0.1 Background theory

The stream function is a function of coordinates and time of an inviscid liquid. It allows to determine the components of velocity by differentiating the stream function with respect to the space coordinates. A family of curves $\Psi = const$ represent *streamlines*, i.e. the stream function remains constant along a streamline. Although also valid in 3D, this approach is mostly used in 2D because of its relative simplicity REFERENCES.

In two dimensions the velocity is obtained as follows:

$$\boldsymbol{v} = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x} \right) \tag{1}$$

Provided the function Ψ is a smooth enough function, this automatically insures that the flow is incompressible:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \Psi}{\partial xy} - \frac{\partial^2 \Psi}{\partial xy} = 0 \tag{2}$$

Assuming constant viscosity, the Stokes equation writes:

$$-\boldsymbol{\nabla} p + \mu \Delta \boldsymbol{v} = \rho \boldsymbol{g} \tag{3}$$

Let us introduce the vector $\boldsymbol{W}$ for convenience such that in each dimension:

$$W_x = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^y} \right) \quad = \quad \rho g_x \tag{4}$$

$$W_y = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^y} \right) \quad = \quad \rho g_y \tag{5}$$

Taking the curl of the vector $\boldsymbol{W}$ and only considering the component perpendicular to the xy-plane:

$$\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \frac{\partial \rho g_y}{\partial x} - \frac{\partial \rho g_x}{\partial y} \tag{6}$$

The advantage of this approach is that the pressure terms cancel out (the curl of a gradient is always zero), so that:

$$\frac{\partial}{\partial x} \mu \left((\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^y} \right) - \frac{\partial}{\partial y} \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^y} \right) = \frac{\partial \rho g_y}{\partial x} - \frac{\partial \rho g_x}{\partial y} \tag{7}$$

and then replacing u, v by the their stream function derivatives yields (for a constant viscosity):

$$\mu \left(\frac{\partial^4 \Psi}{\partial x^4} + \frac{\partial^4 \Psi}{\partial y^4} + 2 \frac{\partial^4 \Psi}{\partial x^2 y^2} \right) = \frac{\partial \rho g_y}{\partial x} - \frac{\partial \rho g_x}{\partial y} \tag{8}$$

or,

$$\boldsymbol{\nabla}^4 \Psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi = \frac{\partial \rho g_y}{\partial x} - \frac{\partial \rho g_x}{\partial y} \tag{9}$$

These equations are also to be found in the geodynamics literature, eee Eq. 1.43 of Tackley book, p 70-71 of Gerya book.

0.2 A simple application

I wish to arrive at an analytical formulation for a 2D incompressible flow in the square domain $[-1:1] \times [-1:1]$ The fluid has constant viscosity $\mu = 1$ and is subject to free slip boundary conditions on all sides. For reasons that will become clear in what follows I postulate the following stream function:

$$\Psi(x, y) = \sin(m\pi x) \sin(n\pi y) \tag{10}$$

We have the velocity being defined as:

$$\boldsymbol{v} = (u, v) = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x} \right) = (n\pi \sin(m\pi x) \cos(n\pi y), -m\pi \cos(m\pi x) \sin(n\pi y)) \tag{11}$$

The strain rate components are then:

$$
\begin{aligned}
\dot{\varepsilon}_{xx} &= \frac{\partial u}{\partial x} = mn\pi^2 \cos(m\pi x)\cos(n\pi y) && (12)\\
\dot{\varepsilon}_{yy} &= \frac{\partial v}{\partial y} = -mn\pi^2 \cos(m\pi x)\cos(n\pi y) && (13)\\
2\dot{\varepsilon}_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} && (14)\\
&= \frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2} && (15)\\
&= -n^2\pi^2\Psi + m^2\pi^2\Psi && (16)\\
&= (m^2-n^2)\pi^2 \sin(m\pi x)\sin(n\pi y) && (17)
\end{aligned}
$$

Note that if $m = n$ the last term is identically zero, which is not desirable (flow is too 'simple') so in what follows we will prefer $m \neq n$.

It is also easy to verify that $u = 0$ on the sides and $v = 0$ at the top and bottom and that the term $\dot{\varepsilon}_{xy}$ is nul on all four sides, thereby garanteeing free slip.

Our choice of stream function yields:

$$
\nabla^4\Psi = \frac{\partial^4\Psi}{\partial x^4} + \frac{\partial^4\Psi}{\partial y^4} + 2\frac{\partial^2\Psi}{\partial x^2 y^2} = \pi^4(m^4\Psi + n^4\Psi + 2m^2n^2\Psi) = (m^4+n^4+2m^2n^2)\pi^4\Psi
$$

We assume $g_x = 0$ and $g_y = -1$ so that we simply have

$$
(m^4+n^4+2m^2n^2)\pi^4\Psi = -\frac{\partial \rho}{\partial x} \tag{18}
$$

so that (assuming the integration constant to be zero):

$$
\rho(x,y) = \frac{m^4+n^4+2m^2n^2}{m}\pi^3\cos(m\pi x)\sin(n\pi y)
$$

The x-component of the momentum equation is

$$
-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial p}{\partial x} - m^2n\pi^3\sin(m\pi x)\cos(n\pi y) - n^3\pi^3\sin(m\pi x)\cos(n\pi y) = 0
$$

