This fieldstone was developed in collaboration with Lukas van de Wiel.

The domain is an annulus with inner radius R_1 and outer radius R_2. It is filled with a single elastic material characterised by a Young's modulus E and a Poisson ratio ν, a density ρ_0. The gravity $\boldsymbol{g} = -g_0 \boldsymbol{e}_r$ is pointing towards the center of the domain.

The problem at hand is axisymmetric so that the tangential component of the displacement vector v_θ is assumed to be zero as well as all terms containing ∂_θ. The components of the strain tensor are

$$\varepsilon_{rr} = \frac{\partial v_r}{\partial r} \tag{1}$$

$$\varepsilon_{\theta\theta} = \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = \frac{v_r}{r} \tag{2}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) = 0 \tag{3}$$

so that the tensor simply is

$$\boldsymbol{\varepsilon} = \left(\begin{array}{cc} \varepsilon_{rr} & \varepsilon_{r\theta} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} \end{array}\right) = \left(\begin{array}{cc} \frac{\partial v_r}{\partial r} & 0 \\ 0 & \frac{v_r}{r} \end{array}\right) \tag{4}$$

The pressure is

$$p = -\lambda \boldsymbol{\nabla}\cdot\boldsymbol{v} = -\lambda\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right) \tag{5}$$

and finally the stress tensor:

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\boldsymbol{\varepsilon} = \left(\begin{array}{cc} \lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial v_r}{\partial r} & 0 \\ 0 & \lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{v_r}{r} \end{array}\right) \tag{6}$$

The divergence of the stress tensor is given by [**?**]:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \left(\begin{array}{c} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} \\ \\ \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\sigma_{\theta\theta}}{\partial\theta} + \frac{\sigma_{r\theta}+\sigma_{\theta r}}{r} \end{array}\right) \tag{7}$$

Given the diagonal nature of the stress tensor this simplifies to (also remember that $\partial_\theta = 0$):

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \left(\begin{array}{c} \frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} \\ \\ 0 \end{array}\right) \tag{8}$$

Focusing on the r-component of the stress divergence:

$$\begin{aligned}
(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})_r &= \frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} && (9)\\
&= \frac{\partial}{\partial r}\left[\lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial v_r}{\partial r}\right] + \frac{1}{r}\left[\lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial v_r}{\partial r} - \lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} - 2\mu\frac{v_r}{r}\right] && (10)\\
&= \lambda\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial^2 v_r}{\partial r^2} + \lambda\frac{1}{r^2}\frac{\partial(rv_r)}{\partial r} + \frac{2\mu}{r}\frac{\partial v_r}{\partial r} - \lambda\frac{1}{r^2}\frac{\partial(rv_r)}{\partial r} - \frac{2\mu v_r}{r^2} && (11)\\
&= \lambda(-\frac{v_r}{r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial r^2}) + 2\mu\frac{\partial^2 v_r}{\partial r^2} + \frac{2\mu}{r}\frac{\partial v_r}{\partial r} - \frac{2\mu v_r}{r^2} && (12)\\
&= -(2\mu+\lambda)\frac{v_r}{r^2} + (2\mu+\lambda)\frac{1}{r}\frac{\partial v_r}{\partial r} + (2\mu+\lambda)\frac{\partial^2 v_r}{\partial r^2} && (13)
\end{aligned}$$

So the momentum conservation in the r direction is

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho_0\boldsymbol{g})_r = -(2\mu+\lambda)\frac{v_r}{r^2} + (2\mu+\lambda)\frac{1}{r}\frac{\partial v_r}{\partial r} + (2\mu+\lambda)\frac{\partial^2 v_r}{\partial r^2} - \rho_0 g_0 = 0 \tag{14}$$

or,

$$\boxed{\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} = \frac{\rho_0 g_0}{\lambda+2\mu}} \tag{15}$$

We now look at the boundary conditions. On the inner boundary we prescribe $v_r(r = R_1) = 0$ while free surface boundary conditions are prescribed on the outer boundary, i.e. $\boldsymbol{\sigma}\cdot\boldsymbol{n} = 0$ (i.e. there is no force applied on the surface).

The general form of the solution can then be obtained:

$$v_r(r) = C_1 r^2 + C_2 r + \frac{C_3}{r} \tag{16}$$

with

$$C_1 = \frac{\rho_0 g_0}{3(\lambda + 2\mu)} \qquad C_2 = -C_1 R_1 - \frac{C_3}{R_1^2} \qquad C_3 = \frac{k_1 + k_2}{(R_1^2 + R_2^2)(2\mu + \lambda) + (R_2^2 - R_1^2)\lambda} \tag{17}$$

and

$$k_1 = (2\mu + \lambda) C_1 (2 R_1^2 R_2^3 - R_1^3 R_2^2) \qquad k_2 = \lambda C_1 (R_1^2 R_2^3 - R_1^3 R_2^2) \tag{18}$$

Pressure can then be computed as follows:

$$p = -\lambda \boldsymbol{\nabla} \cdot \boldsymbol{v} = -\lambda \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) = -\lambda \left(\frac{1}{r} (3 C_1 r^2 + 2 C_2 r) \right) = -\lambda (3 C_1 r + 2 C_2) \tag{19}$$

We choose $R_1 = 2890\text{km}$, $R_2 = 6371\text{km}$, $g_0 = 9.81\text{ms}^{-2}$, $\rho_0 = 3300$, $E = 6 \cdot 10^{10}$, $\nu = 0.49$.

radial profiles of the displacement and pressure fields

displacement and pressure fields in the domain