When using $Q_1 \times P_0$ elements, this benchmark fails because of the Dirichlet b.c. on all 6 sides and all three components. However, as we will see, it does work well with $Q_2 \times Q_1$ elements. .

This benchmark begins by postulating a polynomial solution to the 3D Stokes equation [?]:

$$
v = \begin{pmatrix} x + x^2 + xy + x^3 y \\ y + xy + y^2 + x^2 y^2 \\ -2z - 3xz - 3yz - 5x^2 yz \end{pmatrix}
\tag{1}
$$

and

$$
p = xyz + x^3 y^3 z - 5/32
\tag{2}
$$

While it is then trivial to verify that this velocity field is divergence-free, the corresponding body force of the Stokes equation can be computed by inserting this solution into the momentum equation with a given viscosity μ (constant or position/velocity/strain rate dependent). The domain is a unit cube and velocity boundary conditions simply use Eq. (1). Following [?], the viscosity is given by the smoothly varying function

$$
\mu = \exp(1 - \beta(x(1 - x) + y(1 - y) + z(1 - z)))
\tag{3}
$$

One can easily show that the ratio of viscosities $\mu^\star$ in the system follows $\mu^\star = \exp(-3\beta/4)$ so that choosing $\beta = 10$ yields $\mu^\star \simeq 1808$ and $\beta = 20$ yields $\mu^\star \simeq 3.269 \times 10^6$.

We start from the momentum conservation equation:

$$
-\nabla p + \nabla \cdot (2\mu\dot{\varepsilon}) = f
$$

The x-component of this equation writes

$$
f_x = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(2\mu\dot{\varepsilon}_{xx}) + \frac{\partial}{\partial y}(2\mu\dot{\varepsilon}_{xy}) + \frac{\partial}{\partial z}(2\mu\dot{\varepsilon}_{xz})
\tag{4}
$$

$$
= -\frac{\partial p}{\partial x} + 2\mu\frac{\partial}{\partial x}\dot{\varepsilon}_{xx} + 2\mu\frac{\partial}{\partial y}\dot{\varepsilon}_{xy} + 2\mu\frac{\partial}{\partial z}\dot{\varepsilon}_{xz} + 2\frac{\partial\mu}{\partial x}\dot{\varepsilon}_{xx} + 2\frac{\partial\mu}{\partial y}\dot{\varepsilon}_{xy} + 2\frac{\partial\mu}{\partial z}\dot{\varepsilon}_{xz}
\tag{5}
$$

Let us compute all the block separately:

$$
\begin{aligned}
\dot{\varepsilon}_{xx} &= 1 + 2x + y + 3x^2 y \\
\dot{\varepsilon}_{yy} &= 1 + x + 2y + 2x^2 y \\
\dot{\varepsilon}_{zz} &= -2 - 3x - 3y - 5x^2 y \\
2\dot{\varepsilon}_{xy} &= (x + x^3) + (y + 2xy^2) = x + y + 2xy^2 + x^3 \\
2\dot{\varepsilon}_{xz} &= (0) + (-3z - 10xyz) = -3z - 10xyz \\
2\dot{\varepsilon}_{yz} &= (0) + (-3z - 5x^2 z) = -3z - 5x^2 z
\end{aligned}
$$

In passing, one can verify that $\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = 0$. We further have

$$
\begin{aligned}
\frac{\partial}{\partial x}2\dot{\varepsilon}_{xx} &= 2(2 + 6xy) \\
\frac{\partial}{\partial y}2\dot{\varepsilon}_{xy} &= 1 + 4xy \\
\frac{\partial}{\partial z}2\dot{\varepsilon}_{xz} &= -3 - 10xy \\
\frac{\partial}{\partial x}2\dot{\varepsilon}_{xy} &= 1 + 2y^2 + 3x^2 \\
\frac{\partial}{\partial y}2\dot{\varepsilon}_{yy} &= 2(2 + 2x^2) \\
\frac{\partial}{\partial z}2\dot{\varepsilon}_{yz} &= -3 - 5x^2 \\
\frac{\partial}{\partial x}2\dot{\varepsilon}_{xz} &= -10yz \\
\frac{\partial}{\partial y}2\dot{\varepsilon}_{yz} &= 0 \\
\frac{\partial}{\partial z}2\dot{\varepsilon}_{zz} &= 2(0)
\end{aligned}
$$

