The Drucker-Prager yield function is given by the function f:

$$f = p\sin\phi + c\cos\phi - \tau$$

where τ is the square root of the second invariant of the deviatoric stress. We have

$$p = \frac{1}{2}(\sigma_1 + \sigma_3)$$

and

$$\tau = \frac{1}{2}(\sigma_1 - \sigma_3)$$

Inserting these into f yields:

$$f = \frac{1}{2}(\sigma_1 + \sigma_3)\sin\phi + c\cos\phi - \frac{1}{2}(\sigma_1 - \sigma_3)$$

The yield condition $f = 0$ can be reworked as follows:

$$\sigma_1 - \frac{1+\sin\phi}{1-\sin\phi}\sigma_3 - 2\frac{\cos\phi}{1-\sin\phi}c = 0$$

The third term can further be modified as follows:

$$\frac{\cos\phi}{1-\sin\phi} = \frac{\sqrt{1-\sin^2\phi}}{\sqrt{(1-\sin\phi)^2}} = \frac{\sqrt{(1-\sin\phi)(1+\sin\phi)}}{\sqrt{(1-\sin\phi)^2}} = \sqrt{\frac{1+\sin\phi}{1-\sin\phi}}$$

Finally, we define N_ϕ as follows

$$N_\phi = \frac{1+\sin\phi}{1-\sin\phi}$$

so that the yield condition becomes:

$$\sigma_1 - N_\phi\sigma_3 - 2\sqrt{N_\phi}c = 0$$

which is Eq. 3 of the article by Choi & Petersen [**?**].

This paper offers a solution to the problem of the angle of shear bands in geodynamic models. The underlying idea is based on simple modifications brought to existing incompressible flow codes. Note that the codes featured in that paper also implemented elastic behaviour but this can be easily switched off by setting $Z = 1$ in their equations.

Their plasticity implementation starts with a modification of the continuity equation:

$$\vec{\nabla}\cdot\vec{\text{v}} = R = 2\sin\psi\,\dot{\varepsilon}_p$$

where R is the dilation rate, Ψ is the dilation angle and $\dot{\varepsilon}_p$ is the square root of the second invariant of the plastic strain rate.

Under this assumption, the deviatoric strain rate tensor is given by

$$\dot{\boldsymbol{\varepsilon}}^d(\vec{\text{v}}) = \dot{\boldsymbol{\varepsilon}}(\vec{\text{v}}) - \frac{1}{3}Tr[\dot{\boldsymbol{\varepsilon}}(\vec{\text{v}})]\mathbf{1} = \dot{\boldsymbol{\varepsilon}}(\vec{\text{v}}) - \frac{1}{3}\vec{\nabla}\cdot\vec{\text{v}}\,\mathbf{1} = \dot{\boldsymbol{\varepsilon}}(\vec{\text{v}}) - \frac{1}{3}R\,\mathbf{1}$$

Turning now to the momentum conservation equation:

$$\begin{aligned}
-\vec{\nabla}p + \vec{\nabla}\cdot\boldsymbol{\tau} &= -\vec{\nabla}p + \vec{\nabla}\cdot(2\eta\dot{\boldsymbol{\varepsilon}}^d(\vec{\text{v}})) \\
&= -\vec{\nabla}p + \vec{\nabla}\cdot\left[2\eta\left(\dot{\boldsymbol{\varepsilon}}(\vec{\text{v}}) - \frac{1}{3}R\,\mathbf{1}\right)\right] \\
&= -\vec{\nabla}p + \vec{\nabla}\cdot(2\eta\dot{\boldsymbol{\varepsilon}}(\vec{\text{v}})) - \frac{2}{3}\vec{\nabla}(\eta R) \qquad (1)
\end{aligned}$$

The last term is then an addition to the right hand side of the momentum equation and its weak form is as follows:

$$\vec{f}' = \int_\Omega N_v\frac{2}{3}\vec{\nabla}(\eta R)dV = \frac{4}{3}\sin\Psi\int_\Omega N_v\vec{\nabla}(\eta\dot{\varepsilon}_p)dV$$

