

Calculus

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Mathematics and Informatics Pre-session for Business Analytics

2019

- ▶ A finite sequence has a finite number of elements

- ▶ A finite sequence has a finite number of elements
- ▶ An infinite sequence has infinitely many elements

- ▶ A sequence is monotonically increasing if $a_{n+1} \geq a_n \quad \forall n$

- A sequence is monotonically decreasing if $a_{n+1} \leq a_n \quad \forall n$

- If $\exists N$ such that $a_n \leq M \quad \forall n$ the sequence is bounded from above

- If $\exists M$ such that $a_n > M \quad \forall n$ the sequence is bounded from below

Give an example for each type!

$$a_n \rightarrow A$$
$$\lim_{n \rightarrow \infty} a_n = A$$

11. *Journal of the American Medical Association*, 277: 1001-1002, 1997.

► $a_n = 5 \implies a_n \rightarrow 5$

Formal definition of convergence

A sequence a_n converges to A if $\forall \varepsilon > 0 \quad \exists N$ such that $\forall n > N$ it holds that

Examples:

- ▶ $a_n \equiv n$ is divergent

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n^2 - 3}{n^3 - 2} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n^3}(n^2 - 3)}{\frac{1}{n^3}(n^3 - 2)} \\ \lim_{n \rightarrow \infty} \frac{\frac{1}{n^3}(n^2 - 3)}{\frac{1}{n^3}(n^3 - 2)} &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} - \frac{3}{n^3}\right)}{\left(1 - \frac{2}{n^3}\right)} \\ \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} - \frac{3}{n^3}\right)}{\left(1 - \frac{2}{n^3}\right)} &= \frac{\lim_{n \rightarrow \infty} \left(\frac{1}{n} - \frac{3}{n^3}\right)}{\lim_{n \rightarrow \infty} \left(1 - \frac{2}{n^3}\right)} \\ \frac{\lim_{n \rightarrow \infty} \left(\frac{1}{n} - \frac{3}{n^3}\right)}{\lim_{n \rightarrow \infty} \left(1 - \frac{2}{n^3}\right)} &= \frac{\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) - \lim_{n \rightarrow \infty} \left(\frac{3}{n^3}\right)}{\lim_{n \rightarrow \infty} (1) - \lim_{n \rightarrow \infty} \left(\frac{2}{n^3}\right)} \\ \frac{\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) - \lim_{n \rightarrow \infty} \left(\frac{3}{n^3}\right)}{\lim_{n \rightarrow \infty} (1) - \lim_{n \rightarrow \infty} \left(\frac{2}{n^3}\right)} &= \frac{0 - 0}{1 - 0} = 0\end{aligned}$$

Thus this sequence is divergent.

$$\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{2}$$

Notice that there are two alternating terms: 0 and 1. Thus this sequence doesn't have a limit.

$$\lim \left(1 + \frac{1}{n}\right)^n = e$$

- ▶ Stolz–Cesàro theorem
- ▶ L'Hôpital's rule

$$1. \lim_{n \rightarrow \infty} \frac{n^4 + 5n^3 + 3n^2 - 2}{3n^4 - 6}$$

2. $\lim_{n \rightarrow \infty} \frac{5}{n+1} + \frac{n}{n+1}$

3. $\lim_{n \rightarrow \infty} b^n$ depending on the value of b .

4. $\lim_{n \rightarrow \infty} \frac{1}{n(\sqrt{n^2-1}-n)}$

5. $\lim_{n \rightarrow \infty} \sqrt[n]{5}$

6. $\lim_{n \rightarrow \infty} \ln \left(\frac{1}{n} \right)$

7. $\lim_{n \rightarrow \infty} e^{-n}$

Sidenote: Series

Roughly speaking a series is the sum of the elements of a sequence.

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \dots$$

There is one series that you should remember: the geometric series. The sum of a sequence defined by

$$a_n = a \cdot b^n$$

where $0 < b < 1$ is given by

$$\sum_{i=1}^{\infty} a_i = \frac{ab}{1-b}$$

What is the sum of the following sequences?

