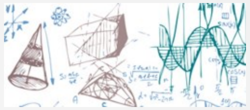

Introduction to probability, statistics and data handling

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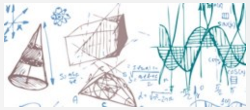


Joint distributions (I)

- If X and Y are two discrete RVs, we can define P.D.F. of X and Y as follow:

$$p(X = x, Y = y) = f(x, y)$$

$$f(x, y) \geq 0$$

$$\sum_x \sum_y f(x, y) = 1$$

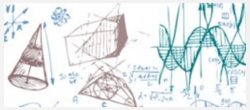
- If we assume that: $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, then the probability of the event that $X = x_i$ and $Y = y_j$ is given by:

$$p\left(X = x_i, Y = y_j\right) = f(x_i, y_j)$$

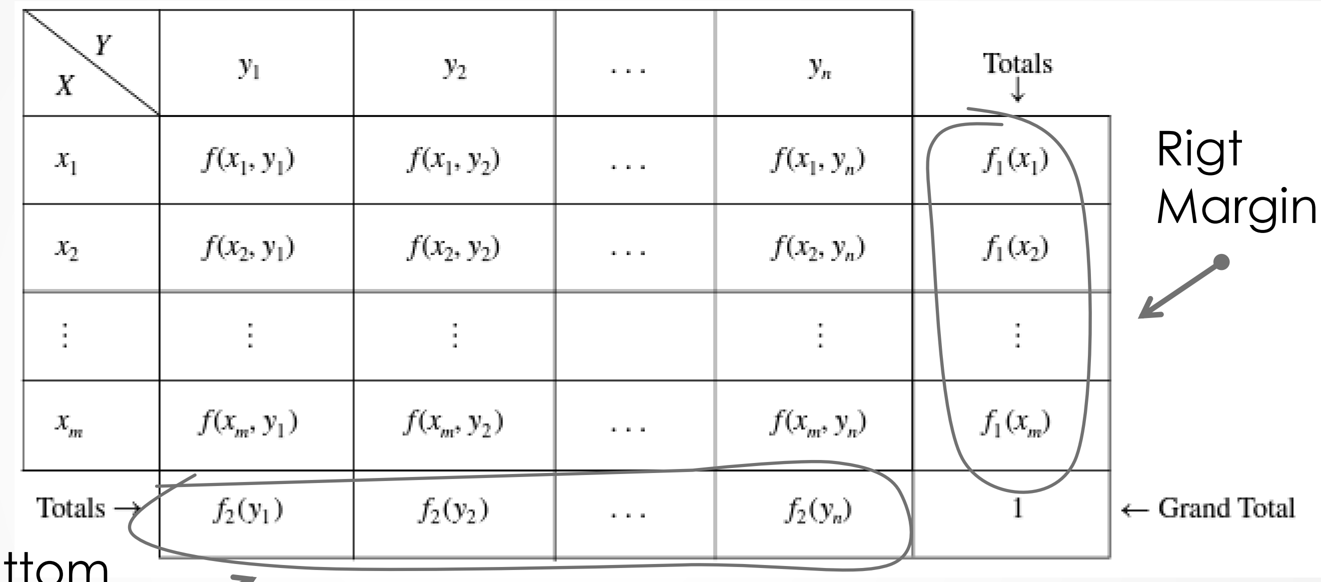
- Respective probabilities for $X = x_i$ and $Y = y_j$ are given by:

$$p(X = x_i) = f_1(x_i) = f_x(x_i) = \sum_k f(x_i, y_k)$$

$$p\left(Y = y_j\right) = f_2\left(y_j\right) = f_y\left(y_j\right) = \sum_l f\left(x_l, y_j\right)$$



Joint distributions (II)



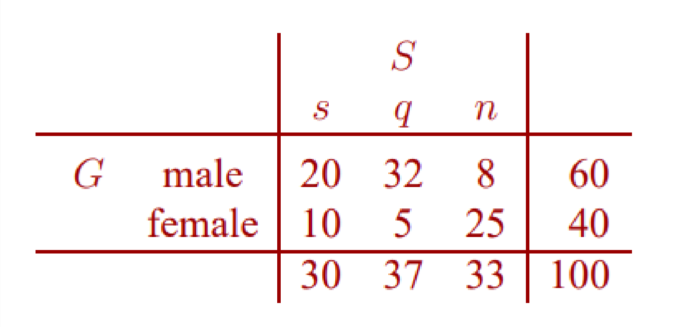
$X \backslash Y$	y_1	y_2	$\dots$	y_n	Totals ↓
x_1	$f(x_1, y_1)$	$f(x_1, y_2)$	$\dots$	$f(x_1, y_n)$	$f_1(x_1)$
x_2	$f(x_2, y_1)$	$f(x_2, y_2)$	$\dots$	$f(x_2, y_n)$	$f_1(x_2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_m	$f(x_m, y_1)$	$f(x_m, y_2)$	$\dots$	$f(x_m, y_n)$	$f_1(x_m)$
Totals →	$f_2(y_1)$	$f_2(y_2)$	$\dots$	$f_2(y_n)$	1 ← Grand Total

