# Cloud droplets growth in adiabatic steady-state updrafts

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**Abstract**

In this exercise the growth of cloud droplets was estimated in an adiabatic steady-state updraft, in which ambient water vapour saturation was known. Vertical profiles of droplet radii were calculated for droplets with different dry radii and hygroscopicity parameters. The evolution of width of cloud droplet size spectra was assessed. It was assumed that the equilibrium saturation of water vapour can be described by the Köhler equation. Calculations in which equilibrium saturation was equal to that over a flat surface of pure water were performed and compared to the results obtained with the Köhler equation. Simplified assumptions about equilibrium saturation led to a narrower spectrum of droplet sizes, compared to the more accurate estimations which took into account the Köhler equation.

## 1 Introduction

The goal of this exercise was to estimate the evolution of width of cloud droplet spectra in an adiabatic, steady-state updraft. It was considered that cloud droplets form in an air parcel which rises with constant velocity. During this process, droplets grow by condensation of water vapour. It was assumed that saturation profile for an adiabatic steady-state updraft is known. The goal was to estimate the change of droplet radii with height, for droplets formed on different types of condensation nuclei. The purpose of this work was to find the width of droplet size spectra. In cloud models it is often impossible to accurately represent equilibrium saturation of droplets, which results in narrower size spectra than the spectra which can be obtained by a more accurate model. This exercise was meant to investigate this problem. Calculations were done in two parts, in the first part the equilibrium saturation was calculated from the Köhler equation, and in the second part it was assumed that the equilibrium saturation of activated droplets was the same as over a flat surface of pure water.

## 2 Köhler theory

For cloud droplets to form, aerosol particles must be present. Aerosol particles on which cloud droplets form are called condensation nuclei. They are often soluble. Equilibrium saturation over a droplet depends on its size and the amount and type of the solute. The amount of solute is represented by a dry radius - a radius of a sphere which volume corresponds to the volume of dry solute particle. The type of solute is represented by a parameter $\kappa$, called hygroscopicity parameter. Equilibrium saturation over a droplet is described by the $\kappa$-Köhler curve [1]:

$$S_{eq}(r,\kappa,r_d,T) = \frac{r^3 - r_d^3}{r^3 - r_d^3(1-\kappa)} exp\left(\frac{A(T)}{r}\right), \tag{1}$$

where

$$A(T) = \frac{2\sigma}{\rho_l R_v T}\ , \tag{2}$$

where r is the droplet radius, T is the temperature, $R_v = 461.5\ \frac{J}{kgK}$ is the individual gas constant for water vapor, $\sigma = 75.64 \times 10^{-3} \frac{N}{m}$ is the surface tension of the solution/air interface.

If the radius of a droplet is much larger than its dry radius, equation 1 can be written in an approximate form:

$$S_{eq} = 1 + \frac{A(T)}{r} - \frac{\kappa r_d^3}{r^3}, \tag{3}$$

where $\frac{A(T)}{r}$ is called the curvature term and $\frac{\kappa r_d^3}{r^3}$ is called the solute term. If the solute and curvature terms are neglected, the equilibrium saturation is the same as over a flat surface of pure water, $S_{eq} = 1$.

In order to accurately model the growth of a droplet, it is necessary to know its equilibrium saturation and thus to have information about the type and amount of solute. Usually it is impossible to know the composition of atmospheric aerosol and therefore many cloud models with bin representation use simplifications, such as neglecting solute and curvature terms.

Condensational growth of a cloud droplet by water vapor diffusion is described by the following equation:

$$r\frac{dr}{dt} = \frac{S - S_{eq}}{F_d + F_k}, \tag{4}$$

where

$$F_d = \frac{\rho_l R_v T}{D e_s(T)}, \tag{5}$$

$$F_k = \left(\frac{L}{R_v T} - 1\right)\frac{\rho_l L}{KT}, \tag{6}$$

where $S$ is the ambient saturation of air, $L$ is the latent heat, $\rho_l$ is the density of water, $e_s(T)$ is the saturated partial water vapor pressure, $D$ is the diffusion coefficient of water vapor (it depends on temperature and pressure: $D = 2.11 \times 10^{-5}\left(\frac{T}{T_0}\right)^{1.94}\frac{p_0}{p}\frac{m^2}{s}$, where $p_0 = 101\ 325$ Pa, $T_0 = 273$ K). $K$ is the thermal conductivity coefficient: $K = 4.1868 \times 10^{-3}[5.69 + 0.017(T - T_0)]\frac{W}{mK}$.

