

This fieldstone was developed in collaboration with Lukas van de Wiel.

The domain is an annulus with inner radius R_1 and outer radius R_2 . It is filled with a single elastic material characterised by a Young's modulus E and a Poisson ratio ν , a density ρ_0 . The gravity $\mathbf{g} = -g_0 \mathbf{e}_r$ is pointing towards the center of the domain.

The problem at hand is axisymmetric so that the tangential component of the displacement vector v_θ is assumed to be zero as well as all terms containing ∂_θ . The components of the strain tensor are

$$\varepsilon_{rr} = \frac{\partial v_r}{\partial r} \quad (1)$$

$$\varepsilon_{\theta\theta} = \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = \frac{v_r}{r} \quad (2)$$

$$\varepsilon_{r\theta} = \frac{1}{2} \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) = 0 \quad (3)$$

so that the tensor simply is

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_r}{\partial r} & 0 \\ 0 & \frac{v_r}{r} \end{pmatrix} \quad (4)$$

The pressure is

$$p = -\lambda \nabla \cdot \mathbf{v} = -\lambda \left(\frac{1}{r} \frac{\partial(rv_r)}{\partial r} \right) \quad (5)$$

and finally the stress tensor:

$$\boldsymbol{\sigma} = -p \mathbf{1} + 2\mu \boldsymbol{\varepsilon} = \begin{pmatrix} \lambda \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + 2\mu \frac{\partial v_r}{\partial r} & 0 \\ 0 & \lambda \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + 2\mu \frac{v_r}{r} \end{pmatrix} \quad (6)$$

The divergence of the stress tensor is given by [?]:

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\sigma_{r\theta} + \sigma_{\theta r}}{r} \end{pmatrix} \quad (7)$$

Given the diagonal nature of the stress tensor this simplifies to (also remember that $\partial_\theta = 0$):

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \\ 0 \end{pmatrix} \quad (8)$$

Focusing on the r -component of the stress divergence:

$$(\nabla \cdot \boldsymbol{\sigma})_r = \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \quad (9)$$

$$= \frac{\partial}{\partial r} \left[\lambda \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + 2\mu \frac{\partial v_r}{\partial r} \right] + \frac{1}{r} \left[\lambda \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + 2\mu \frac{\partial v_r}{\partial r} - \lambda \frac{1}{r} \frac{\partial(rv_r)}{\partial r} - 2\mu \frac{v_r}{r} \right] \quad (10)$$

$$= \lambda \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + 2\mu \frac{\partial^2 v_r}{\partial r^2} + \lambda \frac{1}{r^2} \frac{\partial(rv_r)}{\partial r} + \frac{2\mu}{r} \frac{\partial v_r}{\partial r} - \lambda \frac{1}{r^2} \frac{\partial(rv_r)}{\partial r} - \frac{2\mu v_r}{r^2} \quad (11)$$

$$= \lambda \left(-\frac{v_r}{r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial r^2} \right) + 2\mu \frac{\partial^2 v_r}{\partial r^2} + \frac{2\mu}{r} \frac{\partial v_r}{\partial r} - \frac{2\mu v_r}{r^2} \quad (12)$$

$$= -(2\mu + \lambda) \frac{v_r}{r^2} + (2\mu + \lambda) \frac{1}{r} \frac{\partial v_r}{\partial r} + (2\mu + \lambda) \frac{\partial^2 v_r}{\partial r^2} \quad (13)$$

So the momentum conservation in the r direction is

$$(\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{g})_r = -(2\mu + \lambda) \frac{v_r}{r^2} + (2\mu + \lambda) \frac{1}{r} \frac{\partial v_r}{\partial r} + (2\mu + \lambda) \frac{\partial^2 v_r}{\partial r^2} - \rho_0 g_0 = 0 \quad (14)$$

or,

$$\boxed{\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} = \frac{\rho_0 g_0}{\lambda + 2\mu}} \quad (15)$$

We now look at the boundary conditions. On the inner boundary we prescribe $v_r(r = R_1) = 0$ while free surface boundary conditions are prescribed on the outer boundary, i.e. $\boldsymbol{\sigma} \cdot \mathbf{n} = 0$ (i.e. there is no force applied on the surface).