Rigt Margin

Bottom Margin

- Because the respective probabilities: $p(X = x_i)$ and $p(Y = y_j)$ are found on the margins of the joint probability table, we call both functions $f_1(x_i)$ and $f_2(y_j)$ the **marginal probability functions** of X and Y

Example: Students in a class of 100 were classified according to gender (G) and smoking (*S*) as follows:



		S			
		s	q	n	
G	male	20	32	8	60
	female	10	5	25	40
		30	37	33	100

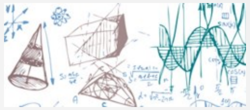
s: “now smokes”,
q: “did smoke but quit”
n: “never smoked”.

Identify:

- joint distribution function
- marginal distributions

Find the probability that a randomly selected student

1. is a male;
2. is a male smoker;
3. is either a smoker or did smoke but quit;
4. is a female who is a smoker or did smoke but quit.



Joint distributions (III)

- It is essential to note that for both marginal density functions we have:

$$\sum_i f_1(x_i) = 1, \sum_j f_2(y_j) = 1$$

- The two above are equivalent of:

$$\sum_i \sum_j f(x_i, y_j) = 1$$

- Since all of these functions represent P.D.F. they must be normalised, or in other words the probability of all entries is 1
- The **joint distribution function** of RVs X and Y is given by

$$F(x, y) = p(X \leq x, Y \leq y) = \sum_{u \leq x} \sum_{v \leq y} f(u, v)$$

- So, to get a value of $F(x, y)$ for a given pair (x, y) we need to sum-up all the entries for which $x_i \leq x$ and $y_j \leq y$

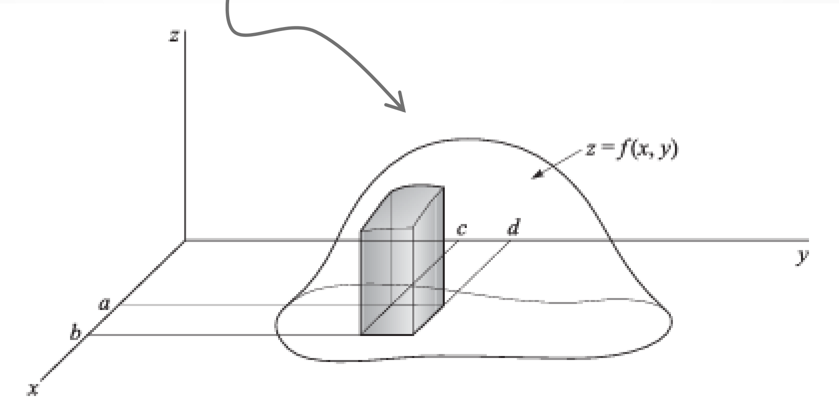
Joint distributions (IV)

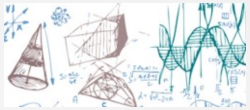
- Again, by analogy we can easily obtain the joint probability function for continuous RVs X and Y:

$$f(x,y) \geq 0, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

- The probability can be estimated using the joint P.D.F. as follow:

$$p(a < X < b, c < Y < d) = \int_{a}^{b}\int_{c}^{d} f(x,y)dxdy$$





Joint distributions (V)

- Now, we can define the joint distribution function:

$$F(x,y) = p(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)dudv$$

- We also have the following:

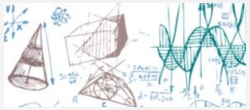
$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$$

- By analogy, the respective marginal functions can be defined for both density, $f(x,y)$, and distribution, $F(x,y)$ functions

$$f_1(x) = \int_{-\infty}^{\infty} f(x,v)dv \, , f_2(y) = \int_{-\infty}^{\infty} f(u,y)du$$

$$F_1(x) = p(X \leq x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,v)dudv$$

$$F_2(y) = p(Y \leq y) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f(u,v)dudv$$