Saturated partial water vapor pressure can be calculated from the following equation:

$$e_s(T) = A\ exp(-B/T), \tag{7}$$

where $A = 2.53 \times 10^{11}$ Pa, $B = 5.42 \times 10^3$ K.

In the case when it is assumed that the equilibrium saturation over a droplet is the same as over a surface of pure water, the right-hand side of equation 4 is independent of droplet radius. The rate of change of droplet radius is inversely proportional to the radius – smaller droplets grow faster than larger ones. Therefore, assuming that $S_{eq} = 1$ results in a narrower spectrum of droplet sizes than calculating $S_{eq}$ from the Köhler equation.

This is important for predictions of precipitation, which forms when droplets grow by collision-coalescense. Difference in fall velocities between larger and smaller droplets allows them to collide. Precipitation is therefore more likely to form when droplet size spectrum is wide.

One can find the maximum of the Köhler curve (activation radius $r_{act}$ and activation saturation $S_{act}$) from equation 3 :

$$r_{act} = \sqrt{\frac{3\kappa r_d^3}{A(T)}}\ , \tag{8}$$

$$S_{act} = S_{eq}(r_{act}) = 1 + \frac{2\kappa r_d^3}{\left(\frac{3\kappa r_d^3}{A(T)}\right)^{3/2}}\ . \tag{9}$$

One can also derive the relation between dry radius and activation saturation for a given hygroscopicity parameter and temperature:

$$r_d(S_{act}, \kappa, T) = \left(\frac{2}{S_{act} - 1}\right)^{2/3}\frac{A(T)}{3\kappa^{1/3}}\ . \tag{10}$$

# 3 Köhler curve properties

In order to confirm that the approximated Köhler equation 3 can be used in this problem, the difference between the exact and approximated form of the Köhler equation was estimated. Figure 1 shows the exact and approximated Köhler curves and the difference between them (approximated results subtracted from exact results).

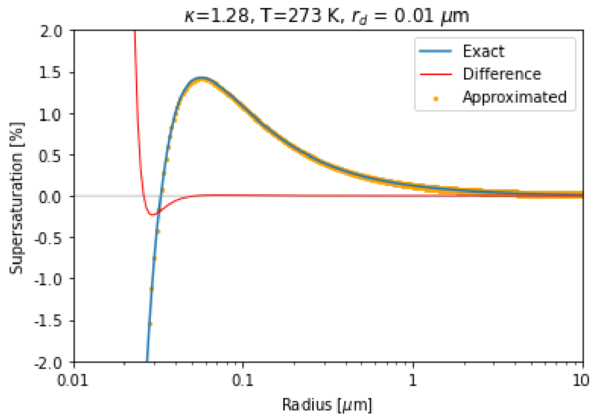


Figure 1: Köhler curves for $r_d = 0.01$ $\mu$m, $\kappa = 1.28$, T=273 K, calculated from the exact equation 1 (blue line), the approximated equation 3 (orange dots) and the difference between them (red line).

The approximated Köhler equation 3 (and specifically the solute term) is defined for weak solutions. It is valid only if the droplet radius is much larger than its dry radius. In figure 1, the dry radius is $r_d = 0.01$ $\mu$m. The approximated Köhler curve is shown only for $r > r_d$.

It can be seen in figure 1 that there is a significant difference between the exact and approximated values of supersaturation, but only for droplets with radii smaller than approximately 0.03 $\mu$m. For droplets with radii in the vicinity of 0.03 $\mu$m, the difference between the two supersaturations is approximately 0.2%. This difference between the exact and approximated results is expected for droplets which radii are not much larger than the dry radius. For droplets with radii much larger than the dry radius, differences between exact and approximated results are negligible and both equations correctly describe the Köhler curve.

Köhler curves were plotted for two values of dry radii: $r_d = 0.03$ $\mu$m and $r_d = 0.06$ $\mu$m and two values of the hygroscopicity parameter: $\kappa = 1.28$ and $\kappa = 0.61$, using the approximated Köhler equation 3. It was assumed that the temperature is constant and equal to 273 K. Values of hygroscopicity parameter correspond to two substances: NaCl ($\kappa = 1.28$) and $NH_4NO_3$ ($\kappa = 0.61$). For each curve parameters of their maximum (activation radius $r_{act}$ and activation saturation $S_{act}$) were calculated.

