(\varphi) - (u * v)(\varphi)| = |(u_j * (v_j - v))(\varphi) + ((u_j - u) * v)(\varphi)| \\
&\leqslant |(u_j * (v_j - v))(\varphi)| + |((u_j - u) * v)(\varphi)| \\
&= |u_j (x \mapsto (v_j (y \mapsto \varphi(x + y)) - v (y \mapsto \varphi (x + y))))| \\
&\quad + |(u_j - u)(x \mapsto v (y \mapsto \varphi(x + y)))| \\
&\leqslant C \max_{|\alpha|\leqslant m} \sup_{x\in K} |v_j (y \mapsto D^\alpha\varphi(x + y)) - v (y \mapsto D^\alpha\varphi (x + y))| \\
&\quad + |(u_j - u)(x \mapsto v (y \mapsto \varphi(x + y)))| . \qquad (4.39)
\end{aligned}$$

The assertion in Theorem 4.7 follows from (4.39). □

We also need the following result.

4.8. Theorem. *Let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ which converges to φ in $\mathcal{D}$, and let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$ which converges to $u \in \mathcal{D}'$. Then $\mathcal{E}\text{-}\lim_{k\to\infty} u_k * \varphi_k = u * \varphi$.*

Proof. Let K be a compact subset of $\mathbb{R}^n$, and $\alpha \in \mathbb{N}^n$. Put $\psi_k = D^\alpha\varphi_k$, $k \in \mathbb{N}$. Then for appropriately chosen $x_k \in K$, $m \in \mathbb{N}$, and compact subset $K' \supset K - K_0$, where $K_0 \supset \operatorname{supp}(\psi_k)$, $k \in \mathbb{N}$, we have for $x \in K$

$$\begin{aligned}
&\sup_{x\in K} |u_k * \psi_k(x) - u * \psi(x)| = \sup_{x\in K} \left|u_k\left(\tau_x\check{\psi}_k\right) - u\left(\tau_x\check{\psi}\right)\right| = \left|u_k\left(\tau_{x_k}\check{\psi}_k\right) - u\left(\tau_{x_k}\check{\psi}\right)\right| \\
&\leqslant \left|u_k\left(\tau_{x_k}\check{\psi}_k\right) - u_k\left(\tau_x\check{\psi}\right)\right| + \left|u_k\left(\tau_x\check{\psi}\right) - u\left(\tau_x\check{\psi}\right)\right| + \left|u\left(\tau_x\check{\psi}\right) - u\left(\tau_{x_k}\check{\psi}\right)\right| \\
&\leqslant C \max_{|\beta|\leqslant m} \sup_{y\in K'} \left|\tau_{x_k}D^\beta\check{\psi}_k(y) - \tau_x D^\beta\check{\psi}(y)\right| + \left|u\left(\tau_x\check{\psi}\right) - u\left(\tau_{x_k}\check{\psi}\right)\right|
\end{aligned}$$

$$+\left|u_k\left(\tau_x\check{\psi}\right)-u\left(\tau_x\check{\psi}\right)\right|, \tag{4.40}$$

where by passing to a subsequence we assume that $x_k \to x'$ in K as $k \to \infty$. From (4.40) we infer

$$\begin{aligned}
&\sup_{x\in K}\left|u_k*\psi_k(x)-u*\psi(x)\right| \\
&\leqslant \inf_{x\in K}\left\{C\max_{|\beta|\leqslant m}\sup_{y\in K'}\left|\tau_{x_k}D^\beta\check{\psi}_k(y)-\tau_x D^\beta\check{\psi}(y)\right|+\left|u\left(\tau_x\check{\psi}\right)-u\left(\tau_{x_k}\check{\psi}\right)\right|\right\} \\
&\quad+\sup_{x\in K}\left|u_k\left(\tau_x\check{\psi}\right)-u\left(\tau_x\check{\psi}\right)\right| \\
&\leqslant C\max_{|\beta|\leqslant m}\sup_{y\in K'}\left|\tau_{x_k}D^\beta\check{\psi}_k(y)-\tau_{x'}D^\beta\check{\psi}(y)\right|+\left|u\left(\tau_{x'}\check{\psi}\right)-u\left(\tau_{x_k}\check{\psi}\right)\right| \\
&\quad+\sup_{x\in K}\left|u_k\left(\tau_x\check{\psi}\right)-u\left(\tau_x\check{\psi}\right)\right|.
\end{aligned} \tag{4.41}$$

Then $\tau_{x_k}D^\beta\check{\psi}_k$ converges to $\tau_x D^\beta\check{\psi}$ in $\mathcal{D}_{K'}$, and hence the first term in the right-hand-side of (4.40) converges to 0, and so do the second and third terms.

This completes the proof of Theorem 4.8. □

4.9. Proposition. *Let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ which converges to $\varphi\in\mathcal{D}$ in the topology of $\mathcal{D}$. If $\lim_{k\to\infty}x_k = x$ in $\mathbb{R}^n$. Then $\mathcal{D}\text{-}\lim_{k\to\infty}\tau_{x_k}\varphi_k=\tau_x\varphi$.*

A similar result is true in the spaces $\mathcal{E}$ and $\mathcal{S}$.

Proof of Proposition 4.9. Let the notation and hypotheses be as in Proposition 4.9, and let $y \in \mathbb{R}^n$. The equality

$$\tau_{x_k}\varphi_k(y) - \tau_x\varphi(y) = \tau_{x_k}(\varphi_k - \varphi)(y) + (\tau_{x_k} - \tau_x)\varphi(y)$$

yields the inequalities

$$\begin{aligned}\|\tau_{x_k}\varphi_k - \tau_x\varphi\|_\infty &\leqslant \|\tau_{x_k}(\varphi_k - \varphi)\|_\infty + \|(\tau_{x_k} - \tau_x)\varphi\|_\infty \\ &\leqslant \|\varphi_k - \varphi\|_\infty + \|(\tau_{x_k} - \tau_x)\varphi\|_\infty . \end{aligned} \tag{4.42}$$

From (4.42) we infer

$$\lim_{k\to\infty} \|\tau_{x_k}\varphi_k - \tau_x\varphi\|_\infty = 0. \tag{4.43}$$

The above argument can be repeated with $D^\alpha\varphi_k$ and $D^\alpha\varphi$ instead of φ_k and φ. These observations complete the first part of the proof of Proposition 4.9. The second part is left as an exercise for the reader. □

4.10. Proposition. *Let the sequence $(u_j)_{j\in\mathbb{N}} \subset \mathcal{E}'$ be such that $\mathcal{E}'\text{-}\lim_{j\to\infty} u_j = u$, let $(v_j)_{j\in\mathbb{N}} \subset \mathcal{D}'$ be such that $\mathcal{D}'\text{-}\lim_{j\to\infty} v_j = v$, and let $(\varphi_j)_{j\in\mathbb{N}} \subset \mathcal{D}$ be such that $\mathcal{D}\text{-}\lim_{j\to\infty} \varphi_j = \varphi$. Then $\mathcal{E}\text{-}\lim_{j\to\infty} u_j * v_j * \varphi_j = u * v * \varphi$.*

Proof. Let $x \in \mathbb{R}^n$. We write

$$\begin{aligned}&u_j * v_j * \varphi_j(x) - u * v * \varphi(x) = (u_j * v_j)(\tau_x(\check{\varphi}_j)) - (u * v)(\tau_x(\check{\varphi})) \\ &= u_j(\check{v}_j * \tau_x(\check{\varphi}_j)) - u(\check{v} * \tau_x(\check{\varphi})) \\ &= u_j(\check{v}_j * \tau_x(\check{\varphi}_j)) - u_j(\check{v} * \tau_x(\check{\varphi})) + u_j(\check{v} * \tau_x(\check{\varphi})) - u(\check{v} * \tau_x(\check{\varphi})) . \end{aligned} \tag{4.44}$$

Since $\mathcal{E}'\text{-}\lim_{j\to\infty} u_j = u$ there exists a constant C, a compact subset K of $\mathbb{R}^n$, and a positive integer m such that $|u_j(f)| \leqslant C \max_{|\alpha|\leqslant m} \sup_{y\in K} |D^\alpha f(y)|$ for all $f \in \mathcal{E}$. This knowledge combined with (4.44) yields:

$$\begin{aligned}&|u_j * v_j * \varphi_j(x) - u * v * \varphi(x)| \\ &\leqslant |u_j(\check{v}_j * \tau_x(\check{\varphi}_j)) - u_j(\check{v} * \tau_x(\check{\varphi}))| + |u_j(\check{v} * \tau_x(\check{\varphi})) - u(\check{v} * \tau_x(\check{\varphi}))| \\ &\leqslant C \max_{|\alpha|\leqslant m} \sup_{y\in K} |D^\alpha(\check{v}_j * \tau_x(\check{\varphi}_j) - \check{v} * \tau_x(\check{\varphi}))(y)| + |u_j(\check{v} * \tau_x(\check{\varphi})) - u(\check{v} * \tau_x(\check{\varphi}))| \\ &= C \max_{|\alpha|\leqslant m} \sup_{y\in K-x} |D^\alpha(\check{v}_j * \check{\varphi}_j - \check{v} * \check{\varphi})(y)| + |u_j(\check{v} * \tau_x(\check{\varphi})) - u(\check{v} * \tau_x(\check{\varphi}))| . \end{aligned} \tag{4.45}$$

Let K' be an arbitrary compact subset of $\mathbb{R}^n$, and put $K'' = K - K'$. Employing (4.45) we obtain for $x \in K'$

$$\begin{aligned}&\sup_{x\in K'} |u_j * v_j * \varphi_j(x) - u * v * \varphi(x)| \\ &\leqslant C \max_{|\beta|\leqslant m} \sup_{y\in K-K'} |D^\alpha(\check{v}_j * \check{\varphi}_j - \check{v} * \check{\varphi})(y)| + \sup_{x\in K'} |u_j(\check{v} * \tau_x(\check{\varphi})) - u(\check{v} * \tau_x(\check{\varphi}))| \\ &= C \max_{|\beta|\leqslant m} \sup_{y\in K-K'} |D^\alpha(\check{v}_j * \check{\varphi}_j - \check{v} * \check{\varphi})(y)| + \left|u_j\left(\check{v} * \tau_{x_j}(\check{\varphi})\right) - u\left(\check{v} * \tau_{x_j}(\check{\varphi})\right)\right| \end{aligned} \tag{4.46}$$

for an appropriate sequence $(x_j)_{j\in\mathbb{N}} \subset K'$. By Theorem 4.8 the first term in the right-hand-side of (4.46) tends to 0 as $j \to \infty$. The second term also tends to zero, because otherwise by passing to a subsequence we may assume that $x_j \to x$ for some

$x \in K'$. Then $\mathcal{E}\text{-}\lim_{j\to\infty} \check{v} * \tau_{x_j}(\check{\varphi})\, \check{v} * \tau_x(\check{\varphi})$, so that the second term in (4.46) tends to 0 as $j \to \infty$.

The above arguments can be repeated with derivatives of the functions φ_k and φ to obtain the result in Proposition 4.10. □

2. Fourier transform and its inverse

Let f be a function in $L^1(\mathbb{R}^n)$. Then its Fourier transform $\widehat{f}$ is defined by $\widehat{f}(\xi) = \int e^{-i\xi\cdot x} f(x)dx$, $\xi \in \mathbb{R}^n$. The symbol $\xi \cdot x$ stands for the inner-product of the vector $x = (x_1, \dots, x_n)$ and the vector $\xi = (\xi_1, \dots, \xi_n)$:

$$\xi \cdot x = \langle \xi, x \rangle = \sum_{j=1}^{n} \xi_j \overline{x}_j.$$

4.11. Theorem. *The following assertions are true.*

(1) *If f and g belong to $L^1(\mathbb{R}^n)$, then*

$$\int \widehat{f}(y)g(y)dy = \int f(y)\widehat{g}(y)dy. \tag{4.47}$$

(2) *If φ belongs to $\mathcal{S}(\mathbb{R}^n)$, then its Fourier transform $\widehat{\varphi}$ belongs to $\mathcal{S}(\mathbb{R}^n)$ as well. Moreover, the mapping $\varphi \mapsto \widehat{\varphi}$ is continuous as a mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.*

(3) *Every function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written as the Fourier transform of a unique function ψ in $\mathcal{S}(\mathbb{R}^n)$. In fact, put*

$$\psi(\xi) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \varphi(x)dx. \tag{4.48}$$

Then

$$\varphi(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \widehat{\varphi}(\xi)d\xi, \quad \text{and} \tag{4.49}$$

$$\varphi(x) = \widehat{\psi}(x) = \int e^{-i\xi\cdot x} \psi(\xi)d\xi. \tag{4.50}$$

(4) *The mapping $\mathcal{F} : \varphi \mapsto \widehat{\varphi}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a linear homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself. Its inverse is given by the mapping*

$$\mathcal{F}^{-1}(\psi) = \frac{1}{(2\pi)^n} \widetilde{\mathcal{F}(\psi)} = \frac{1}{(2\pi)^n} \check{\widehat{\psi}} = \frac{1}{(2\pi)^n} \widehat{\check{\psi}} = \frac{1}{(2\pi)^n} \mathcal{F}\left(\check{\psi}\right). \tag{4.51}$$

Moreover, $\mathcal{F}^2(\varphi) = (2\pi)^n \check{\varphi}$, and $\mathcal{F}^4(\varphi) = (2\pi)^{2n} \varphi$.

(5) *(Plancherel's formula) Let the functions φ and ψ belong to $\mathcal{S}(\mathbb{R}^n)$. Then*

$$\int \varphi(x)\overline{\psi(x)}dx = \frac{1}{(2\pi)^n} \int \widehat{\varphi}(z)\overline{\widehat{\psi}(z)}dz. \tag{4.52}$$

(6) *Put* $\Phi(\varphi)(z) = \dfrac{1}{(2\pi)^{n/2}}\widehat{\varphi}(z)$, $z \in \mathbb{R}^n$. *Then the mapping* $\varphi \mapsto \Phi(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, *extends to a unitary operator on the space* $L^2(\mathbb{R}^n)$.

(7) *Let* f *and* g *be functions in* $\mathcal{S}$. *Then*

$$\widehat{f * g} = \widehat{f}\widehat{g}. \tag{4.53}$$

(8) *Let* f *and* g *be functions in* $\mathcal{S}$. *Then*

$$\widehat{f} * \widehat{g} = (2\pi)^n \widehat{f \cdot g}. \tag{4.54}$$

PROOF. (1) This assertion is a direct consequence of Fubini's theorem.

(2) This assertion follows from the following two equalities:

$$\xi_j\widehat{\varphi}(\xi) = \widehat{D_j\varphi}(\xi), \quad \text{and}$$

$$D_j\widehat{\varphi}(\xi) = \text{Fourier transform of the function } x \mapsto -x_j\varphi(x) \text{ evaluated at } \xi.$$

Here $D_j\varphi(x) = \dfrac{\partial}{i\partial x_j}\varphi(x)$ or $D_j\widehat{\varphi}(\xi) = \dfrac{\partial}{i\partial \xi_j}\widehat{\varphi}(\xi)$. Moreover, if φ belongs to $\mathcal{S}(\mathbb{R}^n)$, then so does $D_j\varphi$ and also the function $x \mapsto x_j\varphi(x)$. Moreover, these operations are continuous as linear operators on the space $\mathcal{S}(\mathbb{R}^n)$. So the continuity of the Fourier transform also follows.

(3) First we prove the equality in (4.49). The equality in (4.50) will be a consequence of (4.49): see (4.63) below. In order to prove (3) we proceed as follows. Put $f_\lambda(x) = \psi\left(\lambda^{-1}x\right)$, $\lambda > 0$. Then

$$\widehat{f_\lambda}(y) = \int e^{-iy\cdot x}\psi\left(\lambda^{-1}x\right)dx = \lambda^n\widehat{\psi}\left(\lambda y\right). \tag{4.55}$$

Then by (4.47) we have

$$\begin{aligned}\int \varphi\left(\lambda^{-1}y\right)\widehat{\psi}(y)dy &= \lambda^n\int\varphi(y)\widehat{\psi}(\lambda y)dy = \int\varphi(y)\widehat{f_\lambda}(y)dy\\ &= \int\widehat{\varphi}(y)f_\lambda(y)dy = \int\widehat{\varphi}(y)\psi\left(\lambda^{-1}y\right)dy.\end{aligned} \tag{4.56}$$

In (4.56) we let λ tend to ∞ to obtain:

$$\varphi(0)\int\widehat{\psi}(y)dy = \psi(0)\int\widehat{\varphi}(y)dy. \tag{4.57}$$

From (4.57) we infer

$$\varphi(0) = \frac{\psi(0)}{\int\widehat{\psi}(y)dy}\int\widehat{\varphi}(y)dy \tag{4.58}$$

provided $\int\widehat{\psi}(y)dy \neq 0$. The equality in (4.58) holds for all $\psi \in \mathcal{S}\left(\mathbb{R}^n\right)$ for which $\int\widehat{\psi}(y)dy \neq 0$. We choose $\psi(y) = e^{-\frac{1}{2}|y|^2}$. Then we have

$$\widehat{\psi}(x) = \int e^{-ix\cdot y-\frac{1}{2}|y|^2}dy = e^{-\frac{1}{2}|x|^2}\prod_{j=1}^n\int_{-\infty}^{\infty}e^{-\frac{1}{2}(y+ix_j)^2}dy$$

(Cauchy's theorem for holomorphic functions)

$$= e^{-\frac{1}{2}|x|^2}\left(\int_{-\infty}^{\infty}e^{-\frac{1}{2}y^2}dy\right)^n = e^{-\frac{1}{2}|x|^2}\left(\sqrt{2\pi}\right)^n. \tag{4.59}$$

Hence we see

$$\int\widehat{\psi}(x)dx = (2\pi)^{n/2}\int e^{-\frac{1}{2}|x|^2}dx = (2\pi)^n. \tag{4.60}$$

From (4.58), (4.59), and (4.60) it follows that

$$\varphi(0)\left(2\pi\right)^n = \varphi(0)\int_{\mathbb{R}^n}\widehat{\psi}(y)dy = \psi(0)\int_{\mathbb{R}^n}\widehat{\varphi}(y)dy = \int_{\mathbb{R}^n}\widehat{\varphi}(y)dy. \tag{4.61}$$

From (4.61) it follows that

$$\begin{aligned}\varphi(x) = \tau_{-x}\varphi(0) &= \frac{1}{(2\pi)^n}\int\left(\widehat{\tau_{-x}\varphi}\right)(y)dy = \frac{1}{(2\pi)^n}\iint e^{-iz\cdot y}\tau_{-x}\varphi(z)dz\,dy\\ &= \frac{1}{(2\pi)^n}\iint e^{-i(z+x)\cdot y}e^{ix\cdot y}\varphi(z+x)dz\,dy = \frac{1}{(2\pi)^n}\int e^{ix\cdot y}\widehat{\varphi}(y)dy.\end{aligned} \tag{4.62}$$

The result (4.49) in (3) follows from (4.62). The second equality (4.50) in (3) can be obtained from (4.49) as follows. By the definition of $\psi(\xi)$ we see

$$\int e^{-i\xi\cdot x}\psi(\xi)d\xi = \int e^{-i\xi\cdot x}\frac{1}{(2\pi)^n}\int e^{i\xi\cdot y}\varphi(y)dy\,d\xi = \frac{1}{(2\pi)^n}\int e^{i\xi\cdot x}\int e^{-i\xi\cdot y}\varphi(y)dy\,d\xi$$
$$= \frac{1}{(2\pi)^n}\int e^{i\xi\cdot x}\widehat{\varphi}(\xi)d\xi = \varphi(x) \tag{4.63}$$

where in the final equality of (4.63) we employed (4.49).

(4) The equalities in (4.51) follow from (3). The equalities in (4.51) imply the equality $\mathcal{F}^2(\check{\psi}) = (2\pi)^n\,\psi$, and hence $\mathcal{F}^2(\psi) = (2\pi)^n\,\check{\psi}$. The assertion about $\mathcal{F}^4$ is a direct consequence of the fact that $\mathcal{F}^2$ is, up to the constant $(2\pi)^n$ a point reflection.

(5) The proof of (5) requires again (4.49). More precisely, (4.49) applied to the function ψ yields:

$$\int \varphi(x)\overline{\psi(x)}dx = \frac{1}{(2\pi)^n}\int \varphi(x)\overline{\int e^{iz\cdot x}\widehat{\psi}(z)dz}dx = \frac{1}{(2\pi)^n}\int \varphi(x)\int e^{-iz\cdot x}\overline{\widehat{\psi}(z)}dz\,dx$$

(Fubini)

$$= \frac{1}{(2\pi)^n}\int\int \varphi(x)e^{-iz\cdot x}dx\overline{\widehat{\psi}(z)}dz = \frac{1}{(2\pi)^n}\int \widehat{\varphi}(z)\overline{\widehat{\psi}(z)}dz. \tag{4.64}$$

(6) From (4.52) and the definition of Φ we obtain

$$\langle\varphi,\psi\rangle_{L^2(\mathbb{R}^n)} = \langle\Phi\varphi,\Phi\psi\rangle_{L^2(\mathbb{R}^n)}, \quad \varphi,\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{4.65}$$

The same equality holds for Φ^{-1} instead of Φ. Next let f belong to $L^2(\mathbb{R}^n)$. Then there exists a sequence of functions $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ such that $f = L^2\text{-}\lim_{k\to\infty} f_k$. From (4.65) and the linearity of Φ it follows that

$$\|\Phi(f_k) - \Phi(f_\ell)\|_{L^2(\mathbb{R}^n)} = \left\|\Phi^{-1}(f_k) - \Phi^{-1}(f_\ell)\right\|_{L^2(\mathbb{R}^n)} = \|f_k - f_\ell\|_{L^2(\mathbb{R}^n)}. \tag{4.66}$$

Since the sequence $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the space $L^2(\mathbb{R}^n)$, from (4.66) it follows that the sequences $(\Phi(f_k))_{k\in\mathbb{N}}$ and $(\Phi^{-1}(f_k))_{k\in\mathbb{N}}$ are Cauchy sequences in $L^2(\mathbb{R}^n)$ as well. It follows that it makes sense to define the function $\Phi(f) \in L^2(\mathbb{R}^n)$ and $\Phi^{-1}(f) \in L^2(\mathbb{R}^n)$ by $\Phi(f) = L^2\text{-}\lim_{k\to\infty}\Phi(f_k)$, and $\Phi^{-1}(f) = L^2\text{-}\lim_{k\to\infty}\Phi^{-1}(f_k)$. In addition, from (4.65) we infer that this equality remains true for φ and $\psi \in L^2(\mathbb{R}^n)$. It also follows that Φ is a surjective linear map from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. It follows that the map Φ can be considered as a unitary map from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$.

(7) In order to prove the equality in (4.53) we apply Fubini's theorem to obtain:

$$\widehat{f*g}(x) = \int e^{-ix\cdot y}(f*g)(y)\,dy = \int e^{-ix\cdot y}\int f(y-z)g(z)\,dz\,dy$$
$$= \int e^{-ix\cdot z}\int e^{-ix\cdot(y-z)}f(y-z)\,dyg(z)\,dz$$

(translation invariance of the Lebesgue measure)

$$= \int e^{-ix\cdot z} \int e^{-ix\cdot y} f\,(y)\; dy\, g(z)\, dz$$
$$= \widehat{f}(x)\widehat{g}(x),$$

which proves the equality in (4.53).