- ▶ $a_n = \frac{3}{5^n}$
- ▶ $a_n = 0.5^n$

Definition

A function $f(x)$ has a limit L when x approaches to p IF for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x that satisfies $|x - p| < \delta$ it holds that $|f(x) - L| < \varepsilon$. The notation is

$$\lim_{x \rightarrow p} f(x) = L$$

$$S(\lambda) = \frac{1}{2} \lambda^2$$

Limits of functions

Example: $f(x) = 3x$. Calculate $\lim_{x \rightarrow 3} f(x)$. Now let's understand the definition.

- ▶ We claim that $\lim_{x \rightarrow 3} f(x) = 9$
- ▶ Let's have any positive number ε
- ▶ There should exist a δ for any ε that if we are in the δ neighborhood of 3, the function value is always closer to 9 than ε
- ▶ We can compute this δ depending on ε .

$$|f(x) - 9| < \varepsilon \implies -\varepsilon < f(x) - 9 < \varepsilon \implies -\varepsilon + 9 < f(x) < \varepsilon + 9 \implies$$

$$-\varepsilon + 9 < 3x < \varepsilon + 9 \implies -\frac{\varepsilon}{3} + 3 < x < \frac{\varepsilon}{3} + 3 \implies |x - 3| < \frac{\varepsilon}{3} = \delta$$

- ▶ Let's say $\varepsilon = 6$. It implies that $\delta = \frac{6}{3} = 2$, that is, if we are in the $(3 - 2, 3 + 2)$ interval, the function value should always be closer to 9 than 6.

$$\lim_{x \rightarrow p} (f(x) + g(x)) = \lim_{x \rightarrow p} f(x) + \lim_{x \rightarrow p} g(x)$$

$$\lim_{x \rightarrow p} (f(x) - g(x)) = \lim_{x \rightarrow p} f(x) - \lim_{x \rightarrow p} g(x)$$

$$\lim_{x \rightarrow p} (f(x) \cdot g(x)) = \lim_{x \rightarrow p} f(x) \cdot \lim_{x \rightarrow p} g(x)$$

$$\lim_{x \rightarrow p} (f(x)/g(x)) = \lim_{x \rightarrow p} f(x) / \lim_{x \rightarrow p} g(x)$$

Examples

Find $\lim_{x \rightarrow 5} e^{x-3}$. Notice that this is a standard exponential function, which is continuous.

Thus

$$\lim_{x \rightarrow 5} e^{x-3} = e^{5-3} = e^2$$

Find $\lim_{x \rightarrow 0} \ln(x)$. Now notice, that $\ln(0)$ is not defined. However the $\ln(x)$ function is monotonically increasing, thus as we get closer and closer to zero, it's value gets closer and closer to minus infinity. Thus

$$\lim_{x \rightarrow 0} \ln(x) = -\infty$$

Examples

Find $\lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + x - 3}{x^5 - 2x}$

$$\lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + x - 3}{x^5 - 2x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^5}(x^4 - 2x^3 + x - 3)}{\frac{1}{x^5}(x^5 - 2x)}$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x^5}(x^4 - 2x^3 + x - 3)}{\frac{1}{x^5}(x^5 - 2x)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^4} - \frac{3}{x^5}}{1 - \frac{2}{x^4}}$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^4} - \frac{3}{x^5}}{1 - \frac{2}{x^4}} = \frac{\lim_{x \rightarrow \infty} \frac{1}{x} - \lim_{x \rightarrow \infty} \frac{2}{x^2} + \lim_{x \rightarrow \infty} \frac{1}{x^4} - \lim_{x \rightarrow \infty} \frac{3}{x^5}}{\lim_{x \rightarrow \infty} 1 - \lim_{x \rightarrow \infty} \frac{2}{x^4}}$$

$$\frac{\lim_{x \rightarrow \infty} \frac{1}{x} - \lim_{x \rightarrow \infty} \frac{2}{x^2} + \lim_{x \rightarrow \infty} \frac{1}{x^4} - \lim_{x \rightarrow \infty} \frac{3}{x^5}}{\lim_{x \rightarrow \infty} 1 - \lim_{x \rightarrow \infty} \frac{2}{x^4}} = \frac{0 - 0 + 0 - 0}{1 - 0} = 0$$

$$\lim_{x \rightarrow 2} \frac{3(x^2 + x - 6)}{x - 2} = \lim_{x \rightarrow 2} \frac{3(x + 3)(x - 2)}{x - 2}$$

$$\lim_{x \rightarrow 2} \frac{3(x+3)(x-2)}{x-2} = \lim_{x \rightarrow 2} 3(x+3) = 3 \cdot 5 = 15$$