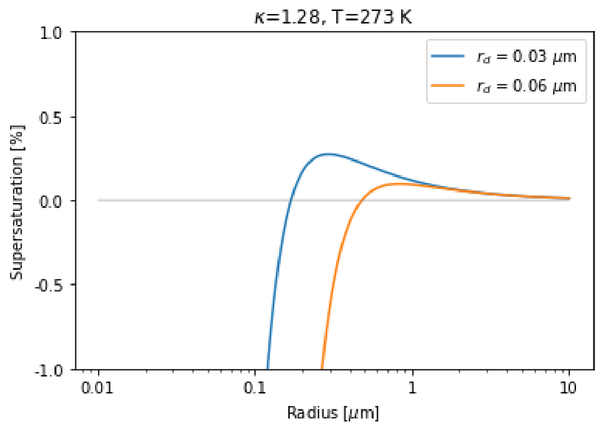


Figure 2: Köhler curves for $\kappa$=1.28, T=273 K and two values of dry radius: $r_d$=0.03 $\mu$m (blue line) and $r_d$=0.06 $\mu$m (orange line).

For curves shown in figure 2, parameters of activation are $r_{act}$=0.29 $\mu$m, $S_{act}$=1.003 for $r_d$=0.03 $\mu$m and $r_{act}$=0.83 $\mu$m, $S_{act}$=1.001 for $r_d$=0.06 $\mu$m.

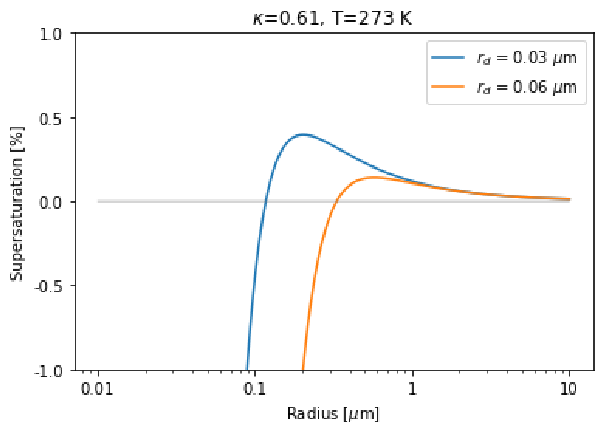


Figure 3: Köhler curves for $\kappa$=0.61, T=273 K and two values of dry radius: $r_d$=0.03 $\mu$m (blue line) and $r_d$=0.06 $\mu$m (orange line).

For curves shown in figure 3, parameters of activation are $r_{act}$=0.20 $\mu$m, $S_{act}$=1.004 for $r_d$=0.03 $\mu$m and $r_{act}$=0.57 $\mu$m, $S_{act}$=1.001 for $r_d$=0.06 $\mu$m.

It can be seen in figures 2 and 3 that droplets with $r_d$=0.06 $\mu$m activate in smaller saturations and their radii of activation are larger than for droplets with $r_d$=0.03 $\mu$m. Droplets with hygroscopicity parameter $\kappa$=1.28 activate in slightly smaller air saturations than droplets with $\kappa$=0.61.

# 4 Condensational growth in a steady-state updraft

In order to estimate the growth of cloud droplets, it is assumed that saturation profile in a steady-state updraft is known. Profiles of saturation are presented in figure 4 for two different velocities: $v = 1\frac{m}{s}$ and $v = 0.5\frac{m}{s}$.

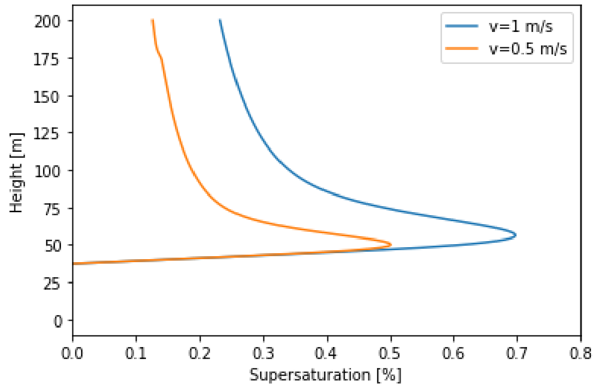


Figure 4: Saturation profiles for steady-state updrafts with velocity $v = 1\frac{m}{s}$ (blue line) and $v = 0.5\frac{m}{s}$ (orange line).

It is also assumed that profiles of temperature and pressure are known. It is considered that cloud droplets form on particles with dry radii 0.02 $\mu m$, 0.03 $\mu m$, 0.05 $\mu m$ and 0.1 $\mu m$, with two values of hygroscopity parameter $\kappa$: 0.61 and 1.28.

