Analysis of the Instability of Free Rotation for Rigid Bodies

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The Dzhanibekov effect, otherwise known as the Tennis Racket theorem or Intermediate Axis theorem, describes the unstable rotation of a free rigid body rotating around its intermediate axis. This system will be modelled mathematically to show the origin of this instability with a phase sphere visually describing the orbits of rotation around each principal axis, as well as how this instability disappears when the moments of inertia along each principal axis are not unique.

Keywords: Dzhanibekov effect, Tennis Racket theorem, Intermediate Axis theorem, Free Rigid Body, Unstable Rotation

I. INTRODUCTION

Throw a three-dimensional object into the air and you may find that it rotates stably around two of its axes, but seems to tumble chaotically about the third. This phenomenon was first discovered by astronaut Vladimir Dzhanibekov who observed the unstable tumbling of a wingnut rotating aboard a spacecraft [1]. As such, it is refered to as the Dzhanibekov Effect. The effect is such that a rigid body rotating through free space, where no external torques are acting on the body, will experience unstable rotation around its intermediate axis [2]. The characteristic equations describing the trajectories of the rotation of this body about its axes can be analyzed to determine stability, or lack thereof, around each principal axis [3].

To gain a more intuitive understanding of the rotation of such a rigid body, take some object, such as a phone or a book, and toss it about its principal axes. The object will rotate smoothly about its shortest and longest sides, but tumble chaotically along its intermediate length side. This is the intermediate axis theorem at work. This phenomenon can likewise be observed quantitatively by experiment [4].

II. MODELLING THE PROBLEM

Note that the following derivation follows the steps laid out by Colley in her paper "The Tumbling Box" [3].

Considering the rigid body in both an inertial frame of reference and a rotating one, the time rate of change of some arbitrary vector $\vec{A}$ in the inertial frame can be described by the following relation, where $\vec{\omega}$ is the angular velocity with which the rotating frame of reference is rotating in relation to the inertial frame:

$$(\frac{d\vec{A}}{dt})_{inertial} = (\frac{d\vec{A}}{dt})_{rotating} + \vec{\omega} \times \vec{A} \tag{1}$$

Taking the vector $\vec{A}$ to be the angular momentum, $\vec{L}$, the equation can be simplified by considering the rigid body to be spinning such that it has a constant angular momentum, so that $(\frac{d\vec{L}}{dt})_{inertial} = 0$. As such,

$$(\frac{d\vec{L}}{dt})_{rotating} = \vec{L} \times \vec{\omega} \tag{2}$$

Assuming henceforth that all measurements are made in the rotating frame of reference, the subscript "rotating" will be implied but not shown. Then, the angular momentum is defined as $\vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, where ω_i is the i^{th} component of angular velocity, and I_i is the momentum of inertia around the i^{th} principal axis. Preliminary, the relation between the moments of inertia will be such that $I_1 > I_2 > I_3 > 0$, but this restriction will be overturned in the latter part of the paper.

Given this definition, the system of equations becomes

$$\begin{aligned} \dot{L_1} &= (I_2 - I_3)\omega_2\omega_3 \\ \dot{L_2} &= (I_3 - I_1)\omega_1\omega_3 \\ \dot{L_3} &= (I_2 - I_2)\omega_1\omega_2 \end{aligned} \tag{3}$$

Taking $\omega_i = \frac{L_i}{I_i}$,

$$\begin{aligned} \dot{L_1} &= (\frac{1}{I_3} - \frac{1}{I_2})L_2L_3 \\ \dot{L_2} &= (\frac{1}{I_1} - \frac{1}{I_3})L_1L_3 \\ \dot{L_3} &= (\frac{1}{I_2} - \frac{1}{I_1})L_1L_2 \end{aligned} \tag{4}$$

It can be seen by inspection that this system has isolated critical points at the momenta coordinates $(\pm C, 0, 0), (0, \pm C, 0)$, and $(0, 0, \pm C)$, where C is some constant. For example, the point $(\pm C, 0, 0)$ satisfies each of the above equations:

$$\begin{aligned} \dot{C} &= 0 \\ \dot{0} &= 0 \\ \dot{0} &= 0 \end{aligned}$$

The equivalency $L_1\dot{L_1} + L_2\dot{L_2} + L_3\dot{L_3} = 0$ can also be easily verified by this system of equations. Integrating this equivalency gives

$$L_1{}^2 + L_2{}^2 + L_3{}^2 = C \tag{5}$$

Where C is some constant.

Given equation 5, it can be determined that the trajectories described by equation 4 must lie on a sphere centered at the origin. Choosing C to be 1, this reduces the problem to finding solutions on a phase sphere of radius 1.

