In what follows we will be re-doing the numerical experiments presented in Zhong et al. [?].

The first benchmark showcases a unit square domain with free slip boundary conditions prescribed on all sides. The resolution is fixed to 64×64 $Q_1 \times P_0$ elements. The flow is isoviscous and the buoyancy force $\boldsymbol{f}$ is given by

$$\begin{aligned} f_x &= 0 \\ f_y &= \rho_0 \alpha T(x,y) \end{aligned}$$

with the temperature field given by

$$T(x,y) = \cos(kx)\delta(y-y_0)$$

where $k = 2\pi/\lambda$ and λ is a wavelength, and y_0 represents the location of the buoyancy strip. We set $g_y = -1$ and prescribe $\rho(x,y) = \rho_0\alpha\cos(kx)\delta(y-y_0)$ on the nodes of the mesh.

One can prove ([?] and refs. therein) that there is an analytic solution for the surface stress σ_{zz} [1]

$$\frac{\sigma_{yy}}{\rho \alpha g h} = \frac{\cos(kx)}{\sinh^2(k)} \left[k(1-y_0)\sinh(k)\cosh(ky_0) - k\sinh(k(1-y_0)) + \sinh(k)\sinh(ky_0) \right]$$

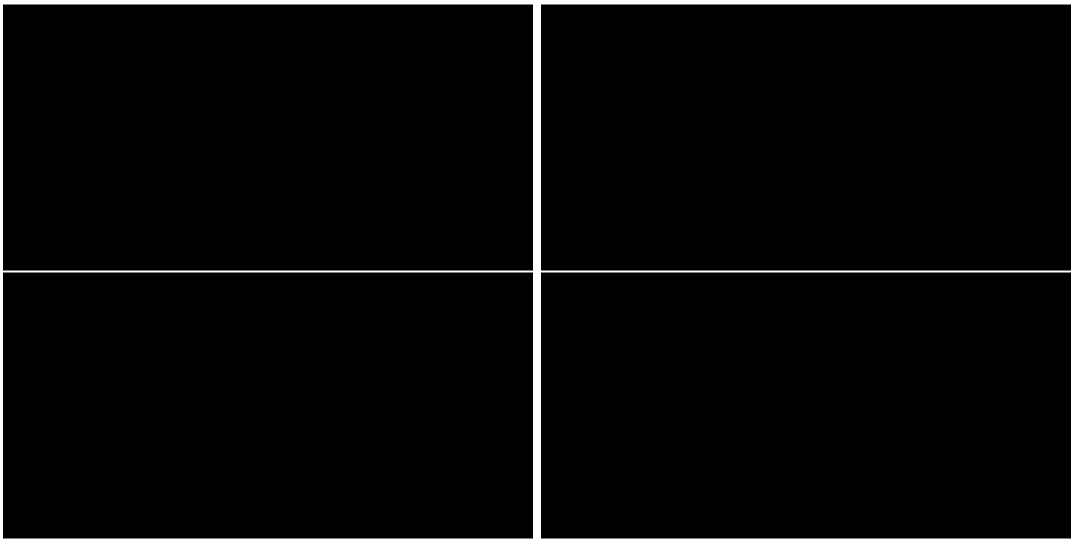
We choose $\rho_0\alpha = 64$, $\eta = 1$ (note that in this case the normalising coefficient of the stress is exactly 1 (since $h = L_x/nelx = 1/64$) so it is not implemented in the code). $\lambda = 1$ is set to 1 and we explore $y_0 = \frac{63}{64}, \frac{62}{64}, \frac{59}{64}$ and $y_0 = 32/64$. Under these assumptions the density field for $y_0 = 59/64$ is:

We can recover the stress at the boundary by computing the yy component of the stress tensor in the top row of elements:

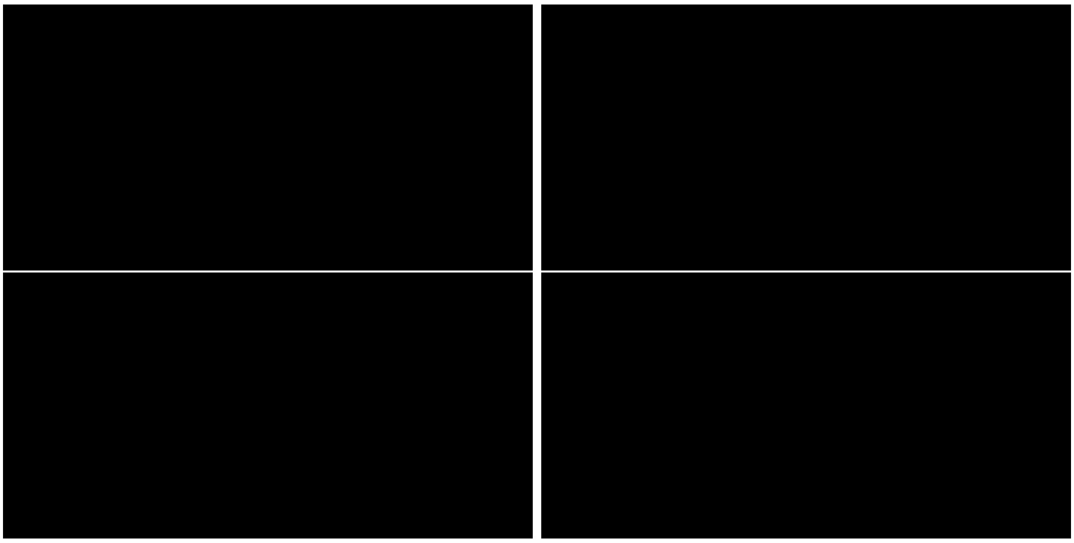
$$\sigma_{yy} = -p + 2\eta\dot{\epsilon}_{yy}$$

Note that pressure is by definition elemental, and that strain rate components are then also computed in the middle of each element.

These elemental quantities can be projected onto the nodes (see section **??**) by means of the C→N algorithm or a least square algorithm (LS).



[1]Note that in the paper the authors use $\rho\alpha g$ which does not have the dimensions of a stress



The consistent boundary flux (CBF) method allows us to compute traction vectors $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$ on the boundary of the domain. On the top boundary, $\boldsymbol{n} = (0,1)$ so that $\boldsymbol{t} = (\sigma_{xy}, \sigma_{yy})^T$ and t_y is the quantity we need to consider and compare to other results.

In the following table are shown the results presented in [?] alongside the results obtained with Fieldstone:

Method	$y_0 = 63/64$	$y_0 = 62/64$	$y_0 = 59/64$ [2]	$y_0 = 32/64$
Analytic solution	0.995476	0.983053	0.912506	0.178136
Pressure smoothing [?]	1.15974	1.06498	0.911109	n.a.
CBF [?]	0.994236	0.982116	0.912157	n.a.
fieldstone: elemental	0.824554 (-17.17 %)	0.978744 (-0.44%)	0.909574 (-0.32 %)	0.177771 (-0.20 %)
fieldstone: nodal (C→N)	0.824554 (-17.17 %)	0.978744 (-0.44%)	0.909574 (-0.32 %)	0.177771 (-0.20 %)
fieldstone: LS	1.165321 (17.06 %)	1.070105 (8.86%)	0.915496 (0.33 %)	0.178182 (0.03 %)
fieldstone: CBF	0.994236 (-0.13 %)	0.982116 (-0.10%)	0.912157 (-0.04 %)	0.177998 (-0.08 %)

We see that we recover the published results with the same exact accuracy, thereby validating our implmentation.

On the following figures are shown the velocity, pressure and traction fields for two cases $y_0 = 32/64$ and $y_0 = 63/64$.