If environmental saturation is lower than the activation saturation for a given droplet and the droplet's radius is smaller than its activation radius ($r < r_{act}$), then it is considered that the droplet is in equilibrium with its environment, which means that saturation of air is equal to equilibrium saturation of the droplet. Otherwise equation 4 is solved.

Calculations are performed in two versions. In the first version, equilibrium saturation is calculated from the Köhler equation 3. In the second version, it is assumed that for droplets with $r > r_{act}$, curvature and solute terms can be neglected and $S_{eq}$ is equal to 1 in the Köhler equation 3. When a droplet is in equilibrium with environmental saturation, its radius is calculated using Newton's method. It is a numerical algorithm for finding roots of a function. It is used to find radii of droplets for given values of saturation, using equation 3.

Figures 5, 6 and 7 show vertical profiles of droplet radii, for droplets growing in an environment with given saturation values, with different dry radii, higroscopicity parameters and velocities.

Figure 5 shows the droplet radius in logarithmic scale. Figures 6 and 7 show the droplet radius in linear scale, in which differences in growth of different droplets above the cloud base are more distinguishable. These figures allow for comparison between growth of droplets with different assumptions about their equilibrium saturation.

Results presented in figures 5 and 6 were obtained with equilibrium saturation $S_{eq}$ calculated from the Köhler equation 3. Results presented in figure 7 were obtained with $S_{eq}$=1 for activated droplets and $S_{eq}$ calculated from the Köhler equation for not activated droplets.

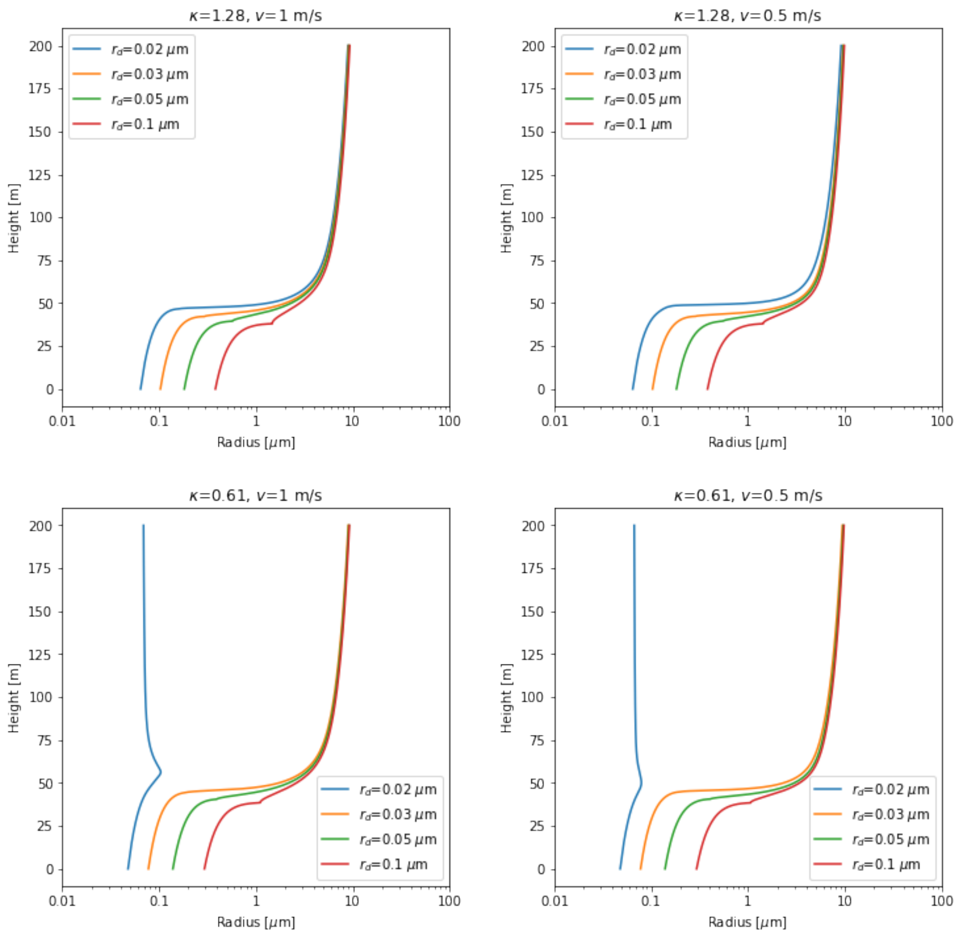


Figure 5: Growth of cloud droplets in logarithmic scale, with $S_{eq}$ calculated from the Köhler equation. Droplets have velocities 1 $\frac{m}{s}$ (left panels) or 0.5 $\frac{m}{s}$ (right panels) and parameters $\kappa$=1.28 (top panels) or $\kappa$=0.61 (bottom panels). Different colors correspond to different values of dry radius.