(8). Let f and g be members of $\mathcal{S}$. Then by Fubini and (4.49) we obtain

$$\begin{aligned}
\widehat{f} * \widehat{g}(x) &= \int \widehat{f}(x-y)\widehat{g}(y)\,dy = \int \left(\int e^{-i(x-y)\cdot z} f(z)\,dz \right) \widehat{g}(y)\,dy \\
&= \int e^{-ix\cdot z} f(z) \left(\int e^{iy\cdot z}\widehat{g}(y)\,dy \right) dz = (2\pi)^n \int e^{-ix\cdot z} f(z) g(z)\,dz \\
&= (2\pi)^n \widehat{f\cdot g}(x), \quad x \in \mathbb{R}^n. \qquad (4.67)
\end{aligned}$$

The equality in (4.67) proves item (7) of Theorem 4.11. $\square$

2.1. Riesz-Thorin interpolation. We conclude this section with a formulation of the Riesz-Thorin interpolation theorem, a proof of which can be found in Reed and Simon [**105**], Theorem IX.21, page 40. In Theorem 6.68 of Section 3 in Chapter 6 we formulate and prove a version of Stein's interpolation theorem. Stein's interpolation theorem is a generalization of the Riesz-Thorin interpolation theorem, which we now formulate.

4.12. THEOREM. *Let $(E_j, \mathcal{A}_j, m_j)$, $j = 0, 1$, be σ-finite measure spaces, and fix $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. Put*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}. \tag{4.68}$$

Let

$$T : L^{p_0}(E_0, \mathcal{A}_0, m_0) + L^{p_1}(E_0, \mathcal{A}_0, m_0) \to L^{q_0}(E_1, \mathcal{A}_1, m_1) + L^{q_1}(E_1, \mathcal{A}_1, m_1)$$

be a linear operator. Suppose that

$$M_0 := \|T\|_{q_0,p_0} < \infty, \quad \text{and} \quad M_1 := \|T\|_{q_1,p_1} < \infty.$$

Then, for each $0 < t < 1$, $T : L^{p_t}(E_0, \mathcal{A}_0, m_0) \to L^{q_t}(E_1, \mathcal{A}_1, m_1)$ and

$$\|T\|_{q_t,p_t} \leqslant M_0^{1-t} M_1^t. \tag{4.69}$$

The space $L^{p_0}(E_0, \mathcal{A}_0, m_0) + L^{p_1}(E_0, \mathcal{A}_0, m_0)$ is supplied with the norm

$$f \mapsto$$
$$\inf \left\{ \|f_0\|_{p_0} + \|f_1\|_{p_1} : f = f_0 + f_1, f_0 \in L^{p_0}(E_0, \mathcal{A}_0, m_0),\ f_1 \in L^{p_1}(E_0, \mathcal{A}_0, m_0) \right\}.$$

4.13. REMARK. Let, for $0 < t < 1$, p_t be determined by (4.68), and suppose $1 \leqslant p_0 < p_1$. Then $p_0 < p_t < p_1$, and

$$L^{p_t}(E_0, \mathcal{A}_0, m_0) \subset L^{p_0}(E_0, \mathcal{A}_0, m_0) + L^{p_1}(E_0, \mathcal{A}_0, m_0).$$

Let $f \in L^{p_t}(E_0, \mathcal{A}_0, m_0)$, then $f_0 := f\mathbf{1}_{\{|f|>1\}}$ belongs to $L^{p_0}(E_0, \mathcal{A}_0, m_0)$, and $f_1 := f\mathbf{1}_{\{|f|\leqslant 1\}}$ belongs to $L^{p_1}(E_0, \mathcal{A}_0, m_0)$. Moreover, $f = f_0 + f_1$.

Next let $Tf = \widehat{f}$, $f \in \mathcal{S}(\mathbb{R}^n)$. This means that T is the Fourier transform. Then T extends to a linear mapping from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with norm $\|T\|_{\infty,1} = 1$. In fact, if $f \in L^1(\mathbb{R}^n)$, then $Tf \in C_0(\mathbb{R}^n)$. In addition, T also can be considered as a mapping from $L^2(\mathbb{R}^n)$ onto itself with norm $\|T\|_{2,2} = \left(\sqrt{2\pi}\right)^n$. If $1 \leqslant p \leqslant 2 \leqslant q$, with $\frac{1}{p} + \frac{1}{q} = 1$, then T extends as a continuous linear mapping from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with norm

$$\|T\|_{q,p} \leqslant \|T\|_{\infty,1}^{1-2/q} \|T\|_{2,2}^{2/q} = (2\pi)^{n/q}. \tag{4.70}$$

This result is due to Titchmarsh. Inequality (4.70) follows from Theorem 4.12 in Theorem 4.12 by observing that the choice $t = 2 - \frac{2}{p} = \frac{2}{q}$ satisfies:

$$\frac{1}{p} = \frac{1}{p_t} = \frac{1-t}{1} + \frac{t}{2}, \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{2},$$

where $p_0 = 1$, $q_0 = \infty$, $p_1 = 2$, $q_1 = 2$. Hence if $1 \leqslant p \leqslant 2 \leqslant q$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$, and if $f \in L^p(\mathbb{R}^n)$, then we have

$$\left\|\widehat{f}\right\|_{L^q(\mathbb{R}^n)} \leqslant (2\pi)^{n/q} \|f\|_{L^q(\mathbb{R}^n)}.$$

Suppose that the function f belongs to $L^p(\mathbb{R}^n)$ where $1 \leqslant p \leqslant 2$. Then the function f belongs to the vector sum $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. In fact, put $f_1 = f\mathbf{1}_{\{|f|>1\}}$ and $f_2 = f\mathbf{1}_{\{|f|\leqslant 1\}}$. Then, on the set $\{|f| > 1\}$ the inequality $|f| \leqslant |f|^p$ holds, and hence $f_1 \in L^1(\mathbb{R}^n)$. On the set $\{|f| \leqslant 1\}$ the inequality $|f|^2 \leqslant |f|^p$ is valid, and hence $f_2 \in L^2(\mathbb{R}^n)$. It follows that, for $1 \leqslant p \leqslant 2$, the inclusion

$$L^p(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$$

holds.

Essentially speaking Theorem 1.50 is the same as the one below, but its proof is quite different.

4.14. THEOREM. *Suppose* $0 < \Re\lambda < n$, *and put*

$$c_\lambda = \frac{1}{2^\lambda} \frac{1}{(\sqrt{\pi})^n} \frac{\Gamma\left(\frac{1}{2}(n-\lambda)\right)}{\Gamma\left(\frac{1}{2}\lambda\right)}, \tag{4.71}$$

and let u_λ *be the tempered distribution given by*

$$u_\lambda(x) = \frac{c_\lambda}{|x|^{n-\lambda}}. \tag{4.72}$$

Then its Fourier transform is given by

$$\widehat{u}_\lambda(\xi) = \frac{1}{|\xi|^\lambda}. \tag{4.73}$$

In particular if $\lambda = 2$ *and* $n \geqslant 3$, *then*

$$u_2(x) = \frac{1}{4} \frac{1}{\pi^{n/2}} \frac{\Gamma\left(\frac{1}{2}(n-2)\right)}{|x|^{n-2}} \quad \textit{and} \quad \widehat{u}_2(\xi) = \frac{1}{|\xi|^2}. \tag{4.74}$$

PROOF. We write $u \mapsto \mathcal{F}(u)$, $u \in \mathcal{S}'(\mathbb{R}^n)$, for the mapping $u \mapsto \widehat{u}$. By definition $\mathcal{F}(u)$ is given by $\mathcal{F}(u)(\varphi) = u(\mathcal{F}(\varphi))$, where $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int e^{-i\xi\cdot x}\varphi(x)dx.$$

Fix $v \in \mathcal{S}'(\mathbb{R}^n)$, and $\varphi = \mathcal{F}(\psi) = \widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}\mathcal{F}^{-1}v(\varphi) &= \mathcal{F}^{-1}v(\mathcal{F}(\psi)) = \mathcal{F}^{-1}v\left(\widehat{\psi}\right) = \mathcal{F}^{-1}\widehat{v}(\psi) = \left[\mathcal{F}^{-1}\mathcal{F}v\right](\psi) = v(\psi) \\ &= v\left(\mathcal{F}^{-1}(\varphi)\right) = v\left(\xi \mapsto \frac{1}{(2\pi)^n}\int e^{i\xi\cdot x}\varphi(x)dx\right). \end{aligned} \tag{4.75}$$

With $v(\xi) = v_\lambda(\xi) = \dfrac{1}{|\xi|^\lambda}$ as in (4.73) viewed as a distribution we see that the equality in (4.75) can be rewritten as

$$\left[\mathcal{F}^{-1}v_\lambda\right](\varphi) = \frac{1}{(2\pi)^n}\int \frac{1}{|\xi|^\lambda}\int e^{i\xi\cdot x}\varphi(x)dx\,d\xi. \tag{4.76}$$

Next we investigate the invariance properties of the distribution $\mathcal{F}^{-1}v_\lambda$. First let $U : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation. Then $UU^* = U^*U = I$ and hence $\det(U) = \pm 1$. By definition we know that the U-transform of a distribution v is given by $v^U(\varphi) = v\left(\varphi^{U^{-1}}|\det U|\right) = v\left(\varphi^{U^{-1}}\right)$, where $\varphi^{U^{-1}}(x) = \varphi(Ux)$, $x \in \mathbb{R}^n$. Hence (4.76) implies

$$\begin{aligned}\left[\mathcal{F}^{-1}v_\lambda\right]^U(\varphi) &= \frac{1}{(2\pi)^n}\int \frac{1}{|\xi|^\lambda}\int e^{i\xi\cdot x}\varphi(Ux)\,dx\,d\xi \\ &= \frac{1}{(2\pi)^n}\int \frac{1}{|U\xi|^\lambda}\int e^{iU\xi\cdot Ux}\varphi(Ux)\,d(Ux)\;d(U\xi) \\ &= \frac{1}{(2\pi)^n}\int \frac{1}{|\xi|^\lambda}\int e^{i\xi\cdot x}\varphi(x)\,dx\,d\xi = \left[\mathcal{F}^{-1}(v_\lambda)\right](\varphi).\end{aligned} \tag{4.77}$$

From (4.77) we infer $\left[\mathcal{F}^{-1}v_\lambda\right]^U = \mathcal{F}^{-1}v_\lambda$ for all orthogonal transformations $U : \mathbb{R}^n \to \mathbb{R}^n$. Let $T_t : \mathbb{R}^n \to \mathbb{R}^n$ be the linear mapping $T_t x = tx$, $t > 0$. Then, as above, $v^{T_t}(\varphi) = v\left(\varphi^{T_t^{-1}} \left|\det T_t\right|\right) = t^n v\left(\varphi^{T_t^{-1}}\right)$ where $\varphi^{T_t^{-1}}(x) = \varphi(tx)$, $x \in \mathbb{R}^n$. A similar argument with which we obtained (4.77) shows the equality $\left[\mathcal{F}^{-1}v_\lambda\right]^{T_t} = t^{n-\lambda}\mathcal{F}^{-1}v_\lambda$. Next we will show that, for $\frac{1}{2}n < \Re\lambda < n$, the distribution $\mathcal{F}^{-1}v_\lambda$ is in fact a function. We do this by showing that, for such λ, the function v_λ is a function which belongs to the vector sum $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Then the distribution $\mathcal{F}^{-1}v_\lambda$ can be considered as a function in $L^\infty(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Let $B_1 = \{|\xi| \leqslant 1\}$ be the unit ball in $\mathbb{R}^n$. Let $|S_{n-1}|$ denote the area of the unit sphere in $\mathbb{R}^n$. Since $\Re\lambda < n$, the function $v_\lambda \mathbf{1}_{B_1}$ belongs to $L^1(\mathbb{R}^n)$:

$$\int \left|v_\lambda(\xi)\mathbf{1}_{B_1}(\xi)\right| \, d\xi = \int_{B_1} \left|v_\lambda(\xi)\right| \, d\xi = \int_{B_1} \frac{1}{|\xi|^{\Re\lambda}} d\xi \tag{4.78}$$

$$= |S_{n-1}| \int_0^1 r^{n-\Re\lambda-1} dr = \frac{|S_{n-1}|}{n - \Re\lambda}. \tag{4.79}$$

Since $\frac{1}{2}n < \Re\lambda$, the function $v_\lambda \mathbf{1}_{\mathbb{R}^n\setminus B_1}$ belongs to $L^2(\mathbb{R}^n)$:

$$\int \left|v_\lambda(\xi)\mathbf{1}_{\mathbb{R}^n\setminus B_1}(\xi)\right|^2 \, d\xi = \int_{\mathbb{R}^n\setminus B_1} \left|v_\lambda(\xi)\right|^2 \, d\xi = \int_{\mathbb{R}^n\setminus B_1} \frac{1}{|\xi|^{2\Re\lambda}} d\xi$$

$$= |S_{n-1}| \int_1^\infty r^{n-2\Re\lambda-1} dr = \frac{|S_{n-1}|}{2\Re\lambda - n}. \tag{4.80}$$

From (4.79) and (4.80) it follows that, for $\frac{1}{2}n < \Re\lambda < n$, the function v_λ belongs to vector space $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Hence, by properties of (inverse) Fourier transforms we see that the function $\mathcal{F}^{-1}v_\lambda$ belongs to the space $C_0(\mathbb{R}^n) + L^2(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. The equalities

$$\mathcal{F}^{-1}v_\lambda = \left[\mathcal{F}^{-1}v_\lambda\right]^{U^{-1}} = \frac{1}{t^{n-\lambda}} \left[\mathcal{F}^{-1}v_\lambda\right]^{T_t} \tag{4.81}$$

where U is an arbitrary orthogonal matrix and $T_t x = tx$, $t > 0$, $x \in \mathbb{R}^n$, entail

$$\mathcal{F}^{-1}v_\lambda(x) = \left[\mathcal{F}^{-1}v_\lambda\right](Ux) = \frac{1}{t^{n-\lambda}} \left[\mathcal{F}^{-1}v_\lambda\right]\left(t^{-1}x\right). \tag{4.82}$$

From (4.82) with $t = |x|$ we deduce

$$\mathcal{F}^{-1}v_\lambda(x) = \frac{1}{|x|^{n-\lambda}} \left[\mathcal{F}^{-1}v_\lambda\right]\left(\frac{x}{|x|}\right) = \frac{1}{|x|^{n-\lambda}} \left[\mathcal{F}^{-1}v_\lambda\right]\left(\frac{Ux}{|x|}\right). \tag{4.83}$$

for all orthogonal transformations U. Consequently, we see that for some constant c_λ' we have

$$\mathcal{F}^{-1}v_\lambda(x) = \frac{c_\lambda'}{|x|^{n-\lambda}}. \tag{4.84}$$

We shall prove that $c_\lambda' = c_\lambda$ where c_λ is as in (4.71). We will use (4.76) with $\varphi(x) = e^{-\frac{1}{2}|x|^2}$. Then we have

$$c_\lambda' \int \frac{1}{|x|^{n-\lambda}} e^{-\frac{1}{2}|x|^2} dx = \left[\mathcal{F}^{-1}v_\lambda\right](\varphi) = \frac{1}{(2\pi)^n} \int \frac{1}{|\xi|^\lambda} \int e^{i\xi\cdot x}\varphi(x) dx \, d\xi$$

$$= \frac{1}{(2\pi)^n} \int \frac{1}{|\xi|^\lambda} \int e^{i\xi\cdot x} e^{-\frac{1}{2}|x|^2} dx\, d\xi = \frac{1}{(2\pi)^{n/2}} \int \frac{1}{|\xi|^\lambda} e^{-\frac{1}{2}|\xi|^2} d\xi. \tag{4.85}$$

Let us calculate the extreme terms in (4.85). Since both are very similar it suffices to calculate $\int \frac{1}{|\xi|^\lambda} e^{-\frac{1}{2}|\xi|^2} d\xi$. By inserting polar coordinates we get

$$\int \frac{1}{|\xi|^\lambda} e^{-\frac{1}{2}|\xi|^2} d\xi = |S_{n-1}| \int_0^\infty r^{n-\lambda-1} e^{-\frac{1}{2}r^2} dr$$

(substitute $r = \sqrt{2s}$)

$$= |S_{n-1}|\, 2^{\frac{1}{2}(n-\lambda)-1} \int_0^\infty s^{\frac{1}{2}(n-\lambda)-1} e^{-s} ds = |S_{n-1}|\, 2^{\frac{1}{2}(n-\lambda)-1} \Gamma\left(\frac{1}{2}(n-\lambda)\right). \tag{4.86}$$

We also have by replacing λ with $n-\lambda$:

$$\int \frac{1}{|\xi|^{n-\lambda}} e^{-\frac{1}{2}|\xi|^2} d\xi = |S_{n-1}|\, 2^{\frac{1}{2}\lambda-1} \int_0^\infty s^{\frac{1}{2}\lambda-1} e^{-s} ds = |S_{n-1}|\, 2^{\frac{1}{2}\lambda-1} \Gamma\left(\frac{1}{2}\lambda\right). \tag{4.87}$$

From (4.85), (4.87) and (4.86) we infer:

$$\begin{aligned} |S_{n-1}|\, c'_\lambda 2^{\frac{1}{2}\lambda-1} \Gamma\left(\frac{1}{2}\lambda\right) &= |S_{n-1}| \frac{1}{(2\pi)^{n/2}} 2^{\frac{1}{2}(n-\lambda)-1} \Gamma\left(\frac{1}{2}(n-\lambda)\right) \\ &= |S_{n-1}| \frac{1}{\pi^{n/2}} \frac{1}{2^{\frac{1}{2}\lambda+1}} \Gamma\left(\frac{1}{2}(n-\lambda)\right), \end{aligned} \tag{4.88}$$

and hence

$$c'_\lambda = \frac{1}{\pi^{n/2}} \frac{1}{2^\lambda} \frac{\Gamma\left(\frac{1}{2}(n-\lambda)\right)}{\Gamma\left(\frac{1}{2}\lambda\right)} = c_\lambda.$$

This completes the proof of Theorem 4.14. □

Put

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|y-x|^2}{2t}} \quad \text{and} \quad E_n(x) = \frac{\Gamma\left(\frac{1}{2}n-1\right)}{2\pi^{n/2}} \frac{1}{|x|^{n-2}}.$$

For $n \geqslant 3$ we notice the following identities:

$$\frac{\partial}{\partial t} p(t,x,y) = \frac{1}{2} \Delta_x p(t,x,y), \tag{4.89}$$

$$\int_0^\infty p(t,x,y) dt = \frac{\Gamma\left(\frac{1}{2}n-1\right)}{2\pi^{n/2}} \frac{1}{|y-x|^{n-2}} = E_n(y-x). \tag{4.90}$$

The proof of the following Proposition follows Exercises 6.4 and 6.5 in Trèves [**136**] Chapter 1. The result should be compared to Theorem 2.18.

4.15. Proposition. *The function E_n is the fundamental solution to the partial differential equation $-\frac{1}{2}\Delta E_n = \delta$.*

PROOF. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then by (4.89) and (4.90) we have

$$\begin{aligned}
&-\frac{1}{2}\Delta E_n(\varphi) = -\frac{1}{2}\int E_n(x)\Delta\varphi(x)dx = -\frac{1}{2}\int_{\mathbb{R}^n}\int_0^\infty p(t,x,0)\,dt\Delta\varphi(x)dx \\
&= -\frac{1}{2}\int_0^\infty\int_{\mathbb{R}^n} p(t,x,0)\,\Delta\varphi(x)dx\,dt
\end{aligned}$$

(integration by parts)

$$= -\frac{1}{2}\int_0^\infty\int_{\mathbb{R}^n} \Delta_x p(t,x,0)\,\varphi(x)dx\,dt$$

(apply (4.89))

$$\begin{aligned}
&= -\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty\int_{\mathbb{R}^n}\frac{\partial}{\partial t}p(t,x,0)\,\varphi(x)dx\,dt = -\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty\frac{\partial}{\partial t}\int_{\mathbb{R}^n} p(t,x,0)\,\varphi(x)dx\,dt \\
&= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} p(\varepsilon,x,0)\,\varphi(x)dx
\end{aligned}$$

(make the substitution $x = y\sqrt{\varepsilon}$)

$$= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} p(1,y,0)\,\varphi\left(y\sqrt{\varepsilon}\right)dy = \varphi(0)\int_{\mathbb{R}^n} p(1,y,0)\,dy = \varphi(0). \tag{4.91}$$

This completes the proof Proposition 4.15. $\square$

In the following proposition $H(t)$ denotes the Heaviside function.

4.16. PROPOSITION. *Put $E_a(t,x) = ap(at,x,0)\,H(t)$, $\Re a \geqslant 0$, $a \neq 0$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. Then*

$$\frac{\partial}{a\partial t}E_a(t,x) - \frac{1}{2}\Delta_x E_a(t,x) = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \tag{4.92}$$

and E_a is the fundamental solution of the equation:

$$\frac{\partial}{a\partial t}E_a - \frac{1}{2}\Delta E_a = \delta, \quad \Re a \geqslant 0, \quad a \neq 0. \tag{4.93}$$

PROOF. The equality in (4.92) can easily be checked. First we assume that $\Re a > 0$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We use (4.92) to prove the second equality:

$$\begin{aligned}
&\left(\frac{\partial}{a\partial t}E_a - \frac{1}{2}\Delta E_a\right)(\varphi) = -E_a\left(\frac{\partial}{a\partial t}\varphi + \frac{1}{2}\Delta\varphi\right) \\
&= -\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\int_\varepsilon^\infty p(at,x,0)\,\frac{\partial}{\partial t}\varphi(t,x)dt\,dx - \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\int_\varepsilon^\infty ap(at,x,0)\,\frac{1}{2}\Delta_x\varphi(t,x)dt\,dx
\end{aligned}$$

(integration by parts)

$$= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} p(a\varepsilon,x,0)\,\varphi(\varepsilon,x)dx$$

$$
\begin{aligned}
&+\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\int_{\varepsilon}^{\infty}\left(\frac{\partial}{a\partial t}ap\left(at,x,0\right)-\frac{1}{2}\Delta_x ap\left(at,x,0\right)\right)\varphi(t,x)dt\,dx\\
=&\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}p\left(a,x,0\right)\varphi\left(\varepsilon,x\sqrt{\varepsilon}\right)dx\\
&+\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\int_{\varepsilon}^{\infty}\left(\frac{\partial}{a\partial t}E_a(t,x)-\frac{1}{2}\Delta_x E_a\left(t,x\right)\right)\varphi(t,x)dt\,dx
\end{aligned}
$$

(employ the equality in (4.92))

$$
=\int_{\mathbb{R}^n}p\left(a,x,0\right)\varphi\left(0,0\right)dx=\varphi(0,0). \tag{4.94}
$$

The equality in (4.94) proves Proposition 4.90 in the case that $\Re a > 0$. If $\Re a = 0$ we replace a by $a+\varepsilon$, $\varepsilon > 0$. Then we have:

$$
\frac{\partial}{(a+\varepsilon)\partial t}E_{a+\varepsilon}-\frac{1}{2}\Delta E_{a+\varepsilon}=\delta,\quad \Re a \geqslant 0. \tag{4.95}
$$

In (4.95) we let ε tend to 0 to obtain (4.93) for $\Re a \geqslant 0$ and $a \neq 0$. $\square$

3. Theorem of Malgrange and Ehrenpreis

In this section we present the preliminaries which are needed to prove the existence of fundamental solutions for partial differential operators with constant coefficients, which is known as the theorem of Malgrange-Ehrenpreis. We follow the proof given in Rudin [**113**] Theorem 8.5. A more elementary proof is given by Rosay [**111**]. In fact in the sections 4, 5 and 6 we also follow the proofs of Rudin.