III. FINDING SOLUTIONS ON A PHASE SPHERE

Choosing C to be 1 determines the critical points of equation 4 to be $(\pm 1, 0, 0), (0, \pm 1, 0)$, and $(0, 0, \pm 1)$. These critical points represent points along the principal axes of the rigid body. While equation 4 will be used to numerically analyze the system using MATLAB later in the paper, an approximation is necessary to analyze the equations mathematically. As, along a principal axis, the angular momentum component along that axis is constant, equation 4 can be approximated to the following linear systems near each critical point using Taylor's theorem:

$$\begin{gathered} near(\pm 1, 0, 0): \\ \dot{L_1} = 0 \\ \dot{L_2} = \pm(\frac{1}{I_1} - \frac{1}{I_3})L_3 \\ \dot{L_3} = \pm(\frac{1}{I_2} - \frac{1}{I_1})L_2 \end{gathered}$$

$$\begin{gathered} near(0, \pm 1, 0): \\ \dot{L_1} = \pm(\frac{1}{I_3} - \frac{1}{I_2})L_3 \\ \dot{L_2} = 0 \\ \dot{L_3} = \pm(\frac{1}{I_2} - \frac{1}{I_1})L_1 \end{gathered}$$

$$\begin{gathered} near(0, 0, \pm 1): \\ \dot{L_1} = \pm(\frac{1}{I_3} - \frac{1}{I_2})L_2 \\ \dot{L_2} = \pm(\frac{1}{I_1} - \frac{1}{I_3})L_1 \\ \dot{L_3} = 0 \end{gathered}$$

Since, in each equation, one of L_1, L_2, L_3 is constant to the first order, we can ignore this constant variable to obtain a two-dimensional linear system near each critical point.

With this approximation, the characteristic equation near each critical point can be determined.

Near $(\pm 1, 0, 0)$, the characteristic equation is $x^2 - \alpha = 0$, where

$$\alpha = (1/I_1 - 1/I_3)(1/I_2 - 1/I_1) < 0 \tag{6}$$

Near $(0, 0, \pm 1)$, the characteristic equation is $x^2 - \beta = 0$, where

$$\beta = (1/I_1 - 1/I_3)(1/I_3 - 1/I_2) < 0 \tag{7}$$

And near $(0, \pm 1, 0)$, the characteristic equation is $x^2 - \gamma = 0$, where

$$\gamma = (1/I_2 - 1/I_1)(1/I_3 - 1/I_2) > 0 \tag{8}$$

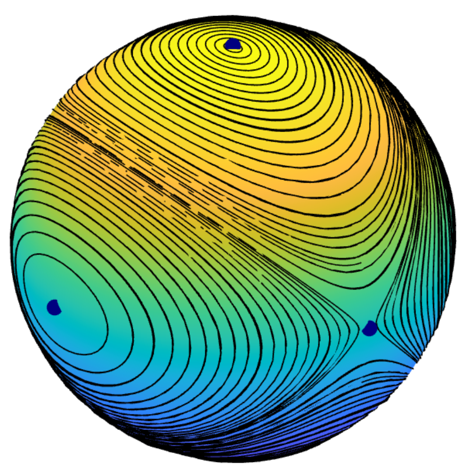
As the variables α and β are negative given the relationship between the moments of inertia, where $I_1 > I_2 > I_3 > 0$, the corresponding characteristic equations' eigenvalues are purely imaginary, and thus the critical points are centers. As such, the rotation around the axes that correspond to these critical points will be stable.

In contrast, the variable γ is positive, and thus the eigenvalues of the characteristic equation are real and of opposite sign, meaning the points $(0, \pm 1, 0)$ are saddle points, and rotation around the corresponding axis will be unstable.

IV. DETERMINING THE STABILITY OF THE CRITICAL POINTS

To visually confirm the stability and instability of trajectories near the critical points, the trajectories of the rigid body's motion described by equation 4, which lie on the phase sphere, will be plotted using MATLAB.

The moments of inertia are set to $I_1 = 1$, $I_2 = 1/2$, $I_3 = 1/3$ in the program, to preserve the relation $I_1 > I_2 > I_3 > 0$ first set in this paper.



Taking z to be in the positive upwards direction, the stable shortest and longest sides can thus be seen to be represented by the x and z direction, while the intermediate unstable side is represented by the y direction. As can be seen in the visual, rotation around either the shortest or longest side will result in stable rotation as those trajectories around those axes follow closed curves. This means that any deviation off axis will not have significant effects on the path of rotation of the object. However, rotation around the intermediate axis will be unstable because of the asymptotic nature of the trajectories off the y axis. Any slight perturbation off axis will cause the rotation of the object to temporarily follow the stable path along one of the stable axes, resulting in a flip over that stable axis during the rotation about the unstable axis. This is what causes the apparent tumbling of the object when it is flipped over the unstable intermediate axis.

If the object were flipped perfectly along the intermediate axis, it would follow the path that the stable trajectories approach asymptotically. But in the real world, where humans cannot flip perfectly and the slightest interaction with air molecules would deviate the rotation's path, such a perfect rotation is impossible, and thus the chaotic tumbling along the intermediate axis is always observed.