Independent RVs

- We learned how to calculate probability of independent events:

$$\boxed{p(\mathbb{A} \cap \mathbb{B}) = p(\mathbb{B}|\mathbb{A})p(\mathbb{A}) = p(\mathbb{B})p(\mathbb{A})}$$

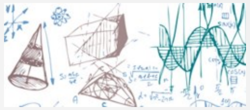
- This definition can also be used for probability functions. Say, X and Y are RVs. If the events $X = x$ and $Y = y$ are independent for all x and y, then we say that X and Y are independent RVs. We also have:

$$p(X = x, Y = x) = p(X = x) \cdot p(Y = y)$$

$$f(x, y) = f_1(x)f_2(y)$$

- Similarly, we say that X and Y are independent RVs if the events $X \leq x$ and $Y \leq y$ are independent for all x and y. We can write:

$$p(X \leq x, Y \leq y) = p(X \leq x)p(Y \leq y) \rightarrow F(x, y) = F_1(x)F_2(y)$$



Conditional P.D.F.s

- Let’s assume that X and Y are CRVs. We define the conditional density function of Y given X, as:

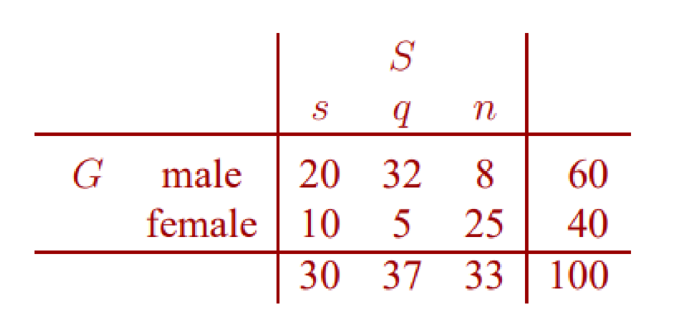
$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$f(x|y) = \frac{f(x,y)}{f_2(y)}$$

- So, to define the conditional P.D.F. we need a joint P.D.F. and a marginal one to calculate an appropriate probability we do:

$$p(c < Y < d|x < X < x + dx) = \int_c^d f(y|x)dy$$

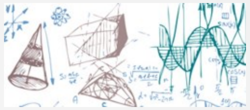
Example: Students in a class of 100 were classified according to gender (G) and smoking (S) as follows:



		S			
		s	q	n	
G	male	20	32	8	60
	female	10	5	25	40
		30	37	33	100

Calculate the probability that a randomly selected student is

1. a smoker given that he is a male;
2. female, given that the student smokes.



Covariance

- Next step, as usual, lead to more RVs. Let's see what's new if we consider two RVs X and Y with joint density function $f(x, y)$:

$$\mu_X = E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy$$

$$\mu_Y = E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy$$

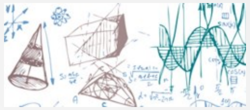
$$\sigma_X^2 = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)^2 f(x,y)dxdy$$

$$\sigma_Y^2 = E[(Y-\mu_Y)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\mu_Y)^2 f(x,y)dxdy$$

- And what about the mixed terms? Analysis leads to the **covariance**

$$\boxed{\sigma_{XY} = Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]}$$

$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y)f(x,y)dxdy$$



Theorems regarding covariance

- **Theorem 1**. For any RVs the following is true:

$$\sigma_{XY} = Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - E[X]E[Y]$$

- **Theorem 2**. In case the RVs X and Y are independent:

$$Cov[X,Y] = 0$$

- **Theorem 3**. For any two RVs we have:

$$V[X \pm Y] = V[X] + V[Y] \pm 2Cov[X,Y]$$

- **Theorem 4**. For any two RVs we have:

$$|\sigma_{XY}| \leq \sigma_X \sigma_Y$$

- NOTE, that the converse of Theorem 9 is not necessarily true!

Correlation coefficient

- The covariance gives us a strong hint on how to measure **the dependence** of RVs. If X and Y are independent, then:

$$Cov[X,Y] = \sigma_{XY} = 0$$

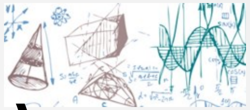
- On the other hand, if they are completely dependent (e.g., $X = Y$), then:

$$Cov[X,Y] = \sigma_{XY} = \sigma_X \sigma_Y$$

- So, one can use the following to measure the dependence of RVs:

$$\boxed{\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}}$$

- We call it the correlation coefficient and it is easy to note that its values vary between $[-1,1]$
- In case where the CC is equal zero, we call the RVs linearly uncorrelated. In general, however, the variables may or may not be independent.