1

$$\frac{\partial p}{\partial x} = yz + 3x^2y^3z \tag{6}$$

$$\frac{\partial p}{\partial y} = xz + 3x^3y^2z \tag{7}$$

$$\frac{\partial p}{\partial z} = xy + x^3y^3 \tag{8}$$

Pressure normalisation Here again, because Dirichlet boundary conditions are prescribed on all sides the pressure is known up to an arbitrary constant. This constant can be determined by (arbitrarily) choosing to normalised the pressure field as follows:

$$\int_\Omega p \, d\Omega = 0 \tag{9}$$

This is a single constraint associated to a single Lagrange multiplier λ and the global Stokes system takes the form

$$\begin{pmatrix} \mathbb{K} & \mathbb{G} & 0 \\ \mathbb{G}^T & 0 & \mathcal{C} \\ 0 & \mathcal{C}^T & 0 \end{pmatrix} \begin{pmatrix} V \\ P \\ \lambda \end{pmatrix}$$

In this particular case the constraint matrix $\mathcal{C}$ is a vector and it only acts on the pressure degrees of freedom because of Eq.(9). Its exact expression is as follows:

$$\int_\Omega p \, d\Omega = \sum_e \int_{\Omega_e} p \, d\Omega = \sum_e \int_{\Omega_e} \sum_i N_i^p p_i \, d\Omega = \sum_e \sum_i \left(\int_{\Omega_e} N_i^p \, d\Omega \right) p_i = \sum_e \mathcal{C}_e \cdot \boldsymbol{p}_e$$

where $\boldsymbol{p}_e$ is the list of pressure dofs of element e. The elemental constraint vector contains the corresponding pressure basis functions integrated over the element. These elemental constraints are then assembled into the vector $\mathcal{C}$.

0.0.1 Constant viscosity

Choosing $\beta = 0$ yields a constant velocity $\mu(x, y, z) = \exp(1) \simeq 2.718$ (and greatly simplifies the right-hand side) so that

$$\frac{\partial}{\partial x}\mu(x, y, z) = 0 \tag{10}$$

$$\frac{\partial}{\partial y}\mu(x, y, z) = 0 \tag{11}$$

$$\frac{\partial}{\partial z}\mu(x, y, z) = 0 \tag{12}$$

and

$$\begin{aligned}
f_x &= -\frac{\partial p}{\partial x} + 2\mu\frac{\partial}{\partial x}\dot{\epsilon}_{xx} + 2\mu\frac{\partial}{\partial y}\dot{\epsilon}_{xy} + 2\mu\frac{\partial}{\partial z}\dot{\epsilon}_{xz} \\
&= -(yz + 3x^2y^3z) + 2(2 + 6xy) + (1 + 4xy) + (-3 - 10xy) \\
&= -(yz + 3x^2y^3z) + \mu(2 + 6xy) \\
f_y &= -\frac{\partial p}{\partial y} + 2\mu\frac{\partial}{\partial x}\dot{\epsilon}_{xy} + 2\mu\frac{\partial}{\partial y}\dot{\epsilon}_{yy} + 2\mu\frac{\partial}{\partial z}\dot{\epsilon}_{yz} \\
&= -(xz + 3x^3y^2z) + \mu(1 + 2y^2 + 3x^2) + \mu 2(2 + 2x^2) + \mu(-3 - 5x^2) \\
&= -(xz + 3x^3y^2z) + \mu(2 + 2x^2 + 2y^2) \\
f_z &= -\frac{\partial p}{\partial z} + 2\mu\frac{\partial}{\partial x}\dot{\epsilon}_{xz} + 2\mu\frac{\partial}{\partial y}\dot{\epsilon}_{yz} + 2\mu\frac{\partial}{\partial z}\dot{\epsilon}_{zz} \\
&= -(xy + x^3y^3) + \mu(-10yz) + 0 + 0 \\
&= -(xy + x^3y^3) + \mu(-10yz)
\end{aligned}$$