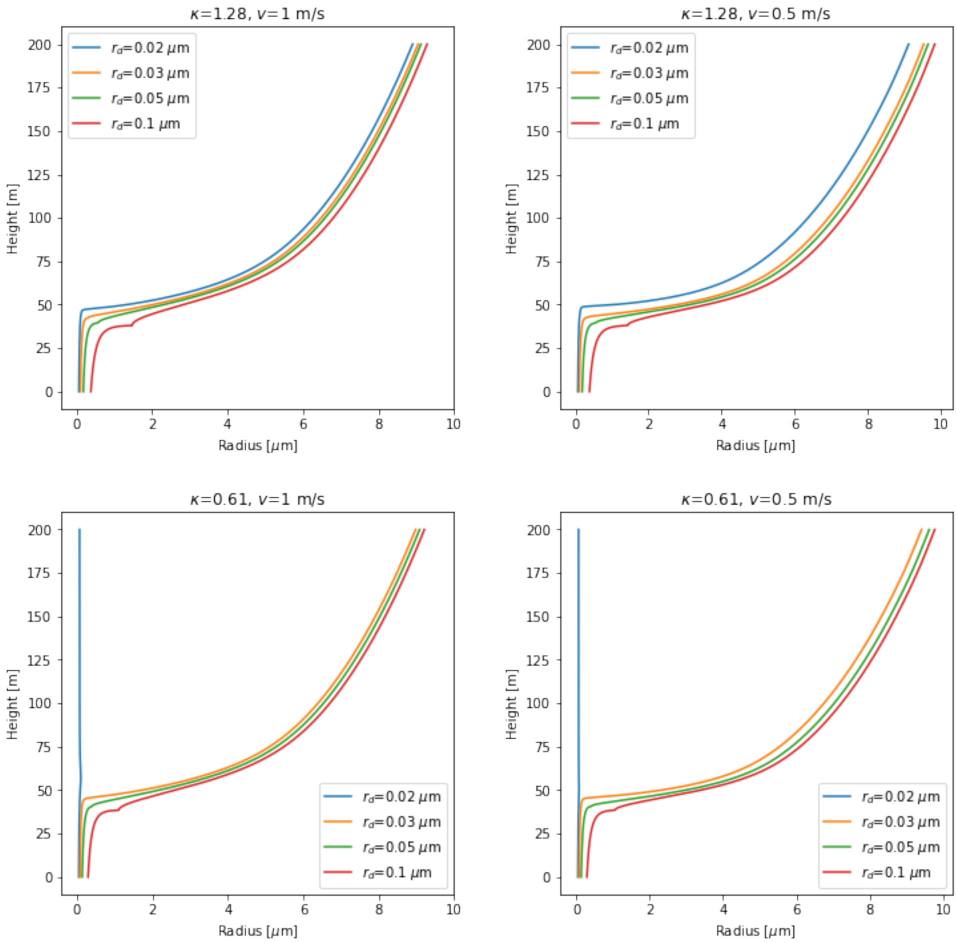


Figure 6: Growth of cloud droplets in linear scale, with $S_{eq}$ calculated from the Köhler equation. Droplets have velocities 1 $\frac{m}{s}$ (panels on the left) or 0.5 $\frac{m}{s}$ (panels on the right) and parameters $\kappa$=1.28 (top panels) or $\kappa$=0.61 (bottom panels). Different colors correspond to different values of dry radius.

Almost all droplets activate approximately on the height of 50 m. Only droplets with dry radius $r_d$=0.02 $\mu$m and $\kappa$=0.61 don't activate, but they are very close to activation. In particular, for droplet with $\kappa$=0.61, $v$=1 $\frac{m}{s}$, $r_d$=0.02 $\mu$m, activation supersaturation is approximately 0.7%, while maximum supersaturation of ambient air reaches 0.698%.

Droplets with larger dry radius activate lower in the cloud base and attain larger sizes than droplets with smaller dry radius. This is because droplets with larger dry radius have lower values of the activation supersaturation. Droplets activate when environmental supersaturation reaches the value of their activation supersaturation. For large droplets this happens at lower altitudes than for small droplets.