Solve the following problems

1. $\lim_{x \rightarrow 0} (3 + 2x^2)$

2. $\lim_{x \rightarrow -1} \frac{3+2x}{x-1}$

3. $\lim_{x \rightarrow 1} \frac{x^2+7x-8}{x-1}$

4. $\lim_{x \rightarrow \infty} \frac{x^3-3x^2+x-5}{3x^3+5x^2-2}$

5. $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$

6. $\lim_{h \rightarrow 0} \frac{\sqrt{h+1}-1}{h}$

7. $\lim_{x \rightarrow 5} \frac{3x^2-9x-30}{x-5}$

- $$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x) - f(1)}{h}$$

$$\begin{aligned} f'(1) &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \\ \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} \\ \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} &= \lim_{x \rightarrow 1} x + 1 = 2 \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x) - f(2)}{x - 2}$$

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x + 2) = 4 \end{aligned}$$

1. $f'(5)$

- A bit more difficult problem: Consider now $g(x) = x^3$.

1. First find $g'(2)$

- $$\frac{\mathrm{d} f(x)}{\mathrm{d} x}$$

$$\frac{d f(x)}{d x}$$

- ▶ $c' = 0 \quad \forall c \in \mathbb{R}$
- ▶ $(af)' = af'$
- ▶ $(af + bg)' = af' + bg'$
- ▶ $(fg)' = f'g + fg'$
- ▶ $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

- ▶ $f(x) = x^3 + 2x^2 - x$
- ▶ $g(x) = (x^2 + 2)(x - 4)$
- ▶ $h(x) = \frac{x^{12} - 15x^2}{x - 5}$

► $h(x) = \frac{x^{12}-15x^2}{x-5}$

Some additional useful derivatives

- ▶ $\frac{d}{dx} e^x = e^x$
- ▶ $\frac{d}{dx} a^x = a^x \ln(a)$
- ▶ $\frac{d}{dx} \ln(x) = \frac{1}{x}, \quad \forall x > 0$
- ▶ $\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$

Solve the following problems

Find the derivative of the following functions:

► $f(x) = \frac{x^2}{\ln x}$

► $g(x) = e^x(x^3 - x^2)$

► $h(x) = \frac{5^x}{x^2 - 2}$

The chain rule

You can take the derivative of a function of a function the following way:

$$(f(g(x)))' = f'(g(x))g'(x)$$

Example: e^{-x^2} . Here $f(x) = e^x$ and $g(x) = -x^2$. Thus:

$$(e^{-x^2})' = -2xe^{-x^2}$$

Find the derivative of $\ln(x^2 + 2x)$

Unconstrained optimization

We often want to solve so-called unconstrained optimization problems. Examples:

- ▶ What is the optimal quantity to produce in order to maximize your profit?
- ▶ What is the optimal length of sleep if you want to be as productive as possible?

If we can characterize these problems with functions, we can optimize them.

- ▶ We want to find their minima/maxima
- ▶ At these points, the tangent line should be horizontal
- ▶ Thus the derivative should be equal to zero

