The components of the velocity are obtained from the stream function as follows:

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \qquad v = -\frac{\partial \Psi}{\partial r}$$

where u is the radial component and v is the tangential component. In polar coordinates the curl of a vector $\boldsymbol{A}$ is:

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)$$

and the Laplacian of any scalar function f is

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

The Stokes equations are

$$A_r = -\frac{\partial p}{\partial r} + \eta \Delta u \quad = \quad \rho g_r \tag{1}$$

$$A_\theta = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \eta \Delta v \quad = \quad \rho g_\theta \tag{2}$$

Taking the curl of vector $\boldsymbol{A}$ yields:

$$\frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right) = \frac{1}{r}\left(\frac{\partial (r\rho g_\theta)}{\partial r} - \frac{\partial (\rho g_r)}{\partial \theta}\right)$$

Multiplying on each side by r and assuming the gravity vector is radial ($g_\theta = 0$):

$$\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} = -\frac{\partial \rho g_r}{\partial \theta}$$

If we now replace A_r and A_θ by their expressions as a function of velocity and pressure, we see that the pressure terms cancel out so that only the Laplacian terms remain:

$$\frac{\partial (r\eta\Delta v)}{\partial r} - \frac{\partial (\eta \Delta u)}{\partial \theta} = -\frac{\partial \rho g_r}{\partial \theta}$$

Assuming the viscosity and the gravity vector to be constant:

$$\frac{\partial (r\Delta v)}{\partial r} - \frac{\partial \Delta u}{\partial \theta} = -\frac{1}{\eta}\frac{\partial \rho}{\partial \theta} g_r$$

Then

$$\begin{aligned}
\frac{\partial (r\Delta v)}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 v}{\partial \theta^2}\right) && (3)\\
&= \frac{\partial^2}{\partial r^2}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 v}{\partial \theta^2}\right) && (4)\\
&= \frac{\partial^2}{\partial r^2}\left(r\frac{\partial}{\partial r}(-\frac{\partial \Psi}{\partial r})\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2}{\partial \theta^2}(-\frac{\partial \Psi}{\partial r})\right) && (5)\\
&= -\frac{\partial^2}{\partial r^2}\left(r\frac{\partial^2 \Psi}{\partial r^2}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^3 \Psi}{\partial \theta^2 \partial r}\right) && (6)\\
&= -2\frac{\partial^3 \Psi}{\partial r^3} - r\frac{\partial^4 \Psi}{\partial r^4} + \frac{1}{r^2}\frac{\partial^3 \Psi}{\partial \theta^2 \partial r} - \frac{1}{r}\frac{\partial^4 \Psi}{\partial \theta^2 \partial r^2} && (7)\\
\frac{\partial \Delta u}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) && (8)\\
&= \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^3 u}{\partial \theta^3} && (9)\\
&= \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \Psi}{\partial \theta})\right)\right) + \frac{1}{r^2}\frac{\partial^3}{\partial \theta^3}(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}) && (10)\\
&= \frac{1}{r^3}\frac{\partial^2 \Psi}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^3 \Psi}{\partial r \partial \theta^2} + \frac{1}{r}\frac{\partial^4 \Psi}{\partial r^2 \partial \theta^2} + \frac{1}{r^3}\frac{\partial^4 \Psi}{\partial \theta^4} && (11)
\end{aligned}$$

Assuming the following separation of variables $\boxed{\Psi(r,\theta) = \phi(r)\xi(\theta)}$:

$$\frac{\partial(r\Delta v)}{\partial r} = -2\phi'''\xi - r\phi''''\xi + \frac{1}{r^2}\phi'\xi'' - \frac{1}{r}\phi''\xi'' \tag{12}$$

$$\frac{\partial \Delta u}{\partial \theta} = \frac{1}{r^3}\phi\xi'' - \frac{1}{r^2}\phi'\xi'' + \frac{1}{r}\phi''\xi'' + \frac{1}{r^3}\phi\xi'''' \tag{13}$$

so that

$$\frac{\partial(r\Delta v)}{\partial r} - \frac{\partial \Delta u}{\partial \theta} = -2\phi'''\xi - r\phi''''\xi + \frac{1}{r^2}\phi'\xi'' - \frac{1}{r}\phi''\xi'' - \frac{1}{r^3}\phi\xi'' + \frac{1}{r^2}\phi'\xi'' - \frac{1}{r}\phi''\xi'' - \frac{1}{r^3}\phi\xi''''$$

