Analyzing Quantum Systems Graphically

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A series of Hamiltonians are studied through their eigenvalues and eigenvectors. They are graphically analyzed to find the phase transitions between the lowest-energy states. Interactive plots are provided to the reader to allow exploration of the effect of the variables on the phase transitions of the system.

Keywords: Hamiltonian, Phase Transition

I. INTRODUCTION

A paper by K. Berrada, A. Sabik, and H. Eleuch[1] describes a model for two interacting qubits in a common dephasing environment. Using their model, we will explore the properties of this system and similar systems. By finding the eigenvalues and eigenvectors for the systems' Hamiltonians, we can graphically analyze the properties of the system. Using these graphs, we can find the lowest energy points for the system, which is the state that the system will stay in. The points at which the lowest-energy eigenvalue changes describe the phase transitions of the system. The interactive graphs and provided for the reader, so that they may explore how changing parameters can affect the phase transitions.

II. CLOSED QUANTUM SYSTEMS

First we will analyze a series of Hamiltonians describing closed quantum systems, which do not have a dissipation term. We use the basis $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$ in describing the state of the system.

The Hamiltonian for two interactive qubits given in the paper[1] is

$$\hat{H}_1 = \hbar\omega(\hat{S}_1^z \otimes \hat{I}_2) + \hbar\omega(\hat{I}_1 \otimes \hat{S}_2^z) + \hbar\lambda(\hat{S_1^-}\hat{S_2^+} + \hat{S_1^+}\hat{S_2^-})$$

where, in matrix form, $\hat{I} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\hat{S^z} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\hat{S^-} = \frac{1}{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$, and $\hat{S^+} = \frac{1}{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$. Also note that, for simplicity, we will be setting $\hbar = 1$. This is because we want to examine the phase changes, which will remain invariant under this change. Therefore we have

$$\hat{H}_1 = \omega(\hat{S}_1^z \otimes \hat{I}_2) + \omega(\hat{I}_1 \otimes \hat{S}_2^z) + \lambda(\hat{S_1^-}\hat{S_2^+} + \hat{S_1^+}\hat{S_2^-}) \quad (1)$$

In matrix form,

$$\hat{H}_1 = \begin{pmatrix}\omega & 0 & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & \lambda & 0 & 0\\ 0 & 0 & 0 & -\omega\end{pmatrix}$$

We can find the eigenvalues by solving $det(H - eI) = (e^2 - \omega^2)(e^2 - \lambda^2) = 0$, where e_i is the i^{th} eigenvalue. We find that the eigenvalues are

$$e_1 = \omega, e_2 = -\omega, e_3 = \lambda, e_4 = -\lambda$$

To find the eigenvectors, we solve $(H - e_i I)|v_i\rangle = |0\rangle$.

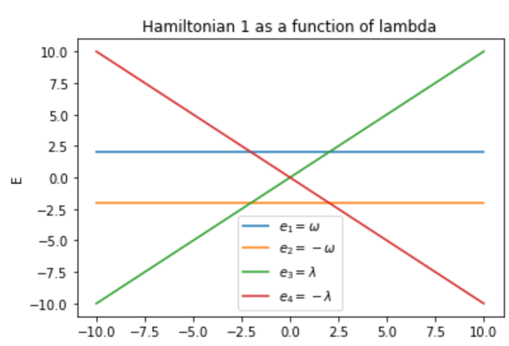
$$|e_1\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |e_2\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$|e_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, |e_4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

Which can also be written as

$$|e_1\rangle = \uparrow\uparrow, |e_2\rangle = \downarrow\downarrow, |e_3\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), |e_4\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

Plotting the eigenvalues as a function of λ will allow us to explore the lowest energy states of this system. For this particular graph, $\omega = 2$.