This formulation proves to be problematic since in order to compute the gradient, we would need the viscosity and the plastic strain rate on the mesh nodes and both these quantities are effectively computed on the quadrature points. One option could be to project those quadrature values onto the nodes, which may introduce interpolation errors/artefacts and/or smoothing. Another option is to resort to integration by parts:

$$\int_\Omega N_v\vec{\nabla}(\eta\dot{\varepsilon}_p)dV = [N_v\eta\dot{\varepsilon}_p]_\Gamma - \int_\Omega \vec{\nabla}N_v(\eta\dot{\varepsilon}_p)dV$$

The last term is now trivial to compute since the shape function derivatives, the viscosity and the plastic strain rate are known at the quadrature points. Remains the surface term. We will neglect it for now to simplify our implementation and note that a) it will not directly affect what happens inside the domain, b) it could be somewhat important when shear bands intersect with the free surface.

$$\vec{f'} = -\frac{4}{3}\sin\psi \int_\Omega \vec{\nabla} N_v (\eta \dot{\varepsilon}_p) dV = -\frac{2}{3} \int_\Omega \vec{\nabla} N_v (\eta R) dV$$

Although the authors do indicate that they add a term in each rhs, it is not very clear how they deal with the implementation issue above. We then propose an alternative: instead of explicitely removing the deviatoric part of the strain rate as in Eq. 1 and replace the trace of the tensor by R, one could leave the term inside the matrix, thereby using a compressible form of the viscous block of the Stokes matrix. We will recover the same converged solution as before, but the path to convergence will be different that the first approach. In what follows, we denote the original approach by Choi & Petersen 'method 1' and the latter 'method 2'.

Finally, we need to define what the plastic strain rate tensor is. When using a rigid plastic rheology, the only deformation mechanism *is* plasticity so that the plastic strain rate *is* the strain rate. When using a visco-plastic rheology, the plastic strain rate is the strain rate of the zones above/at yield (the shear bands, where the vrm is active).

The setup is similar to the one in [**?**]. It is a 2D Cartesian domain filled with a single rigid-plastic material characterised by a cohesion $c = 10$MPa, an angle of friction ϕ, a dilation angle ψ and a density $\rho = 2800\text{kg/m}^3$. Extensional boundary conditions are as follows:

