The system is a layer of fluid between $y = 0$ and $y = 1$, with boundary conditions $T(x, y = 0) = 1$ and $T(x, y = 1) = 0$, characterized by ρ, c_p, k, η_0. The Rayleigh number of the system is

$$\mathrm{Ra} = \frac{\rho_0 g_0 \alpha \Delta T h^3}{\eta_0 \kappa}$$

We have $\Delta T = 1$, $h = 1$ and choose $\kappa = 1$ so that the Rayleigh number simplifies to $\mathrm{Ra} = \rho_0 g_0 \alpha / \eta_0$.

The Stokes equation is $\vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{b} = \vec{0}$ with $\vec{b} = \rho \vec{g}$. Then the components of the this equation on the x- and $y-$axis are:

$$(\vec{\nabla} \cdot \boldsymbol{\sigma})_x = -\rho \vec{g} \cdot \vec{e}_x = 0 \qquad (1)$$

$$(\vec{\nabla} \cdot \boldsymbol{\sigma})_y = -\rho \vec{g} \cdot \vec{e}_y = \rho g_0 \qquad (2)$$

since $\vec{g}$ and $\vec{e}_y$ are in opposite directions ($\vec{g} = -g_0 \vec{e}_y$, with $g_0 > 0$). The stream function formulation of the incompressible isoviscous Stokes equation is then

$$\nabla^4 \Psi = \frac{g_0}{\eta_0} \frac{\partial \rho}{\partial x}$$

Assuming a linearised density field with regards to temperature $\rho(T) = \rho_0(1 - \alpha T)$ we have

$$\frac{\partial \rho}{\partial x} = -\rho_0 \alpha \frac{\partial T}{\partial x}$$

and then

$$\boxed{\nabla^4 \Psi = -\frac{\rho_0 g_0 \alpha}{\eta_0} g \frac{\partial T}{\partial x} = -Ra \frac{\partial T}{\partial x}} \qquad (3)$$

For small perturbations of the conductive state $T_0(y) = 1 - y$ we define the temperature perturbation $T_1(x, y)$ such that

$$T(x, y) = T_0(y) + T_1(x, y)$$

The temperature perturbation T_1 satisfies the homogeneous boundary conditions $T_1(x, y = 0) = 0$ and $T_1(x, y = 1) = 0$. The temperature equation is

$$\rho c_p \frac{DT}{Dt} = \rho c_p \left(\frac{\partial T}{\partial t} + \vec{\mathrm{v}} \cdot \vec{\nabla} T \right) = \rho c_p \left(\frac{\partial T_0 + T_1}{\partial t} + \vec{\mathrm{v}} \cdot \vec{\nabla}(T_0 + T_1) \right) = k \Delta (T_0 + T_1)$$

and can be simplified as follows:

$$\rho c_p \left(\frac{\partial T_1}{\partial t} + \vec{\mathrm{v}} \cdot \vec{\nabla} T_0 \right) = k \Delta T_1$$

since T_0 does not depend on time, $\Delta T_0 = 0$ and we assume the nonlinear term $\vec{\mathrm{v}} \cdot \vec{\nabla} T_1$ to be second order (temperature perturbations and coupled velocity changes are assumed to be small). Using the relationship between velocity and stream function $v_y = -\partial_x \Psi$ we have $\boldsymbol{v} \cdot \boldsymbol{\nabla} T_0 = -v_y = \partial_x \Psi$ and since $\kappa = k/\rho c_p = 1$ we get

$$\boxed{\frac{\partial T_1}{\partial t} - \kappa \Delta T_1 = -\frac{\partial \Psi}{\partial x}} \qquad (4)$$

Looking at these equations, we immediately think about a separation of variables approach to solve these equations. Both equations showcase the Laplace operator Δ, and the eigenfunctions of the biharmonic operator and the Laplace operator are the same. We then pose that Ψ and T_1 can be written:

$$\Psi(x, y, t) = A_\Psi \exp(pt) \exp(\pm i k_x x) \exp(\pm i k_y y) = A_\Psi E_\psi(x, y, t) \qquad (5)$$

$$T_1(x, y, t) = A_T \exp(pt) \exp(\pm i k_x x) \exp(\pm i k_y y) = A_T E_T(x, y, t) \qquad (6)$$

where k_x and k_y are the horizontal and vertical wave number respectively. Note that we then have

$$\nabla^2 \Psi = -(k_x^2 + k_y^2) \Psi \qquad \nabla^2 T_1 = -(k_x^2 + k_y^2) T_1$$

The boundary conditions on T_1, coupled with a choice of a real function for the x dependence yields:

$$E_T(x, y, t) = \exp(pt) \cos(k_x x) \sin(n \pi y).$$

from here onwards check for minus signs!

The velocity boundary conditions are $v_y(x, y = 0) = 0$ and $v_y(x, y = 1) = 0$ which imposes conditions on $\partial\Psi/\partial x$ and we find that we can use the same y dependence as for T_1. Choosing again for a real function for the x dependence yields:

$$E_\Psi(x, y, t) = \exp(pt)\sin(k_x x)\sin(n\pi z)$$

We then have

$$\begin{aligned} \Psi(x, y, t) &= A_\Psi \exp(pt)\sin(k_x x)\sin(n\pi z) = A_\Psi E_\psi(x, y, t) && (7) \\ T_1(x, y, t) &= A_T \exp(pt)\cos(k_x x)\sin(n\pi z) = A_T E_T(x, y, t) && (8) \end{aligned}$$

In what follows we simplify notations: $k = k_x$. Then the two PDEs become:

$$pT_1 + \kappa(k^2 + n^2\pi^2) - kA_\Psi \exp(pt)\cos(k_x x)\sin(n\pi z) = kA_\Psi E_\theta \tag{9}$$

$$-RaA_T \cos(kx)\sin(n\pi z) + \kappa(k^2 + n^2\pi^2)^2 A_\Psi = -RaA_T E_\Psi + \kappa(k^2 + n^2\pi^2)^2 A_\Psi = 0 \tag{10}$$

These equations must then be verified for all ... which leads to write:

$$\begin{pmatrix} p + (k^2 + n^2\pi^2) & -k \\ -Ra\ k & (k^2 + n^2\pi^2)^2 \end{pmatrix} \begin{pmatrix} A_\theta \\ A_\Psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant of such system must be nul otherwise there is only a trivial solution to the problem, i.e. $A_\theta = 0$ and $A_\Psi = 0$ which is not helpful. CHECK/REPHRASE

$$D = [p + (k^2 + n^2\pi^2)](k^2 + n^2\pi^2)^2 - Ra\ k^2 = 0$$

or,

$$p = \frac{Ra\ k^2 - (k^2 + n^2\pi^2)^3}{(k^2 + n^2\pi^2)^2}$$

The coefficient p determines the stability of the system: if it is negative, the system is stable and both Ψ and T_1 will decay to zero (return to conductive state). If $p = 0$, then the system is meta-stable, and if $p > 0$ then the system is unstable and the perturbations will grow. The threshold is then $p = 0$ and the solution of the above system is