We can see that, as we increase λ, the lowest energy state goes from $|e_4\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ to $|e_2\rangle = \downarrow\downarrow$ to $|e_3\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$. The phase transitions occur at the points $\lambda = -\omega$ and $\lambda = \omega$. So if $\omega = 0$, then the

system will never be in the state $\downarrow\downarrow$.
The states $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$ and $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$ are completely non-polarized, with spin 0, and are therefore antiferromagnetic. $\downarrow\downarrow$, on the other hand, is fully polarized with spin 1, and is therefore ferromagnetic.
Now, we want to explore what happens when we add a term that defines interaction between the two qubits:

$$\hat{H}_2 = \hat{H}_1 + \mu(\hat{S}_1^z \otimes \hat{S}_2^z) \tag{2}$$

In matrix form,

$$\hat{H}_2 = \begin{pmatrix} \omega+\mu & 0 & 0 & 0 \\ 0 & -\mu & \lambda & 0 \\ 0 & \lambda & -\mu & 0 \\ 0 & 0 & 0 & -\omega+\mu \end{pmatrix}$$

The eigenvalues are

$$e_1 = \omega + \mu, e_2 = -\omega + \mu, e_3 = \lambda - \mu, e_4 = -\lambda - \mu$$

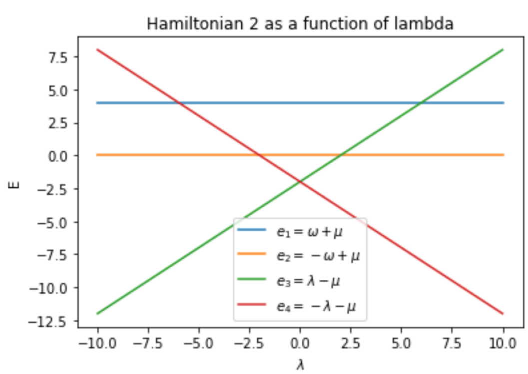
However, the eigenvectors remain unchanged:

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|e_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, |e_4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$|e_1\rangle = \uparrow\uparrow, |e_2\rangle = \downarrow\downarrow, |e_3\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), |e_4\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

In the following figure, once again we see the snapshot where $\omega = 2$.



The strength of the μ term affects the phase transitions of the system. A phase transition to $|e_2\rangle = \downarrow\downarrow$ occurs at $\lambda = -\omega + \mu$ and $\lambda = \omega + \mu$ only if $\omega > \mu$. Otherwise the system transitions directly from $|e_4\rangle$ to $|e_3\rangle$, as in the figure above.

Finally, we want to consider weighting the S^z operator on the two qubits differently. Our Hamiltonian will look as below:

$$\hat{H}_3 = \omega_1(\hat{S}_1^z \otimes \hat{I}_2) + \omega_2(\hat{I}_1 \otimes \hat{S}_2^z) + \lambda(\hat{S_1^-}\hat{S_2^+} + \hat{S_1^+}\hat{S_2^-}) \tag{3}$$

where ω_1 and ω_2 can be tweaked individually. Our Hamiltonian in matrix form is

$$\hat{H}_3 = \frac{1}{2}\begin{pmatrix} \omega_1+\omega_2 & 0 & 0 & 0 \\ 0 & -\omega_1+\omega_2 & 2\lambda & 0 \\ 0 & 2\lambda & \omega_1-\omega_2 & 0 \\ 0 & 0 & 0 & -\omega_1-\omega_2 \end{pmatrix}$$

Our eigenvalues becomes

$$e_1 = \frac{1}{2}(\omega_1+\omega_2), e_2 = -\frac{1}{2}(\omega_1+\omega_2)$$

$$e_3 = \frac{1}{2}\sqrt{(\omega_1-\omega_2)^2 + 4\lambda^2}, e_4 = -\frac{1}{2}\sqrt{(\omega_1-\omega_2)^2 + 4\lambda^2}$$

And our eigenvectors are

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|e_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -\frac{\omega_1-\omega_2-\sqrt{(\omega_1-\omega_2)^2+4\lambda^2}}{2\lambda} \\ 1 \\ 0 \end{pmatrix}$$

$$|e_4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -\frac{\omega_1-\omega_2+\sqrt{(\omega_1-\omega_2)^2+4\lambda^2}}{2\lambda} \\ 1 \\ 0 \end{pmatrix}$$

