

This fieldstone was developed in collaboration with Lukas van de Wiel.

The domain is an annulus with inner radius R_1 and outer radius R_2 . It is filled with a single elastic material characterised by a Young's modulus E and a Poisson ratio ν , a density ρ_0 . The gravity $\mathbf{g} = -g_0 \mathbf{e}_r$ is pointing towards the center of the domain.

2μ is missing!

Given these assumptions, the momentum Stokes equations in the annulus are [?]

$$\mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) - \frac{\partial p}{\partial r} = \rho g_r \quad (1)$$

$$\mu \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \right) - \frac{1}{r} \frac{\partial p}{\partial \theta} = 0 \quad (2)$$

Equations (1) and (2) are the momentum equations in polar coordinates. The problem at hand is axisymmetric so that the tangential component of the displacement vector v_θ is assumed to be zero as well as all terms containing ∂_θ . Under these assumptions the second equation is automatically fulfilled. Of the first one remain the following terms:

$$\mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right) - \frac{\partial p}{\partial r} = \rho g_r$$

As we have seen before we have

$$p = -\lambda \nabla \cdot \mathbf{v} = -\lambda \left(\frac{1}{r} \frac{\partial(rv_r)}{\partial r} \right)$$

Inserting this expression into the first equation yields:

$$\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} = \frac{\rho g_0}{2\mu + \lambda}$$

We now look at the boundary conditions. On the inner boundary we prescribe $v_r(r = R_1) = 0$ while free surface boundary conditions are prescribed on the outer boundary, i.e. $\boldsymbol{\sigma} \cdot \mathbf{n} = 0$ (i.e. there is no force applied on the surface).

The components of the strain tensor are

$$\varepsilon_{rr} = \frac{\partial v_r}{\partial r} \quad (3)$$

$$\varepsilon_{\theta\theta} = \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = \frac{v_r}{r} \quad (4)$$

$$\varepsilon_{r\theta} = \frac{1}{2} \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) = 0 \quad (5)$$

The stress tensor is given by

$$\boldsymbol{\sigma} = -p \mathbf{1} + 2\mu \boldsymbol{\varepsilon} = \lambda (\nabla \cdot \mathbf{v}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}$$

which is a diagonal tensor so that the free surface condition is:

$$\sigma_{rr} = (2\mu + \lambda) \frac{\partial v_r}{\partial r} + \lambda \frac{v_r}{r} = 0 \quad r = R_2$$