V. VARIATION OF THE PROBLEM SETUP

Consider the situation where the moments of inertia along the principal axes are not unique. If, for example, $I_1 = I_2$ equal some I, where $I > I_3 > 0$, equation 4 simplifies to

$$\dot{L_1} = (\frac{1}{I_3} - \frac{1}{I})L_2 L_3$$
$$\dot{L_2} = (\frac{1}{I} - \frac{1}{I_3})L_1 L_3$$
$$\dot{L_3} = 0$$

As L_3 is constant, the equations can be simplified to

$$\begin{aligned} \dot{L_1} &= \Omega L_2 \\ \dot{L_2} &= -\Omega L_1 \end{aligned} \tag{9}$$

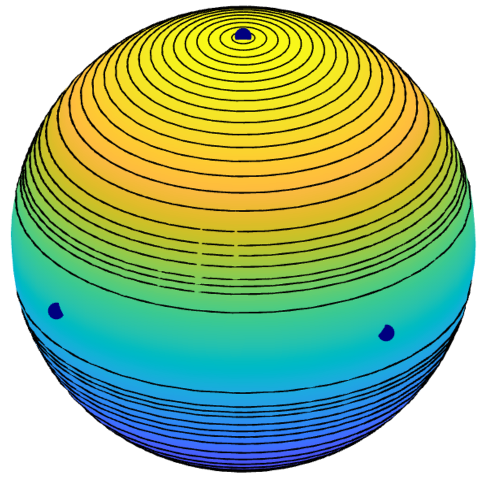
where $\Omega = (\frac{1}{I_3} - \frac{1}{I})L_3 = (\frac{1}{I_3} - \frac{1}{I})I_3\omega_3 = (1 - \frac{I_3}{I})\omega_3 = (I - I_3)\omega_3/I$, which represents the fixed angular velocity of precession Ω.

The analytical solution of the system is

$$L_1(t) = L \sin \Omega t$$
$$L_2(t) = L \cos \Omega t$$
$$L_3(t) = L_3 = constant$$

which represents stable, periodic rotation. As such, if the moments of inertia along the principal axes are not unique, there will not be any unstable rotation observed for the rigid body in free space.

This can be further confirmed by creating a phase sphere such that $I_1 = I_2 = 1$, $I_3 = 1/3$, generating the following figure:



As the trajectories of rotation are not asymptotically approaching any of the principal axes, the rotation will thus be stable along any of these axes. In addition, the trajectories match the analytic solution of the system, which describes circular trajectories at constant heights in the z direction.

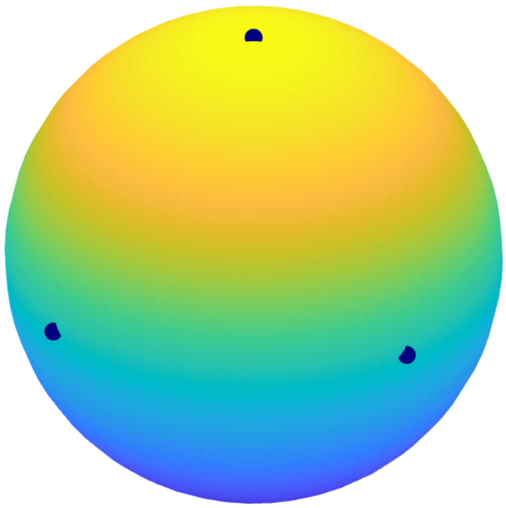
The situation where all three moments of inertia are equivalent gives a trivial solution. In this set-up,

$$\begin{aligned} \dot{L_1} &= 0 \\ \dot{L_2} &= 0 \\ \dot{L_3} &= 0 \end{aligned} \tag{10}$$

And thus

$$\begin{aligned} L_1 &= C_1 \\ L_2 &= C_2 \\ L_3 &= C_3 \end{aligned} \tag{11}$$

Where C_1, C_2, C_3 are all constants. This similarly represents stable rotation. The phase sphere, where $I_1 = I_2 = I_3 = 1$, is shown below.



Because the angular momentum along each principal axis is 0, there are no trajectories that appear on the phase sphere. As a consequence, the lack of asymptotic trajectories confirms the stability of rotation along any principal axis in this situation.

VI. CONCLUSIONS

Understanding the rotation of such bodies is imperative in order to create better models and predictions. For one such instance, the stability of rotation is important in the prediction of rockfall, a specific type of dangerous landslide. Correctly predicting the path of and trajectories of these rockfalls can improve preparations for this dangerous natural phenomena [5].

The rotation of a rigid body rotating in space, free of any external torques, is determined by the relationship between the moments of inertia of the principal axes. If these three moments of inertia are all distinct from one another, the body will experience stable rotation only around the shortest and the longest axes of rotation. When flipped about the intermediate axis, the body will rotate unstably, wobbling between rotating about the longer axis and shorter one. This instability, however, disappears if the moments of inertia are made non-unique. In this case, rotation about each axis will be stable, as there are not two distinct axes of rotation that the third axis can switch between.

CODE AVAILABILITY

Code can be found at `https://github.com/Agnes-Kov/unstable_rotation`.

Code from Cleve Moler's tumbling_box.m was used as source code for this project [6]. Using this code as a base, set initial conditions were chosen and inserted, as well as a variable to determine the relationship between the moments of inertia.

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