Change of variables (I)

- Let's assume we know distribution functions of one or more RVs. In practice, we are often interested in finding distributions of other RVs that depend on them (here we focus on CRV)
- **Theorem 1.** Let X be a CRV with P.D.F. given by $f(x)$. Next, define RV $U = \varphi(X)$, where $X = \omega(U)$. The P.D.F. of U is given by $g(u)$ where:

$$g(u)|du| = f(x)|dx|$$

$$g(u) = f(x)\left|\frac{dx}{du}\right| = f\big(\omega(u)\big)\omega'(u)$$

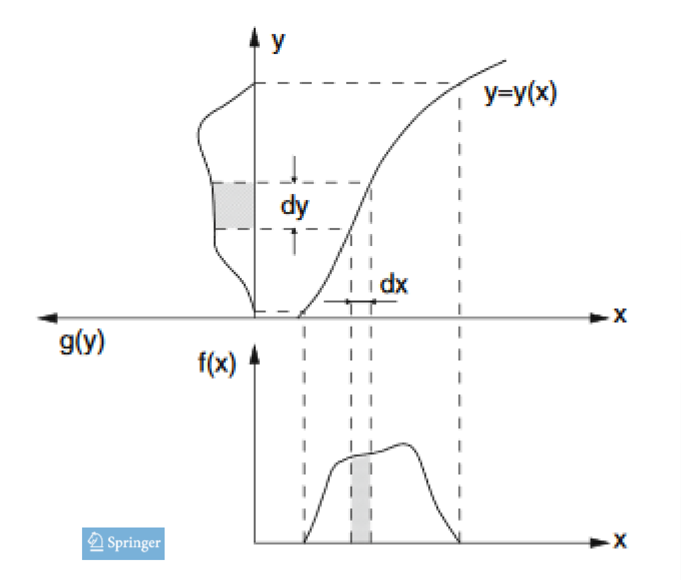
- For more than one variable things getting a bit more difficult…
- **Theorem 2.** Let X and Y be CRVs having joint P.D.F. $f(x,y)$. Let's define new variables $U = \varphi_1(X,Y)$ and $V = \varphi_2(X,Y)$, where $X = \omega_1(U,V)$ and $Y = \omega_2(U,V)$. Then the joint density function of U and V is given as:

$$g(u,v)|dudv| = f(x,y)|dxdy|$$

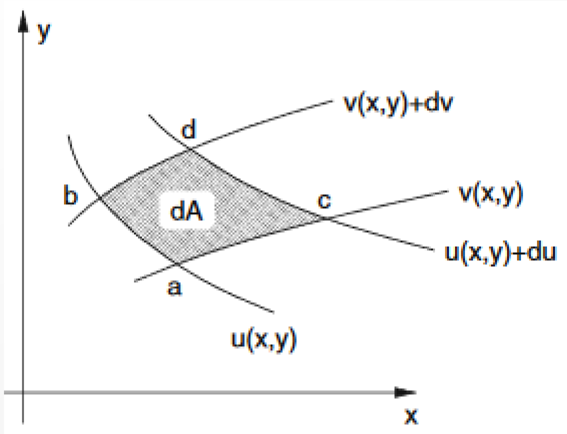
Change of variables (II)

$$g(u,v) = f(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = f(\omega_1(u,v),\omega_1(u,v))|J|$$

- For multi-dimensional case we have something brand new – Jacobian determinant or Jacobian



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$



Standarised RVs

- Let X be a RV with the mean value μ and standard deviation σ. We can define an associated RV that is called **standardised random variable**:

$$\boxed{X^* = \frac{X-\mu}{\sigma}}$$

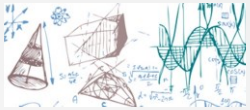
- Note, that X^* has a mean of zero and a variance of 1 – this is why we call it standardised in the first place!

$$\boxed{E[X^*] = 0, V[X^*] = 1}$$

$$E[X^*] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}E[(X-\mu)] = \frac{1}{\sigma}(E[X]-\mu) = 0$$

$$V[X^*] = V\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma^2}E[(X-\mu)^2] = 1$$

- We will be using the SRV all the time – it makes comparison of different distributions possible.