Finally

$$\boldsymbol{f} = -\begin{pmatrix} yz + 3x^2y^3z \\ xz + 3x^3y^2z \\ xy + x^3y^3 \end{pmatrix} + \mu \begin{pmatrix} 2 + 6xy \\ 2 + 2x^2 + 2y^2 \\ -10yz \end{pmatrix}$$

Note that there seems to be a sign problem with Eq.(26) in [?].

0.0.2 Variable viscosity

The spatial derivatives of the viscosity are then given by

$$
\begin{aligned}
\frac{\partial}{\partial x}\mu(x,y,z) &= -(1-2x)\beta\mu(x,y,z) \\
\frac{\partial}{\partial y}\mu(x,y,z) &= -(1-2y)\beta\mu(x,y,z) \\
\frac{\partial}{\partial z}\mu(x,y,z) &= -(1-2z)\beta\mu(x,y,z)
\end{aligned}
$$

and thr right-hand side by

$$
\begin{aligned}
\boldsymbol{f} = &-\begin{pmatrix} yz + 3x^2y^3z \\ xz + 3x^3y^2z \\ xy + x^3y^3 \end{pmatrix} + \mu \begin{pmatrix} 2 + 6xy \\ 2 + 2x^2 + 2y^2 \\ -10yz \end{pmatrix} \\
&-(1-2x)\beta\mu(x,y,z)\begin{pmatrix} 2\dot{\epsilon}_{xx} \\ 2\dot{\epsilon}_{xy} \\ 2\dot{\epsilon}_{xz} \end{pmatrix} - (1-2y)\beta\mu(x,y,z)\begin{pmatrix} 2\dot{\epsilon}_{xy} \\ 2\dot{\epsilon}_{yy} \\ 2\dot{\epsilon}_{yz} \end{pmatrix} - (1-2z)\beta\mu(x,y,z)\begin{pmatrix} 2\dot{\epsilon}_{xz} \\ 2\dot{\epsilon}_{yz} \\ 2\dot{\epsilon}_{zz} \end{pmatrix} \\
= &-\begin{pmatrix} yz + 3x^2y^3z \\ xz + 3x^3y^2z \\ xy + x^3y^3 \end{pmatrix} + \mu \begin{pmatrix} 2 + 6xy \\ 2 + 2x^2 + 2y^2 \\ -10yz \end{pmatrix} \\
&- (1-2x)\beta\mu\begin{pmatrix} 2 + 4x + 2y + 6x^2y \\ x + y + 2xy^2 + x^3 \\ -3z - 10xyz \end{pmatrix} - (1-2y)\beta\mu\begin{pmatrix} x + y + 2xy^2 + x^3 \\ 2 + 2x + 4y + 4x^2y \\ -3z - 5x^2z \end{pmatrix} - (1-2z)\beta\mu\begin{pmatrix} -3z - 10xyz \\ -3z - 5x^2z \\ -4 - 6x - 6y - 10x^2y \end{pmatrix}
\end{aligned}
\tag{13}
$$

Note that at $(x, y, z) = (0, 0, 0)$, $\mu = \exp(1)$, and at $(x, y, z) = (0.5, 0.5, 0.5)$, $\mu = \exp(1 - 3\beta/4)$ so that the maximum viscosity ratio is given by

$$
\mu^\star = \frac{\exp(1 - 3\beta/4)}{\exp(1)} = \exp(-3\beta/4)
$$

By varying β between 1 and 22 we can get up to 7 orders of magnitude viscosity difference.

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features

- $Q_1 \times P_0$ element
- incompressible flow
- saddle point system
- Dirichlet boundary conditions (free-slip)
- direct solver
- isothermal
- non-isoviscous
- 3D
- elemental b.c.
- analytical solution