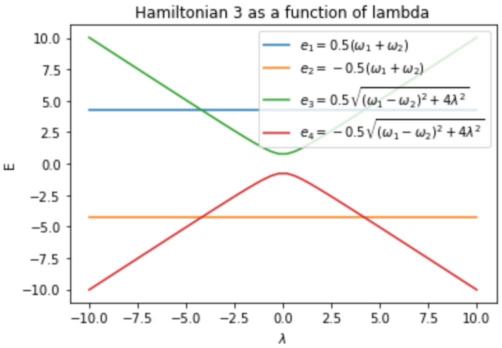
Which can be written as

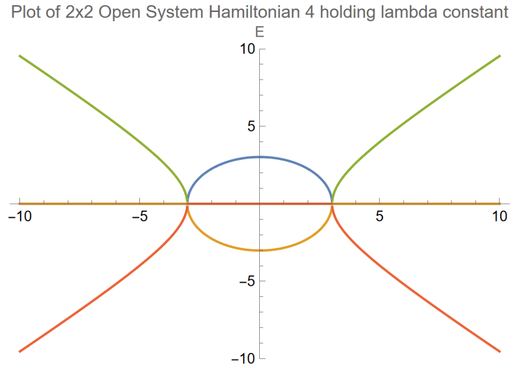
$$|e_1\rangle = \uparrow\uparrow, |e_2\rangle = \downarrow\downarrow$$

$$|e_3\rangle = \frac{1}{\sqrt{2}}(-\frac{\omega_1-\omega_2-\sqrt{(\omega_1-\omega_2)^2+4\lambda^2}}{2\lambda}\uparrow\downarrow + \downarrow\uparrow)$$

$$|e_4\rangle = \frac{1}{\sqrt{2}}(-\frac{\omega_1-\omega_2+\sqrt{(\omega_1-\omega_2)^2+4\lambda^2}}{2\lambda}\uparrow\downarrow - \downarrow\uparrow)$$

We can verify that the eigenvectors $|e_3\rangle, |e_4\rangle$ simplify to $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$, $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ when $\omega_1 = \omega_2$. The snapshot is shown for $\omega_1 = 5, \omega_2 = 3.5$.





$|e_3\rangle$ and $|e_4\rangle$ are no longer completely non-polarized. The values of ω_1 and ω_2 will affect both the location of the phase transitions, as they happen at

$$-\frac{1}{2}\sqrt{(\omega_1-\omega_2)^2+4\lambda^2} = -\frac{1}{2}(\omega_1+\omega_2)$$

$$\lambda = \pm\sqrt{\omega_1\omega_2}$$

III. OPEN QUANTUM SYSTEMS

We will now explore open quantum systems, which will have a dissipation term that describes their interaction with the environment.
First we have a simple 2x2 Hamiltonian.

$$\hat{H}_4 = \lambda\hat{\sigma_x} + i\gamma\hat{\sigma_z} \tag{4}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In matrix form,

$$\hat{H}_4 = \begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix}$$

The eigenvalues are

$$e_1 = \lambda\sqrt{1-\left(\frac{\gamma}{\lambda}\right)^2}, e_2 = -\lambda\sqrt{1-\left(\frac{\gamma}{\lambda}\right)^2}$$

And the eigenvectors are

$$|e_1\rangle = \begin{pmatrix} 1 \\ -\frac{i\gamma-\lambda\sqrt{1-(\frac{\gamma}{\lambda})^2}}{\lambda} \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 1 \\ -\frac{i\gamma+\lambda\sqrt{1-(\frac{\gamma}{\lambda})^2}}{\lambda} \end{pmatrix}$$

Which can also be written as

$$|e_1\rangle = \uparrow - \frac{i\gamma-\lambda\sqrt{1-(\frac{\gamma}{\lambda})^2}}{\lambda}\downarrow, |e_2\rangle = \uparrow - \frac{i\gamma+\lambda\sqrt{1-(\frac{\gamma}{\lambda})^2}}{\lambda}$$

In the following figure, the real part of e_1 is blue, the imaginary part of e_1 is green, the real part of e_2 is yellow, and the imaginary part of e_2 is red. The snapshot is shown for $\lambda = 3$.