4.17. LEMMA. *Let $P(z) = \sum_{\alpha\in\mathbb{N}^n, |\alpha|\leqslant N} c_\alpha z^\alpha$, $z \in \mathbb{C}^n$, be a polynomial of exact degree N. Let $P_N(z) = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=N} c_\alpha z^\alpha$ be its principal part. Furthermore, let $f : \mathbb{C}^n \to \mathbb{C}$ be an entire function. Then $\int_{\mathbb{T}^n} |P_N(w)|\, d\sigma_n(w) \neq 0$. Put $A = \left(\int_{\mathbb{T}^n} |P_N(w)|\, d\sigma_n(w)\right)^{-1}$. Then the following inequality holds for all $z \in \mathbb{C}^n$ and $r > 0$:*

$$|f(z)| \leqslant \frac{A}{r^N} \int_{\mathbb{T}^n} |P(z + rw) f(z + rw)|\, d\sigma_n(w). \tag{4.96}$$

In Lemma 4.17 the symbol $\mathbb{T}$ stands for the one-dimensional torus:

$$\mathbb{T} = \left\{ z = e^{i\vartheta} \in \mathbb{C} : |z| = 1 \right\},$$

which is a compact abelian group, and as such it carries a unique Haar measure σ_1 of total mass one: $d\sigma_1(\vartheta) = \dfrac{d\vartheta}{2\pi}$. Then $\sigma_n = \sigma_1 \times \cdots \times \sigma_1$ (n times).

PROOF. Put $Q(\lambda) = P(z + r\lambda w)$, and $F(\lambda) = f(z + r\lambda w)$, $z \in \mathbb{C}^n$, $r > 0$, $w \in \mathbb{T}^n$, and $\lambda \in \mathbb{C}$. Then $Q(\lambda) = c\prod_{j=1}^N (\lambda + a_j)$, where $c = r^N P_N(w)$. We also introduce the polynomial $Q_0(\lambda) = c\prod_{j=1}^N (1 + \overline{a}_j\lambda)$. For $|\lambda| = 1$ we have $|Q(\lambda)| = |Q_0(\lambda)|$. Since

$$cF(0) = F(0)Q_0(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(e^{i\vartheta}\right) Q_0\left(e^{i\vartheta}\right) d\vartheta \tag{4.97}$$

we have

$$\begin{aligned} |c|\,|F(0)| = |F(0)Q_0(0)| &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F\left(e^{i\vartheta}\right) Q_0\left(e^{i\vartheta}\right)\right| d\vartheta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F\left(e^{i\vartheta}\right) Q\left(e^{i\vartheta}\right)\right| d\vartheta. \end{aligned} \tag{4.98}$$

From (4.98) we infer

$$r^N |P_N(w)|\,|f(z)| = |cF(0)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(z + re^{i\vartheta}w\right) P\left(z + re^{i\vartheta}w\right)\right| d\vartheta. \tag{4.99}$$

We integrate the expressions in (4.99) relative to $d\sigma_n(w)$ to obtain

$$r^N \int_{\mathbb{T}^n} |P_N(w)|\, d\sigma_n(w)\,|f(z)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{T}^n} \left|f\left(z + re^{i\vartheta}w\right) P\left(z + re^{i\vartheta}w\right)\right| d\sigma_n(w)\, d\vartheta$$

(translation invariance of the measure σ_n)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{T}^n} |f(z+rw)\,P(z+rw)|\,d\sigma_n(w)\,d\vartheta$$
$$= \int_{\mathbb{T}^n} |f(z+rw)\,P(z+rw)|\,d\sigma_n(w). \tag{4.100}$$

There remains to be shown that $\int_{\mathbb{T}^n} |P_N(w)|\,d\sigma_n(w) \neq 0$. This assertion follows from the inequalities:

$$\left(\int_{\mathbb{T}^n} |P_N(w)|\,d\sigma_n(w)\right)^2 \leqslant \int_{\mathbb{T}^n} |P_N(w)|^2\,d\sigma_n(w)$$
$$\leqslant \sup_{w\in\mathbb{T}^n} |P_N(w)| \int_{\mathbb{T}^n} |P_N(w)|\,d\sigma_n(w).$$

Since $\int_{\mathbb{T}^n} |P_N(w)|^2\,d\sigma_n(w) = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=N} |c_\alpha|^2 > 0$, we see that $\int_{\mathbb{T}^n} |P_N(w)|\,d\sigma_n(w) \neq 0$. This completes the proof of Lemma 4.17. □

As a consequence of Lemma 4.17 we get inequality (4.101) in the following lemma.

4.18. Lemma. *Let the polynomial P be as in Lemma 4.17. Let $\varphi \in \mathcal{D} = \mathcal{D}(\mathbb{R}^n)$. Then*

$$|\varphi(x)| \leqslant \frac{A}{(2\pi)^n r^N}\int_{\mathbb{T}^n}\int_{\mathbb{R}^n} |P(t+rw)\,\widehat{\varphi}(t+rw)|\,dt\,d\sigma_n(w). \tag{4.101}$$

Let ψ, $\varphi \in \mathcal{D}$ be such that $\psi = P(D)\varphi$. Then $\widehat{\psi}(\xi) = P(\xi)\widehat{\varphi}(\xi)$, and the inequality in (4.101) implies

$$|\varphi(x)| \leqslant \frac{A}{(2\pi)^n r^N}\int_{\mathbb{T}^n}\int_{\mathbb{R}^n} \left|\widehat{\psi}(t+rw)\right|\,dt\,d\sigma_n(w). \tag{4.102}$$

Proof. The proof of inequality (4.101) employs the formula for inverse Fourier transform in combination with inequality (4.96), to wit

$$|\varphi(x)| = \frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n} e^{i\langle x,t\rangle}\widehat{\varphi}(t)\,dt\right| \leqslant \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\widehat{\varphi}(t)|\,dt$$
$$\leqslant \frac{A}{(2\pi)^n r^N}\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} |P(t+rw)\,\widehat{\varphi}(t+rw)|\,d\sigma_n(w)\,dt. \tag{4.103}$$

The inequality in (4.103) implies (4.101). □

4.19. Theorem. *There exists a distribution $u \in \mathcal{D}'$ with the following properties:*

$$u(P(D)\varphi) = \varphi(0),\ \varphi \in \mathcal{D}, \quad \textit{and} \tag{4.104}$$
$$|u(\psi)| \leqslant \frac{A}{(2\pi)^n r^N}\int_{\mathbb{T}^n}\int_{\mathbb{R}^n} \left|\widehat{\psi}(t+rw)\right|\,dt\,d\sigma_n(w), \quad \psi \in \mathcal{D}. \tag{4.105}$$

Proof. Let M be the subspace of $\mathcal{D}$ given by the range of the operator $P(D)$: $M = P(D)\mathcal{D}$. Define the functional $u_M : M \to \mathbb{C}$ by $u_M(P(D)\varphi) = \varphi(0)$. Suppose that $\psi = P(D)\varphi$. From (4.102) we see that

$$|u_M(\psi)| \leqslant \frac{A}{(2\pi)^n r^N} \int_{\mathbb{T}^n} \int_{\mathbb{R}^n} \left| \widehat{\psi}(t + rw) \right| dt\, d\sigma_n(w), \quad \psi \in M. \tag{4.106}$$

By the Hahn-Banach theorem (see Theorem 8.2) there exists a linear functional $u : \mathcal{D} \to \mathbb{C}$ such that (4.104) and (4.105) are satisfied. Note that this implies that u is an extension of u_M. The functional

$$\psi \to \int_{\mathbb{T}^n} \int_{\mathbb{R}^n} \left| \widehat{\psi}(t + rw) \right| dt\, d\sigma_n(w) = \int_{\mathbb{T}^n} \int_{\mathbb{R}^n} \left| \widehat{e_{-rw}\psi}(t) \right| dt\, d\sigma_n(w) \tag{4.107}$$

is in fact a continuous norm on $\mathcal{D}$. Here $e_{-rw}(x) = e^{-ir\sum_{j=1}^n x_j w_j}$. Consequently, the functional u is a member of $\mathcal{D}'$. The proof of the continuity of the norm in (4.107) can be found in the proof of Theorem 8.5 in Rudin [**113**]. In fact the continuity of the functional in (4.107) can be proved as follows. Fix $m \in \mathbb{N}$ so large that $4m > n$, and look at the estimates:

$$\left(\int_{\mathbb{R}^n} \left| \widehat{e_{-rw}\psi}(t) \right| dt \right)^2$$

$$= \left(\int_{\mathbb{R}^n} \left(1 + |t|^2\right)^m \left| \widehat{e_{-rw}\psi}(t) \right| \frac{1}{\left(1 + |t|^2\right)^m} dt \right)^2$$

$(\left(1 + |t|^2\right)^m \widehat{\chi}(t) = \widehat{(I - \Delta)^m \chi}(t)$; $\chi \in \mathcal{D}$ is arbitrary)

$$= \left(\int_{\mathbb{R}^n} \left| \widehat{(I - \Delta)^m (e_{-rw}\psi)}(t) \right| \frac{1}{\left(1 + |t|^2\right)^m} dt \right)^2$$

(inequality of Cauchy-Schwartz)

$$\leqslant \int \left| \widehat{(I - \Delta)^m (e_{-rw}\psi)}(t) \right|^2 dt \int \frac{1}{\left(1 + |t|^2\right)^{2m}} dt$$

(Plancherel's theorem)

$$= (2\pi)^n \int \left| (I - \Delta)^m (e_{-rw}\psi)(x) \right|^2 dx \int \frac{1}{\left(1 + |t|^2\right)^{2m}} dt. \tag{4.108}$$

From (4.108) it easily follows that the functional in (4.107) is continuous on the space $\mathcal{D}(\mathbb{R}^n)$. This completes the proof of Theorem 4.19. □

The theorem of Malgrange-Ehrenpreis is a corollary to Theorem 4.19.

4.20. Theorem. *Let the polynomial P be as in Lemma 4.17. Then there exists $E \in \mathcal{D}'$ such that $P(D)E = \delta$.*

Proof. Let $u \in \mathcal{D}'$ satisfy (4.104) and (4.105) in Theorem 4.19. Put $E = \check{u}$. Then

$$\begin{aligned} P(D)E(\varphi) &= E\left(P(-D)\varphi\right) = \check{u}\left(P(-D)\varphi\right) = u\left(\left(\widecheck{P(-D)\varphi}\right)\right) \\ &= u\left(P(D)\check{\varphi}\right) = \check{\varphi}(0) = \varphi(0) = \delta(\varphi). \end{aligned} \tag{4.109}$$

The assertion in Theorem 4.20 is a consequence of (4.109). □

4. Sobolev theory

We begin with Sobolev's lemma. It needs some introduction.

4.21. Definition. Let Ω be an open subset $\mathbb{R}^n$. A Borel measurable function $f : \Omega \to \mathbb{C}$ is said to be locally L^2 in Ω if $\int_K |f(x)|^2 \, dx < \infty$ for all compact subsets K of Ω. For such a function the notation $f \in L^2_{\text{loc}}(\Omega)$ is often used. Similarly, a distribution $u \in \mathcal{D}'(\Omega)$ is said to be locally L^2 if there exists a function $g \in L^2_{\text{loc}}(\Omega)$ such that $u(\varphi) = \int_\Omega g(x)\varphi(x)\, dx$ for all test functions $\varphi \in \mathcal{D}(\Omega)$. To say that a distribution $u \in \mathcal{D}'(\Omega)$ has a distributional derivative $D^\alpha u$ which is locally L^2 refers to the distribution $D^\alpha u$, and means explicitly that there exists a function $g \in L^2_{\text{loc}}(\Omega)$ with the property that

$$\int_\Omega g(x)\varphi(x)\, dx = (-1)^{|\alpha|} u\left(D^\alpha \varphi\right) \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{4.110}$$

A priori knowing that $D^\alpha u$ belongs to $L^2_{\text{loc}}(\Omega)$ does not give any information about the differentiability in the classical sense of the distribution u (even if it is function). Sobolev's lemma gives some information about this issue.

For p a non-negative integer the space $C^{(p)}(\Omega)$ consists of those functions $f : \Omega \to \mathbb{C}$ whose derivatives $D^\alpha f$ exist in the classical sense and belong $C(\Omega)$ for all multi-indices α with length not exceeding p, *i.e.* $|\alpha| \leqslant p$. Fix $u \in \mathcal{D}'(\Omega)$ Let $D^k_j u$, $1 \leqslant j \leqslant n$, $k = 0, 1, \dots$, denote the distributional derivative given by $D^k_j u = \dfrac{\partial^k u}{\partial x^k_j}$.

4.22. Theorem (Sobolev's lemma). *Let n, p, r be integers such that $n \geqslant 1$, $p \geqslant 0$, and $r > p + \frac{1}{2}n$. Suppose that $f : \Omega \to \mathbb{C}$ is a function whose distribution derivatives $D^k_j f$ belong to $L^2_{\text{loc}}(\Omega)$ for all $1 \leqslant j \leqslant n$, and all $0 \leqslant k \leqslant r$. Then there exists a function $f_0 \in C^{(p)}(\Omega)$ such that $f_0(x) = f(x)$ for almost all $x \in \Omega$.*

Proof. By hypothesis there exist functions $g_{j,k} \in L^2_{\text{loc}}(\Omega)$ such that

$$\int_\Omega g_{j,k}(x)\varphi(x)\, dx = (-1)^k \int_\Omega f(x) D^k_j \varphi(x)\, dx, \quad \varphi \in \mathcal{D}(\Omega), \tag{4.111}$$

for all $1 \leqslant j \leqslant n$, and $0 \leqslant k \leqslant r$.

Let Ω_1 be an open subset whose closure K_1 is a compact subset of Ω. Choose $\psi \in \mathcal{D}(\Omega)$ so that $\psi = 1$ on K_1, and define the function $F : \mathbb{R}^n \to \mathbb{C}$ by

$$F(x) = \begin{cases} \psi(x) f(x) & \text{for } x \in \Omega; \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.112}$$

In Ω the formula of Leibniz gives:

$$D^r_j F = \sum_{\ell=0}^{r} \binom{r}{\ell} \left(D^{r-\ell}_j \psi\right) \left(D^\ell_j f\right) = \sum_{\ell=0}^{r} \binom{r}{\ell} \left(D^{r-\ell}_j \psi\right) g_{j,\ell}. \tag{4.113}$$

In the complement Ω_0 of the support of the function ψ we have $D_j^r F = 0$. It follows that the distributions $D_j^r F$ and 0 coincide on $\Omega \cap \Omega_0$. Hence $D_j^r F$, originally defined as a distribution in $\mathbb{R}^n$, belongs actually to $L^2(\mathbb{R}^n)$, for $1 \leqslant j \leqslant n$, because the functions $\left(D_j^{r-\ell}\psi\right) g_{j,\ell}$ are in $L^2(\mathbb{R}^n)$, for $1 \leqslant j \leqslant n$, and $0 \leqslant \ell \leqslant r$. Having compact support, $D_j^r F$ is therefore also in $L^1(\mathbb{R}^n)$, for $1 \leqslant j \leqslant n$. The Plancherel theorem, applied to F and to $D_j^r F$, $1 \leqslant j \leqslant n$, shows that

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\widehat{F}(\xi)\right|^2 d\xi &= (2\pi)^n \int_{\mathbb{R}^n} |F(x)|^2 \, dx < \infty, \quad \text{and} \\ \int_{\mathbb{R}^n} \xi_j^{2r} \left|\widehat{F}(\xi)\right|^2 d\xi &= (2\pi)^n \int_{\mathbb{R}^n} \left|D_j^r F(x)\right|^2 dx < \infty, \quad \text{for } 1 \leqslant j \leqslant n. \end{aligned} \tag{4.114}$$

Since

$$\left(1+|\xi|^2\right)^r \leqslant \frac{(1+n^2)^r}{n+1}\left(1+\xi_1^{2r}+\cdots+\xi_n^{2r}\right) \tag{4.115}$$

the inequalities in (4.114) imply

$$J := \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^r \left|\widehat{F}(\xi)\right|^2 d\xi < \infty. \tag{4.116}$$

By the inequality of Cauchy-Schwarz we see

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{p/2} \left|\widehat{F}(\xi)\right| d\xi\right)^2 = \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{r/2} \left|\widehat{F}(\xi)\right| \frac{1}{\left(1+|\xi|^2\right)^{r/2-p/2}} \, d\xi\right)^2 \\ &\leqslant \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^r \left|\widehat{F}(\xi)\right|^2 d\xi \cdot \int_{\mathbb{R}^n} \frac{1}{\left(1+|\xi|^2\right)^{r-p}} \, d\xi \\ &= J \cdot |S_{n-1}| \int_0^\infty \frac{\rho^{n-1}}{(1+\rho^2)^{r-p}} \, d\rho < \infty, \end{aligned} \tag{4.117}$$

where $|S_{n-1}|$ is the volume of the $n-1$-dimensional unit sphere. The final integral in (4.117) is finite because, by hypothesis, $r > p + \frac{1}{2}n$. So from (4.117) it follows that

$$\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{p/2} \left|\widehat{F}(\xi)\right| d\xi < \infty. \tag{4.118}$$

Define the function $F_{\Omega_1} : \mathbb{R}^n \to \mathbb{C}$ by

$$F_{\Omega_1}(x) = \frac{1}{(2\pi)^n} \int e^{i\xi\cdot x} \widehat{F}(\xi) \, d\xi, \ x \in \mathbb{R}^n. \tag{4.119}$$

Then the function F_{Ω_1} is almost everywhere equal to F. In addition, for $|\alpha| \leqslant p$ the derivatives $D^\alpha F_{\Omega_1}$ exist and are continuous functions, because by (4.118) the integrals $\int_{\mathbb{R}^n} \left|\xi^\alpha \widehat{F}(\xi)\right| d\xi$, $|\alpha| \leqslant p$, are finite. Notice that $|\xi^\alpha|^2 \leqslant \left(1+|\xi|^2\right)^{|\alpha|}$. These arguments suffice to conclude that F_{Ω_1} belongs to $C^{(p)}(\mathbb{R}^n)$. Since, in a neighborhood of x the function f and F_{Ω_1} coincide almost everywhere, the conclusion in Theorem 4.22 follows by gluing restrictions of functions F_{Ω_1} as in (4.119) together, *i.e.* $f_0(x) = F_{\Omega_1}(x)$ if $x \in \Omega_1$. $\square$

5. Elliptic operators

Let Ω be an open subset of $\mathbb{R}^2$. If $u : \Omega \to \mathbb{R}$ is a twice continuously differentiable function that satisfies

$$\Delta u(x,y) = \frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0, \quad (x,y) \text{ in } \Omega, \tag{4.120}$$

then it is well known that u is a C^∞-function. In fact if Ω is simply connected, then there exists a function $f \in \mathrm{Hol}\left(\widetilde{\Omega}\right)$ such that $u(x,y) = \Re f(x+iy)$, $(x,y) \in \Omega$, where $\widetilde{\Omega} = \{x+iy \in \mathbb{C} : (x,y) \in \Omega\}$. Twice continuously differentiable functions which satisfy (4.120) are called harmonic functions. Any theorem stating that a solution to a differential equation is more regular, or has stronger smoothness properties, than the input data (or function, or distribution) is called a "*regularity theorem*". In Theorem 4.30 below we shall formulate and prove a rather general regularity theorem for so-called elliptic differential operators (of which the Laplace operator is a prototype). First let us look at an differential operator which is typically non-elliptic. Put, for $\varphi \in \mathcal{D}(\mathbb{R}^2)$,

$$P_1\varphi(x) = \int_{-\infty}^{\infty} \varphi(x,t)\,dt = (\delta_x \otimes 1)(\varphi), \text{ and } P_2\varphi(y) = \int_{-\infty}^{\infty} \varphi(s,y)\,ds = (1 \otimes \delta_y)(\varphi).$$

The following proposition shows that a distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ satisfying $\dfrac{\partial^2 u}{\partial x \partial y} = 0$ can be written as a sum of two distributions, one which only depends on the first coordinate x, and one which only depends on the second coordinate y. The result should be compared to Lemma 3.2.

4.23. Proposition. *Let $u \in \mathcal{D}'(\mathbb{R}^2)$. Then the following assertions are equivalent:*

(1) *the distribution u satisfies $D_1 D_2 u = 0$ where*

$$D_1 = \frac{\partial}{\partial x}, \quad \text{and} \quad D_2 = \frac{\partial}{\partial y}.$$

(2) *the distribution u can be written as*

$$u(\varphi) = u_1(P_1\varphi) + u_2(P_2\varphi) = (u_1 \otimes 1)(\varphi) + (1 \otimes u_2)(\varphi), \quad \varphi \in \mathcal{D}\left(\mathbb{R}^2\right),$$

where the distributions u_1 and u_2 belong to $\mathcal{D}'(\mathbb{R})$.

Here we use the following notation. Let, for $j = 1, 2$, Ω_j be an open subset of $\mathbb{R}^{n_j}$, and let $u_j \in \mathcal{D}'(\Omega_j)$. Then the distribution $u_1 \otimes u_2 \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ is defined by

$$u_1 \otimes u_2(\varphi) = u_1(x \mapsto u_2(y \mapsto \varphi(x, y))) = u_2(y \mapsto u_1(x \mapsto \varphi(x, y))), \qquad (4.121)$$

where x varies over Ω_1, and y over Ω_2. The first equality is a definition, and the second equality is based on the following argument. If $\varphi = \varphi_1 \otimes \varphi_2$, $\varphi_j \in \mathcal{D}(\Omega_j)$, $j = 1, 2$, the second equality in (4.121) is a triviality. A general test function $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ is approximated by linear combinations of such tensor products, which are dense in $\mathcal{D}(\Omega_1 \times \Omega_2)$ by an adapted Stone-Weierstrass theorem. It also follows that $\operatorname{supp}(u_1 \otimes u_2) = \operatorname{supp}(u_1) \times \operatorname{supp}(u_2)$. Hence, in assertion (2) of Proposition 4.23 $\operatorname{supp}(u_1 \otimes 1) = \operatorname{supp}(u_1) \times \mathbb{R}$, and $\operatorname{supp}(1 \otimes u_2) = \mathbb{R} \times \operatorname{supp}(u_2)$.

4.24. Remark. In addition, notice that the function $(x, y) \mapsto E(x, y) = H(x)H(y)$, where $x \mapsto H(x)$ is the Heaviside function, is a fundamental solution for the operator $L = D_1 D_2$. From Proposition 4.23 it follows that the function $E(x, y) = H(x)H(y)$, $(x, y) \in \mathbb{R}^2$ is the only fundamental solution of the operator $D_1 D_2$ whose support is contained in the closed subset $\{(x, y) \in \mathbb{R}^2 : y + x \geqslant 0\}$.