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1999, 2000, 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030, 2031, 2032, 2033, 2034, 2035, 2036, 2037, 2038, 2039, 2040, 2041, 2042, 2043, 2044, 2045, 2046, 2047, 2048, 2049, 2050, 2051, 2052, 2053, 2054, 2055, 2056, 2057, 2058, 2059, 2060, 2061, 2062, 2063, 2064, 2065, 2066, 2067, 2068, 2069, 2070, 2071, 2072, 2073, 2074, 2075, 2076, 2077, 2078, 2079, 2080, 2081, 2082, 2083, 2084, 2085, 2086, 2087, 2088, 2089, 2090, 2091, 2092, 2093, 2094, 2095, 2096, 2097, 2098, 2099, 2100, 2101, 2102, 2103, 2104, 2105, 2106, 2107, 2108, 2109, 2110, 2111, 2112, 2113, 2114, 2115, 2116, 2117, 2118, 2119, 2120, 2121, 2122, 2123, 2124, 2125, 2126, 2127, 2128, 2129, 2130, 2131, 2132, 2133, 2134, 2135, 2136, 2137, 2138, 2139, 2140, 2141, 2142, 2143, 2144, 2145, 2146, 2147, 2148, 2149, 2150, 2151, 2152, 2153, 2154, 2155, 2156, 2157, 2158, 2159, 2160, 2161, 2162, 2163, 2164, 2165, 2166, 2167, 2168, 2169, 2170, 2171, 2172, 2173, 2174, 2175, 2176, 2177, 2178, 2179, 2180, 2181, 2182, 2183, 2184, 2185, 2186, 2187, 2188, 2189, 2190, 2191, 2192, 2193, 2194, 2195, 2196, 2197, 2198, 2199, 2200, 2201, 2202, 2203, 2204, 2205, 2206, 2207, 2208, 2209, 2210, 2211, 2212, 2213, 2214, 2215, 2216, 2217, 2218, 2219, 2220, 2221, 2222, 2223, 2224, 2225, 2226, 2227, 2228, 2229, 2230, 2231, 2232, 2233, 2234, 2235, 2236, 2237, 2238, 2239, 2240, 2241, 2242, 2243, 2244, 2245, 2246, 2247, 2248, 2249, 2250, 2251, 2252, 2253, 2254, 2255, 2256, 2257, 2258, 2259, 2260, 2261, 2262, 2263, 2264, 2265, 2266, 2267, 2268, 2269, 2270, 2271, 2272, 2273, 2274, 2275, 2276, 2277, 2278, 2279, 2280, 2281, 2282, 2283, 2284, 2285, 2286, 2287, 2288, 2289, 2290, 2291, 2292, 2293, 2294, 2295, 2296, 2297, 2298, 2299, 2300, 2301, 2302, 2303, 2304, 2305, 2306, 2307, 2308, 2309, 2310, 2311, 2312, 2313, 2314, 2315, 2316, 2317, 2318, 2319, 2320, 2321, 2322, 2323, 2324, 2325, 2326, 2327, 2328, 2329, 2330, 2331, 2332, 2333, 2334, 2335, 2336, 2337, 2338, 2339, 2340, 2341, 2342, 2343, 2344, 2345, 2346, 2347, 2348, 2349, 2350, 2351, 2352, 2353, 2354, 2355, 2356, 2357, 2358, 2359, 2360, 2361, 2362, 2363, 2364, 2365, 2366, 2367, 2368, 2369, 2370, 2371, 2372, 2373, 2374, 2375, 2376, 2377, 2378, 2379, 2380, 2381, 2382, 2383, 2384, 2385, 2386, 2387, 2388, 2389, 2390, 2391, 2392, 2393, 2394, 2395, 2396, 2397, 2398, 2399, 2400, 2401, 2402, 2403, 2404, 2405, 2406, 2407, 2408, 2409, 2410, 2411, 2412, 2413, 2414, 2415, 2416, 2417, 2418, 2419, 2420, 2421, 2422, 2423, 2424, 2425, 2426, 2427, 2428, 2429, 2430, 2431, 2432, 2433, 2434, 2435, 2436, 2437, 2438, 2439, 2440, 2441, 2442, 2443, 2444, 2445, 2446, 2447, 2448, 2449, 2450, 2451, 2452, 2453, 2454, 2455, 2456, 2457, 2458, 2459, 2460, 2461, 2462, 2463, 2464, 2465, 2466, 2467, 2468, 2469, 2470, 2471, 2472, 2473, 2474, 2475, 2476, 2477, 2478, 2479, 2480, 2481, 2482, 2483, 2484, 2485, 2486, 2487, 2488, 2489, 2490, 2491, 2492, 2493, 2494, 2495, 2496, 2497, 2498, 2499, 2500, 2501, 2502, 2503, 2504, 2505, 2506, 2507, 2508, 2509, 2510, 2511, 2512, 2513, 2514, 2515, 2516, 2517, 2518, 2519, 2520, 2521, 2522, 2523, 2524, 2525, 2526, 2527, 2528, 2529, 2530, 2531, 2532, 2533, 2534, 2535, 2536, 2537, 2538, 2539, 2540, 2541, 2542, 2543, 2544, 2545, 2546, 2547, 2548, 2549, 2550, 2551, 2552, 2553, 2554, 2555, 2556, 2557, 2558, 2559, 2560, 2561, 2562, 2563, 2564, 2565, 2566, 2567, 2568, 2569, 2570, 2571, 2572, 2573, 2574, 2575, 2576, 2577, 2578, 2579, 2580, 2581, 2582, 2583, 2584, 2585, 2586, 2587, 2588, 2589, 2590, 2591, 2592, 2593, 2594, 2595, 2596, 2597, 2598, 2599, 2600, 2601, 2602, 2603, 2604, 2605, 2606, 2607, 2608, 2609, 2610, 2611, 2612, 2613, 2614, 2615, 2616, 2617, 2618, 2619, 2620, 2621, 2622, 2623, 2624, 2625, 2626, 2627, 2628, 2629, 2630, 2631, 2632, 2633, 2634, 2635, 2636, 2637, 2638, 2639, 2640, 2641, 2642, 2643, 2644, 2645, 2646, 2647, 2648, 2649, 2650, 2651, 2652, 2653, 2654, 2655, 2656, 2657, 2658, 2659, 2660, 2661, 2662, 2663, 2664, 2665, 2666, 2667, 2668, 2669, 2670, 2671, 2672, 2673, 2674, 2675, 2676, 2677, 2678, 2679, 2680, 26