The exceptional point occurs at $\gamma = \lambda$, which is where the system goes from being fully real to fully imaginary. We can analyze a larger system by taking the original $H_1 = \omega(\hat{S_1^z}\otimes\hat{I_2}) + \omega(\hat{I_1}\otimes\hat{S_2^z}) + \lambda(\hat{S_1^-}\hat{S_2^+} + \hat{S_1^+}\hat{S_2^-})$ and adding a dissipating term to both qubits.

$$\hat{H}_5 = \hat{H}_1 + i\gamma(\hat{S_1^z} + \hat{S_2^z}) \tag{5}$$

In matrix form,

$$\hat{H}_5 = \begin{pmatrix} \omega+i\gamma & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & -\omega-i\gamma \end{pmatrix}$$

The eigenvalues are

$$e_1 = \omega + i\gamma, e_2 = -\omega - i\gamma, e_3 = \lambda, e_4 = -\lambda$$

And the eigenvectors are

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

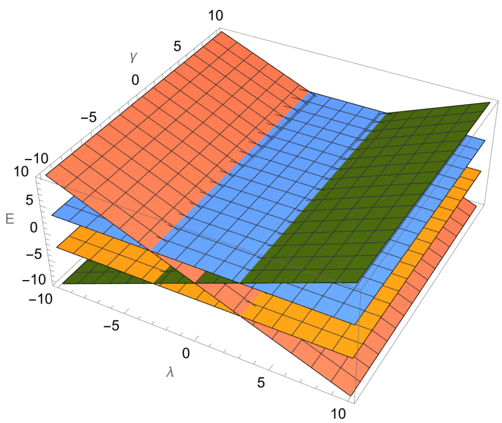
$$|e_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, |e_4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Once again, the eigenvectors are the same as for the original Hamiltonian, but the eigenvalues are affected by the value of the dissipating term.

$$|e_1\rangle = \uparrow\uparrow, |e_2\rangle = \downarrow\downarrow, |e_3\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), |e_4\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

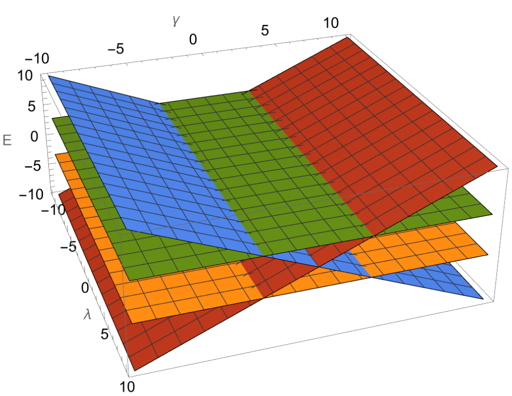
The following two figures are shown for the snapshot $\omega = 3$. The first figure depicts the real part of all the eigenvalues of the system. e_1 is depicted in blue, e_2 in yellow, e_3 in green, and e_4 in red. The second figure shows both the real and imaginary parts of e_1 and e_2. The real part of e_1 is shown in green, the imaginary part of e_1 is shown in red, the real part of e_2 is yellow, and the imaginary part of e_2 is red.

Plot of 4x4 Open System Hamiltonian 5 holding omega constant



We can see that the behavior of the real eigenvalues mimics the original Hamiltonian H_1, where we have apparent phase transitions between e_4, e_2, and e_3. However we must inspect the imaginary parts of the eigenvalues for the whole picture.