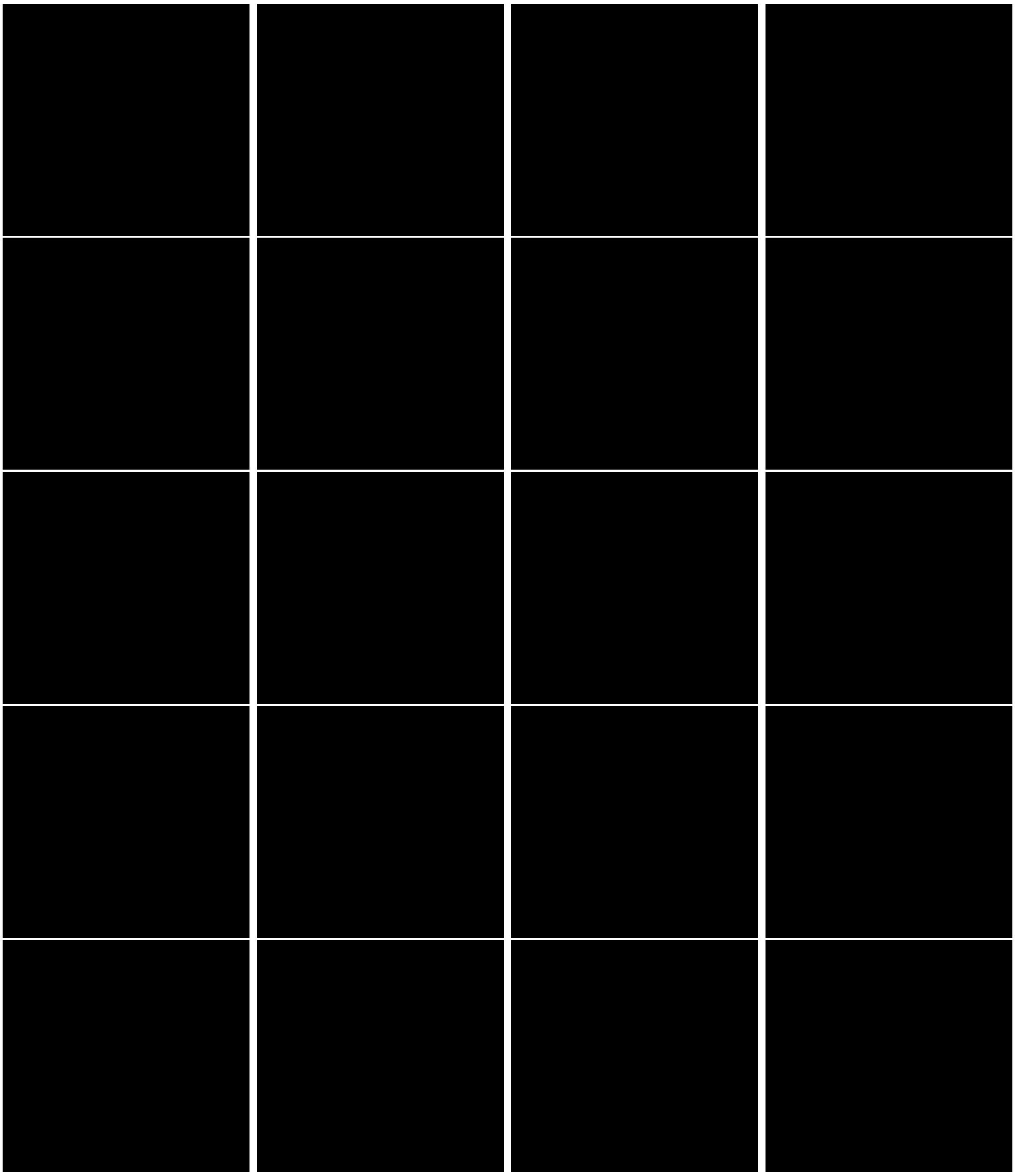
so

$$
\frac{\partial p}{\partial x} = -(m^2n + n^3)\pi^3\sin(m\pi x)\cos(n\pi y)
$$

and the pressure field is then (once again neglecting the integration constant):

$$
p(x,y) = \frac{m^2n+n^3}{m}\pi^2\cos(\pi x)\cos(\pi y)
$$

Note that in this case $\int p dV = 0$ so that volume normalisation of the pressure is turned on (when free slip boundary conditions are prescribed on all sides the pressure is known up to a constant and this undeterminacy can be lifted by adding an additional constraint to the pressure field).



Top to bottom: Velocity components u and v, pressure p, density ρ and strain rate $\dot{\varepsilon}_{xy}$. From left to right: $(m,n)=(1,1)$, $(m,n)=(1,2)$, $(m,n)=(2,1)$, $(m,n)=(2,2)$

[redacted]

Errors for velocity and pressure for $(m, n) = (1,1), (1,2), (2,1), (2,2)$

[redacted]

Velocity arrows for $(m, n) = (2,1)$