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- ▶ Notice that if it is a minimum point, the function has to be convex around the point
- ▶ For a maximum point, the function has to be concave around the point
- ▶ In case of an inflection point, the function is convex on one side but concave on the other side
- ▶ We should look at convexity

How to decide convexity?

- ▶ Notice that for convex functions the slope of the tangent line is continuously increasing (or at least not decreasing).
- ▶ For concave functions, this is the opposite. The slope of the tangent line is continuously decreasing (or at least not increasing).
- ▶ We already know a method to show whether a function is increasing or decreasing: taking it's derivative
- ▶ Thus if the derivative shows the slope of the function (how the function values change), the derivative of the derivative shows how the slope of the function changes (convexity).
- ▶ Therefore we will need to check the sign of the second derivative denoted by $f''(x)$ or $\frac{d^2 f(x)}{dx^2}$

An example

Find the minima/maxima of $f(x) = \frac{1}{3}x^3 - 1.5x^2 - 4x + 10$

- First find the points of minima/maxima:

$$\frac{df(x)}{dx} = 0$$

$$3x^2 - 3x - 4 = 0$$

$$x_1 = 4 \quad x_2 = -1$$

- Take the second derivative and substitute these values

$$\frac{d^2f(x)}{dx^2} = 6x - 3$$

$$f''(4) = 21 \quad f''(-1) = -9$$

Thus the function is concave at $x = -1$, and that point should be a local maximum. It is convex at $x = 4$, and it should be a local minimum. Check on WolframAlpha!

Plotting functions

- ▶ Using the tools we have studied, we can easily plot even quite difficult functions.
- ▶ We can decide whether they are increasing or decreasing using the first derivative
- ▶ We can also find local minima and maxima using the first derivative
- ▶ We can find out their convexity using the second derivative
- ▶ Let's try to plot $f(x) = \frac{\ln x}{x}$

$$\ln x = 0$$

- $$d'(\gamma) = -2; \quad \gamma = -1, -1, \dots, -2, (-1, -1, \dots)$$

$$x^{-2}(1 - \ln x) = 0 \quad \implies \ln x = 1$$

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The following table shows the results of the regression analysis:

Plotting functions

The second derivative is

$$\begin{aligned}
 f''(x) &= [x^{-2}(1 - \ln x)]' = [x^{-2} - x^{-2} \ln x]' \\
 [x^{-2} - x^{-2} \ln x]' &= -2x^{-3} - (-2x^{-3} \ln x + x^{-2}x^{-1}) \\
 -2x^{-3} - (-2x^{-3} \ln x + x^{-2}x^{-1}) &= x^{-3}(2 \ln x - 3)
 \end{aligned}$$

At e this is

$$f''(e) = e^{-3}(2 \ln e - 3) = -\frac{1}{e^3} < 0$$

Thus at e the function is concave, we have a local maximum.

