0.1 Background theory

The stream function is a function of coordinates and time of an inviscid liquid. It allows to determine the components of velocity by differentiating the stream function with respect to the space coordinates. A family of curves $\Psi = const$ represent *streamlines*, i.e. the stream function remains constant along a stream line.

In two dimensions the velocity is obtained as follows:

$$\boldsymbol{v} = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x} \right) \tag{1}$$

This automatically insures that the flow is incompressible:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \Psi}{\partial xy} - \frac{\partial^2 \Psi}{\partial xy} = 0 \tag{2}$$

Assuming constant viscosity, the Stokes equation writes:

$$-\boldsymbol{\nabla} p + \mu \Delta \boldsymbol{v} = \rho \boldsymbol{g} \tag{3}$$

In each dimension:

$$W_x = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^y} \right) \quad = \quad \rho g_x \tag{4}$$

$$W_y = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^y} \right) \quad = \quad \rho g_y \tag{5}$$

Taking the curl of the vector $\boldsymbol{W}$ and only looking at the component perpendicular to the xy-plane:

$$\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \tag{6}$$

The advantage of this approach is that the pressure terms cancel out (the curl of a gradient is always zero), so that:

$$\frac{\partial}{\partial x} \mu \left((\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^y} \right) - \frac{\partial}{\partial y} \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^y} \right) = \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}$$

and then replacing u, v by the their stream function derivatives yields (for a constant viscosity):

$$\mu \left(\frac{\partial^4 \Psi}{\partial x^4} + \frac{\partial^4 \Psi}{\partial y^4} + 2 \frac{\partial^4 \Psi}{\partial x^2 y^2} \right) = \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}$$

or,

$$\boldsymbol{\nabla}^4 \Psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}$$

See Eq. 1.43 of Tackley book, p 70-71 of Gerya book.

0.2 A simple application

I wish to formulate a 2D incompressible flow in a square bound by $[-1:1] \times [-1:1]$ of constant viscosity $\mu = 1$ and subject to free slip boundary conditions on all sides. I postulate the following stream function:

$$\Psi(x, y) = \sin(m\pi x) \sin(n\pi y) \tag{7}$$

We have the velocity being defined as:

$$\boldsymbol{v} = (u, v) = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x} \right) = (n\pi \sin(m\pi x) \cos(n\pi y), -m\pi \cos(m\pi x) \sin(n\pi y)) \tag{8}$$

The strain rate components are then:

$$\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x} = mn\pi^2 \cos(m\pi x) \cos(n\pi y) \tag{9}$$

$$\dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y} = -mn\pi^2 \cos(m\pi x) \cos(n\pi y) \tag{10}$$

$$2\dot{\varepsilon}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{11}$$

$$= \frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2} \tag{12}$$

$$= -n^2 \pi^2 \Psi + m^2 \pi^2 \Psi \tag{13}$$

$$= (m^2 - n^2) \pi^2 \sin(m\pi x) \sin(n\pi y) \tag{14}$$

Note that if $m = n$ the last term is identically zero, which is not desirable (flow is too 'simple') so in what follows we will assume $m \neq n$. Our choice of stream function yields:

$$\nabla^4 \Psi = \frac{\partial^4 \Psi}{\partial x^4} + \frac{\partial^4 \Psi}{\partial y^4} + 2\frac{\partial^2 \Psi}{\partial x^2 y^2} = \pi^4 (m^4 \Psi + n^4 \Psi + 2m^2 n^2 \Psi) = (m^4 + n^4 + 2m^2 n^2)\pi^4 \Psi$$

We assume $g_x = 0$ and $g_y = -1$ so that we simply have

$$(m^4 + n^4 + 2m^2 n^2)\pi^4 \Psi = -\frac{\partial \rho}{\partial x} \tag{15}$$

so that (assuming the integration constant to be zero):

$$\rho(x,y) = \frac{m^4 + n^4 + 2m^2 n^2}{m} \pi^3 \cos(m\pi x) \sin(n\pi y)$$

The x-component of the momentum equation is

$$-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial p}{\partial x} - m^2 n \pi^3 \sin(m\pi x)\cos(n\pi y) - n^3 \pi^3 \sin(m\pi x)\cos(n\pi y) = 0$$