Proof of Proposition 4.23. (1) $\Longrightarrow$ (2) Let $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R})$ be such that $\int_{\mathbb{R}} \varphi_1(x)\,dx = \int_{\mathbb{R}} \varphi_2(x)\,dx = 1$, and let $u \in \mathcal{D}'(\mathbb{R}^2)$ satisfy (1). Put

$$u_1(\psi_1) = u(\psi_1 \otimes \varphi_2) - u(\varphi_1 \otimes \varphi_2)\int_{\mathbb{R}} \psi_1(x)\,dx, \quad \text{and} \quad u_2(\psi_2) = u(\varphi_1 \otimes \psi_2), \qquad (4.122)$$

where ψ_1 and ψ_2 belong to $\mathcal{D}(\mathbb{R})$. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ be given, and put

$$\chi_1(x, y) = \int_{-\infty}^{y} (\varphi(x, t) - P_1\varphi(x)\varphi_2(t))\,dt, \text{ and}$$
$$\chi_2(x, y) = \int_{-\infty}^{x} (\varphi(s, y) - \varphi_1(s)P_2\varphi(y))\,ds. \qquad (4.123)$$

From (4.123) it follows that

$$\varphi = D_2\chi_1 + P_1\varphi \otimes \varphi_2 = D_1\chi_2 + \varphi_1 \otimes P_2\varphi. \tag{4.124}$$

Here $D_1 = \dfrac{\partial}{\partial x}$ and $D_2 = \dfrac{\partial}{\partial y}$. Since, by assumption (1) $D_1D_2u = D_2D_1u = 0$, a consequence of (4.124) is that

$$\begin{aligned} D_1u(\varphi) &= D_1u\left(P_1\varphi \otimes \varphi_2\right) = D_1v_1(\varphi), \quad \text{and} \\ D_2u(\varphi) &= D_2u\left(P_1\varphi \otimes \varphi_2\right) = D_2v_2(\varphi) \end{aligned} \tag{4.125}$$

where

$$v_1(\varphi) = u\left(P_1\varphi \otimes \varphi_2\right), \quad \text{and} \quad v_2(\varphi) = u\left(\varphi_1 \otimes P_2\varphi\right).$$

Since $D_1v_2 = D_2v_1 = 0$, and since $\varphi \in \mathcal{D}\left(\mathbb{R}^2\right)$ is arbitrary, from (4.125) we obtain

$$D_1\left(u - v_1 - v_2\right) = D_2\left(u - v_1 - v_2\right) = 0, \tag{4.126}$$

and hence, in distributional sense, $u - v_1 - v_2$ is equal to a constant C. This means that

$$\left(u - v_1 - v_2\right)(\varphi) = \int C\varphi(x, y)\, dx\, dy, \quad \varphi \in \mathcal{D}\left(\mathbb{R}^2\right). \tag{4.127}$$

From (4.127) we infer

$$\begin{aligned} C &= C\int \varphi_1 \otimes \varphi_2(x, y)\, dx\, dy = u\left(\varphi_1 \otimes \varphi_2\right) - v_1\left(\varphi_1 \otimes \varphi_2\right) - v_2\left(\varphi_1 \otimes \varphi_2\right) \\ &= u\left(\varphi_1 \otimes \varphi_2\right) - u\left(P_1\left(\varphi_1 \otimes \varphi_2\right) \otimes \varphi_2\right) - u\left(\varphi_1 \otimes P_2\left(\varphi_1 \otimes \varphi_2\right)\right) \\ &= u\left(\varphi_1 \otimes \varphi_2\right) - u\left(\varphi_1 \otimes \varphi_2\right) - u\left(\varphi_1 \otimes \varphi_2\right) = -u\left(\varphi_1 \otimes \varphi_2\right), \end{aligned} \tag{4.128}$$

because

$$\varphi_1 \otimes P_2\left(\varphi_1 \otimes \varphi_2\right) = P_1\left(\varphi_1 \otimes \varphi_2\right) \otimes \varphi_2 = \varphi_1 \otimes \varphi_2.$$

Whence assertion (2) follows by the choices in (4.122).

(2) $\Longrightarrow$ (1) This implication follows from the equalities:

$$D_2P_1\varphi = D_1P_2\varphi = 0, \quad \varphi \in \mathcal{D}\left(\mathbb{R}^2\right).$$

Altogether this completes the proof of Proposition 4.23. □

4.25. Definition. Suppose Ω is open in $\mathbb{R}^n$, N is a positive integer, $f_\alpha \in C^\infty(\Omega)$ for every multi-index $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant N$, and at least one f_α with $|\alpha| = N$ is not identically 0. These data determine a linear differential operator $L = \sum_{|\alpha| \leqslant N} f_\alpha D^\alpha$ which acts on distributions $u \in \mathcal{D}'(\Omega)$ by $Lu = \sum_{|\alpha| \leqslant N} f_\alpha D^\alpha u$. The order of L is N. The operator $\sum_{|\alpha| = N} f_\alpha D^\alpha$ is called the principal part of the operator L. The characteristic polynomial (symbol) of L is $p(x, \xi) = \sum_{|\alpha| = N} f_\alpha(x)\xi^\alpha$, $x \in \Omega$, $\xi \in \mathbb{R}^n$. This is a homogenous polynomial of degree N in the variables $\xi = (\xi_1, \ldots, \xi_n)$, with coefficients in $C^\infty(\Omega)$.

The operator L is said to be elliptic if $p(x, \xi) \neq 0$ for every $x \in \Omega$, and for $\xi \in \mathbb{R}^n$, except, of course, $\xi = 0$. Notice that ellipticity is defined in terms of the principal part of the operator L; the lower terms that appear in L don't play a role.

For example, the characteristic polynomial of the Laplacian $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$ is given by $p(x,\xi) = -\left(\xi_1^2 + \cdots + \xi_n^2\right)$, and so Δ is elliptic. On the characteristic polynomial of the operator $L = \dfrac{\partial^2}{\partial y_1 \partial y_2}$ is $p(x,\xi) = -\xi_1\xi_2$, and so L is not elliptic.

The main result we are aiming at, Theorem 4.30, involves some special spaces of tempered distributions, the so-called Sobolev spaces $H^s\left(\mathbb{R}^n\right)$, $s \in \mathbb{R}$.

5.1. Sobolev spaces. Associate to each real number s a positive measure μ_s on (the Borel field of) $\mathbb{R}^n$ by setting

$$\mu_s(B) = \int_B \left(1 + |y|^2\right)^s dy, \quad B \in \mathcal{B} = \mathcal{B}\left(\mathbb{R}^n\right). \tag{4.129}$$

If $f \in L^2\left(\mu_s\right)$, that is, if

$$\int_{\mathbb{R}^n} |f(y)|^2 \left(1 + |y|^2\right)^s d(y) < \infty,$$

then, as distribution f belongs to $\mathcal{S}' = \mathcal{S}\left(\mathbb{R}^n\right)$. Hence there exists a tempered distribution u such that $f = \widehat{u}$. A (tempered) distribution belongs to the Hilbert space H^s provided its Fourier transform $\widehat{u}$, which a priori is a tempered distribution, is in fact a function for which

$$\|u\|_s = \|u\|_{H^s} = \left(\int_{\mathbb{R}^n} |\widehat{u}(y)|^2 \left(1 + |y|^2\right)^s dy\right)^{1/2} = \left(\int_{\mathbb{R}^n} |\widehat{u}(y)|^2 \, d\mu_s(y)\right)^{1/2} < \infty. \tag{4.130}$$

The space H^s is clearly isometric to $L^2(\mu_s)$. The spaces H^s, $s \in \mathbb{R}$, are called Sobolev spaces. The dimension n will be fixed throughout the sequel, and no reference will be made to it in the notation. By Plancherel's theorem $H^0 = L^2(\mathbb{R}^n)$. It is obvious that $H^s \subset H^t$ if $t < s$; the larger the index, the smaller the space. The union $X := \bigcup_s H^s$ of all the Sobolev spaces is therefore a vector space. A linear operator $\Lambda : X \to X$ is said to have order t if the restriction of Λ to H^s is a continuous mapping from H^s to H^{s-t}. If $t > 0$, and Λ has order t, then the regularity of Λu is an order t "worse" than the regularity of $u \in H^s$. Notice that t need not be an integer.

4.26. Remark. Define, $t > 0$, the operator $S(t) : \mathcal{S}' \to \mathcal{S}$ by

$$S(t)u(x) := u\left[y \mapsto \frac{1}{\left(2\sqrt{\pi t}\right)^n} \exp\left(-\frac{|x-y|^2}{4t}\right)\right],$$

and define the operators $(\lambda I - \Delta)^{-s/2}$, $\lambda > 0$, $s > 0$, by the formula

$$(\lambda I - \Delta)^{-s/2} u(x) = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{-\lambda t} S(t)u(x)\, dt$$

for those distributions $u \in \mathcal{S}'$ for which these expressions make sense. Then a distribution $u \in \mathcal{S}'$ belongs to H^{-s}, $s > 0$, if and only the function $x \mapsto (I - \Delta)^{-s/2} u(x)$ belongs to $L^2(\mathbb{R}^n)$. The operator $(I - \Delta)^{s/2}$ is by definition the inverse of the operator $(I - \Delta)^{-s/2}$. The operator $(I - \Delta)^{s/2}$ is defined on all of H^t and maps it as a one to one mapping onto H^{t-s}. The latter is true for all $t \in \mathbb{R}$. In fact, in the context of operator semigroup theory all this can be justified. The case $s = 1$ deserves some extra attention. A distribution $u \in \mathcal{S}'$ belongs to the Sobolev space H^1 if and only u belongs to $L^2(\mathbb{R}^n)$, and the distributional derivatives $D_j u = \dfrac{\partial u}{\partial x_j}$, $1 \leqslant j \leqslant n$, also belong to $L^2(\mathbb{R}^n)$. A proof can be based on the following equalities:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(1 + |y|^2\right) |\widehat{u}(y)|^2\, dy = \int_{\mathbb{R}^n} |\widehat{u}(y)|^2\, dy + \sum_{j=1}^n \int_{\mathbb{R}^n} |y_j \widehat{u}(y)|^2\, dy \\
&= \int_{\mathbb{R}^n} |\widehat{u}(y)|^2\, dy + \sum_{j=1}^n \int_{\mathbb{R}^n} \left|\widehat{D_j u}(y)\right|^2\, dy \\
&= (2\pi)^n \left(\int_{\mathbb{R}^n} |u(x)|^2\, dx + \sum_{j=1}^n \int_{\mathbb{R}^n} |D_j u(x)|^2\, dx\right).
\end{aligned}$$

In the following theorem we list some relevant properties of the Sobolev spaces.

4.27. Theorem. *In the above notation the following assertions are valid.*

(a) *Every distribution with compact support lies in some H^s.*
(b) *If $t \in \mathbb{R}$, then the mapping $u \mapsto v$ given by*

$$\widehat{v}(y) = \left(1 + |y|^2\right)^{t/2} \widehat{u}(y), \quad y \in \mathbb{R}^n,$$

is a linear isometry from H^s onto H^{s-t} and therefore it is an operator of order t whose inverse has order $-t$. In fact $v = \mathcal{F}^{-1}\left(1+|y|^2\right)^{t/2}\mathcal{F}u$.

(c) *If $b \in L^\infty(\mathbb{R}^n)$, the mapping $u \mapsto v$ given by $\widehat{v} = b\widehat{u}$, or $v = \mathcal{F}^{-1}b\mathcal{F}u$, is an operator of order 0.*

(d) *For every multi-index α the operator D^α is an operator of order $|\alpha|$.*

(e) *If $f \in \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, then the operator $u \mapsto fu$ is an operator of order 0.*

Proof. (a) If the distribution u has compact support, in other words, if $u \in \mathcal{E}'$, then there exists a finite constant C and an integer N such that

$$|\widehat{u}(y)| \leqslant C\left(1+|y|^2\right)^{N/2}, \quad y \in \mathbb{R}^n.$$

Hence $u \in H^s$ if $s < -N - n/2$. This shows assertion (a).

The assertions (b) and (c) are clear.

(d) The relation

$$\left|\widehat{D^\alpha u}(y)\right| = |y^\alpha|\,|\widehat{u}(y)| \leqslant \left(1+|y|^2\right)^{|\alpha|/2} |\,|\widehat{u}(y)|$$

implies

$$|D^\alpha u\|_{s-|\alpha|} \leqslant \|u\|_s\,, \quad u \in H^s,$$

and so D^α is of order $|\alpha|$, which is assertion (d).

(e) The proof of (e) depends (partially) on the inequality:

$$\left(1+|x+y|^2\right)^s \leqslant 2^{|s|}\left(1+|x|^2\right)^s\left(1+|y|^2\right)^{|s|}. \tag{4.131}$$

The inequality in (4.131) is valid for x, $y \in \mathbb{R}^n$, $s \in \mathbb{R}$. The case $s = 1$ is easy. From it the case $s = -1$ is obtained by replacing x with $x - y$, and then y by $-y$. The general case of (4.131) is then obtained from these two by raising everything to the power $|s|$. It follows from (4.131) that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |h(x-y)|^2\, d\mu_s(x) = \int_{\mathbb{R}^n} |h(x)|^2\left(1+|x+y|^2\right)^s dx \\
&\leqslant 2^{|s|}\left(1+|y|^2\right)^{|s|}\int_{\mathbb{R}^n} |h(x)|^2\left(1+|x|^2\right)^s dx \\
&= 2^{|s|}\left(1+|y|^2\right)^{|s|}\int_{\mathbb{R}^n} |h(x)|^2\, d\mu_s(x).
\end{aligned} \tag{4.132}$$

Now suppose $u \in H^s$, $f \in \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, and $t > |s| + n/2$. Since $\widehat{f} \in \mathcal{S}$ the function f belongs to H^t; in fact $\mathcal{S} \subset \bigcap_{\rho>0} H^\rho$. By (v) in Theorem 4.5 we then have

$$\begin{aligned}
\|fu\|_s^2 &= \int\left(1+|x|^2\right)^s\left|\widehat{fu}(x)\right|^2 dx = \frac{1}{(2\pi)^{2n}}\int\left(1+|x|^2\right)^s\left|\widehat{u} * \widehat{f}(x)\right|^2 dx \\
&\leqslant \frac{1}{(2\pi)^{2n}}\int\left(1+|x|^2\right)^s\left(\int|\widehat{u}(x-y)|\left|\widehat{f}(y)\right| dy\right)^2 dx
\end{aligned}$$

$$= \frac{1}{(2\pi)^{2n}} \int \left(1+|x|^2\right)^s \left(\int \frac{|\widehat{u}(x-y)|}{\left(1+|y|^2\right)^{t/2}} \left|\widehat{f}(y)\right| \left(1+|y|^2\right)^{t/2} dy \right)^2 dx$$

(apply the inequality of Cauchy-Schwarz)

$$\leqslant \frac{1}{(2\pi)^{2n}} \int \left(1+|x|^2\right)^s \int \frac{|\widehat{u}(x-y)|^2}{\left(1+|y|^2\right)^t} dy \left(\int \left|\widehat{f}\,(y')\right|^2 \left(1+|y'|^2\right)^t dy' \right) dx$$

(Fubini's theorem)

$$= \frac{1}{(2\pi)^{2n}} \int \frac{1}{\left(1+|y|^2\right)^t} \int \left(1+|x|^2\right)^s |\widehat{u}(x-y)|^2 \, dx \, dy \, \|f\|_t^2$$

(employ (4.132) with $h = \widehat{u}$)

$$\leqslant \frac{2^{|s|}}{(2\pi)^{2n}} \int \frac{\left(1+|y|^2\right)^{|s|}}{\left(1+|y|^2\right)^t} dy \int \left(1+|x|^2\right)^s |\widehat{u}(x)|^2 \, dx \, \|f\|_t^2$$

$$= \frac{2^{|s|}}{(2\pi)^{2n}} \mu_{|s|-t}\left(\mathbb{R}^n\right) \|u\|_s^2 \, \|f\|_t^2 . \tag{4.133}$$

Since, for $t > |s| + n/2$, the quantity $\mu_{|s|-t}\left(\mathbb{R}^n\right)$ is finite this proves (e), and completes the proof of Theorem 4.27. $\square$

4.28. Definition. Let Ω be an open subset of $\mathbb{R}^n$. A distribution $u \in \mathcal{D}'(\Omega)$ is said to be locally H^s in Ω, also written as $u \in H^s_{\mathrm{loc}}(\Omega)$, if there corresponds to each point $x \in \Omega$ a distribution $v \in H^s$ such that $u = v$ in an open neighborhood of x.

4.29. Theorem. *If $u \in \mathcal{D}'(\Omega)$ and $s \in \mathbb{R}$, the following two statements are equivalent:*

(a) *the distribution u is locally H^s;*
(b) *the distribution ψu belongs to H^s for every $\psi \in \mathcal{D}(\Omega)$.*

Moreover, if s is a nonnegative integer, assertions (a) and (b) are also equivalent to

(c) *the distribution $D^\alpha u$ is locally L^2 for every α with $|\alpha| \leqslant s$.*

Of course, in (b) the distribution ψu, $\psi \in \mathcal{D}(\Omega)$, is defined by $\psi u(\varphi) = u(\psi\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$; notice that $\psi\varphi \in \mathcal{D}(\Omega)$ whenever $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\Omega)$.

Proof of Theorem 4.29. Assume u is locally H^s. Let K be the support of some $\psi \in \mathcal{D}(\Omega)$. Since $K \subset \Omega$ is compact there exist finitely many open sets $\Omega_j \subset \Omega$ together with distributions $v_j \in H^s$, $1 \leqslant j \leqslant N$, such that u coincides with v_j in Ω_j, and such that $K \subset \cup_{j=1}^N \Omega_j$. In addition, there exist functions $\psi_j \in \mathcal{D}(\Omega_j)$ such that $\sum_{j=1}^N \psi_j = 1$ on K. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ it follows that

$$u(\psi\varphi) = \sum_{j=1}^N u(\psi_j\psi\varphi) = \sum_{j=1}^N v_j(\psi_j\psi\varphi) = \sum_{j=1}^N \psi_j\psi v_j(\varphi),$$

since $\psi_j\psi \in \mathcal{D}(\Omega_j)$. Thus $\psi u = \sum_{j=1}^N \psi_j\psi v_j$. By item (e) of Theorem 4.27 it follows that $\psi_j\psi v_j \in H^s$, for $1 \leqslant j \leqslant N$. Consequently, $\psi u \in H^s$, and (a) implies (b).

If (b) holds, if $x \in \Omega$, and if ψ is 1 in a neighborhood Ω_x of x, then $u = \psi u$ in Ω_x, and $\psi u \in H^s$ by (b). So (a) is a consequence of (b).

Assume again that (b) holds. If $\psi \in \mathcal{D}(\Omega)$, then $\psi u \in H^s$, hence $D^\alpha(\psi u) \in H^{s-|\alpha|}$, by assertion (d) of Theorem 4.27. If $|\alpha| \leqslant s$, then

$$H^{s-|\alpha|} \subset H^0 = L^2(\mathbb{R}^n),$$

and consequently $D^\alpha(\psi u) \in L^2(\mathbb{R}^n)$. By taking ψ identically 1 in a neighborhood of a point $x \in \Omega$ shows that $D^\alpha u \in L^2_{\mathrm{loc}}(\Omega)$. Thus (b) implies (c).

Finally, assume $D^\alpha u \in L^2_{\mathrm{loc}}(\Omega)$ for every α with $|\alpha| \leqslant s$. Fix $\psi \in \mathcal{D}(\Omega)$. The Leibniz formula shows that $D^\alpha(\psi u) \in L^2(\mathbb{R}^n)$ if $|\alpha| \leqslant s$. Hence, by Plancherel's theorem it follows that

$$\int_{\mathbb{R}^n} |y^\alpha|^2 \left|\widehat{\psi u}(y)\right|^2 dy < \infty, \quad |\alpha| \leqslant s. \tag{4.134}$$

If s is a nonnegative integer, (4.134) holds with the monomials $y_1^s, \dots, y_n^s$ in place of y^α. As in the proof of Theorem 4.22 it follows that

$$\int_{\mathbb{R}^n} \left(1 + |y|^2\right)^s \left|\widehat{\psi u}(y)\right|^2 dy < \infty. \tag{4.135}$$

Using (4.135), the definition of the space H^s shows that $\psi u \in H^s$. So that (c) implies (b), and the proof of Theorem 4.29 is complete. □

4.30. THEOREM. *Assume that Ω is an open subset of $\mathbb{R}^n$, and*

(a) *the operator $L = \sum_{\alpha, |\alpha| \leqslant N} f_\alpha D^\alpha$ is a linear elliptic differential operator in Ω, of order $N \geqslant 1$, with coefficients $f_\alpha \in C^\infty(\Omega)$,*
(b) *for each α with $|\alpha| = N$, f_α is constant,*
(c) *the distributions $u, v \in \mathcal{D}'(\Omega)$ satisfy $Lu = v$, and v is locally H^s, or what is the same $v \in H^s_{\mathrm{loc}}(\Omega)$.*

Then u is locally H^{s+N} in Ω, or, what is the same, $u \in H^{s+N}_{\mathrm{loc}}(\Omega)$.

Assumption (b) can be dropped form the theorem, but its presence makes its proof considerably easier. As a corollary to Theorems 4.22, 4.30 and 4.29 (c) we have the following result.

4.31. COROLLARY. *If L satisfies (a) and (b) in Theorem 4.30 and if $v \in C^\infty(\Omega)$, then every solution u of the equation $Lu = v$ belongs to $C^\infty(\Omega)$. In particular, every solution u of the homogeneous equation $Lu = 0$ is in $C^\infty(\Omega)$.*

PROOF OF COROLLARY 4.31. Since $v \in C^\infty(\Omega)$ the function ψv belongs to $\mathcal{D}(\mathbb{R}^n)$ for every $\psi \in \mathcal{D}(\Omega)$; hence v is locally H^s for every $s \in \mathbb{R}$, and Theorem 4.30 implies that u is locally H^s for every $s \in \mathbb{R}$; it follows from Theorems 4.22 and 4.29 that $u \in C^\infty(\Omega)$. This completes the proof of Corollary 4.31. □

PROOF OF THEOREM 4.30. Fix $x \in \Omega$, let $B_0 \subset \Omega$ be closed ball with center at x, and let $\varphi_0 \in \mathcal{D}(\Omega)$ be 1 in some open set containing B_0. By assertion (a) of Theorem 4.27 $\varphi_0 u \in H^t$ for some t. Since H^t becomes larger as t decreases, we may assume that $t = s + N - k$ for some positive integer k. Choose closed balls

$$B_0 \supset B_1 \supset \cdots \supset B_{k-1} \supset B_k,$$

each centered at x, and each properly inclosed in the preceding one. Choose functions $\varphi_1, \dots, \varphi_k \in \mathcal{D}(\Omega)$ so that $\varphi_j = 1$ on some open set containing B_j, and $\varphi_j = 0$ off B_{j-1}. Since $\varphi_0 u \in H^t$, the "bootstrap" Proposition 4.32 below implies that

$$\varphi_1 u \in H^{t+1}, \dots, \varphi_k u \in H^{t+k}.$$

It therefore leads to the conclusion that u is locally $H^{t+k} = H^{s+N}$, because $t + k = s + N$, and $\varphi_k = 1$ on (a neighborhood of) B_k. □

4.32. PROPOSITION. *If, in addition to the hypotheses of Theorem 4.30, $\psi u \in H^t$ for some $t \leqslant s + N - 1$ and for some $\psi \in \mathcal{D}(\Omega)$ which is 1 on an open set containing the support of a function $\varphi \in \mathcal{D}(\Omega)$, then $\varphi u \in H^{t+1}$.*

PROOF OF PROPOSITION 4.32. We begin by showing that $L(\varphi u) \in H^{t-N+1}$. Consider the distribution

$$\Lambda = L(\varphi u) - \varphi L u = L(\varphi u) - \varphi v. \tag{4.136}$$

Since the support of Λ lies in the support of φ, u can be replaced by ψu in (4.136) without changing Λ:

$$\begin{aligned}\Lambda = L(\varphi\psi u) - \varphi L(\psi u) &= \sum_{|\alpha|\leqslant N} f_\alpha \left[D^\alpha (\varphi\psi u) - \varphi D^\alpha (\psi u)\right] \\ &= \sum_{1\leqslant|\alpha|\leqslant N} f_\alpha \left[D^\alpha (\varphi\psi u) - \varphi D^\alpha (\psi u)\right] = \sum_{1\leqslant|\alpha|\leqslant N} f_\alpha \sum_{\beta<\alpha} \binom{\alpha}{\beta} \left[D^{\alpha-\beta}(\varphi)\right] D^\beta (\psi u) \\ &= \sum_{|\beta|\leqslant N-1} \sum_{\alpha>\beta,\,|\alpha|\leqslant N} f_\alpha \binom{\alpha}{\beta} \left[D^{\alpha-\beta}(\varphi)\right] D^\beta (\psi u) = \sum_{|\beta|\leqslant N-1} \chi_\beta D^\beta (\psi u)\,. \end{aligned} \tag{4.137}$$

where

$$\chi_\beta = \sum_{\alpha>\beta,\,|\alpha|\leqslant N} f_\alpha \binom{\alpha}{\beta} \left[D^{\alpha-\beta}(\varphi)\right].$$

The third equality in (4.137) is a consequence of Leibniz' rule applied to $D^\alpha(\varphi\cdot\psi u)$. Therefore Λ is a linear combination, with coefficients in $\mathcal{D}(\mathbb{R}^n)$, of derivatives of ψu. Since $\psi u \in H^t$, assertions (d) and (e) of Theorem 4.27 imply that $\Lambda \in H^{t-N+1}$. By assumption $v \in H^s_{\mathrm{loc}}(\Omega)$, and so by assertion (b) of Theorem 4.29 we have $\varphi v \in H^{t-N+1}$ because $t - N + 1 \leqslant s$. By (4.136) we have $L(\varphi u) = \Lambda + \varphi v = \Lambda + \varphi L u \in H^{t-N+1}$.