Plotting functions

Let's check whether the function changes convexity anywhere. If it does, the second derivative should change from positive to negative at that point (or the other way around), thus it has to be zero.

$$\begin{aligned} f''(x) &= x^{-3}(2 \ln x - 3) = 0 \\ 2 \ln x - 3 &= 0 \\ x &= e^{3/2} \end{aligned}$$

We know that if $x < e^{3/2}$ (for example e), the second derivative is negative, and the function is concave. What if it is larger? Let's check e^2 .

$$f''(e^2) = (e^2)^{-3}(2 \ln e^2 - 3) = \frac{1}{e^6} > 0$$

Thus if $x > e^{3/2}$, the function is convex.

Plotting functions

We should also find the limits at the ends of the domain. Since $\ln x$ requires $x > 0$, the function's domain is \mathbb{R}^+ . Finding these limits is quite difficult without using L'Hôpital's rule, but we can use a trick and some intuition. Since $x \in \mathbb{R}^+$, we can write any x as $x = e^y$, where $y = \ln x$. Thus we can transform the limit we are looking for a bit:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{y \rightarrow \infty} \frac{\ln e^y}{e^y} = \lim_{y \rightarrow \infty} \frac{y}{e^y} = 0$$

Notice that in the last step we divide a linear function with an exponential, and the exponential grows much faster. This is why the limit is zero. With the other limit:

$$\lim_{x \rightarrow 0} \frac{\ln x}{x} = \lim_{y \rightarrow -\infty} \frac{\ln e^y}{e^y} = \lim_{y \rightarrow -\infty} \frac{y}{e^y} = -\infty$$

Notice that since we defined $x = e^y$, $x \rightarrow 0$ is the same as $y \rightarrow -\infty$.

Plotting functions

We can summarize everything in a table:

| x | <1 | 1 | >1 and $<e$ | e | $>e$ and $<e^{3/2}$ | $e^{3/2}$ | $>e^{3/2}$ |
|-----------|------------|------------|---------------|--------|---------------------|------------|------------|
| $f(x)$ | - | 0 | + | + | + | + | + |
| $f'(x)$ | + | + | + | 0 | - | - | - |
| Slope | \nearrow | \nearrow | \nearrow | MAX | \searrow | \searrow | \searrow |
| $f''(x)$ | - | - | - | - | - | 0 | + |
| Convexity | \cap | \cap | \cap | \cap | \cap | INF | \cup |

Now we know everything to plot it! Let's do the same with $f(x) = (x - 1)(x + 3)^2$

- ▶ The partial derivative w.r.t. x is $f'_x(x, y)$ or $\frac{\partial f(x, y)}{\partial x}$
- ▶ The partial derivative w.r.t. y is $f'_y(x, y)$ or $\frac{\partial f(x, y)}{\partial y}$

All the rules of differentiation still hold!

$$\frac{\partial f(x, y)}{\partial x} = 2x + 2y$$

$$\frac{\partial f(x, y)}{\partial y} = 2x + 2y$$

Calculus

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$$\frac{\partial f(x, y, z)}{\partial y} = xz + e^y \ln(x)z^5$$

$$\frac{\partial f(x, y, z)}{\partial z} = xy + e^y \ln(x) 5z^4$$

1. $g(x, y) = 42x + 42y$
2. $f(x, y) = x^2 \ln(y) + \frac{e^y}{\ln x}$
3. $h(x, y, z) = \frac{z^5 e^y}{y^2 \ln(z)x}$

3. $h(x, y, z) = \frac{z^5 e^y}{y^2 \ln(z)x}$

Multivariate unconstrained optimization

Just like in the one variable case, the derivatives at the minima/maxima have to be equal to zero. The difference is that now we need all partial derivatives to be equal to zero. If we put all these partial derivatives in a vector, it is called the **gradient**. You don't have to use it right now, but it is good to know, as it will come up later (E.g.: Gradient descent in ML courses).