Further assuming $\boxed{\xi(\theta) = \cos(k\theta)}$, then $\xi'' = -k^2\xi$ and $\xi'''' = k^4\xi$ then

$$\frac{\partial(r\Delta v)}{\partial r} - \frac{\partial \Delta u}{\partial \theta} = -2\phi'''\xi - r\phi''''\xi - k^2\frac{1}{r^2}\phi'\xi + k^2\frac{1}{r}\phi''\xi + k^2\frac{1}{r^3}\phi\xi - k^2\frac{1}{r^2}\phi'\xi + k^2\frac{1}{r}\phi''\xi - k^4\frac{1}{r^3}\phi\xi$$

By choosing ρ such that $\rho = f(r)g(\theta)$ and such that $\partial_\theta g = \xi$ then we have

$$-2\phi'''\xi - r\phi''''\xi - k^2\frac{1}{r^2}\phi'\xi + k^2\frac{1}{r}\phi''\xi + k^2\frac{1}{r^3}\phi\xi - k^2\frac{1}{r^2}\phi'\xi + k^2\frac{1}{r}\phi''\xi - k^4\frac{1}{r^3}\phi\xi = -\frac{1}{\eta}f\xi g_r$$

and then dividing by ξ: (IS THIS OK ?)

$$-2\phi''' - r\phi'''' - k^2\frac{1}{r^2}\phi' + k^2\frac{1}{r}\phi'' + k^2\frac{1}{r^3}\phi - k^2\frac{1}{r^2}\phi' + k^2\frac{1}{r}\phi'' - k^4\frac{1}{r^3}\phi = -\frac{1}{\eta}fg_r$$

$$-2\phi''' - r\phi'''' - 2k^2\frac{1}{r^2}\phi' + 2k^2\frac{1}{r}\phi'' + (k^2 - k^4)\frac{1}{r^3}\phi = -\frac{1}{\eta}fg_r$$

so

$$\boxed{f(r) = \frac{\eta}{g}\left(2\phi''' + r\phi'''' + 2k^2\frac{1}{r^2}\phi' - 2k^2\frac{1}{r}\phi'' - (k^2 - k^4)\frac{1}{r^3}\phi\right)}$$

0.1 No slip boundary conditions

No-slip boundary conditions inside and outside impose that all components of the velocity must be zero on both boundaries, i.e.

$$\boldsymbol{v}(r = R_1) = \boldsymbol{v}(r = R_2) = \mathbf{0}$$

Due to the separation of variables, and since $\xi(\theta) = \cos(k\theta)$ we have

$$u(r,\theta) = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{1}{r}\phi\xi' = -\frac{1}{r}\phi(r)k\sin(k\theta) \qquad v(r,\theta) = -\frac{\partial \Psi}{\partial r} = -\phi'(r)\xi = -\phi'(r)\cos(k\theta)$$

It is obvious that the only way to insure no-slip boundary conditions is to have

$$\phi(R_1) = \phi(R_2) = \phi'(R_1) = \phi'(R_2) = 0$$

We could then choose

$$\phi(r) = (r - R_1)^2(r - R_2)^2\mathcal{F}(r) \tag{14}$$

$$\phi'(r) = 2(r - R_1)(r - R_2)^2\mathcal{F}(r) + 2(r - R_1)^2(r - R_2)\mathcal{F}(r) + (r - R_1)^2(r - R_2)^2\mathcal{F}'(r) \tag{15}$$

which are indeed identically zero on both boundaries. Here $\mathcal{F}(r)$ is any (smooth enough) function of r. We would then have

$$\boxed{\Psi(r,\theta) = (r - R_1)^2(r - R_2)^2\mathcal{F}(r)\cos(k\theta)}$$

0.2 Free slip boundary conditions

Before postulating the form of $\phi(r)$, let us now turn to the boundary conditions that the flow must fulfill, i.e. free-slip on both boundaries. Two conditions must be met:

