This fieldstone was developed in collaboration with Lukas van de Wiel.

The domain is an annulus with inner radius R_1 and outer radius R_2. It is filled with a single elastic material characterised by a Young's modulus E and a Poisson ratio ν, a density ρ_0. The gravity $\boldsymbol{g} = -g_0 \boldsymbol{e}_r$ is pointing towards the center of the domain.

2μ is missing!

Given these assumptions, the momentum Stokes equations in the annulus are [?]

$$\mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} \right) - \frac{\partial p}{\partial r} = \rho g_r \tag{1}$$

$$\mu \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \right) - \frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \tag{2}$$

Equations (1) and (2) are the momentum equations in polar coordinates. The problem at hand is axisymmetric so that the tangential component of the displacement vector v_θ is assumed to be zero as well as all terms containing ∂_θ. Under these assumptions the second equation is automatically fulfilled. Of the first one remain the following terms:

$$\mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right) - \frac{\partial p}{\partial r} = \rho g_r$$

As we have seen before we have

$$p = -\lambda \boldsymbol{\nabla} \cdot \boldsymbol{v} = -\lambda \left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} \right)$$

Inserting this expression into the first equation yields:

$$\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} = \frac{\rho g_0}{2\mu + \lambda}$$

We now look at the boundary conditions. On the inner boundary we prescribe $v_r(r = R_1) = 0$ while free surface boundary conditions are prescribed on the outer boundary, i.e. $\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$ (i.e. there is no force applied on the surface).

The components of the strain tensor are

$$\varepsilon_{rr} = \frac{\partial v_r}{\partial r} \tag{3}$$

$$\varepsilon_{\theta\theta} = \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = \frac{v_r}{r} \tag{4}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta} \right) = 0 \tag{5}$$

The stress tensor is given by

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\boldsymbol{\varepsilon} = \lambda(\boldsymbol{\nabla} \cdot \boldsymbol{v})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

which is a diagonal tensor so that the free surface condition is:

$$\sigma_{rr} = (2\mu + \lambda)\frac{\partial v_r}{\partial r} + \lambda \frac{v_r}{r} = 0 \qquad r = R_2$$

The solution is then

$$v_r(r) = ar^2 + br + \frac{c}{r} \qquad \text{with} \qquad a = \frac{\rho g_0}{3(2\mu + \lambda)}$$

where b, c are determined by means of the boundary conditions.

Pressure can be computed as follows:

$$p = -\lambda \boldsymbol{\nabla} \cdot \boldsymbol{v} = -\lambda \left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} \right) = -\lambda \left(\frac{1}{r}(3ar^2 + 2br) \right) = -\lambda(3ar + 2b)$$


[redacted]

0.1 Alternative derivation of the solution

We start from the strain tensor:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_r}{\partial r} & 0 \\ 0 & \frac{v_r}{r} \end{pmatrix}$$

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\boldsymbol{\varepsilon} = \begin{pmatrix} \lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial v_r}{\partial r} & 0 \\ 0 & \lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{v_r}{r} \end{pmatrix}$$

The divergence of the stress tensor is given by:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} \\ \\ \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\sigma_{\theta\theta}}{\partial\theta} + \frac{\sigma_{r\theta}+\sigma_{\theta r}}{r} \end{pmatrix}$$

Since the stress tensor is diagonal this simplifies to (also $\partial_\theta = 0$):

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} \\ \\ 0 \end{pmatrix}$$

So

$$\begin{aligned}
(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})_r &= \frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} && (6)\\
&= \frac{\partial}{\partial r}\left[\lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial v_r}{\partial r}\right] + \frac{1}{r}\left[\lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial v_r}{\partial r} - \lambda\frac{1}{r}\frac{\partial(rv_r)}{\partial r} - 2\mu\frac{v_r}{r}\right] && (7)\\
&= \lambda\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + 2\mu\frac{\partial^2 v_r}{\partial r^2} + \lambda\frac{1}{r^2}\frac{\partial(rv_r)}{\partial r} + \frac{2\mu}{r}\frac{\partial v_r}{\partial r} - \lambda\frac{1}{r^2}\frac{\partial(rv_r)}{\partial r} - \frac{2\mu v_r}{r^2} && (8)\\
&= \lambda(-\frac{v_r}{r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial r^2}) + 2\mu\frac{\partial^2 v_r}{\partial r^2} + \frac{2\mu}{r}\frac{\partial v_r}{\partial r} - \frac{2\mu v_r}{r^2} && (9)\\
&= -(2\mu+\lambda)\frac{v_r}{r^2} + (2\mu+\lambda)\frac{1}{r}\frac{\partial v_r}{\partial r} + (2\mu+\lambda)\frac{\partial^2 v_r}{\partial r^2} && (10)
\end{aligned}$$

So we then have

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})_r = -(2\mu+\lambda)\frac{v_r}{r^2} + (2\mu+\lambda)\frac{1}{r}\frac{\partial v_r}{\partial r} + (2\mu+\lambda)\frac{\partial^2 v_r}{\partial r^2} = \rho_0 g_r$$

or,

$$\boxed{\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} = \frac{\rho_0 g_r}{\lambda + 2\mu}}$$