All droplets reach size of approximately 9-10 $\mu$m at the height of 200 m. For droplets which move with velocity 1 $\frac{m}{s}$, environmental saturation reaches approximately 1.5 times higher values, as presented in figure 4, but they move faster and thus have less time to grow than droplets with velocity 0.5 $\frac{m}{s}$. This results in the fact that all droplets reach similar sizes.

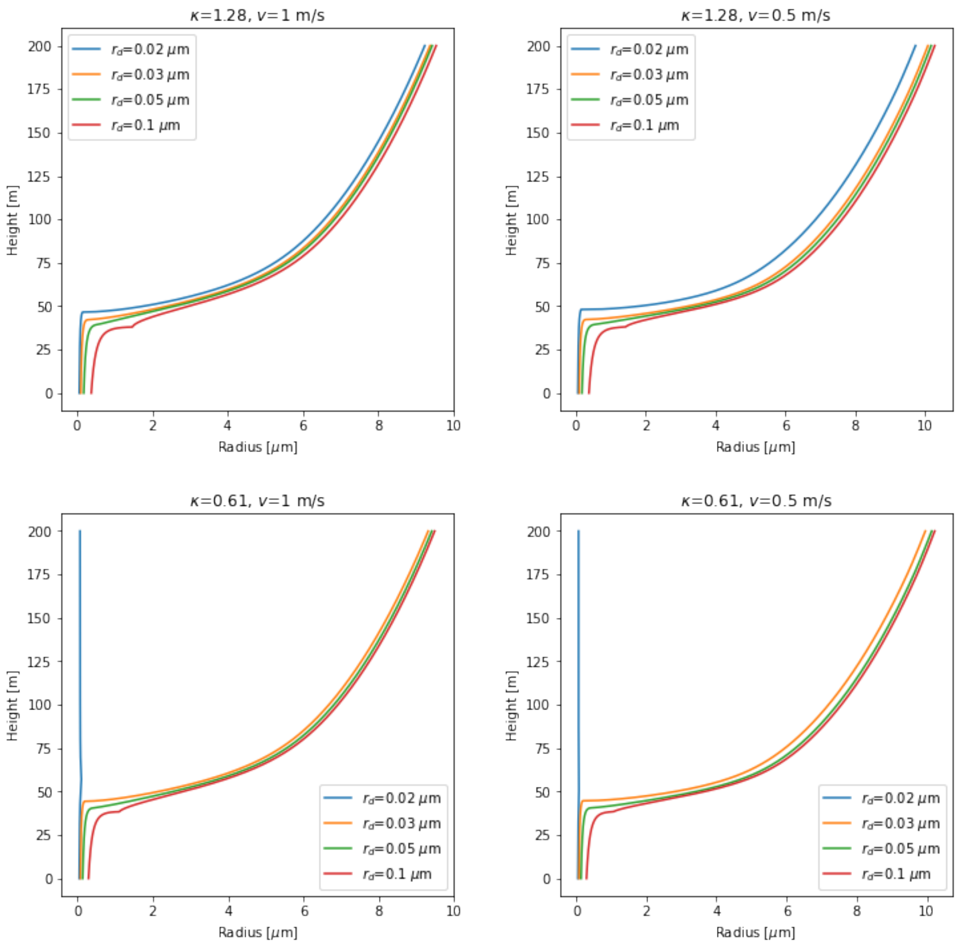


Figure 7: Growth of cloud droplets in linear scale, with $S_{eq}$=1 for activated droplets. Droplets have velocities 1 $\frac{m}{s}$ (panels on the left) or 0.5 $\frac{m}{s}$ (panels on the right) and parameters $\kappa$=1.28 (top panels) or $\kappa$=0.61 (bottom panels). Different colors correspond to different values of dry radius.

In the case when it is assumed that $S_{eq} = 1$ after activation, droplets reach similar sizes to droplets for which $S_{eq}$ is calculated from the Köhler equation. However, it can be seen by comparing figures 6 and 7, that in figure 6 the differences between droplets with different dry radii are more pronounced than in figure 7.

The width of cloud droplet spectrum can be defined as a difference between the radius of the largest and the smallest droplet. Figure 8 shows widths of droplet spectra for different velocities and hygroscopicity parameters. The differences are calculated between droplets with dry radius 0.1 $\mu$m and 0.03 $\mu$m, because the smallest droplets with dry radius 0.02 $\mu$m don't always activate.