Since the operator L is elliptic its characteristic polynomial

$$p(y) = \sum_{|\alpha|=N} f_\alpha y^\alpha, \quad y \in \mathbb{R}^n, \tag{4.138}$$

has no zero in $\mathbb{R}^n$, except at $y = 0$. Define the functions

$$q(y) = \frac{p(y)}{|y|^N} = p\left(\frac{y}{|y|}\right), \quad r(y) = \left(1 + |y|^N\right) q(y) = \left(1 + \frac{1}{|y|^N}\right) p(y), \tag{4.139}$$

for $y \in \mathbb{R}^n \backslash \{0\}$, and define the operators Q, R, S on the union of the Sobolev spaces by

$$\widehat{Qw} = q\widehat{w} \text{ or } Qw = \mathcal{F}^{-1} q \mathcal{F} w, \quad \widehat{Rw} = r\widehat{w} \text{ or } Rw = \mathcal{F}^{-1} r \mathcal{F} w, \text{ and} \tag{4.140}$$

$$S = \sum_{|\alpha| \leqslant N-1} f_\alpha D^\alpha. \tag{4.141}$$

Since p is a homogenous polynomial of order N, $q(\lambda y) = q(y)$ if $\lambda > 0$ (and $y \in \mathbb{R}^n$), and since p vanishes only at the origin, the compactness of the unit sphere in $\mathbb{R}^n$ implies that both q and $\dfrac{1}{q}$ are bounded functions. It follows from (c) of Theorem 4.27 that both Q and Q^{-1} are operators of order 0. Since both $\left(1 + |y|^2\right)^{-N/2} \left(1 + |y|^N\right)$ and its reciprocal are bounded functions on $\mathbb{R}^n$, it follows from the preceding arguments, combined with (b) and (c) of Theorem 4.27, that R is an operator of order N whose inverse R^{-1} has order $-N$.

Since $p = r - q$, and since p is assumed to have constant coefficients f_α, we have

$$\sum_{|\alpha|=N} \widehat{f_\alpha D^\alpha} w = p\widehat{w} = (r - q)\,\widehat{w} = \widehat{Rw - Qw} \tag{4.142}$$

if w belongs to some Sobolev space. Hence, (4.142) implies

$$(R - Q + S)(\varphi u) = L(\varphi u). \tag{4.143}$$

In the first part of this proof it was shown that $L(\varphi u) \in H^{t-N+1}$. Since $\psi u \in H^t$ and $\psi\varphi = \varphi$, (e) of Theorem 4.27 implies that $\varphi u = \varphi\psi u \in H^t$. Hence

$$(Q - S)(\varphi u) \in H^{t-N+1},$$

because the operator Q has order 0 and the operator S has order $N - 1 \geqslant 0$. It now follows from (4.143) that $R(\varphi u) \in H^{t-N+1}$, and since R^{-1} has order $-N$, we finally conclude that $\varphi u \in H^{t+1}$. This completes the proof of Theorem 4.30. □

4.33. Example. Suppose that L is an elliptic operator with constant coefficients, and let E be a fundamental solution of L. In the complement of the origin the equation $LE = \delta$ reduces to $LE = 0$. Theorem 4.30 implies therefore that, except at the origin, E is a C^∞-function. The nature of the singularity of E at the origin depends on L. In particular this holds for the Laplace operator: see Proposition 4.15 where a fundamental solution for the Laplace operator is given in $\mathbb{R}^n$, $n \geqslant 3$.

4.34. EXAMPLE. The origin in $\mathbb{R}^2$ is the only zero of the complex polynomial $p(y_1, y_2) = y_1 + iy_2$. If Ω is open in $\mathbb{R}^2$, and if $u \in \mathcal{D}'(\Omega)$ is a distribution solution of the Cauchy-Riemann equation $Lu := \dfrac{\partial u}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2}\right) u = 0$, Theorem 4.30 implies that $u \in C^\infty(\Omega)$. It follows that u can be considered as a holomorphic function $\widetilde{u}$ on $\widetilde{\Omega} = \{z = x_1 + ix_2 : (x_1, x_2) \in \Omega\}$: $\widetilde{u}(z) = u(x_1, x_2)$, $z = x_1 + ix_2 \in \widetilde{\Omega}$. In other words, holomorphic distributions are holomorphic functions. A fundamental solution E of L is given by $E(x_1, x_2) = \dfrac{1}{\pi(x_1 + ix_2)}$. The proof of this fact relies on the equality

$$\frac{\partial \varphi}{\partial \bar{z}}(r\cos\vartheta, r\sin\vartheta) = \frac{e^{i\vartheta}}{2}\left(\frac{\partial \varphi(r\cos\vartheta, r\sin\vartheta)}{\partial r} + \frac{i}{r}\frac{\partial \varphi(r\cos\vartheta, r\sin\vartheta)}{\partial \vartheta}\right). \tag{4.144}$$

Hence

$$\frac{\partial E}{\partial \bar{z}}(\varphi) = -E\left(\frac{\partial \varphi}{\partial \bar{z}}\right) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{x_1 + ix_2}(D_1 + iD_2)\varphi(x_1, x_2)\, dx_1\, dx_2$$

(pass to polar coordinates)

$$= -\frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \frac{1}{re^{i\vartheta}}(D_1 + iD_2)\varphi(r\cos\vartheta, r\sin\vartheta)\, r\, d\vartheta\, dr$$

(employ (4.144))

$$= -\frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \left(\frac{\partial \varphi(r\cos\vartheta, r\sin\vartheta)}{\partial r} + \frac{i}{r}\frac{\partial \varphi(r\cos\vartheta, r\sin\vartheta)}{\partial \vartheta}\right) d\vartheta\, dr$$

(the second integral vansishes)

$$\begin{aligned} &= -\frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \frac{\partial \varphi(r\cos\vartheta, r\sin\vartheta)}{\partial r}\, d\vartheta\, dr \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \int_0^\infty \frac{\partial \varphi(r\cos\vartheta, r\sin\vartheta)}{\partial r}\, dr\, d\vartheta = \varphi(0,0). \end{aligned} \tag{4.145}$$

The equality in (4.145) shows that E is a fundamental solution for the operator $\bar{\partial} = \dfrac{\partial}{\partial \bar{z}}$. A fundamental solution E for the Cauchy-Riemann operator is unique provided it is assumed that the function $(x_1, x_2) \mapsto (x_1 + ix_2) E(x_1, x_2)$ is bounded on $\mathbb{R}^2$. This is a consequence of Liouville's theorem applied to bounded function $z \mapsto zh(x_1, x_2)$, $z = x_1 + ix_2 \in \mathbb{C}$, with $h = E_2 - E_1$, where E_1 are E_2 are fundamental solutions for the Cauchy-Riemann operator with the property that the function $z \mapsto zh(x_1, x_2)$, $z = x_1 + ix_2 \in \mathbb{C}$, is bounded on $\mathbb{C}$.

4.35. EXAMPLE. The Laplace operator in polar coordinates in $n = 2$ dimensions reads

$$\Delta\varphi(r\cos\vartheta, r\sin\vartheta)$$

$$
\begin{aligned}
&= \frac{\partial^2 \varphi}{\partial r^2}\left(r\cos\vartheta, r\sin\vartheta\right) + \frac{1}{r}\frac{\partial \varphi}{\partial r}\left(r\cos\vartheta, r\sin\vartheta\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \vartheta^2}\left(r\cos\vartheta, r\sin\vartheta\right) \\
&= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)\right)\left(r\cos\vartheta, r\sin\vartheta\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \vartheta^2}\left(r\cos\vartheta, r\sin\vartheta\right). \qquad (4.146)
\end{aligned}
$$

As in the proof for the Cauchy-operator, the equality in (4.146) yields that the function $E\left(x_1, x_2\right) = \dfrac{1}{2\pi}\log\left(x_1^2 + x_2^2\right)^{1/2}$ is a fundamental solution for the Laplace operator Δ in 2 dimensions. Moreover, up to constants we have uniqueness of fundamental solutions provided we assume $E\left(x_1, x_2\right)$ has the property that, for some constant $C < \infty$, $\left|E\left(x_1, x_2\right)\right| \leqslant C\log\left(x_1^2 + x_2^2\right)$ for all $\left(x_1, x_2\right) \in \mathbb{R}^2$ with $x_1^2 + x_2^2$ large. This can be seen as follows. Let E_1 and E_2 be two fundamental solutions. Then their difference $h = E_2 - E_1$ is a harmonic function which does not grow faster than $C\log\left(x_1^2 + x_2^2\right)$ for $\left(x_1, x_2\right) \in \mathbb{R}^2$ large. Then there exists a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $h\left(x_1, x_2\right) = \Re f(z)$, $z = x_1 + ix_2 \in \mathbb{C}$. It follows that the function e^f is polynomially bounded. By expanding e^f into a power series around 0, and using Cauchy estimates for its coefficients we obtain that the function e^f is a polynomial, and hence $e^{f(z)} = P(z)$, $z \in \mathbb{C}$, for some complex polynomial P. This polynomial must be constant, because the function e^f does not have zeros. So we get $e^{f(z)} = P(z) = P(0) = e^{f(0)}$, whence $f(z) = f(0)$, $z \in \mathbb{C}$, and so $E_2 - E_1 = \Re f(0)$.

We have genuine uniqueness provided we assume that the fundamental solution vanishes on the unit circle in $\mathbb{R}^2$. In fact, let E_1 and E_2 be two fundamental solutions for the Laplace operator in $\mathbb{R}^2$ with property that $E_1(x_1, x_2) = E_2(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$ with $x_1^2 + x_2^2 = 1$. Again the difference $h := E_2 - E_1$ is a harmonic function on $\mathbb{R}^2$ which vanishes on the unit circle. However, a harmonic function which is continuous on the closed unit disc attains its maximum and minimum on the unit circle, and so $h = 0$ on the unit disc with center at the origin. But then h is identically 0, and hence $E_2 = E_1$. For completeness we insert a proof of the fact that the function $E : (x_1, x_2) \mapsto \dfrac{1}{2\pi} \log\left(x_1^2 + x_2^2\right)^{1/2}$ is a fundamental solution for the Laplace operator in $\mathbb{R}^2$. Therefore let $\varphi \in \mathcal{D}(\mathbb{R}^2)$, and calculate

$$(\Delta E)\varphi = E(\Delta\varphi) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \log\left(x_1^2 + x_2^2\right)^{1/2} \Delta\varphi(x_1, x_2)\, dx_1 dx_2$$

(use polar coordinates: see (4.146))

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^\infty r \log r \int_{-\pi}^{\pi} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\varphi}{\partial r} \right) (r\cos\vartheta, r\sin\vartheta)\, d\vartheta\, dr \\ &\quad + \frac{1}{2\pi} \int_0^\infty r \log r \int_{-\pi}^{\pi} \frac{1}{r^2} \frac{\partial^2\varphi}{\partial\vartheta^2} (r\cos\vartheta, r\sin\vartheta)\, d\vartheta\, dr \end{aligned}$$

(apply Fubini's theorem; the second integral is 0)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^\infty \log r \frac{\partial}{\partial r} \left(r \frac{\partial\varphi}{\partial r} \right) (r\cos\vartheta, r\sin\vartheta)\, dr\, d\vartheta$$

(integration by parts: notice that $\lim_{r\downarrow 0} r \log r = 0$)

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^\infty \left(\frac{\partial\varphi}{\partial r} \right) (r\cos\vartheta, r\sin\vartheta)\, dr\, d\vartheta = \varphi(0,0). \tag{4.147}$$

The equality in (4.147) shows that the function E is a fundamental solution for the Laplace operator in $\mathbb{R}^2$.

4.36. Example. Any fundamental solution E of the operator $D^2 = \dfrac{d^2}{dx^2}$ is of the form $E(x) = \frac{1}{2}|x| + \alpha x + \beta$, $x \in \mathbb{R}$, where α and β are constants.

5.2. Quadratic forms and a compact embedding result. In this subsection we will discuss Friedrichs extension of a symmetric operator which is bounded below: see Theorem 4.41 below. We will also prove a compact embedding result of Sobolev spaces: see Theorem 4.43 below.

4.37. Definition. Let X and Y be Banach spaces and $X \subset Y$. The space X is said to be *compactly embedded* in Y (notation $X \subset\subset Y$) if any X-bounded sequence has a Y-convergent subsequence.

4.38. Definition. Define the subspace $H^1(\Omega)$ of $L^2(\Omega)$ as follows. A (class of) functions $f \in L^2(\Omega)$ belongs to $H_0^1(\Omega)$ provided its distributional gradient $\nabla f = (D_1 f, \dots, D_n f)$ belongs to $L^2(\Omega, \mathbb{C}^n)$. This means that each of the distributions $D_j f$ in fact is an L^2-function on Ω. The norm, determined by

$$\|f\|_{H^1}^2 = \int_\Omega |f(x)|^2\,dx + \sum_{j=1}^n \int_\Omega |D_j f(x)|^2\,dx = \int_\Omega |f(x)|^2\,dx + \int_\Omega |\nabla f(x)|^2\,dx, \tag{4.148}$$

turns the space $H^1(\Omega)$ into a Hilbert space. The space $H_0^1(\Omega)$ consists of those $f \in H^1(\Omega)$ which can be written as an $H^1(\Omega)$-limit of a sequence of functions $(\varphi_k)_k$ in $\mathcal{D}(\Omega)$. Define the quadratic form $Q: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by

$$Q(f,g) = \int_\Omega f(x)\overline{g(x)}\,dx + \int_\Omega \nabla f(x) \cdot \overline{\nabla g(x)}\,dx, \quad f,\, g \in H_0^1(\Omega). \tag{4.149}$$

Then the form Q in (4.149) is closed in the sense that, if $(f_k)_k$ is a sequence in $H_0^1(\Omega)$ which converges to $f \in L^2(\Omega)$, and is such that $\lim_{k,\ell\to\infty} Q(f_k - f_\ell, f_k - f_\ell) = 0$, then f belongs to $H_0^1(\Omega)$ and $\lim_{k\to\infty} Q(f_k - f, f_k - f) = 0$. Next, consider the subspace $L \subset H_0^1(\Omega)$ defined as the set of those functions $g \in H_0^1(\Omega)$ which possess the property that there exists a finite constant $C = C_g$ such that

$$|Q(f,g)| \leqslant C\,\|f\|_{L^2(\Omega)} \text{ for all } f \in H_0^1(\Omega).$$

In other words a function $g \in H_0^1(\Omega)$ belongs to L if and only if the (linear) functional $f \mapsto Q(f,g)$, $f \in H_0^1(\Omega)$, is continuous for the L^2-norm. By the Riesz-Fisher representation theorem there exists a function $h = Tg \in L^2(\Omega)$ such that

$$Q(f,g) = \langle f, h\rangle_{L^2} = \langle f, Tg\rangle_{L^2}, \; f \in H_0^1(\Omega).$$

Observe that the operator T is a closed linear operator with domain $L \subset H_0^1(\Omega)$. It extends the operator $I - \Delta$, where Δ is the Laplace operator on $\mathcal{D}(\Omega)$. In fact the operator T is the *Friedrichs extension* of the operator $I - \Delta$.

We will elaborate on this issue in a more abstract setting. Let H with inner-product $\langle \cdot, \cdot \rangle$ be a Hilbert space, and let Q be a symmetric quadratic form on a subspace $D = D(Q)$ of H. The quadratic form Q is called symmetric if $\overline{Q(x,y)} = Q(y,x)$ for all $x,\, y \in D$. If Q is symmetric, then the polarization formula holds true:

$$\begin{aligned} &4Q(x,y) \\ &= Q(x+y, x+y) - Q(x-y, x-y) + iQ(x+iy, x+iy) - iQ(x-iy, x-iy), \end{aligned}$$

for $x,\, y \in D$.

4.39. Definition. If the subspace D is dense in H, then Q is said to be densely defined. If for every sequence $(x_n)_n \subset D$ for which there exists $x \in H$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$, and for which $\lim_{n,m\to\infty} Q(x_n - x_m, x_n - x_m) = 0$, it follows that $x \in D$ and that $\lim_{n\to\infty} Q(x_n - x, x_n - x) = 0$, then Q is said to be closed. The form Q is said to be closable if for all $(x_n)_n \subset D$ such that $\lim_{n\to\infty} x_n = 0$ in H and $\lim_{n,m\to\infty} Q(x_n - x_m, x_n - x_m) = 0$ implies that $\lim_{n\to\infty} Q(x, x_n) = 0$.

Let $Q : D \times D \to \mathbb{C}$ be a closable quadratic form. Then its closure $\widehat{Q} : \widehat{D} \times \widehat{D} \to \mathbb{C}$ can be described as follows. Its domain $\widehat{D} = D\left(\widehat{Q}\right)$ consists of those $x \in H$ for which there exists a sequence $(x_n)_n \subset D$ such that $\lim_{n\to\infty} x_n = x$ in H and $\lim_{n,m\to\infty} Q\left(x_n - x_m, x_n - x_m\right) = 0$. Such a sequence $(x_n)_n$ is said to converge in form sense. If x, y belong to $D\left(\widehat{Q}\right)$ and if the sequence $(x_n)_n \subset D$ converges to $x \in H$ in form sense, and the sequence $(y_n)_n$ converges to $y \in H$ in form sense, then $\widehat{Q}\left(x, y\right) = \lim_{n\to\infty} Q\left(x_n, y_n\right)$. Since Q is symmetric the latter equality holds whenever a sequence $(x_n)_n \subset D$ converges to $x \in H$ in form sense it follows that $\widehat{Q}\left(x, x\right) = \lim_{n\to\infty} Q\left(x_n, x_n\right)$. Let Q_1 and Q_2 be two quadratic forms with domains D_1 respectively D_2 in H. Then Q_2 is said to be greater than Q_1 (notation $Q_1 \leqslant Q_2$) provided $D_1 \subset D_2$ and $Q_1\left(x, x\right) \leqslant Q_2\left(x, x\right)$ for all $x \in D_1$. If an operator T with domain $D(T)$ and range $R(T)$ in H is symmetric in the sense that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all x, $y \in D(T)$, then the quadratic form $Q : D(T) \times D(T) \to \mathbb{C}$ defined by $Q(x, y) = \langle Tx, y\rangle = \langle x, Ty\rangle$, x, $y \in D(T)$ is symmetric.

4.40. Lemma. *Let T be a symmetric operator, and define the quadratic form $Q : D(T) \times D(T) \to \mathbb{C}$ by $Q(x, y) = \langle Tx, y\rangle$, x, $y \in D(T)$. Then the quadratic form Q is closable.*

Proof of Lemma 4.40. Let the sequence $(x_n)_n \subset D(T)$ converge to 0 in form sense. This means that $\lim_{n\to\infty} x_n = 0$ in H, and that $\lim_{n,m\to\infty} \langle T(x_n - x_m), x_n - x_m\rangle = 0$. We shall prove that $\lim_{n\to\infty} Q(x_n, x_n) = 0$. This conclusion follows from the equalities:

$$\begin{aligned}\langle Tx_n, x_n\rangle + \langle Tx_m, x_m\rangle &= \langle T(x_n - x_m), x_n - x_m\rangle + \langle Tx_m, x_n\rangle + \langle x_n, Tx_m\rangle \\ &= \langle T(x_n - x_m), x_n - x_m\rangle + \langle x_m, Tx_n\rangle + \langle Tx_n, x_m\rangle .\end{aligned} \tag{4.150}$$

Upon letting n and m tend to ∞ in (4.150) we see that $\lim_{n\to\infty} \langle Tx_n, x_n\rangle = 0$. This proves the claim in Lemma 4.40. □

In the following theorem we present some results on closed quadratic forms. This theorem is taken from [**44**].

4.41. Theorem. *Let Q be a densely defined closed quadratic form with $Q \geqslant 1$. Then there exists an operator A such that*

(1) *The inclusion $D(A) \subset D(Q)$ holds;*
(2) *The equality $Q(x,z) = \langle x, Az\rangle$ is true for all $x \in D(Q)$, $z \in D(A)$;*
(3) *The operator A is self-adjoint;*
(4) *If B is another operator satisfying (1) and (2), then $B \subset A$;*
(5) *If B satisfies (1), (2) and is also self-adjoint then $B = A$;*
(6) *$A \geqslant I$ in the sense that $\langle Ax, x\rangle \geqslant \|x\|^2$, $x \in D(A)$;*
(7) *The domain $D(A)$ is a core for Q;*
(8) *$D(Q) = D\left(A^{1/2}\right)$, and*
(9) *$Q(x,y) = \left\langle A^{1/2}x, A^{1/2}y\right\rangle$ for all x, $y \in D(Q)$.*

For a discussion on square roots of positive operators the reader is referred to Theorem 5.41 in Chapter 5.

Proof. Formally, here is how we will proceed. We want to find A such that $Q(x,z) = \langle x, Az\rangle$. If A were invertible in some sense, we could take $z = A^{-1}y$ and have $Q(x, A^{-1}y) = \langle x, y\rangle$. Working backwards, we will find an operator T that will fit in for A^{-1}, and then take T^{-1} to get A. If $x \in D(Q)$, $y \in H$, notice that

$$|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\| \leqslant \sqrt{Q(x,x)}\, \|y\| . \tag{4.151}$$

Remember also that $D(Q)$ is a Hilbert space under the Q inner product. So this shows that $x \mapsto \langle x, y\rangle$ is a bounded linear functional on the Hilbert space $D(Q)$. We can apply the Riesz-Fischer representation theorem to find a unique element of $D(Q)$, call it Ty, such that $\langle x, y\rangle = Q(x, Ty)$. Of course T depends linearly on y, so we have a linear operator on H with $R(T) \subset D(Q)$. We will show that T is injective and that $T^{-1} : R(T) \subset H$ satisfies the necessary conditions to be A. To see T is injective, suppose $y \in H$ with $Ty = 0$. Then for any $x \in D(Q)$, $0 = Q(x, Ty) = \langle x, y\rangle$. But $D(Q)$ is dense so this implies $y = 0$. So set $A = T^{-1}$ with $D(A) = R(T) \subset D(Q)$. That is item (1).

Note $AT = I_H$ and $TA = I_{D(A)}$. Now it is clear that, for $x \in D(Q)$ and $y \in D(A)$, $Q(x,y) = Q(x,TAy) = \langle x, Ay \rangle$. That is item (2).