$$\frac{\partial f(x, y)}{\partial x} = 2xy^2 - 5$$

$$\frac{\partial f(x, y)}{\partial y} = 2yx^2 - 5$$

$$2xy^2 - 5 = 0 \quad \text{AND} \quad 2yx^2 - 5 = 0$$

We find that $x = y$, which implies that

Example

Let's find the maxima/minima of the following function: $f(x, y) = -xye^{-x^2-y^2}$ The derivatives are:

$$\frac{\partial f(x, y)}{\partial x} = -ye^{-y^2}(e^{-x^2} - 2x^2e^{-x^2}) = e^{-x^2-y^2}y(2x^2 - 1)$$

$$\frac{\partial f(x, y)}{\partial y} = -xe^{-x^2}(e^{-y^2} - 2y^2e^{-y^2}) = e^{-x^2-y^2}x(2y^2 - 1)$$

Thus we need to solve

$$e^{-x^2-y^2}y(2x^2 - 1) \quad \text{AND} \quad e^{-x^2-y^2}x(2y^2 - 1)$$

We find 5 solutions: $(x, y) = (0, 0)$, $(x, y) = (1, 1)$, $(x, y) = (-1, -1)$, $(x, y) = (-1, 1)$, $(x, y) = (1, -1)$

Are these minima or maxima?

- ▶ That, again, depends on the convexity
- ▶ But it is significantly more difficult to check the convexity here
- ▶ We would need to check whether the so-called Hessian matrix (see below) is positive or negative definite, which we can not do without a decent knowledge on linear algebra

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

1. $f(x, y) = x^2 + 9x + y^2 - 6y + 15$
2. $h(x, y) = e^{-x^2-y^2}$

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- ▶ Sometimes we want to maximize/minimize a function, but we face a constraint
- ▶ For example: You have 20m of fence-material and you want to fence the largest possible rectangle area
- ▶ In this case the sides of the rectangle are a and b , thus we want to maximize $f(a, b) = ab$. But $2a + 2b = 20$ also has to hold.
- ▶ To solve such problems we will use a function called the Lagrangian

$$\frac{\partial \mathcal{L}}{\partial x} = f'_x(x, y) - \lambda g'_x(x, y) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = f'_y(x, y) - \lambda g'_y(x, y) = 0$$

We can rearrange them as:

$$\frac{f'_x(x, y)}{g'_x(x, y)} = \lambda$$

$$\frac{f'_y(x, y)}{g'_y(x, y)} = \lambda$$

How does this method work?

It means that at the solution:

$$\frac{f'_x(x, y)}{g'_x(x, y)} = \frac{f'_y(x, y)}{g'_y(x, y)}$$

- ▶ $f'_x(x, y)$ shows how much the objective function would increase if we managed to increase x marginally
- ▶ $g'_x(x, y)$ shows how much we would violate the constraint by raising x marginally
- ▶ Thus $\frac{f'_x(x, y)}{g'_x(x, y)}$ basically shows how much we can increase the objective function by violating the constraint marginally and increasing x .
- ▶ The same way $\frac{f'_y(x, y)}{g'_y(x, y)}$ shows how much we can increase the objective function by violating the constraint marginally and increasing y .
- ▶ Thus we basically make sure that the effect of a marginal increase in x on the value of the objective function is the same as the effect of a marginal increase in y

Example

Let's solve $f(x, y) = xy \rightarrow \max$ s.t. $x + y = 10$.

- ▶ The Lagrangian is $\mathcal{L} = xy - \lambda(x + y - 10)$
- ▶ The derivatives have to be zero:

$$\frac{\partial \mathcal{L}}{\partial x} = y - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + y - 10 = 0$$

- ▶ From the first two $x = y$, which implies $x = y = 5$.

1. $g(x, y) = x^2 y^4 \rightarrow \max$ s.t. $x + y = 9$
2. $f(x, y) = e^{xy} \rightarrow \max$ s.t. $x + y = 2$

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