More than μ and σ

- Sometimes we are more interested in the most probable value of RV instead of the mean (especially important for asymmetric distributions)
- The MPV, also called the mode is the value of the random variable that corresponds to the highest probability:

$$\mathcal{P}(X = x_m) = max$$

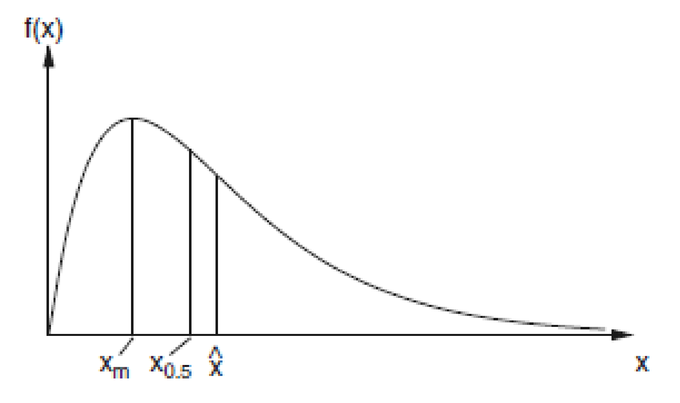
- In case we have a regular function representing the P.D.F. then the mode can be easily found:

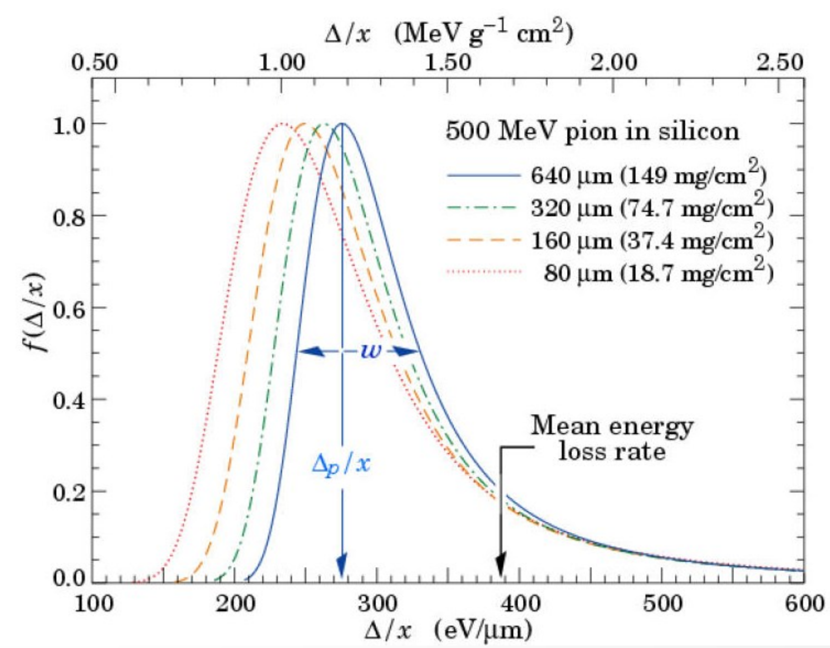
$$\frac{d}{dx} f(x) = 0, \frac{d^2}{dx^2} f(x) < 0$$

- If a given P.D.F has just one maximum, we call it a unimodal.
- The median can also be defined using the distribution function:

$$F(x_{0.5}) = \mathcal{P}(X < x_{0.5}) = 0.5$$

More than μ and σ



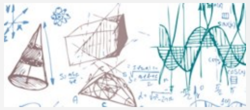


- It is also useful to define **quantiles**:

$$F(x_{0.25}) = 0.25, F(x_{0.75}) = 0.75$$

- And deciles…

$$F(x_q) = \int_{-\infty}^{x_q} f(x)dx = q$$



Chebyshev's Inequality

- There is an extraordinary theorem related to the fundamental properties of RV (both discrete and continuous). We just need both the expectation value and variance to be finite.
- **Theorem 5**. Suppose that X is a random variable. Let the mean and variance of this RV be μ and σ^2 respectively. If we assume that they are both finite, then if ϵ is any positive number:

$$p(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

$$\epsilon = k\sigma \rightarrow p(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

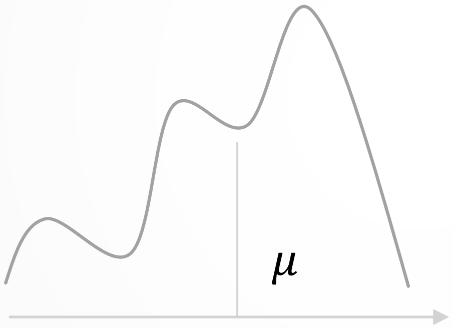
- For instance, let $k = 2$:

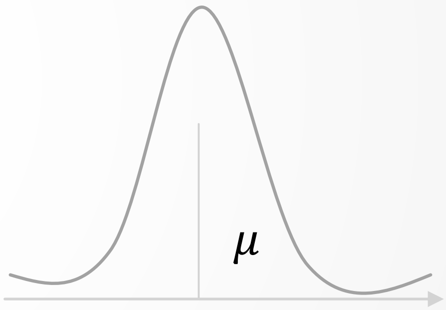
$$p(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2} \rightarrow p(|X-\mu| \geq 2\sigma) \leq \frac{1}{4}$$

$$p(|X-\mu| < 2\sigma) \geq \frac{3}{4}$$

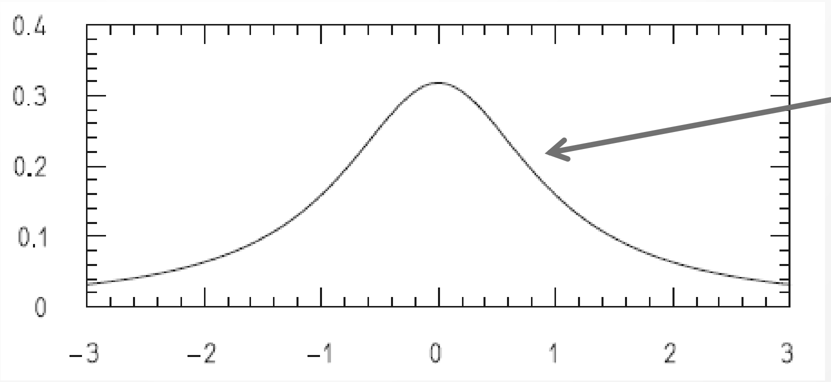
Chebyshev’s Inequality

- This simple rule is actually quite incredible – **without** making any assumptions regarding the probability distribution!
- For any RV the probability of X being different from its mean value by more that two „standard deviations” is less than 25%.



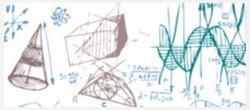


$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$



Cauchy P.D.F

The variance does not exist!



The Binomial Distribution

- We already know this distribution – it emerges when we consider an experiment such as tossing a coin, rolling a die or choosing a marble from a box repeatedly.
- So, we considering trials. Each outcome will have constant probability assigned (that should not change in time, and is the parameter of the Bernoulli prob. model family).
  - Sometimes we are also interested in processes where the probability is not constant (out of the scope of our lecture, however)
- We then say that p is a success and q is a failure (in a Bernoulli sense) and can compose the following P.D.F.

$$\boldsymbol{f(x) = B(n,p) = p(X = x) = \binom{n}{x} p^{x} q^{n-x} = \frac{n!}{x!\,(n-x)!} p^{x} q^{n-x}}$$

- The RV denote the number of successes x in n trials, $x = 0, 1, \dots, n$

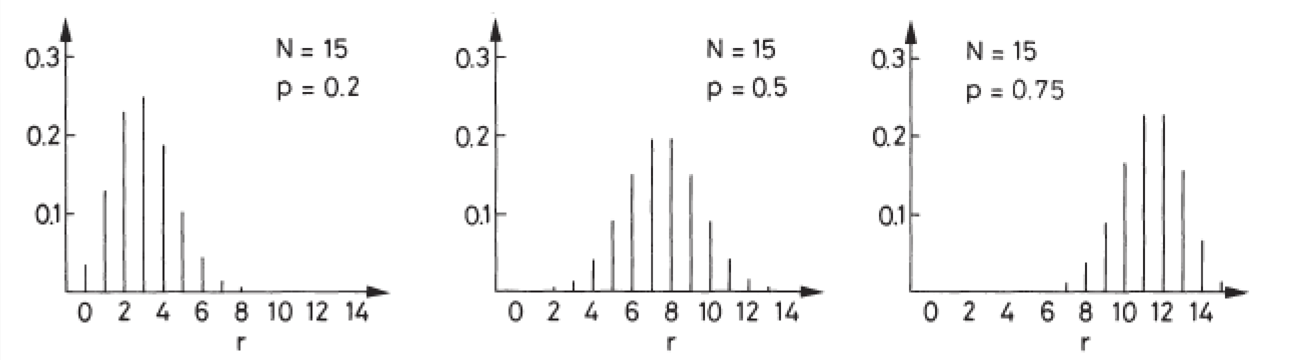
The Binomial Distribution

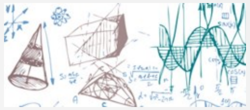
- The mean and variance can be fairly easy calculated:

$$\mu = \sum_x xP(x) = np$$

$$\sigma^2 = \sum_x (x-\mu)^2 P(x) = np(1-p)$$

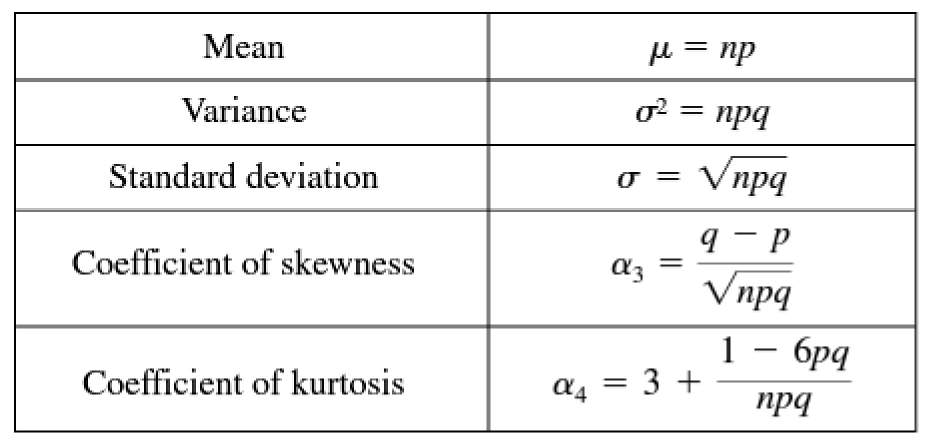
- In the limit of „large” n and „no too small” p we can very accurately approximate the Binomial distribution with Gaussian one





The Binomial Distribution

- **Properties of the Binomial P.D.F.**



Mean	$\mu = np$
Variance	$\sigma^2 = npq$
Standard deviation	$\sigma = \sqrt{npq}$
Coefficient of skewness	$\alpha_3 = \dfrac{q - p}{\sqrt{npq}}$
Coefficient of kurtosis	$\alpha_4 = 3 + \dfrac{1 - 6pq}{npq}$

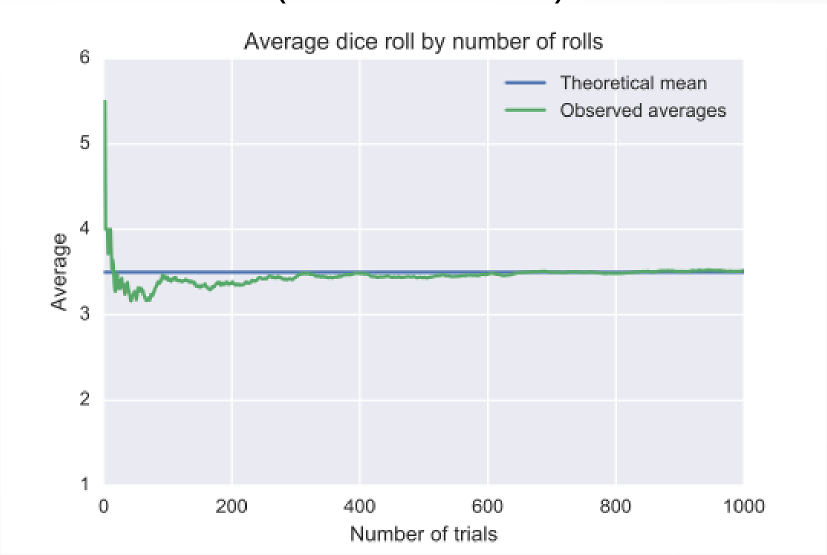
- We can mote something interesting here
- **Theorem 6**. Let X be the RV giving the number of successes in n Bernouli trials, so that $\frac{X}{n}$ is the proportion of successes. Then if p is the probability of success and ϵ is any positive number:

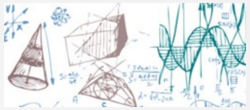
$$\lim_{n\to\infty} prob\left(\left|\frac{X}{n} - p\right| \geq \epsilon\right) = 0$$

Law of large numbers

- Building on the knowledge we gained today, we can formulate a very advanced theorem, that is considered fundamental for statistics.
- **Theorem 7**. *Let $X_1, X_2, \cdots, X_n$ be mutually independent RV (discrete or continuous), each having finite mean μ and variance σ^2. Then if we take into consideration a new RV: $\boldsymbol{S_n = X_1 + X_2 + \cdots + X_n}$, then:*

$$\lim_{n\to\infty} p\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) = 0$$