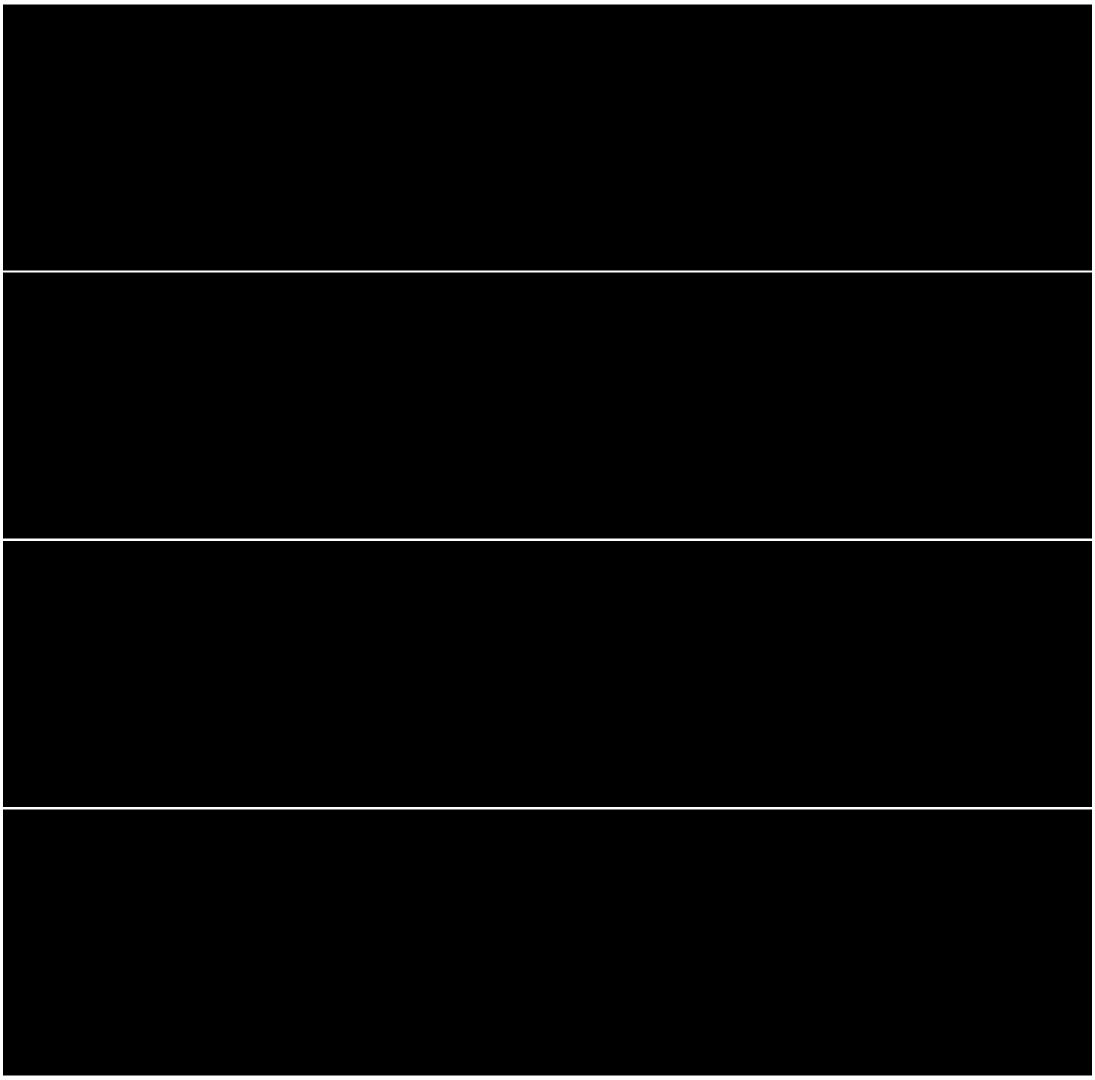
- left boundary: $u = -v_{bc}$;
- right boundary: $u = +v_{bc}$;
- bottom boundary: $v = 0$, $u = -v_{bc}$ for $x < L_x/2$, $u = +v_{bc}$ for $x > L_x/2$, and $u = 0$ if $x = L_x/2$;
- top boundary: zero traction.

For compressional boundary conditions the signs of all horizontal velocities should be reversed. The nonlinear tolerance is set to tol $= 10^{-6}$. Nonlinear iterations stop when maximum of the normalised nonlinear residual reaches the desired tolerance.

Following Choi & Petersen [**?**], we run the experiment with an associative ($\phi = \psi$) plasticity and a non associative one ($\psi = 0$, i.e. $R = 0$). This second approach is essentially what many codes do.

The velocity, pressure, strain rate, dilation rate, and velocity divergence are shown hereunder both in extension and compression.

Extension. 1st row: Non-associative plasticity; 2nd and 3rd row: associative plasticity ($\psi = \phi$) with method 1 for two resolutions 120x12 and 240; 4th and 5th row: associative plasticity ($\psi = \phi$) with method 2 for two re solutions 120x12 and 240



Compression. 1st row: Non-associative plasticity; 2nd and 3rd row: associative plasticity ($\psi = \phi$) with method 1 for two resolutions 120x12 and 240; 4th and 5th row: associative plasticity ($\psi = \phi$) with method 2 for two re solutions 120x12 and 240

One can also run the extension model for $\phi = \psi = 0, 5, 10, 15, 20, 25, 30^\circ$

Per row of elements, and per half of the domain (left and right) we find the element with the highest strain-rate and record their center coordinates. These elements are shown in the following figure for $\phi = \psi = \{0, 10, 20, 30\}^\circ$.

Measured shear bands

Note that benchmarking this in not easy. One solution Timo and I found was to add a velocity field $\underline{\vec{v}} = (x, y, z)$ (with $\vec{\nabla} \cdot \underline{\vec{v}} = 3$) to an existing analytical problem, e.g. the Burstedde benchamrk.