Next, note that T is symmetric: for x, $y \in H$ we have

$$\langle Tx, y \rangle = Q(Tx, Ty) = \overline{Q(Ty, Tx)} = \overline{\langle Ty, x \rangle} = \langle x, Ty \rangle .$$

Since it is symmetric and everywhere defined, it has to be self-adjoint, hence closed, hence bounded (by the closed graph theorem, we could also show this directly), hence Hermitian. It is known that this implies $A = T^{-1}$ is self-adjoint (in particular, densely defined). So the assertion in item (3) follows. Suppose now that B is another operator satisfying items (1) and (2). Then if $x \in D(A)$, $y \in D(B)$, we have

$$\langle y, Ax \rangle = Q(y,x) = \overline{Q(x,y)} = \overline{\langle x, By \rangle} = \langle By, x \rangle .$$

The right side is continuous in x, so $y \in D(A^*)$ and $A^*y = By$. Thus $B \subset A^* = A$. That is item (4).

If B is also self-adjoint, then take adjoint operators above to see $B = A$. That is item (5) follows.

Certainly $A \geqslant I$ because $Q \geqslant 1$. That is item (6).

Now we show $D(A)$ is a core for Q. Equipping $D(Q)$ with the Q inner product gives us a Hilbert space, of which $D(A)$ is a subspace. We will show $D(A)$ is Q-dense. For suppose there is a $x \in D(Q)$ with $Q(x,y) = 0$ for all $y \in D(A)$. If we take $y = Tx \in R(T) = D(A)$, we have $0 = Q(x,Tx) = \langle x, x \rangle$ so $x = 0$. That is item (7).

In other words, Q is the closure of the form $Q_0(x,y) = \langle x, Ay \rangle$, $D(Q_0) = D(A)$, which is just the restriction of Q to $D(A)$ (by item 2). But it can be shown that the closure of Q_0 is the form $Q'(x,y) = \left\langle A^{1/2}x, A^{1/2}y \right\rangle$ with $D(Q') = D\left(A^{1/2}\right)$. So $Q = Q'$. These are items (8) and (9).

The proof of Theorem 4.41 is complete now. □

4.42. Corollary (Friedrichs extension theorem). *Suppose B is a densely defined symmetric operator with $B \geqslant I$. Set $Q_0(x,y) = \langle x, By \rangle$ with $D(Q_0) = D(B)$, and let $Q = \overline{Q_0}$. Then B has exactly one self-adjoint extension A that has $D(A) \subset D(Q)$. Moreover, $A \geqslant I$.*

Proof. Let A be the operator supplied by the previous theorem; A is self-adjoint, $D(A) \subset D(Q)$, and $A \geqslant I$. To see A is an extension of B, we use item (4) of the previous theorem; we must check B satisfies items (1) (which is obvious) and (2). We certainly have $Q(x,y) = \langle x, By \rangle$ for x, $y \in D(B)$; we also need it for $x \in D(Q)$. But by construction $D(B)$ is a core for Q; this means $D(B)$ is a Q-dense subspace of the Hilbert space $(D(Q), Q)$. Fix $y \in D(B)$. Define two linear functionals f_1, f_2 on $D(Q)$, where $f_1(x) = Q(x,y)$ and $f_2(x) = \langle x, By \rangle$. The functional f_1 is certainly Q-bounded (use Cauchy-Schwarz on Q), and so is f_2, because $|\langle x, By \rangle| \leqslant \|x\| \cdot \|By\| \leqslant \sqrt{Q(x,x)}\, \|By\|$ (recall $Q \geqslant 1$). Since these two Q-bounded linear functionals agree on the Q-dense subspace $D(B)$, $f_1 = f_2$ on $D(Q)$;

i.e. $Q(x, y) = \langle x, By \rangle$ for all $x \in D(Q)$. That is item 2, and so we must have $B \subset A$. If A_0 were another self-adjoint extension of B with $D(A_0) \subset D(Q)$, then A_0 would satisfy items (1), (2), (3) of Theorem 4.41; by item (5) we would have $A_0 = A$. The proof of Corollary 4.42 is complete now. $\square$

Note that we can replace the $\geqslant 1$ in this theorem by $\geqslant c$, by applying it to $B_0 = B + (1 - c)I$. Adding a multiple of the identity doesn't affect anything relevant.

Let Ω be a bounded open subset of $\mathbb{R}^n$. The following theorem says that the Sobolev space $H_0^1(\Omega)$, which is a Hilbert space with respect to the norm defined in (4.152), is compactly embedded in the space $L^2(\Omega)$. It is a version of the Rellich-Kondrachov theorem. A more general type of this result can be found in, *e.g.*, [**24**] Theorem 9.16. Our proof is based on a proof of a similar result in [**8**], Proposition 500.

4.43. THEOREM. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Define the subspace $H_0^1(\Omega)$ of $L^2(\Omega)$ as follows. A (class of) functions $f \in L^2(\Omega)$ belongs to $H_0^1(\Omega)$ provided its distributional gradient $\nabla f = (D_1 f, \dots, D_n f)$ belongs to $L^2(\Omega, \mathbb{C}^n)$, and if there exists a sequence $(f_m)_m \subset \mathcal{D}(\Omega)$ such that $\lim_{m\to\infty} \|f_m - f\|_{H_0^1} = 0$. This means that each of the distributions $D_j f$ in fact is an L^2-function on Ω. Equipped with the norm, determined by*

$$\|f\|_{H_0^1}^2 = \int_\Omega |f(x)|^2 \, dx + \sum_{j=1}^n \int_\Omega |D_j f(x)|^2 \, dx = \int_\Omega |f(x)|^2 \, dx + \int_\Omega |\nabla f(x)|^2 \, dx, \tag{4.152}$$

the space $H_0^1(\Omega)$ is a Hilbert space. Moreover, this Hilbert space is compactly embedded in the space $L^2(\Omega)$.

PROOF. Let $(u_m)_m$ be a sequence in $H_0^1(\Omega)$ such that $\sup_m \|u_m\|_{L^2} \leqslant C_1 < \infty$, and such that $\sup_m \|\nabla u_m\|_{L^2} \leqslant C_2 < \infty$. We have to prove that there exists a subsequence $\left(u_{m_j}\right)_j$ which converges in $L^2(\Omega)$. Without loss of generality we assume that the functions u_m vanish outside of Ω, and that they are continuously differentiable. Otherwise we replace the functions u_m with $\widetilde{u}_m \in C^1(\Omega) \cap H_0^1(\Omega)$ such that $\|u_m - \widetilde{u}_m\|_{H_0^1} \leqslant 2^{-m}$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a non-negative C^∞-function whose support is contained in the Euclidean unit ball B and which is such that $\int_B \eta(y)\, dy = 1$. For $\varepsilon > 0$ and $m \in \mathbb{N}$ we define the functions $u_m^\varepsilon : \mathbb{R}^n \to \mathbb{C}$ by

$$\begin{aligned} u_m^\varepsilon(x) &= \frac{1}{\varepsilon^n} \int_{\varepsilon B} \eta\left(\frac{y}{\varepsilon}\right) u_m(x-y)\, dy = \frac{1}{\varepsilon^n} \int_{x - \varepsilon B} \eta\left(\frac{x-y}{\varepsilon}\right) u_m(y)\, dy \\ &= \int_B \eta(y) u_m(x - \varepsilon y)\, dy. \end{aligned} \tag{4.153}$$

Then we have

$$\begin{aligned} u_m^\varepsilon(x) - u_m(x) &= \int_B \eta(y) \left\{u_m(x - \varepsilon y) - u_m(x)\right\} \\ &= -\varepsilon \int_B \int_0^1 \eta(y) y \cdot \nabla u_m(x - s\varepsilon y)\, ds\, dy. \end{aligned} \tag{4.154}$$

From (4.154) we infer, with $\tau_y \nabla u_m(x) = \nabla u_m(x-y)$,

$$\begin{aligned} &\int_{\mathbb{R}^{\ltimes}} |u_m^\varepsilon(x) - u_m(x)|^2 \, dx \\ &= \varepsilon^2 \int_B \int_B \int_0^1 \int_0^1 \eta(y)\, \eta(y') \int_{\mathbb{R}^n} \{y \cdot \tau_{s\varepsilon y} \nabla u_m(x)\} \left\{y' \cdot \overline{\tau_{s'\varepsilon y'} \nabla u_m(x)}\right\} dx\, ds' ds\, dy' dy \\ &\leqslant \varepsilon^2 \int_B \int_B \int_0^1 \int_0^1 |y|\, \eta(y)\, |y'|\, \eta(y') \int_{\mathbb{R}^n} |\tau_{s\varepsilon y} \nabla u_m(x)| \cdot |\tau_{s'\varepsilon y'} \nabla u_m(x)| \, dx\, ds' ds\, dy' dy \\ &\leqslant \varepsilon^2 \left(\int_B |y|\, \eta(y)\, dy\right)^2 \int_\Omega |\nabla u_m(x)|^2 \, dx \leqslant \varepsilon^2 \left(\int_B |y|\, \eta(y)\, dy\right)^2 C_2^2 \leqslant \varepsilon^2 C_2^2. \end{aligned} \tag{4.155}$$

From (4.155) we infer:

$$\|u_m^\varepsilon - u_m\| \leqslant C_2 \varepsilon. \tag{4.156}$$

Next we want to show that, for every $\varepsilon > 0$, the sequence $(u_m^\varepsilon)_m$ is uniformly bounded and equi-continuous. To this end we will estimate the quantities $\sup_m |u_m^\varepsilon(x)|$ and $\sup_m |\nabla u_m^\varepsilon(x)|$. Put $\eta_\varepsilon(y) = \dfrac{1}{\varepsilon^n}\eta\left(\dfrac{y}{\varepsilon}\right)$. We begin with

$$\begin{aligned}
|u_m^\varepsilon(x)|^2 &= \left|\int_{x-\varepsilon B} \eta_\varepsilon(x-y)u_m(y)\,dy\right|^2 \\
&\leqslant \int_{x-\varepsilon B} \eta_\varepsilon(x-y)\,dy \int_{x-\varepsilon B} \eta_\varepsilon(x-y)\,|u_m(y)|^2\,dy \\
&\leqslant \frac{1}{\varepsilon^n}\sup_{y\in B}\eta(y)\int_\Omega |u_m(y)|^2\,dy \\
&\leqslant \frac{C_1^2}{\varepsilon^n}\sup_{y\in B}\eta(y). \qquad (4.157)
\end{aligned}$$

From (4.157) we get, with $\|\eta\|_\infty = \sup_{y\in B}\eta(y)$,

$$\sup_m \sup_x |u_m^\varepsilon(x)| \leqslant \frac{C_1}{\varepsilon^{n/2}}\|\eta\|_\infty^{1/2}. \qquad (4.158)$$

Similarly we obtain

$$|\nabla u_m^\varepsilon(x)|^2 \leqslant \|\eta\|_\infty \frac{1}{\varepsilon^n}\int_\Omega |\nabla u_m(y)|^2\,dy \leqslant \frac{C_2^2}{\varepsilon^n}\|\eta\|_\infty,$$

and so

$$\sup_m \sup_x |\nabla u_m^\varepsilon(x)| \leqslant \frac{C_2}{\varepsilon^{n/2}}\|\eta\|_\infty^{1/2}. \qquad (4.159)$$

From (4.158) and (4.159) it follows that for every $\varepsilon > 0$ the sequence $(u_m^\varepsilon)_m$ is equi-continuous. So, for $\varepsilon > 0$ fixed, by the theorem of Arzela-Ascoli there exists a subsequence which converges uniformly on Ω. Let $(\varepsilon_k)_k$ be a sequence of strictly positive real numbers which decreases to 0. By the diagonal principle there exists a double sequence $\left\{u_{m_j}^{\varepsilon_k} : k \in \mathbb{N},\, j \in \mathbb{N}\right\}$ such that, for every $k \in \mathbb{N}$, the sequence $\left(u_{m_j}^{\varepsilon_k}\right)_j$ converges uniformly to a bounded continuous function u^{ε_k} on Ω. It follows that, for every $k \in \mathbb{N}$, the sequence $\left(u_{m_j}^{\varepsilon_k}\right)_j$ converges in $L^2(\Omega)$. Finally, we will show that the sequence $\left(u_{m_j}\right)_j$ is a Cauchy sequence in $L^2(\Omega)$. From (4.156) we obtain

$$\begin{aligned}
\left\|u_{m_{j_2}} - u_{m_{j_1}}\right\|_{L^2} &\leqslant \left\|u_{m_{j_2}} - u_{m_{j_2}}^{\varepsilon_k}\right\|_{L^2} + \left\|u_{m_{j_2}}^{\varepsilon_k} - u_{m_{j_1}}^{\varepsilon_k}\right\|_{L^2} + \left\|u_{m_{j_1}}^{\varepsilon_k} - u_{m_{j_1}}\right\|_{L^2} \\
&\leqslant C_2\varepsilon_k + \left\|u_{m_{j_2}}^{\varepsilon_k} - u_{m_{j_1}}^{\varepsilon_k}\right\|_{L^2} + C_2\varepsilon_k \\
&= 2C_2\varepsilon_k + \left\|u_{m_{j_2}}^{\varepsilon_k} - u_{m_{j_1}}^{\varepsilon_k}\right\|_{L^2}. \qquad (4.160)
\end{aligned}$$

Fix $\varepsilon > 0$, and choose $k = k_\varepsilon \in \mathbb{N}$ so large that $4C_2\varepsilon_k \leqslant \varepsilon$. Then choose $j = j_\varepsilon \in \mathbb{N}$ so large that $j_1, j_2 \geqslant j$ implies

$$\left\|u_{m_{j_2}}^{\varepsilon_k} - u_{m_{j_1}}^{\varepsilon_k}\right\|_{L^2} \leqslant \frac{1}{2}\varepsilon. \qquad (4.161)$$

From the choice of k and from (4.161) it follows that, for j_1, $j_2 \geqslant j$, the inequality

$$\left\| u_{m_{j_2}} - u_{m_{j_1}} \right\|_{L^2} \leqslant \varepsilon. \tag{4.162}$$

Consequently, the sequence $\left(u_{m_j}\right)_j$ is a Cauchy sequence in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, this concludes the proof of Theorem 4.43. $\square$

6. Paley-Wiener theorems

In this section we want to characterize functions f which are holomorphic on $\mathbb{C}^n$, and which either are Fourier transforms of functions in $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ or of distributions with compact support, *i.e.* distributions in $\mathcal{E}' = \mathcal{E}'(\mathbb{R}^n)$. For a concise formulation of the results we need some notation and some definitions.

4.44. Definition. If Ω is an open subset of $\mathbb{C}^n$, and if $f : \Omega \to \mathbb{C}$ is continuous, then f is said to be holomorphic in Ω if it is holomorphic in each variable separately. This means that, for every $(a_1, \ldots, a_n) \in \Omega$, and for every $1 \leqslant j \leqslant n$, the function

$$g_j : \Omega_{a_1,\ldots,a_{j-1},a_{j+1}\ldots,a_n} := \{\lambda \in \mathbb{C} : (a_1, \ldots, a_{j-1}, \lambda, a_{j+1}, \ldots, a_n) \in \Omega\} \to \mathbb{C}, \tag{4.163}$$

defined by $g_j(\lambda) = f(a_1, \ldots, a_{j-1}, \lambda, a_{j+1}, \ldots, a_n)$, $\lambda \in \Omega_{a_1,\ldots,a_{j-1},a_{j+1}\ldots,a_n}$, is holomorphic in a neighborhood of a_j. A function that is holomorphic on all of $\mathbb{C}^n$ is called entire.

Points of $\mathbb{C}^n$ will be denoted by $z = (z_1, \ldots, z_n)$, where $z_k = x_k + iy_k \in \mathbb{C}$, x_k, $y_k \in \mathbb{R}$. The vectors $x = (x_1, \ldots, x_n)$ respectively $y = (y_1, \ldots, y_n)$ are called the real and imaginary parts of the vector $z = (z_1, \ldots, z_n)$; they are denoted as $x = \Re z$ and $y = \Im z$. The space $\mathbb{R}^n$ will be thought of as the set of all $z \in \mathbb{C}^n$ with $\Im z = 0$. The notations

$$|z| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2}, \quad z \in \mathbb{C}^n,$$
$$|\Im z| = \left(|y_1|^2 + \cdots + |y_n|^2\right)^{1/2}, \quad z = x + iy \in \mathbb{C}^n,$$
$$z \cdot t = \sum_{j=1}^{n} z_j t_j, \quad e_z(t) = e^{iz\cdot t}, \quad z \in \mathbb{C}^n, \quad t \in \mathbb{R}^n.$$

The following triviality result holds.

4.45. Lemma. *If f is an entire function on $\mathbb{C}^n$ that vanishes on $\mathbb{R}^n$, then $f = 0$.*

Proof. We consider the case $n = 1$ as known. Let P_k be the following property of f: If $z \in \mathbb{C}^n$ has at least k real coordinates, then $f(z) = 0$. The property P_n is given. We have to prove P_0. Assume $1 \leqslant j \leqslant n$ and P_j is true. Take $a_1, \ldots, a_{j-1}$ real, $a_{j+1}, \ldots a_n$ arbitrary, and define the function g_j as in (4.163). The function g_j is 0 on $\mathbb{R}$, hence is 0 for $\lambda \in \mathbb{C}$. It follows that P_{j-1} is true. By induction P_0 is true. This completes the proof of Lemma 4.45. □

In what follows we write $rB = \{x \in \mathbb{R}^n : |x| \leqslant r\}$, $r > 0$.

4.46. Theorem. *The following assertions are valid.*

(a) *If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ has its support in rB, and if*

$$f(z) = \int e^{-iz\cdot t}\varphi(t)\,dt, \quad z \in \mathbb{C}^n, \tag{4.164}$$

then f is entire and there exist finite constants γ_N such that

$$|f(z)| \leqslant \gamma_N (1 + |z|)^{-N} e^{r|\Im z|}, \quad z \in \mathbb{C}^n, \; N = 0, 1, 2, \ldots \tag{4.165}$$

(b) *Conversely, if an entire function f satisfies (4.165), then there exists $\varphi \in \mathcal{D}(\mathbb{R}^n)$, with support in rB, such that (4.164) holds.*

Proof. (a) If $t \in rB$, then $|e^{-iz\cdot t}| = e^{y\cdot t} \leqslant e^{r|\Im z|}$. The integrand in (4.164) is therefore in $L^1(\mathbb{R}^n)$, for every $z \in \mathbb{C}^n$, and f is well defined on $\mathbb{C}^n$. The continuity of f as a function from $\mathbb{C}^n$ to $\mathbb{C}$ is clear, and an application of Morera's theorem, applied to every coordinate separately, shows that f is entire. Integration by parts shows that

$$z^\alpha f(z) = \int e^{-iz\cdot t} D^\alpha \varphi(t)\,dt.$$

Hence

$$|z^\alpha|\,|f(z)| \leqslant \|D^\alpha\varphi\|_1\, e^{r|\Im z|}, \tag{4.166}$$

and (4.165) follows from (4.166).

(b) Suppose that f is an entire function that satisfies (4.165), and define

$$\varphi(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it\cdot x} f(x)\, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it\cdot(x+iy)} f(x+iy)\, dx \tag{4.167}$$

Note first that, by (4.165), the function $x \mapsto (1+|x|)^N f(x)$ belongs to $L^1(\mathbb{R}^n)$ for every N. It easily that $\varphi \in C^\infty(\mathbb{R}^n)$. The second quality in (4.167) is a consequence of Cauchy's theorem and the estimate in (4.165) which entail that integrals over x don't change when x is moved to $x+iy$ for each fixed y.

Fix $t \in \mathbb{R}^n$, and choose $y = \lambda t/|t|$, where $\lambda > 0$. Then $t \cdot y = \lambda |t|$, and $|y| = \lambda$. By (4.165) we see

$$\left|e^{it\cdot(x+iy)} f(x+iy)\right| \leqslant \gamma_N (1+|x|)^{-N} e^{(r-|t|)\lambda},$$

and therefore, by the second equality in (4.167) with $y = \lambda t/|t|$,

$$|\varphi(t)| \leqslant \frac{\gamma_N}{(2\pi)^n} e^{(r-|t|)\lambda} \int \frac{1}{(1+|x|)^N}\, dx \tag{4.168}$$

where N is chosen so large that the integral in (4.168) is finite. Now let $\lambda \to \infty$ to conclude that $\varphi(t) = 0$ whenever $|t| > r$. It follows that φ has its support in rB. Now (4.164) follows, for real z, from the first equality in (4.167) and the inversion formula. Since both sides of (4.164) are entire, they coincide on $\mathbb{C}^n$, by Lemma 4.45.

This completes the proof of Theorem 4.46. □

Let u be a distribution in $\mathbb{R}^n$ with compact support. Then its Fourier transform $\widehat{u}$ is defined, as a tempered distribution, by $\widehat{u}(\varphi) = u(\widehat{\varphi})$, $\varphi \in \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$. However, the definition $\widehat{f}(x) = \int e_{-x}(t) f(t)\, dt$, made for $f \in L^1(\mathbb{R}^n)$, suggests that $\widehat{u}$ ought to be a function, namely,

$$\widehat{u}(x) = u(e_{-x}) = u\left(t \mapsto e^{-ix\cdot t}\right), \quad x \in \mathbb{R}^n,$$

because $e_{-x} \in C^\infty(\mathbb{R}^n)$ and $u(\varphi)$ makes sense for every $\varphi \in C^\infty(\mathbb{R}^n)$. Moreover, $e_{-z} \in C^\infty(\mathbb{R}^n)$ for every $z \in \mathbb{C}^n$, and $z \mapsto u(e_{-z})$ therefore looks like an entire function, whose restriction to $\mathbb{R}^n$ is $\widehat{u}$. That all this is correct is part of the next theorem, which also characterizes the resulting entire function by certain growth conditions.

4.47. Theorem. *The following assertions are valid.*

(a) *If $u \in \mathcal{D}'(\mathbb{R}^n)$ has its support in rB, if u has order N (or $\leqslant N$), and if*

$$f(z) = u\left(t \mapsto e^{-iz\cdot t}\right) = u(e_{-z}), \quad z \in \mathbb{C}^n, \tag{4.169}$$

then f is entire, the restriction of f to $\mathbb{R}^n$ is the Fourier transform of u, and there exist a finite constant γ such that

$$|f(z)| \leqslant \gamma (1+|z|)^N e^{r|\Im z|}, \quad z \in \mathbb{C}^n. \tag{4.170}$$

(b) *Conversely, if f is an entire function which satisfies (4.170) for some N and some constant γ, then there exists $u \in \mathcal{D}'(\mathbb{R}^n)$, with support in rB, such that (4.169) holds.*

The notation $\widehat{u}(z) = u\left(e_{-z}\right)$, $z \in \mathbb{C}^n$, is often used to denote the extension given by (4.169).

