The details of the numerical setup are presented in Section ??.

We will here focus on the $Q_1 \times Q_1$ element, which, unless stabilised, violates the LBB stability condition and therefore is unusable. Stabilisation can be of two types: least-squares (see for example [?]) [?, ?, ?], or by means of an additional term in the weak form as first introduced in [?, ?]. It is further analysed in [?, ?]. Note that an equal-order velocity-pressure formulation that does not exhibit spurious pressure modes (without stabilisaion) has been presented in [?].

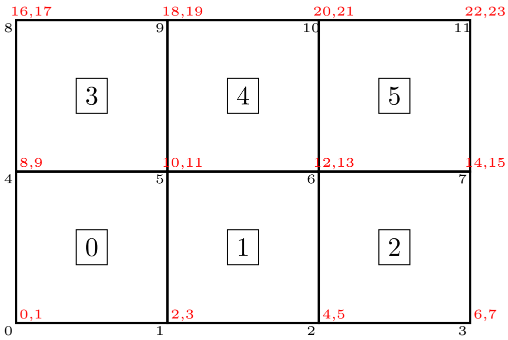
This element corresponds to Bilinear velocities, bilinear pressure (equal order interpolation for both velocity and pressure) which is very convenient in terms of data structures since all dofs are colocated.

In geodynamics, it is used in the Rhea code [?, ?] and in Gale. It is also used in [?] in its stabilised form, in conjunction with AMR.

The stabilisation term $\mathbb{C}$ enters the Stokes matrix in the (2,2) position:

$$\left(\begin{array}{cc} \mathbb{K} & \mathbb{G} \\ \mathbb{G}^T & \mathbb{C} \end{array} \right) \cdot \left(\begin{array}{c} \mathcal{V} \\ \mathcal{P} \end{array} \right) = \left(\begin{array}{c} f \\ h \end{array} \right)$$

Let us take a simple example: a 3x2 element grid.



The $\mathbb{K}$ matrix is of size $NfemV \times NfemV$ with $NfemV = ndofV \times nnp = 2 \times 12 = 24$. The $\mathbb{G}$ matrix is of size $NfemV \times NfemP$ with $NfemP = ndofP \times nnp = 1 \times 12 = 12$. The $\mathbb{C}$ matrix is of size $NfemP \times NfemP$.

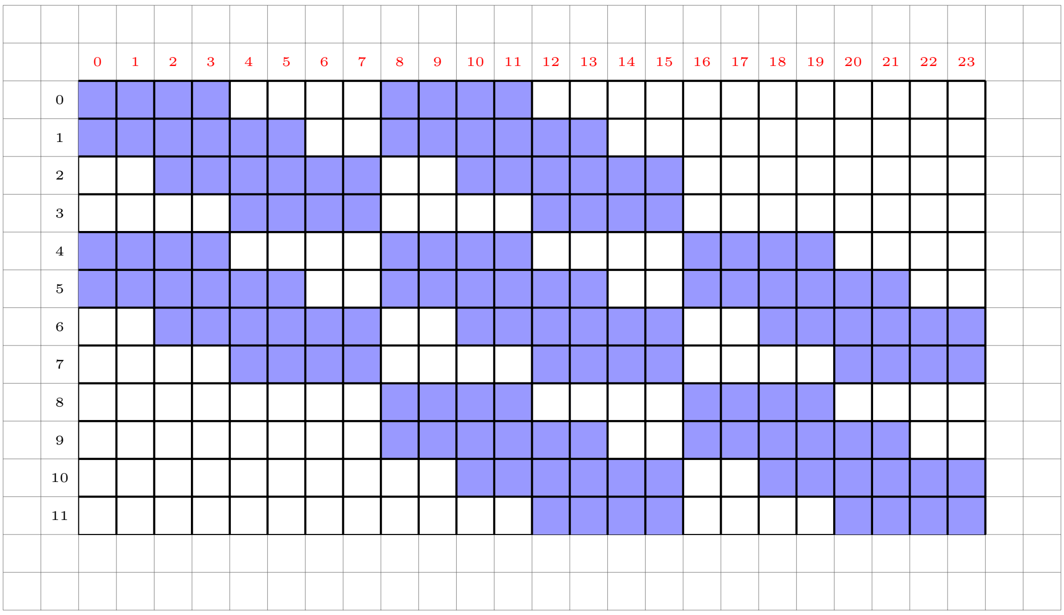
A corner pdof sees 4 vdofs, a side pdof sees 12 vdofs and an inside pdof sees 18 vdofs, so that the total number of nonzeros in $\mathbb{G}$ can be computed as follows:

$$NZ_{\mathbb{G}} = \underbrace{4}_{corners} + \underbrace{2(nnx-2)*12}_{2hor.sides} + \underbrace{2(nny-2)*12}_{2vert.sides} + \underbrace{(nnx-2)(nny-2)*18}_{insidenodes}$$

Concretely,

- pdof #0 sees vdofs 0,1,2,3,8,9,10,11
- pdof #1 sees vdofs 0,1,2,3,4,5,8,9,10,11,12,13
- pdof #5 sees vdofs 0,1,2,3,4,5,8,9,10,11,12,13,16,17,18,19,20,21

so that the $\mathbb{G}^T$ matrix non-zero structure then is as follows:



We will here use the formulation of [**?**], which is very appealing since it does not require any free (user-chosen) parameter. Furthermore the matrix $\mathbb{C}$ is rather simple to compute as we will see hereafter.

We also impose $\int p dV = 0$ which means that the following constraint is added to the Stokes matrix:

$$\left(\begin{array}{ccc} \mathbb{K} & \mathbb{G} & 0 \\ \mathbb{G}^T & \mathbb{C} & \mathbb{L} \\ 0 & \mathbb{L}^T & 0 \end{array}\right) \cdot \left(\begin{array}{c} \mathcal{V} \\ \mathcal{P} \\ \lambda \end{array}\right) = \left(\begin{array}{c} f \\ h \\ 0 \end{array}\right)$$

As in [**?**] we solve the benchmark problem presented in **??**

[redacted]