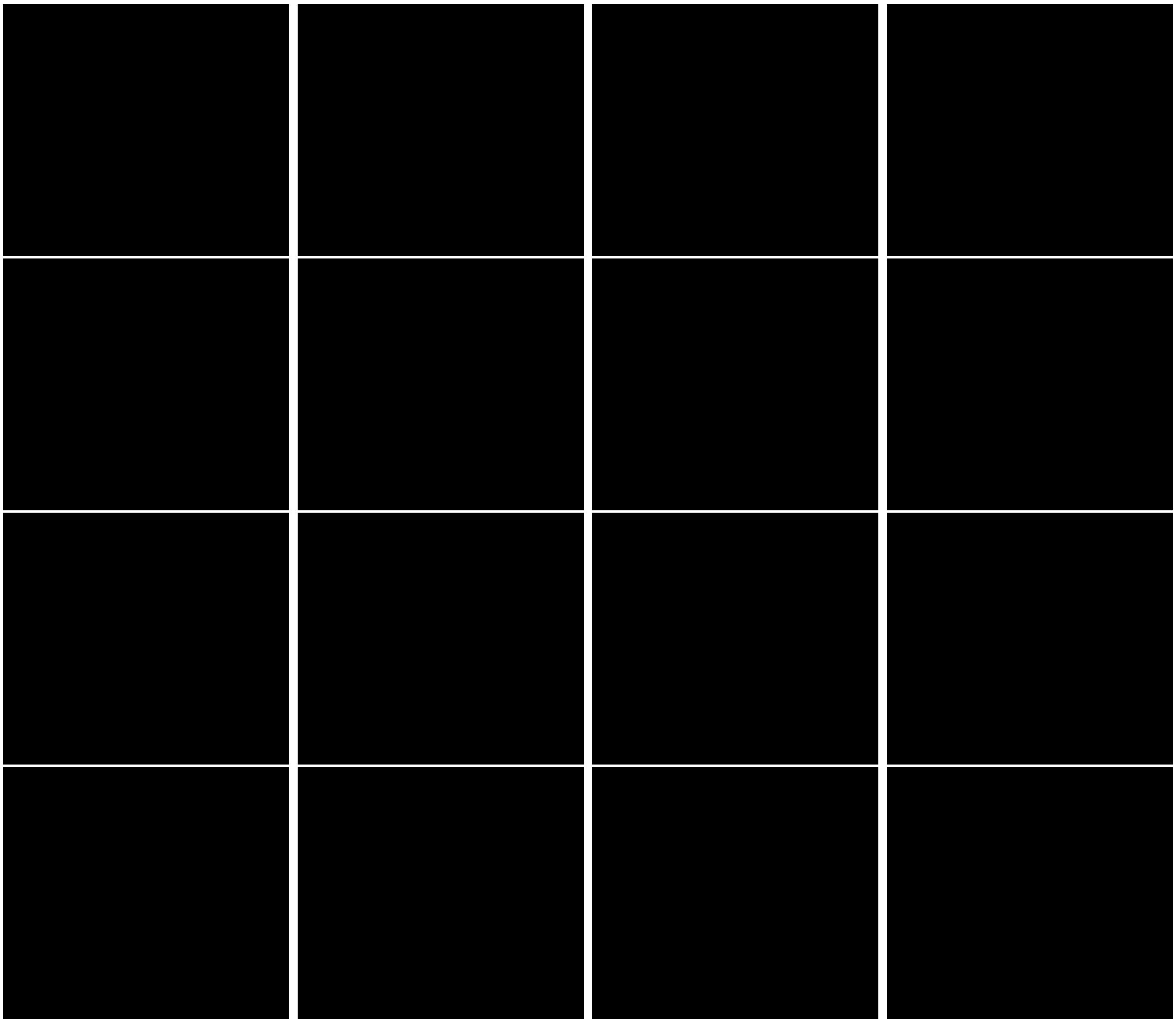
so

$$\frac{\partial p}{\partial x} = -(m^2 n + n^3)\pi^3 \sin(m\pi x)\cos(n\pi y)$$

and the pressure field is then (once again neglecting the integration constant):

$$p(x,y) = \frac{m^2 n + n^3}{m} \pi^2 \cos(\pi x)\cos(\pi y)$$

Note that in this case $\int p dV = 0$ so that volume normalisation of the pressure is turned on (when free slip boundary conditions are prescribed on all sides the pressure is known up to a constant and this undeterminacy can be lifted by adding an additional constraint to the pressure field).



From left to right: $(m,n) = (1,1)$, $(m,n) = (1,2)$, $(m,n) = (2,1)$, $(m,n) = (2,2$