Plot of 4x4 Open System Hamiltonian 5 holding omega constant



Here we see that the imaginary parts of e_1 and e_2 may be the lowest energy state for some combination of γ and λ. Therefore the phase transitions occur between eigenstates along the lines $|\gamma| = |\lambda|$ and $|\gamma| = -|\lambda|$ as well as wherever $-\omega < \lambda$ and $-\omega < \gamma$. The system will either be in a fully complex eigenstate e_1 or e_2 or in the fully real eigenstate e_3 or e_4.
Finally, we will examine the following Hamiltonian.

$$\hat{H_6} = \lambda\hat{\sigma_x} + (\omega + i\gamma)\hat{\sigma_z} \tag{6}$$

In matrix form

$$\bar{H_6} = \begin{pmatrix} \omega + i\gamma & \lambda \\ \lambda & -\omega - i\gamma \end{pmatrix}$$

The eigenvalues are

$$e_1 = \sqrt{\lambda^2 + \omega^2 + 2i\gamma\omega - \gamma^2}, e_2 = -\sqrt{\lambda^2 + \omega^2 + 2i\gamma\omega - \gamma^2}$$

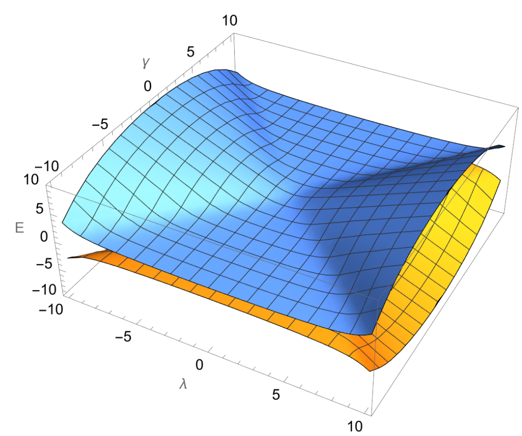
And the eigenvectors are

$$|e_1\rangle = \begin{pmatrix} -\frac{-i\gamma-\omega-\sqrt{\lambda^2+\omega^2+2i\gamma\omega-\gamma^2}}{\lambda} \\ 1 \end{pmatrix}$$

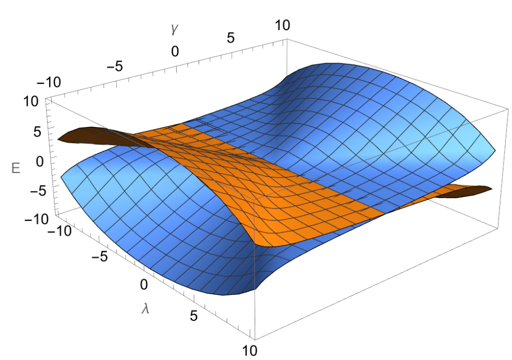
$$|e_2\rangle = \begin{pmatrix} -\frac{-i\gamma-\omega+\sqrt{\lambda^2+\omega^2+2i\gamma\omega-\gamma^2}}{\lambda} \\ 1 \end{pmatrix}$$

We will examine the real and imaginary parts of these eigenvalues in the following respective figures. In both figures, e_1 is depicted in blue and e_2 is depicted in yellow.

Real Plot of 4x4 Open System Hamiltonian 6, omega constant



Imaginary Plot of 4x4 Open System Hamiltonian 6, omega constant



Note that we can write the eigenvectors in the following form:

$$|e_1\rangle = -\frac{-i\gamma - \omega - \sqrt{\lambda^2 + \omega^2 + 2i\gamma\omega - \gamma^2}}{\lambda} \uparrow + \downarrow$$

$$|e_2\rangle = -\frac{-i\gamma - \omega + \sqrt{\lambda^2 + \omega^2 + 2i\gamma\omega - \gamma^2}}{\lambda} \uparrow + \downarrow$$

Once again we see that these states are not completely non-polarized.
The phase transitions and exceptional points are less clear in this case. The reader is encouraged to explore this system's graphical representation, superimposing the graphs and examining the lowest-energy points.

IV. CONCLUSIONS

The phase transitions and exceptional points for described for a number of closed and open quantum systems. Using the graphical representation of the eigenvalues of the Hamiltonians, we demonstrated a method for intuitively understanding the properties of a quantum system in an interactive manner.

CODE AVAILABILITY

Code can be found at `https://github.com/Agnes-Kov/quantum_systems/`.

[1] K. Berrada, A. Sabik, and H. Eleuch, Quantum coherence and nonlocality of two qubits in the presence of a common dephasing environment, Results in Physics **51**, 106666 (2023).