The general form of the solution can then be obtained:

$$v_r(r) = C_1 r^2 + C_2 r + \frac{C_3}{r} \quad (16)$$

with

$$C_1 = \frac{\rho_0 g_0}{3(\lambda + 2\mu)} \quad C_2 = -C_1 R_1 - \frac{C_3}{R_1^2} \quad C_3 = \frac{k_1 + k_2}{(R_1^2 + R_2^2)(2\mu + \lambda) + (R_2^2 - R_1^2)\lambda} \quad (17)$$

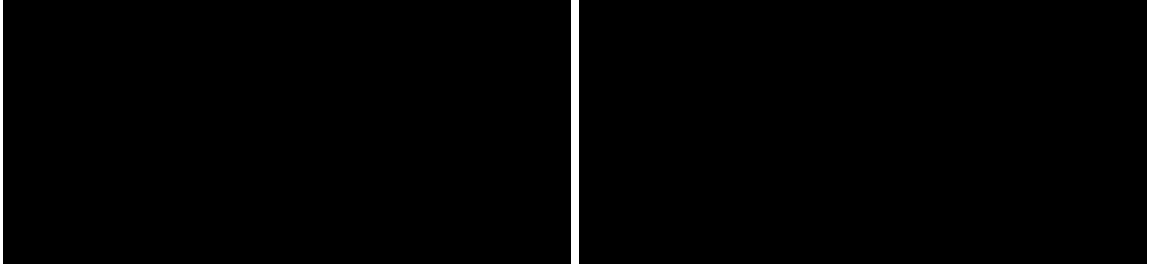
and

$$k_1 = (2\mu + \lambda)C_1(2R_1^2 R_2^3 - R_1^3 R_2^2) \quad k_2 = \lambda C_1(R_1^2 R_2^3 - R_1^3 R_2^2) \quad (18)$$

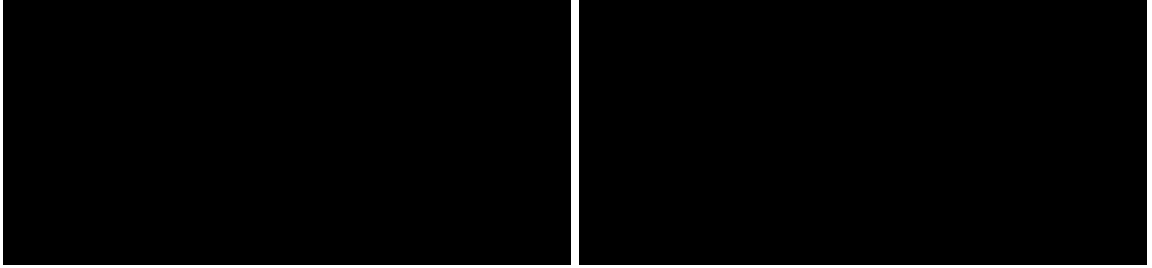
Pressure can then be computed as follows:

$$p = -\lambda \nabla \cdot \mathbf{v} = -\lambda \left(\frac{1}{r} \frac{\partial(r v_r)}{\partial r} \right) = -\lambda \left(\frac{1}{r} (3C_1 r^2 + 2C_2 r) \right) = -\lambda(3C_1 r + 2C_2) \quad (19)$$

We choose $R_1 = 2890\text{km}$, $R_2 = 6371\text{km}$, $g_0 = 9.81\text{ms}^{-2}$, $\rho_0 = 3300$, $E = 6 \cdot 10^{10}$, $\nu = 0.49$.



radial profiles of the displacement and pressure fields



displacement and pressure fields in the domain