Proof. (a) Suppose $u \in \mathcal{D}'\left(\mathbb{R}^n\right)$ has its support in rB. Pick $\psi \in \mathcal{D}\left(\mathbb{R}^n\right)$ so that $\psi = 1$ on $(r+1)B$. Then $u = \psi u$, and (v) of Theorem 4.5 shows that

$$\widehat{u} = \widehat{\psi u} = \frac{1}{(2\pi)^n}\widehat{u} * \widehat{\psi}. \tag{4.171}$$

It follows that $\widehat{u} \in C^{\infty}\left(\mathbb{R}^n\right)$. From item (4) in Theorem 4.11 we infer

$$\begin{aligned}\widehat{u} * \widehat{\psi}(x) &= \widehat{u}\left(\tau_x\left(\widecheck{\mathcal{F}\psi}\right)\right) = \widehat{u}\left[\tau_x\left(\mathcal{F}\left(\check{\psi}\right)\right)\right] \\ &= u\left[\mathcal{F}\left(\tau_x\left(\mathcal{F}\check{\psi}\right)\right)\right] = u\left[e_{-x}\mathcal{F}^2\left(\check{\psi}\right)\right] = (2\pi)^n u\left(e_{-x}\check{\check{\psi}}\right) = (2\pi)^n u\left(e_{-x}\psi\right) \\ &= (2\pi)^n \psi u\left(e_{-x}\right) = (2\pi)^n u\left(e_{-x}\right),\end{aligned} \tag{4.172}$$

and consequently by (4.171)

$$\widehat{u}(x) = u\left(e_{-x}\right), \quad x \in \mathbb{R}^n. \tag{4.173}$$

Note that the equality in (4.173) proves a big chunk of (a). Next our aim is to prove that the function f in (4.169) is holomorphic. The continuity of f poses no problem. If $w \to z$ in $\mathbb{C}^n$, then $e_{-w} \to e_{-z}$ in $\mathcal{E}(\mathbb{R}^n)$, and $\lim_{w\to z} u(e_{-w}) = u(e_{-z})$. Fix a, $b \in \mathbb{C}^n$, and put

$$g(\lambda) = f(a + \lambda b) = u(e_{-a-\lambda b}), \quad \lambda \in \mathbb{C}. \tag{4.174}$$

Then the function g is continuous. Let $\triangle \subset \mathbb{C}$ be any solid triangle. Then, by Fubini's theorem for distributions,

$$\int_{\partial\triangle} g(\lambda)\, d\lambda = \int_{\partial\triangle} u\left(t \mapsto e^{-it\cdot(a+\lambda b)}\right) d\lambda = u\left(t \mapsto \int_{\partial\triangle} e^{-it\cdot(a+\lambda b)}\, d\lambda\right) = u(0) = 0. \tag{4.175}$$

By Morera's theorem the function g is entire.

To complete the proof of (a) the inequality in (4.170) remains to be shown. Choose an auxiliary function h on the real line, infinitely differentiable, such that $h(s) = 1$ when $s < 1$ and $h(s) = 0$ when $s > 2$. With every $z \in \mathbb{C}^n$, $z \neq 0$, we associate the function

$$\varphi_z(t) = e^{-iz\cdot t} h(|t|\,|z| - r\,|z|), \quad t \in \mathbb{R}^n. \tag{4.176}$$

Then $\varphi_z \in \mathcal{D}(\mathbb{R}^n)$, and in fact, for $z \neq 0$,

$$\operatorname{support}(\varphi_z) \subset \left\{ t \in \mathbb{R}^n : |t| \leqslant r + \frac{2}{|z|} \right\}. \tag{4.177}$$

In addition, if $|t| \leqslant r + 1/|z|$, then $h(|t|\,|z| - r\,|z|) = 1$. Hence, we see that $\varphi_z(t) = e^{-iz\cdot t}$ for $|t| \leqslant r + 1/|z|$, and since u has its support in rB, it follows that

$$f(z) = u(e_{-z}) = u(\varphi_z), \quad z \in \mathbb{C}^n. \tag{4.178}$$

Since u has order N (and u has compact support), there exists $\gamma_0 < \infty$ such that $|u(\varphi)| \leqslant \gamma_0 \max_{|\alpha|\leqslant N} \|D^\alpha \varphi\|_\infty$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Put

$$\|\varphi\|_N = \max_{|\alpha|\leqslant N} \|D^\alpha \varphi\|_\infty, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Then (4.178) gives

$$|f(z)| \leqslant \gamma_0 \|\varphi_z\|_N, \quad z \in \mathbb{C}^n. \tag{4.179}$$

On the support of φ_z we have, by (4.177),

$$\left|e^{-iz\cdot t}\right| = e^{y\cdot t} \leqslant e^{2+r|\Im z|} \tag{4.180}$$

An application of Leibniz formula to the product in (4.176) while using (4.180) and (4.179) shows the inequality in (4.170), and finishes the proof of (a).

(b) Since f now satisfies

$$|f(x)| \leqslant \gamma (1 + |x|)^N, \quad x \in \mathbb{R}^n, \tag{4.181}$$

the restriction of f to $\mathbb{R}^n$ is therefore a member of $\mathcal{S}'(\mathbb{R}^n)$, and so it is the Fourier transform of some tempered distribution u. Pick a function $h \in \mathcal{D}(\mathbb{R}^n)$, with support in B, such that $\int h(t)\, dt = 1$, define $h_\varepsilon(t) = \varepsilon^{-n} h(t/\varepsilon)$, $t \in \mathbb{R}^n$, $\varepsilon > 0$, and put

$$f_\varepsilon(z) = f(z)\widehat{h}_\varepsilon(z) = f(z)\widehat{h}(\varepsilon z), \quad z \in \mathbb{C}^n, \tag{4.182}$$

where $\widehat{h_\varepsilon}$ denotes the entire function whose restriction to $\mathbb{R}^n$ is the Fourier transform of h_ε. Statement (a) of Theorem 4.46, applied to h_ε, leads to the conclusion that f_ε satisfies (4.165) of Theorem 4.46 with $r+\varepsilon$ instead of r. Therefore (b) of Theorem 4.46 implies that $f_\varepsilon = \widehat{\varphi_\varepsilon}$ for some $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ whose support lies in $(r+\varepsilon)\,B$.

Consider some $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that the support of $\widehat{\psi}$, which is closed, does not intersect the compact set rB. Then $\widehat{\psi}\varphi_\varepsilon = 0$ for all sufficiently small $\varepsilon > 0$. Since $f\psi \in L^1(\mathbb{R}^n)$ and $\widehat{h}_\varepsilon(x) = \widehat{h}(\varepsilon x) \to \widehat{h}(0) = 1$ boundedly on $\mathbb{R}^n$, we conclude

$$\begin{aligned} u\left(\widehat{\psi}\right) &= \widehat{u}(\psi) = \int f(x)\psi(x)\,dx = \lim_{\varepsilon\downarrow 0}\int f_\varepsilon(x)\psi(x)\,dx \\ &= \lim_{\varepsilon\downarrow 0}\int \widehat{\varphi_\varepsilon}(x)\psi(x)\,dx = \lim_{\varepsilon\downarrow 0}\int \varphi_\varepsilon(t)\widehat{\psi}(t)\,dt = 0. \end{aligned} \tag{4.183}$$

The arbitrariness of $\psi \in \mathcal{S}(\mathbb{R}^n)$ with the property that the support of $\widehat{\psi}$ does not intersect rB and (4.183) yield that the support of u is contained in rB. So the proof of (b) is complete.

This concludes the proof of Theorem 4.47. □

4.48. Definition. If $\varphi \in \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$, then the function $u * \varphi$ is defined by

$$u * \varphi(x) = u\left(\tau_x\check{\varphi}\right), \quad x \in \mathbb{R}^n. \tag{4.184}$$

Note that this convolution product is well defined, since $\tau_x\check{\varphi} \in \mathcal{S}$ for every $x \in \mathbb{R}^n$.

The following theorem largely coincides with Theorem 4.5.

4.49. Theorem. *Let $\varphi \in \mathcal{S}$ and $u \in \mathcal{S}'$. Then the following assertions are valid:*

(a) *$u*\varphi \in C^\infty(\mathbb{R}^n)$, and $D^\alpha(u*\varphi) = (D^\alpha u)*\varphi = u*D^\alpha\varphi$ for every multi-index α,*
(b) *$u*\varphi$ has polynomial growth,hence is a tempered distribution,*
(c) *$\widehat{u*\varphi} = \widehat{\varphi}\widehat{u}$,*
(d) *$(u*\varphi)*\psi = u*(\varphi*\psi)$ for all $\psi \in \mathcal{S}$,*
(e) *$\widehat{u}*\widehat{\varphi} = (2\pi)^n\widehat{\varphi u}$.*

In addition, this convolution product enjoys the following continuity property. If $(u_\ell)_{\ell\in\mathbb{N}}$ converges in $\mathcal{S}'$ to $u \in \mathcal{S}'$, and if $(\varphi_\ell)_{\ell\in\mathbb{N}}$ converges in $\mathcal{S}$ to $\varphi \in \mathcal{S}$, then $u_\ell * \varphi_\ell$ converges to $u*\varphi \in \mathcal{E} = \mathcal{E}(\mathbb{R}^n)$.

7. Multiplicative distributions

In this section we will discuss multiplicative distributions.

4.50. Definition. Let Ω be an open subset of $\mathbb{R}^n$. A linear functional $\Lambda : \mathcal{D}(\Omega) \to \mathbb{C}$ is called multiplicative if $\Lambda(\varphi_1\varphi_2) = \Lambda(\varphi_1)\,\Lambda(\varphi_2)$ for all φ_1 and $\varphi_2 \in \mathcal{D}(\Omega)$.

In fact we have the following general results for multiplicative distributions.

4.51. Proposition. *Let Ω an open subset of $\mathbb{R}^n$. Suppose that $\Lambda : \mathcal{D}(\Omega) \to \mathbb{C}$ is a multiplicative linear functional. Then either $\Lambda \equiv 0$ or else there exists $a \in \Omega$ such that $\Lambda(\varphi) = \varphi(a)$ for all $\varphi \in \mathcal{D}(\Omega)$.*

The proof of Proposition 4.51 requires the following result about proper ideals in $C_0(\Omega)$. The space $C_0(\Omega)$ consists of those complex valued continuous functions $f : \Omega \to \mathbb{C}$ which vanish at infinity; this means that f is continuous, and that for every $\varepsilon > 0$ there exists a compact subset K of Ω such that $|f(x)| \leqslant \varepsilon$ for all $x \in \Omega \backslash K$. The space $C_0(\Omega)$ is equipped with the supremum-norm: $\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$, $f \in C_0(\Omega)$. Notice that $\mathcal{D}(\Omega)$ is a subalgebra of $C_0(\Omega)$, that it separates points of Ω, and that it is closed under taking complex conjugates. Therefore, by the Stone-Weierstrass theorem, it follows that the subspace $\mathcal{D}(\Omega)$ is uniform dense in $C_0(\Omega)$. Let $C_{00}(\Omega)$ denote the space of all complex-valued continuous functions $f : \Omega \to \mathbb{C}$ with compact support in Ω. An ideal J in $C_0(\Omega)$ is a vector subspace of $C_0(\Omega)$ with the property that for all $\varphi \in C_0(\Omega)$ and all $f \in J$ the function φf belongs to J.

4.52. Theorem. *Let J be an ideal in $C_0(\Omega)$. Then either*

(a) *$C_{00}(\Omega) \subset J$, and so the closure of J coincides with $C_0(\Omega)$, or else,*
(b) *there exists $a \in \Omega$ such that $f(a) = 0$ for all $f \in J$,* i.e. *J is contained in a proper closed maximal ideal.*

From the proof it follows that the subspace J needs only to be a module over $C_{00}(\Omega)$ instead of an ideal in $C_0(\Omega)$, *i.e.* it is only required that φf belongs to J whenever $f \in J$ and $\varphi \in C_{00}(\Omega)$.

Proof. Assume that for every $a \in \Omega$ there exists a function $f_a \in J$ such that $f_a(a) \neq 0$. We will show that (a) holds. Let $\varphi \in C_{00}(\Omega)$. Then, by compactness of $\operatorname{supp}(\varphi)$ there exists a finite collection $\left(f_{a_j} : 1 \leqslant j \leqslant N\right) \subset J$ such that $\operatorname{supp}(\varphi) \subset \cup_{j=1}^{N}\left\{f_{a_j} \neq 0\right\}$. The function $f := \sum_{j=1}^{N} f_{a_j}\overline{f}_{a_j}$ belongs to J, and the function g defined by $g(x) := \dfrac{\varphi(x)}{f(x)}$, if $f(x) \neq 0$, and $g(x) = 0$, if $f(x) = 0$, belongs to $C_{00}(\Omega)$, and hence $\varphi = gf$ belongs to J. It follows that $C_{00}(\Omega) \subset J$. Since the space of continuous complex functions with compact support, *i.e.* $C_{00}(\Omega)$, is dense in $C_0(\Omega)$ it follows that the closure of J coincides with $C_0(\Omega)$. This completes the proof of Theorem 4.52. □

Assertion (b) of the next proposition says that a Borel measurable character is in fact a continuous character: see Remark 1.37 in Subsection 2.1 of Chapter 1.

4.53. Proposition. *Let $b : \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function which can be considered as a tempered distribution. Then the following assertions hold.*

(a) *If b is almost everywhere multiplicative in the sense that $b(x+y) = b(x)b(y)$ for almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, then either $b(x) = 0$ for almost all $x \in \mathbb{R}^n$ or else there exists $a \in \mathbb{R}^n$ such that $b(x) = e^{ia\cdot x}$ for almost all $x \in \mathbb{R}^n$.*
(b) *If the function b is multiplicative in the sense that $b(x + y) = b(x)b(y)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then either $b(x) = 0$ for all $x \in \mathbb{R}^n$, or else there exists $a \in \mathbb{R}^n$ such that $b(x) = e^{ia\cdot x}$ for all $x \in \mathbb{R}^n$.*

Proposition 4.53 is applicable if for instance the multiplicative Borel measurable function b is of polynomial growth. A particular situation is the case where b is a bounded function. In the following proposition we present a solution of Cauchy's equation for Borel measurable functions: see [**43**], [**101, 102**], [**115**], [**124**], [**125**].

4.54. Proposition. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function which is additive in the sense that $g(x + y) = g(x) + g(y)$ for almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then there exists $a \in \mathbb{R}^n$ such that $g(y) = a \cdot y$ for almost all $y \in \mathbb{R}^n$.*

4.55. Proposition. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function which is additive in the sense that $g(x+y) = g(x)+g(y)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then there exists $a \in \mathbb{R}^n$ such that $g(y) = a \cdot y$ for all $y \in \mathbb{R}^n$.*

4.56. Remark. The following problem can be posed. Does there exist an additive function like in Proposition 4.55 which is not Borel measurable?

Proof of Proposition 4.51. Suppose that $\Lambda \neq 0$. Then there exists $\varphi \in \mathcal{D}(\Omega)$ such that $\Lambda(\varphi) \neq 0$. Let f be any function in $\mathcal{D}(\Omega)$. By putting it 0 outside of Ω it can be considered as a function in $\mathcal{D}(\mathbb{R}^n)$ with compact support in Ω. Let $\lambda \in \mathbb{C}$ be such that

$$|\lambda| > \sup_{x\in\mathbb{R}^n} |f(x)| = \|f\|_\infty \tag{4.185}$$

Then from (4.185) it follows that there exists a C^∞-function g defined on $\mathbb{R}^n$ such that $(\lambda\mathbf{1} - f)\, g = \mathbf{1}$. Let $\varphi \in \mathcal{D}(\Omega)$ be such that $\Lambda(\varphi) \neq 0$. Then we have

$$0 \neq \Lambda(\varphi) = \Lambda\left((\lambda\mathbf{1} - f)\, g\varphi\right) = (\lambda - \Lambda(f))\, \Lambda\left(g\varphi\right). \tag{4.186}$$

From (4.186) it follows that $\Lambda(f) \neq \lambda$. Since by (4.185) $|\lambda| > \|f\|_\infty$ was arbitrary we conclude that

$$|\Lambda(f)| \leqslant \|f\|_\infty, \quad \text{for all } f \in \mathcal{D}(\Omega). \tag{4.187}$$

But then Λ can be considered as a Borel measure on Ω which is multiplicative on $C_0(\Omega)$, *i.e.* on the space of bounded continuous function on Ω which tend to zero at infinity. The existence of this (canonical) extension Λ_0 from $\mathcal{D}(\Omega)$ to $C_0(\Omega)$ while preserving the inequality in (4.187) is a consequence of the fact that $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$. This density is also used to show that the functional Λ_0 is multiplicative. More precisely, let f_1 and f_2 be functions in $C_0(\Omega)$. Then there exist sequences $(f_{1,k} : k \in \mathbb{N}) \subset \mathcal{D}(\Omega)$ and $(f_{2,k} : k \in \mathbb{N}) \subset \mathcal{D}(\Omega)$ such that

$$\lim_{k\to\infty} \|f_1 - f_{1,k}\|_\infty = \lim_{k\to\infty} \|f_2 - f_{2,k}\|_\infty = 0,$$

so that

$$\lim_{k\to\infty} \|f_1 f_2 - f_{1,k} f_{2,k}\|_\infty = 0,$$

and also

$$\begin{aligned}
&\lim_{k\to\infty} \Lambda_0\left(f_1 f_2\right) = \lim_{k\to\infty} \Lambda_0\left(f_{1,k} f_{2,k}\right) = \lim_{k\to\infty} \Lambda\left(f_{1,k} f_{2,k}\right) \\
&= \lim_{k\to\infty} \Lambda\left(f_{1,k}\right) \Lambda\left(f_{2,k}\right) = \lim_{k\to\infty} \Lambda_0\left(f_{1,k}\right) \Lambda_0\left(f_{2,k}\right) = \Lambda_0\left(f_1\right) \Lambda_0\left(f_2\right).
\end{aligned} \tag{4.188}$$

However, such measures are Dirac measures, *i.e.* there exists $a \in \Omega$ such that $\Lambda(f) = f(a)$ for all $f \in \mathcal{D}(\Omega)$. This can be proved as follows. Let the linear functional $\Lambda_0 : C_0(\Omega) \to \mathbb{C}$ be such that $\Lambda_0(f) = \Lambda(f)$, $f \in \mathcal{D}(\Omega)$, and such that $|\Lambda_0(f)| \leqslant \|f\|_\infty$, for all $f \in C_0(\Omega)$. Put $J = \operatorname{Ker}(\Lambda_0)$. Then J is a closed ideal in $C_0(\Omega)$ with $J \neq C_0(\Omega)$. So that by Theorem 4.52 there exists $a \in \Omega$ such that $\operatorname{Ker}(\Lambda_0) \subset \operatorname{Ker}(\delta_a)$. Choose $\varphi \in C_0(\Omega)$ such that $\varphi(a) = 1$. Then, for $f \in C_0(\Omega)$ arbitrary, we have

$$\Lambda_0\left(f\varphi - \Lambda_0(f)\varphi\right) = \Lambda_0\left(f\varphi\right) - \Lambda_0(f)\Lambda(\varphi) = \Lambda_0\left(f\right)\Lambda_0\left(\varphi\right) - \Lambda_0(f)\Lambda_0(\varphi) = 0,$$

and, consequently,

$$0 = \delta_a\left(f\varphi - \Lambda_0(f)\varphi\right) = f(a)\varphi(a) - \Lambda(f)\varphi(a) = f(a) - \Lambda(f), \quad f \in \mathcal{D}(\Omega).$$

This completes the proof of Proposition 4.51. □

Remark. The extension of the functional $\Lambda : \mathcal{D}(\Omega) \to \mathbb{C}$ to $\Lambda_0 : C_0(\Omega) \to \mathbb{C}$, as in the proof of Proposition 4.51 can also be proved by an appeal to the Hahn-Banach theorem, but then the density of $\mathcal{D}(\Omega)$ in the space $C_0(\Omega)$ has to be employed to prove that the extended linear functional is multiplicative: see the equalities in (4.188). As explained in the comments after Proposition 4.51 the Stone-Weierstrass theorem implies that the subspace $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$ for the uniform topology.

Proof of Proposition 4.53. (a) Define the linear functional $\Lambda : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ by

$$\Lambda(\varphi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} b(y) \int_{\mathbb{R}^n} e^{-iy\cdot x} \varphi(x)\, dx\, dy, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{4.189}$$

Then Λ is a multiplicative distribution on $\mathcal{D}(\mathbb{R}^n)$. Let us check this. To this end we take

$$\varphi_j(x) = \mathcal{F}^{-1}\psi_j(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iy\cdot x} \psi_j(y)\, dy, \quad j = 1, 2$$

in $\mathcal{D}(\mathbb{R}^n)$. Then

$$\psi_j(y) = \mathcal{F}\varphi_j(y) = \int_{\mathbb{R}^n} e^{-iy\cdot x} \varphi_j(x)\, dx, \quad j = 1, 2.$$

Moreover, we have

$$\begin{aligned}\varphi_1(x)\varphi_2(x) &= \mathcal{F}^{-1}\psi_1(x)\mathcal{F}^{-1}\psi_2(x) = \frac{1}{(2\pi)^{2n}}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} e^{ix\cdot(y_1+y_2)}\psi_1\left(y_1\right)\psi_2\left(y_2\right)\,dy_1\,dy_2\\
&= \frac{1}{(2\pi)^{2n}}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} e^{ix\cdot y_2}\psi_1\left(y_1\right)\psi_2\left(y_2-y_1\right)\,dy_1\,dy_2\\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n} e^{ix\cdot y_2}\psi_1*\psi_2\left(y_2\right)\,dy_2 = \frac{1}{(2\pi)^{n}}\mathcal{F}^{-1}\left(\psi_1*\psi_2\right)(x). \qquad (4.190)\end{aligned}$$

From (4.190) and the definition of Λ we obtain:

$$\begin{aligned}\Lambda\left(\varphi_1\varphi_2\right) &= \Lambda\left(\mathcal{F}^{-1}\psi_1\mathcal{F}^{-1}\psi_2\right) = \frac{1}{(2\pi)^{n}}\Lambda\left(\mathcal{F}^{-1}\left(\psi_1*\psi_2\right)\right)\\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n} b(y)\mathcal{F}\left(\mathcal{F}^{-1}\left(\psi_1*\psi_2\right)\right)(y)\,dy\\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n} b(y)\left(\psi_1*\psi_2\right)(y)\,dy\\
&= \frac{1}{(2\pi)^{2n}}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} b(y)\psi_1(z)\psi_2(y-z)\,dy\,dz\\
&= \frac{1}{(2\pi)^{2n}}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} b(y-z+z)\psi_1(z)\psi_2(y-z)\,dy\,dz\end{aligned}$$

(use the multiplicativity property of the function $y \mapsto b(y)$ and translation invariance of the Lebesgue measure)

$$\begin{aligned}&= \frac{1}{(2\pi)^{2n}}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} b(y)b(z)\psi_1(z)\psi_2(y)\,dy\,dz\\
&= \frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^n} b(z)\psi_1(z)\,dz\,\frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^n} b(y)\psi_2(y)\,dy\\
&= \frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^n} b(z)\mathcal{F}\varphi_1(z)\,dz\,\frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^n} b(y)\mathcal{F}\varphi_2(y)\,dy\\
&= \Lambda\left(\varphi_1\right)\cdot\Lambda\left(\varphi_2\right). \qquad (4.191)\end{aligned}$$

The multiplicativity of Λ is a consequence of (4.191). By Proposition 4.51 we either have $\Lambda \equiv 0$ or else there exists $a \in \mathbb{R}^n$ such that $\Lambda(\varphi) = \varphi(a)$, $\varphi \in \mathcal{D}\left(\mathbb{R}^n\right)$. If $\Lambda \neq 0$ it follows that there is $a \in \mathbb{R}^n$ such that:

$$\Lambda(\varphi) = \frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^n} b(y)\int_{\mathbb{R}^n} e^{-iy\cdot x}\varphi(x)\,dx\,dy = \varphi(a), \quad \varphi \in \mathcal{D}\left(\mathbb{R}^n\right). \qquad (4.192)$$

In (4.192) we write $\varphi \in \mathcal{D}\left(\mathbb{R}^n\right)$ in the form

$$\varphi(x) = \mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^n} e^{iy\cdot x}\psi(y)\,dy \qquad (4.193)$$

where ψ belongs to the Schwarz space $\mathcal{S}(\mathbb{R}^n)$. Then inserting (4.193) into (4.192) shows:

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} b(y)\psi(y)\,dy = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{iy\cdot a}\psi(y)\,dy, \quad \psi \in \mathcal{D}(\mathbb{R}^n). \tag{4.194}$$

From (4.194) we see that $b(y) = e^{ia\cdot y}$ for almost all $y \in \mathbb{R}^n$. So the proof of assertion (a) of Proposition 4.53 is complete now.