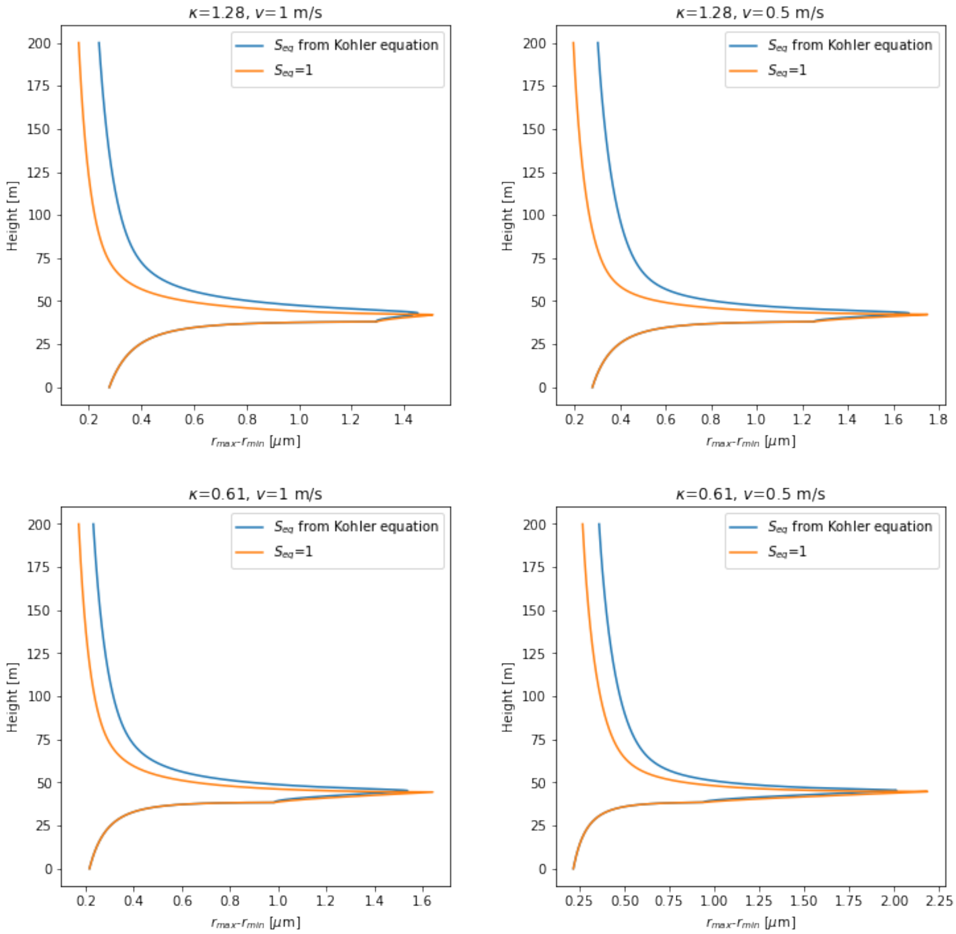


Figure 8: Width of spectrum of droplet radii for droplets with velocities 1 $\frac{m}{s}$ (left panels) or 0.5 $\frac{m}{s}$ (right panels) and parameters $\kappa$=1.28 (top panels) or $\kappa$=0.61 (bottom panels). Blue lines represent results with $S_{eq}$ calculated from Köhler equation. Orange lines represent results with $S_{eq} = 1$ for activated droplets.

Droplet spectra are wider for droplets with equilibrium saturation calculated from the Köhler equation, compared to the case where $S_{eq}$=1. This is true for every set of parameters $\kappa$ and $v$ presented in figure 8. Below the cloud base there is no difference between the two results, since before activation the two results are calculated in the same way. Widths of droplet spectra reach maximum values in the region of the cloud base, where some of the droplets activate and others are not yet activated. Above the cloud base, the width of droplet spectra decreases slightly with height.

Figure 9 shows differences between radius square of the smallest and the largest droplets, for different velocities an hygroscopicity parameters. The difference is calculated between droplets with dry radii 0.1 µm and 0.03 µm.