The Law of Large Numbers

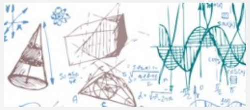
- Imagine that our sample space can be divided into k events (or outcomes that we wish to study): $\{A_j\}, j = 1, \dots, k$. What are the respective probabilities of such events p_j?
- Well, in principle we should conduct an experiment, collect a data sample and then calculate the frequency f_j (we assume that n below means the number of events of type j observed):

$$f_j = \frac{1}{n}\sum_{i/1}^{i/n} X_{ij} = \frac{1}{n}X_j$$

- So, note that X_j is a binomial R.V. that takes the following values:

$$X_j = \begin{cases} 1 \; if \, A_j \; occured \\ 0 \quad otherwise \end{cases}$$

- Now, how is f_j related to the probability p_j? Remember, the probability is just a number, whilst the frequency is a R.V.



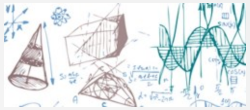
The Law of Large Numbers

- It is essential to understand, the last point – remember frequency will always depend on a particular sample! Different sample will yield a different frequency.
- Having said that, we can however write (remember that X_j is a binomial R.V.!):

$$E[f_j] = E\left[\frac{X_j}{n}\right] = \boldsymbol{p_j}, E[X_j] = np_j$$

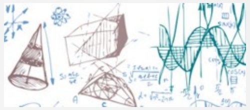
$$\sigma^2(f_j) = \sigma^2\left(\frac{X_j}{n}\right) = \frac{1}{n^2}\sigma^2(X_j) = \frac{1}{n}p_j q_j = \boldsymbol{\frac{1}{n}p_j(1 - p_j)}$$

- In words: the expectation value of the frequency (event A_j) is equal to the probability of success. The variance of the frequency about its mean value can, in turn, be made arbitrarily small – just need to collect enough data! (large n).
- **This, actually, is the law of large numbers**!



The Law of Large Numbers

- Let’s just think about the variance for a second. It is a product of these two elements: $1/n$ and $p_j(1-p_j)$. The latter is always less than unity (the max value is: $max\{p_j(1-p_j) = 1/4\}$), so the „smallness” of departure **will be governed by the number of observed events**.
- Using this argument and the result from previous slide we can justify that the approach, where **respective probabilities** of events that are estimated by **frequencies measured directly** in experiments, is **the right one**!
- The square of the error we make doing so is inversely proportional to the number of measurements in an experiment – this kind of error is called a **statistical** one
- This is essence of, so called, **counting experiments** such as: number of decaying particles, number of animals with a given traits, number of defective items, …



The teaser…

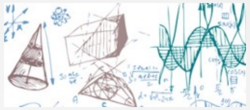
- Imagine that an experiments has been conducted that observed that a certain particle decays in two ways: $X \rightarrow Y_1 + Y_2$ and $X \rightarrow Y_1 + Y_2 + Y_3$. It was estimated that the rate of decays of the second type was factor 200 less that the first one: $R = \frac{Br(X \rightarrow Y_1 + Y_2 + Y_3)}{Br(X \rightarrow Y_1 + Y_2)} = \frac{1}{200}$. Now, a new experiment is being designed to measure the R with accuracy of 1%. How large the data sample must be to achieve that?

$$p_j = 0.005, 1 - p_j \approx 1$$

$$\frac{\sigma(f_j)}{f_j} = \frac{\sigma(f_j)}{R} = 200 \cdot \sigma(f_j) = 0.01$$

$$\sigma(f_j) = 0.00005 \rightarrow \sigma^2(f_j) = 2.5 \cdot 10^{-9}$$

$$2.5 \cdot 10^{-9} = \frac{1}{n} \cdot 0.005 \rightarrow \boldsymbol{n = 2.0 \cdot 10^{6}}$$



The Law of Large Numbers

- And finally to sum up the **LoLN**

Let $X_1, X_2, \dots, X_n$ be mutually independent random variables (no particular P.D.F. is assumed here), each of which have finite mean, μ, and variance, σ^2. Now let's us define new R.V.: $S_n = X_1 + X_2 + \cdots + X_n, n = 1, 2, \dots$ The probability that the arithmetic mean of $X_1, X_2, \dots, X_n$ differs from its expected value more than ϵ approaches zero as $n \to \infty$:

$$\boxed{\lim_{n\to\infty} p\left(\left|\frac{S_n}{n} - E\left[\frac{S_n}{n}\right]\right| \geq \epsilon\right) = \lim_{n\to\infty} p\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) = 0}$$