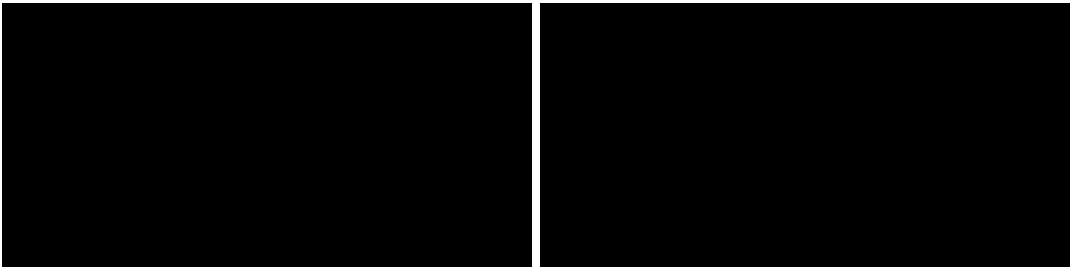
The solution is then

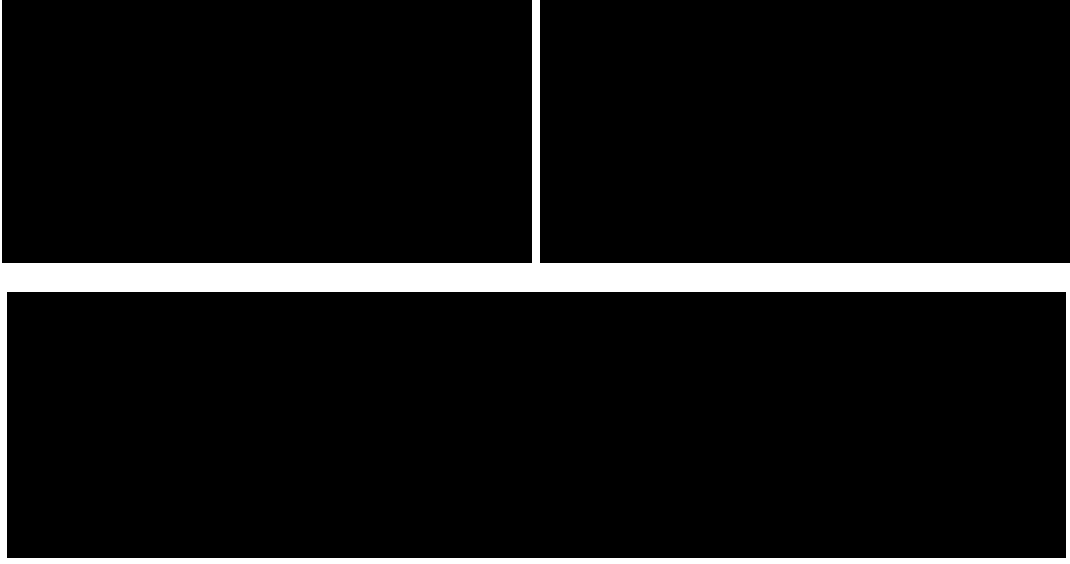
$$v_r(r) = ar^2 + br + \frac{c}{r} \quad \text{with} \quad a = \frac{\rho g_0}{3(2\mu + \lambda)}$$

where b, c are determined by means of the boundary conditions.

Pressure can be computed as follows:

$$p = -\lambda \nabla \cdot \mathbf{v} = -\lambda \left(\frac{1}{r} \frac{\partial(rv_r)}{\partial r} \right) = -\lambda \left(\frac{1}{r} (3ar^2 + 2br) \right) = -\lambda(3ar + 2b)$$





0.1 Alternative derivation of the solution

We start from the strain tensor:

$$\varepsilon = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_r}{\partial r} & 0 \\ 0 & \frac{v_r}{r} \end{pmatrix}$$

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\varepsilon = \begin{pmatrix} \lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial v_r}{\partial r} & 0 \\ 0 & \lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{v_r}{r} \end{pmatrix}$$

The divergence of the stress tensor is given by:

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\sigma_{r\theta} + \sigma_{\theta r}}{r} \end{pmatrix}$$

Since the stress tensor is diagonal this simplifies to (also $\partial_\theta = 0$):

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \\ 0 \end{pmatrix}$$

So

$$(\nabla \cdot \boldsymbol{\sigma})_r = \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \quad (6)$$

$$= \frac{\partial}{\partial r} \left[\lambda \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + 2\mu \frac{\partial v_r}{\partial r} \right] + \frac{1}{r} \left[\lambda \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + 2\mu \frac{\partial v_r}{\partial r} - \lambda \frac{1}{r} \frac{\partial(rv_r)}{\partial r} - 2\mu \frac{v_r}{r} \right] \quad (7)$$

$$= \lambda \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + 2\mu \frac{\partial^2 v_r}{\partial r^2} + \lambda \frac{1}{r^2} \frac{\partial(rv_r)}{\partial r} + \frac{2\mu}{r} \frac{\partial v_r}{\partial r} - \lambda \frac{1}{r^2} \frac{\partial(rv_r)}{\partial r} - \frac{2\mu v_r}{r^2} \quad (8)$$

$$= \lambda \left(-\frac{v_r}{r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial r^2} \right) + 2\mu \frac{\partial^2 v_r}{\partial r^2} + \frac{2\mu}{r} \frac{\partial v_r}{\partial r} - \frac{2\mu v_r}{r^2} \quad (9)$$

$$= -(2\mu + \lambda) \frac{v_r}{r^2} + (2\mu + \lambda) \frac{1}{r} \frac{\partial v_r}{\partial r} + (2\mu + \lambda) \frac{\partial^2 v_r}{\partial r^2} \quad (10)$$

So we then have

$$(\nabla \cdot \boldsymbol{\sigma})_r = -(2\mu + \lambda) \frac{v_r}{r^2} + (2\mu + \lambda) \frac{1}{r} \frac{\partial v_r}{\partial r} + (2\mu + \lambda) \frac{\partial^2 v_r}{\partial r^2} = \rho_0 g_r$$

or,

$$\boxed{\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} = \frac{\rho_0 g_r}{\lambda + 2\mu}}$$