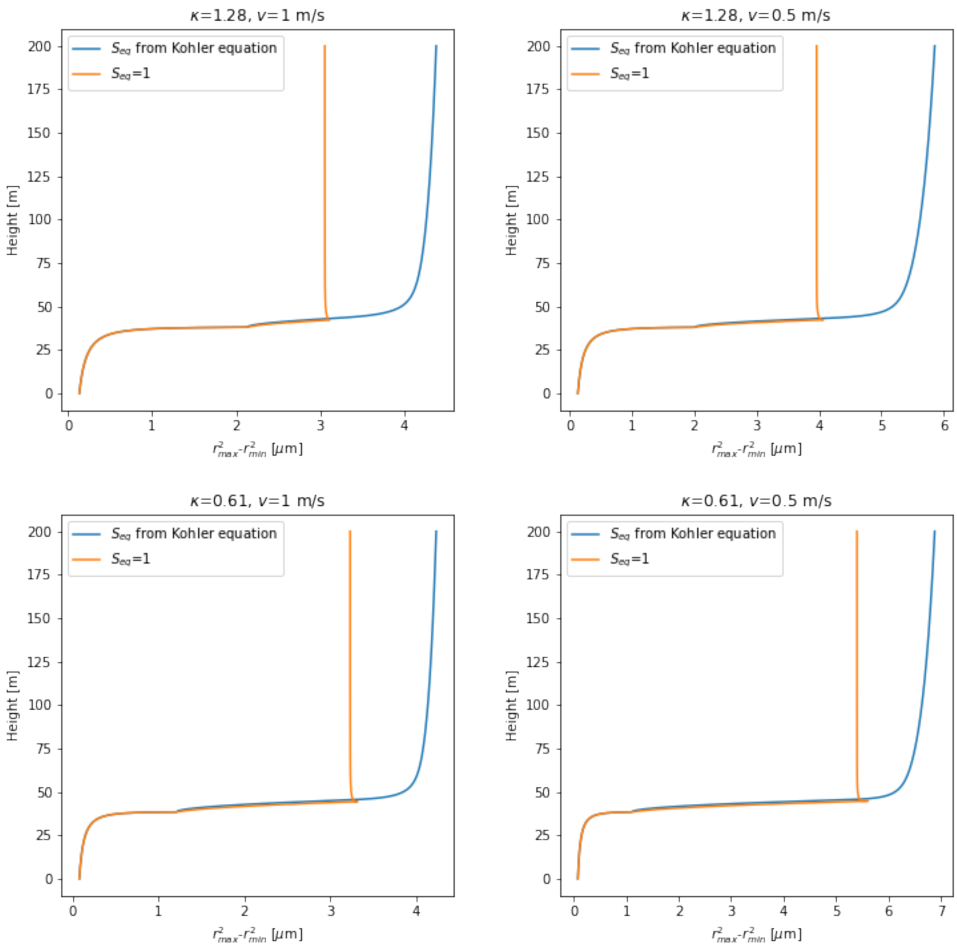


Figure 9: Width of spectrum of squared droplet radii for droplets with velocities 1 $\frac{m}{s}$ (left panels) or 0.5 $\frac{m}{s}$ (right panels) and parameters $\kappa$=1.28 (top panels) or $\kappa$=0.61 (bottom panels). Blue lines represent results with $S_{eq}$ calculated from Köhler equation. Orange lines represent results with $S_{eq} = 1$ for activated droplets.

The difference of squared radii of largest and smallest droplets grows rapidly near the cloud base. In the case when equilibrium saturation is calculated from the Köhler equation, difference of squared radii grows slightly with height above the cloud base. In the case when it is assumed that $S_{eq} = 1$ for activated droplets, it remains constant with height above the cloud base. This stems directly from equation 4. In this case the right-hand side of the equation does not depend on the radius. Therefore the change of radius square is the same for all droplets.

# 5 Conclusions

The growth of cloud droplets in an adiabatic steady-state updraft was estimated for droplets with different dry radii, $\kappa$ parameters and velocities. Droplets with larger dry radii attained larger sizes than droplets with smaller dry radii and activated lower in the cloud base. Activated droplets reached sizes of 9-10 $\mu$m at the height of 200 m. Evolution of cloud droplet spectra was estimated. It was assumed that equilibrium saturation of droplets was described by the Köhler equation, but additional calculations were performed, with equilibrium saturation equal to 1 for activated droplets. It was found that droplet spectra were wider for droplets with equilibrium saturation calculated from the Köhler equation than for equilibrium saturation equal to 1. These results show that in order to accurately estimate spectrum width, it is necessary to correctly describe condensation nuclei on which droplets form, in particular to have knowledge about their dry radii and hygroscopicity parameters. Assuming that equilibrium saturation is the same as over a flat surface of pure water results in a narrower spectrum than the more accurate estimations.

## References


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[3] *Newton's method of finding roots of a function*