- $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ (no flow through the boundaries) which yields $u(r = R_1) = 0$ and $u(r = R_2) = 0$, :

$$\frac{1}{r}\frac{\partial \Psi}{\partial \theta}(r = R_1, R_2) = 0 \qquad \forall \theta$$

  which gives us the first constraint since $\Psi(r, \theta) = \phi(r)\xi(\theta)$:

$$\phi(r = R_1) = \phi(r = R_2) = 0$$

- $(\boldsymbol{\sigma} \cdot \boldsymbol{n}) \times \boldsymbol{n} = \boldsymbol{0}$ (the tangential stress at the boundary is zero) which imposes: $\sigma_{\theta r} = 0$, with

$$\sigma_{\theta r} = 2\eta \cdot \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right) = \eta\left(\frac{\partial}{\partial r}(-\frac{\partial \Psi}{\partial r}) - \frac{1}{r}(-\frac{\partial \Psi}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{1}{r}\frac{\partial \Psi}{\partial \theta})\right)$$

  Finally Ψ must fulfill (on the boundaries!):

$$-\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = 0$$

  or,

$$-\phi''\xi + \frac{1}{r}\phi'\xi + \frac{1}{r^2}\phi\xi'' = 0$$

Since we are looking for a solution ϕ such that $\phi(r = R_1) = \phi(r = R_2) = 0$ then we have to ensure the following equality on the boundary:

$$-\phi''\xi + \frac{1}{r}\phi'\xi = 0$$

or,

$$-r\phi'' + \phi' = 0 \qquad \text{for} \quad r = R_1, R_2$$

I then postulate $\phi(r) = (r - R_1)(r - R_2)\chi(r)$ so that it automatically fulfills $\phi(r = R_1) = \phi(r = R_2) = 0$. We then have

$$\begin{aligned}
\phi(r) &= (r - R_1)(r - R_2)\chi(r) && (16)\\
\phi'(r) &= [2r - (R_1 + R_2)]\chi(r) + (r - R_1)(r - R_2)\chi'(r) && (17)\\
\phi''(r) &= 2\chi(r) + 2[2r - (R_1 + R_2)]\chi'(r) + (r - R_1)(r - R_2)\chi''(r) && (18)\\
-r\phi''(r) &= -2r\chi(r) - 2r[2r - (R_1 + R_2)]\chi'(r) - r(r - R_1)(r - R_2)\chi''(r) && (19)
\end{aligned}$$

$$-2r\chi(r) - 2r[2r - (R_1 + R_2)]\chi'(r) - r(r - R_1)(r - R_2)\chi''(r) + [2r - (R_1 + R_2)]\chi(r) + (r - R_1)(r - R_2)\chi'(r) = 0$$

$$-2r[2r - (R_1 + R_2)]\chi'(r) - (R_1 + R_2)\chi(r) - r(r - R_1)(r - R_2)\chi''(r) + (r - R_1)(r - R_2)\chi'(r) = 0$$

Then the function χ must fulfill:

$$\begin{aligned}
0 = -R_1\phi''(R_1) + \phi'(R_1) &= -2R_1\chi(R_1) - 2R_1[R_1 - R_2]\chi'(R_1) + [R_1 - R_2]\chi(R_1) && (20)\\
&= -2R_1[R_1 - R_2]\chi'(R_1) - [R_1 + R_2]\chi(R_1) && (21)\\
0 = -R_2\phi''(R_2) + \phi'(R_2) &= -2R_2\chi(R_2) + 2R_2[R_1 - R_2]\chi'(R_2) - [R_1 - R_2]\chi(R_2) && (22)\\
&= 2R_2[R_1 - R_2]\chi'(R_2) - [R_1 + R_2]\chi(R_2) && (23)
\end{aligned}$$

http://mathworld.wolfram.com/EulerDifferentialEquation.html