(b) Suppose that $b(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^n$. Then, since $b(0)^2 = b(0)$ we get $b(0) = 1$. As a consequence we see $1 = b(0) = b(x - x) = b(x)b(-x)$, $x \in \mathbb{R}^n$. From assertion (a) it follows that there exists $a \in \mathbb{R}^n$ and a Borel measurable subset A of $\mathbb{R}^n$, with a complement of Lebesgue measure 0, such that $b(x) = e^{ia\cdot x}$ for all $x \in A$. The previous remarks imply that $b(-y) = e^{-ia\cdot y}$ for all $y \in A$. Hence,

$$b\left(k(x-y)\right) = \left(b(x-y)\right)^k = \left(b(x)\right)^k\left(b(-y)\right)^k = e^{ia\cdot kx}e^{-ia\cdot ky} = e^{ia\cdot k(x-y)}, \tag{4.195}$$

for all $k \in \mathbb{N}$ and for $x \in A$, $y \in A$. From Proposition 4.60 below together with (4.195) it follows that $b(x) = e^{ia\cdot x}$ for all $x \in \mathbb{R}^n$. This completes the proof of assertion (b) of Proposition 4.53. $\square$

Proof of Proposition 4.54. Fix $\ell \in \mathbb{N}$. An application of assertion (a) in Proposition 4.53 with $b_\ell(y) = e^{i2^{-\ell}g(y)}$ yields the existence of a vector $a_\ell \in \mathbb{R}^n$ such that $e^{i2^{-\ell}g(y)} = e^{i2^{-\ell}a_\ell\cdot y}$ for Lebesgue almost all $y \in \mathbb{R}^n$. Put $\Omega_\ell = \left\{y \in \mathbb{R}^n : e^{i2^{-\ell}g(y)} = e^{i2^{-\ell}a_\ell\cdot y}\right\}$, and put $\Omega = \bigcap_{\ell=0}^{\infty}\Omega_\ell$. Let $m(\cdot)$ be the Lebesgue measure on $\mathbb{R}^n$. Then $m\left[\mathbb{R}^n\backslash\Omega_\ell\right] = 0$, $\ell = 0, 1, \ldots$ and so $m\left[\mathbb{R}^n\backslash\Omega\right] = 0$. If y belongs to Ω, then $e^{i2^{-\ell}g(y)} = e^{i2^{-\ell}a_\ell\cdot y}$, for $\ell = 0, 1, \ldots$. By raising these expressions to the power 2^ℓ we obtain $e^{ig(y)} = e^{ia_\ell\cdot y}$. In other words for $y \in \Omega$ and for $\ell \in \mathbb{N}$ we have $e^{ia_0\cdot y} = e^{ia_\ell\cdot y}$, or, what amounts to same, $e^{i(a_0-a_\ell)\cdot y} = 1$ for $y \in \Omega$, $\ell \in \mathbb{N}$. But then,

$$e^{i(a_0-a_\ell)\cdot(y_2-y_1)} = 1 \quad \text{for all } y_1,\, y_2 \in \Omega,\ \ell \in \mathbb{N}. \tag{4.196}$$

By Proposition 4.60 below and (4.196) it follows that

$$e^{i(a_0-a_\ell)\cdot y} = 1 \quad \text{for all } y \text{ in an open neighborhood of } 0,\ \ell \in \mathbb{N}. \tag{4.197}$$

But then $a_0 = a_\ell$ for all $\ell \in \mathbb{N}$. It also follows that

$$e^{i2^{-\ell}g(y)} = e^{i2^{-\ell}a_0\cdot y}, \quad \text{for } y \in \Omega, \text{ and } \ell \in \mathbb{N}. \tag{4.198}$$

By raising both sides in (4.198) to the power k we see

$$e^{ik2^{-\ell}g(y)} = e^{ik2^{-\ell}a_0\cdot y}, \quad \text{for } y \in \Omega, \text{ and } \ell \in \mathbb{N},\ k \in \mathbb{N}. \tag{4.199}$$

Since the positive dyadic rational numbers are dense in all positive real numbers, from (4.199) we infer:

$$e^{itg(y)} = e^{ita_0\cdot y}, \quad \text{for } y \in \Omega, \text{ and } t \in \mathbb{R},\ t \geqslant 0. \tag{4.200}$$

Taking the (right) derivative at $t = 0$ the equalities in (4.200) entails: $g(y) = a_0 \cdot y$ for $y \in \Omega$. This completes the proof of Proposition 4.54. $\square$

Proof of Proposition 4.55. Let the notation and hypotheses be as in Proposition 4.55. From Proposition 4.54 it follows that there exists $a \in \mathbb{R}^n$ and a subset A of $\mathbb{R}^n$ such that the Lebesgue measure of $\mathbb{R}^n \backslash A$ is zero, and such that $g(x) = a \cdot x$ for all $x \in A$. In particular it follows that the Lebesgue measure of A is strictly positive, and hence by a theorem from harmonic analysis we see that the vector difference $A - A := \{y_1 - y_2 : y_1, y_2 \in A\}$ contains an open neighborhood U of the origin. For this result see Proposition 4.60 below. However, if y_1 and y_2 belong to A, then

$$\begin{aligned} g(y_1 - y_2) &= g(y_1) + g(-y_2) = g(y_1) - g(y_2) \\ &= a \cdot y_1 - a \cdot y_2 = a \cdot (y_1 - y_2), \end{aligned} \tag{4.201}$$

and consequently, (4.201) together with $A - A \supset U$ implies $g(y) = a \cdot y$ for all $y \in U$. If $y \in \mathbb{R}^n$ is arbitrary we choose $k \in \mathbb{N}$, $k \geqslant 1$, in such a way that $k^{-1}y$ belongs to U. Then we get:

$$g(y) = g\left(k\frac{y}{k}\right) = kg\left(\frac{y}{k}\right) = ka \cdot \frac{y}{k} = a \cdot y.$$

This completes the proof of Proposition 4.55. $\square$

4.57. Theorem. *Let $b : \mathbb{R}^n \to \mathbb{C}\backslash\{0\}$ be a Borel measurable function. Then the following assertions hold.*

(a) *If b is almost everywhere multiplicative in the sense that $b(x+y) = b(x)b(y)$ for almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, then there exists $a \in \mathbb{C}^n$ such that $b(x) = e^{a \cdot x}$ for almost all $x \in \mathbb{R}^n$.*
(b) *If the function b is multiplicative in the sense that $b(x+y) = b(x)b(y)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, then there exists $a \in \mathbb{R}^n$ such that $b(x) = e^{a \cdot x}$ for all $x \in \mathbb{R}^n$.*

Proof. (a) Define the function $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = \log |b(x)|$, $x \in \mathbb{R}^n$. Then for Lebesgue almost all pairs $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have $g(x+y) = g(x) + g(y)$. From Proposition 4.54 it follows that there exists a vector $\alpha \in \mathbb{R}^n$ such that $\log |b(x)| = \alpha \cdot x$ for Lebesgue almost all $x \in \mathbb{R}^n$. Hence, $|b(x)| = e^{\alpha \cdot x}$ for Lebesgue almost all $x \in \mathbb{R}^n$. Then we apply assertion (a) in Proposition 4.53 to the almost everywhere multiplicative function $x \mapsto b(x)e^{-\alpha \cdot x}$, $x \in \mathbb{R}^n$, to obtain a vector $\beta \in \mathbb{R}^n$ such that for Lebesgue almost all $x \in \mathbb{R}^n$ the equality $b(x)e^{-\alpha \cdot x} = e^{i\beta \cdot x}$ holds. Hence assertion (a) follows with $a = \alpha + i\beta$.

(b) Again we employ the function $g(x) = \log |b(x)|$, $x \in \mathbb{R}^n$. Then we apply Proposition 4.54 together with assertion (b) of Proposition 4.55 to obtain $a = \alpha + i\beta$ such that $b(x) = e^{a \cdot x}$ for all $x \in \mathbb{R}^n$.

This competes the proof of Theorem 4.57. $\square$

4.58. Corollary. *Let $\Lambda : L^1(\mathbb{R}^n) \to \mathbb{C}$ be a linear functional which is multiplicative in the sense that $\Lambda(f * g) = \Lambda(f)\Lambda(g)$ for all $f, g \in L^1(\mathbb{R}^n)$. Then $|\Lambda(f))| \leqslant \|f\|_1$ for all $f \in L^1(\mathbb{R}^n)$, and either $\Lambda \equiv 0$, or else there exists $a \in \mathbb{R}^n$ such that $\Lambda(f) = \widehat{f}(a) = \int_{\mathbb{R}^n} e^{-ia \cdot x} f(x)\,dx$ for all $f \in L^1(\mathbb{R}^n)$.*

Proof. Pick $f \in L^1(\mathbb{R}^n)$, and let $\lambda \in \mathbb{C}$ be such that $|\lambda| > \|f\|_1$. Then there exists $g \in L^1(\mathbb{R}^n)$ such that

$$\lambda g - f * g = -\frac{1}{\lambda} f. \tag{4.202}$$

In fact $g = -\frac{1}{\lambda} \sum_{j=1}^{\infty} \frac{1}{\lambda^j} \underbrace{f * \cdots * f}_{j \text{ times}}$. Observe that the series for g converges in L^1-sense, and consequently, g belongs to $L^1(\mathbb{R}^n)$. Since Λ is linear and multiplicative the equality in (4.202) implies

$$\lambda\Lambda(g) - \Lambda(f)\Lambda(g) = \Lambda(\lambda g - f * g) = -\frac{1}{\lambda}\Lambda(f). \tag{4.203}$$

The equality in (4.203) shows that $\Lambda(f) \neq \lambda$. So that $|\Lambda(f)| \leqslant \|f\|_1$, $f \in L^1(\mathbb{R}^n)$. The functional $\varphi \mapsto \Lambda(\mathcal{F}^{-1}(\varphi))$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, is a multiplicative linear functional. For let $\varphi_j = \mathcal{F}(\psi_j) \in \mathcal{D}(\mathbb{R}^n)$, $j = 1, 2$. Then we have

$$\Lambda\left(\mathcal{F}^{-1}(\varphi_1\varphi_2)\right) = \Lambda\left(\mathcal{F}^{-1}(\mathcal{F}(\psi_1)\mathcal{F}(\psi_2))\right) = \Lambda\left(\mathcal{F}^{-1}\mathcal{F}(\psi_1 * \psi_2)\right)$$

$$= \Lambda\left(\psi_1 * \psi_2\right) = \Lambda\left(\psi_1\right)\Lambda\left(\psi_2\right) = \Lambda\left(\mathcal{F}^{-1}\left(\varphi_1\right)\right)\Lambda\left(\mathcal{F}^{-1}\left(\varphi_2\right)\right). \tag{4.204}$$

From Proposition 4.51 it follows then that this functional is either identically zero, or else there exists $a \in \mathbb{R}^n$ such that $\Lambda\left(\mathcal{F}^{-1}(\varphi)\right) = \varphi(a)$ for all $\varphi \in \mathcal{D}\left(\mathbb{R}^n\right)$. It follows that $\Lambda\left(\psi\right) = \mathcal{F}(\psi)(a) = \widehat{\psi}(a)$ for all $\psi \in \mathcal{S}\left(\mathbb{R}^n\right)$ with a Fourier transform of compact support. However, the vector space of such functions is dense in $L^1\left(\mathbb{R}^n\right)$. The L^1-continuity of Λ then implies that either Λ is either identically 0, or else there exists $a \in \mathbb{R}^n$ such that $\Lambda(f) = \widehat{f}(a)$.

This proves Corollary 4.58. □

4.59. Proposition. *Let A be a Borel subset of $\mathbb{R}^n$ with the property that $0 < m(A) < \infty$, where m is the Lebesgue measure on the Borel field of $\mathbb{R}^n$. Then the function*

$$x \mapsto m\left(A \cap (x + A)\right) = \int_{\mathbb{R}^n} \mathbf{1}_A(y - x)\mathbf{1}_A(y)dm(y) \tag{4.205}$$

is continuous. In the particular the set $\{x \in \mathbb{R}^n : m\left(A \cap (x + A)\right) > 0\}$ is an open neighborhood of the origin.

Proof. Let $\varepsilon > 0$ be arbitrary, and choose a continuous function $\varphi : \mathbb{R}^n \to [0, 1]$ with compact support such that $\int_{\mathbb{R}^n} \left|\mathbf{1}_A(y) - \varphi(y)\right| \, dm(y) < \frac{1}{2}\varepsilon$. Since for any $\eta > 0$ there exist a compact subset K and an open subset U, with $K \subset A \subset U$, such that $m\left(U \backslash K\right) < \eta$, such a choice is possible. Then we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \mathbf{1}_A(y)\mathbf{1}_A(y - x)\, dm(y) - \int_{\mathbb{R}^n} \varphi(y)\varphi\left(y - x\right)\, dm(y)\right| \\
&= \left|\int_{\mathbb{R}^n} \left(\mathbf{1}_A(y) - \varphi(y)\right)\mathbf{1}_A(y - x)\, dm(y) + \int_{\mathbb{R}^n} \varphi(y)\left(\mathbf{1}_A(y - x) - \varphi\left(y - x\right)\right)\, dm(y)\right| \\
&\leqslant \left|\int_{\mathbb{R}^n} \left(\mathbf{1}_A(y) - \varphi(y)\right)\mathbf{1}_A(y - x)\, dm(y)\right| + \left|\int_{\mathbb{R}^n} \varphi(y)\left(\mathbf{1}_A(y - x) - \varphi\left(y - x\right)\right)\, dm(y)\right| \\
&\leqslant \int_{\mathbb{R}^n} \left|\mathbf{1}_A(y) - \varphi(y)\right|\mathbf{1}_A(y - x)\, dm(y) + \int_{\mathbb{R}^n} \varphi(y)\left|\mathbf{1}_A(y - x) - \varphi\left(y - x\right)\right|\, dm(y) \\
&\leqslant \int_{\mathbb{R}^n} \left|\mathbf{1}_A(y) - \varphi(y)\right|\, dm(y) + \int_{\mathbb{R}^n} \left|\mathbf{1}_A(y - x) - \varphi\left(y - x\right)\right|\, dm(y) \\
&= 2\int_{\mathbb{R}^n} \left|\mathbf{1}_A(y) - \varphi(y)\right|\, dm(y) \leqslant \varepsilon.
\end{aligned} \tag{4.206}$$

Since $\varepsilon > 0$ is arbitrary by (4.206) we see that the function in (4.205) can be approximated by functions of the form $x \mapsto \int_{\mathbb{R}^n} \varphi(y)\varphi\left(y - x\right)\, dm(y)$ with φ continuous and of compact support. However, such functions are uniformly continuous. By uniform convergence, the function in (4.205) is uniformly continuous as well.

This completes the proof of Proposition 4.59. □

The following result is also true in the context of locally compact abelian groups with a Haar measure in place of the Lebesgue measure. As such it is part of harmonic analysis.

4.60. Proposition. *Let A be a Borel subset of strictly positive Lebesgue measure. Then the vector difference $A - A$ contains an open neighborhood of the origin. In fact*

$$A - A \supset \{x \in \mathbb{R}^n : m(A \cap (x + A)) > 0\},$$

and the latter set is an open subset of $\mathbb{R}^n$ containing the origin.

Proof. As in Proposition 4.59 denote by m the Lebesgue measure on the Borel field of $\mathbb{R}^n$. Let K be a compact subset of A. Then by Proposition 4.59 the function $x \mapsto m(K \cap (x + K))$ is (uniformly) continuous. It follows that the subset

$$\{x \in \mathbb{R}^n : m(K \cap (x + K)) > 0\}$$

is an open neighborhood of the origin. Consequently, the subset U defined by

$$\begin{aligned} U &= \bigcup_{K \subset A,\, K \text{ compact}} \{x \in \mathbb{R}^n : m(K \cap (x + K)) > 0\} \\ &= \{x \in \mathbb{R}^n : m(A \cap (x + A)) > 0\} \end{aligned}$$

is an open neighborhood of the origin. If $x \in U$, then $A \cap (x + A) \neq \varnothing$, and so there exist elements y_1 and $y_2 \in A$ such that $x = y_1 - y_2$. Consequently, $U \subset A - A$, which completes the proof of Proposition 4.60. $\square$

7.1. The representation theorem for the dual of $C_0(X)$. The following theorem, referred to as the Riesz or the Riesz-Markov representation theorem, gives a concrete realisation of the dual space of $C_0(X)$, the set of continuous functions on X which vanish at infinity. The Borel sets in the statement of the theorem refer to the σ-algebra generated by the open sets.

If μ is a complex-valued countably additive Borel measure, μ is regular iff the non-negative countably additive measure $|\mu|$ is regular as defined above. For a thorough discussion of the Riesz (or Riesz-Markov) representation theorem see, *e.g.*, H. König [**81**] and [**80**]. In the latter reference the author discusses the connection between regularity properties of the measure relative to the σ-continuity property of the integral, which is the case in the Daniell-Stone theorem. The lecture notes by B. Driver [**42**] are also of interest for this topic.

4.61. Theorem (Riesz-Markov). *Let X be a locally compact Hausdorff space. For any continuous linear functional ψ on $C_0(X)$, there exists a unique regular countably additive complex Borel measure μ on (the Borel field of) X such that $\psi(f) = \int_X f(x)\,d\mu(x)$ for all f in $C_0(X)$. The norm of ψ as a linear functional is the total variation of μ, that is $\|\psi\| = |\mu|(X)$. Finally, ψ is positive iff the measure μ is non-negative.*

4.62. Remark. Let $C_{00}(X)$ be the vector space of all complex valued continuous function defined on X with compact support. One might expect that by the Hahn-Banach theorem for bounded linear functionals, every bounded linear functional on $C_{00}(X)$ extends in exactly one way to a bounded linear functional on $C_0(X)$, the latter being the closure of $C_{00}(X)$ in the supremum norm, and that for this reason the first statement implies the second. However the first result is for positive linear functionals, not bounded linear functionals, so the two facts are not equivalent.

In fact, a bounded linear functional on $C_{00}(X)$ need not remain so if the locally convex topology on $C_{00}(X)$ is replaced by the supremum norm topology, the norm of $C_0(X)$. An example is the Lebesgue measure on $\mathbb{R}$, which is bounded on $C_{00}(\mathbb{R})$ but unbounded on $C_0(\mathbb{R})$. This fact can also be seen by observing that the total variation of the Lebesgue measure on $\mathbb{R}$ is infinite.

7.2. Runge's theorem. In this subsection we formulate and prove Runge's theorem. Our proof follows that of Grabiner [**57**].

4.63. Theorem. *Let K be a compact subset of $\mathbb{C}$, and S a subset of $\widehat{\mathbb{C}}\backslash K$ that contains at least one point in each component of $\widehat{\mathbb{C}}\backslash K$. Define*

$$B(S) = \{f : f \text{ is a uniform limit on } K \text{ of rational functions whose poles lie in } S\}.$$

Then every function f that is analytic on a neighborhood of K is in $B(S)$. That is, there is a sequence $(R_n)_n$ of rational functions whose poles lie in S such that $R_n \to f$ uniformly on K.

Observe that $\widehat{\mathbb{C}}$ is the one-point compactification of $\mathbb{C}$. In terms of the theory of Riemann surfaces it is identified with the Riemann sphere S^2. Before giving the proof, let us note the conclusion in the special case where $\widehat{\mathbb{C}}\backslash K$ is connected. In this case, we can take $S = \{\infty\}$, and our sequence of rational functions will actually be a sequence of polynomials. The proof given here is due to Sandy Grabiner [**57**]: see Ash and Novinger [**7**] as well. It is based on the following three lemmas.

4.64. LEMMA. *Suppose K is a compact subset of the open set $\Omega \subset \mathbb{C}$. If f is holomorphic on Ω, then f is a uniform limit on K of rational functions whose poles lie in $\Omega\backslash K$.*

The notion of a curves, or a concatenation of curves, which surrounds a given compact subset see Subsection 1.1.

PROOF. Let Γ be a concatenation of curves which surrounds the compact subset K in Ω. Then by Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\lambda)}{\lambda - z}\,d\lambda, \quad z \in K. \tag{4.207}$$

The integral in (4.207) is a Riemann integral, which converges uniformly for $z \in K$. Each of these summands is a rational function in z with poles in $\Omega\backslash K$. This completes the proof of Lemma 4.64. $\square$

The following Lemma should be compared to Proposition 5.7.

4.65. LEMMA. *Let U and V be open subsets of $\mathbb{C}$ with $V \subseteq U$ and* $(\text{boundary}V)\cap U = \varnothing$. *If H is any component of U and $V \cap H \neq \varnothing$, then $H \subseteq V$.*

PROOF. Suppose that H is a component of U which contains a point s in V. Let G be the component of V which contains s. Since H is connected, it must either equal G or contain a boundary point of G. But each boundary point of G is a boundary point of V, and cannot belong to U, which completes the proof of Lemma 4.65 $\square$

4.66. LEMMA. *If K is a compact subset of $\mathbb{C}$ and $\lambda \in \mathbb{C}\backslash K$, then the function $z \mapsto \dfrac{1}{z-\lambda}$ belongs to $B(S)$.*

PROOF. For large enough $|\lambda|$, the Taylor series for

$$z \mapsto \frac{1}{\lambda - z} = \frac{1}{\lambda}\frac{1}{1 - z\lambda^{-1}}$$

converges uniformly on K; so we can assume that S is a subset of the finite complex plane. Let U be the complement of K in the finite plane and let

$$V := \left\{\lambda \in \mathbb{C} : z \mapsto \frac{1}{\lambda - z} \in B(S)\right\}.$$

If λ belongs to V and if $|\mu - \lambda| < \operatorname{dist}(\lambda, K)$, then μ also belongs to V since

$$\frac{1}{\mu - z} = \frac{1}{\lambda - z}\left(1 - \frac{\lambda - \mu}{\lambda - z}\right)^{-1} = \sum_{k=0}^{\infty} \frac{(\lambda - \mu)^k}{(\lambda - z)^{k+1}}$$

is the uniformly converging sum of a series in powers of $\dfrac{1}{\lambda - z}$. Thus V is an open subset of U. Suppose that μ is a boundary point of V, and choose a sequence $(\lambda_n)_n$ in V with limit μ. Since μ does not belong to V we must have $|\lambda_n - \mu| \geqslant \operatorname{dist}(\lambda_n, K)$ for all n. Taking limits we see that $\operatorname{dist}(\mu, K) = 0$, so that μ does not belong to U. Finally, notice that each component of U contains a point of S, and therefore meets V. Hence, by Lemma 4.65, $V = U$, and the proof of Lemma 4.66 is complete. $\square$

Runge's theorem (Theorem 4.63) follows from these three lemmas. First note that if f and g belong to $B(S)$, then so do $f+g$ and fg. Thus by Lemma 4.66 every rational function with poles in $\widehat{\mathbb{C}}\backslash K$ belongs to $B(S)$. Runge's theorem is then a consequence of Lemma 4.64 (Lemma 4.65 is used to prove Lemma 4.66.) In addition, observe that $B(S)$ is a Banach algebra for the supremum norm on K. For more details the reader is referred to [**7**] Chapter 5.

Index

To see Section 5–9 download:

Partial differential equations and operators: Part II