Introduction to probability, statistics and data handling

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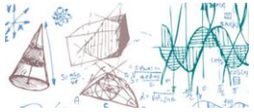
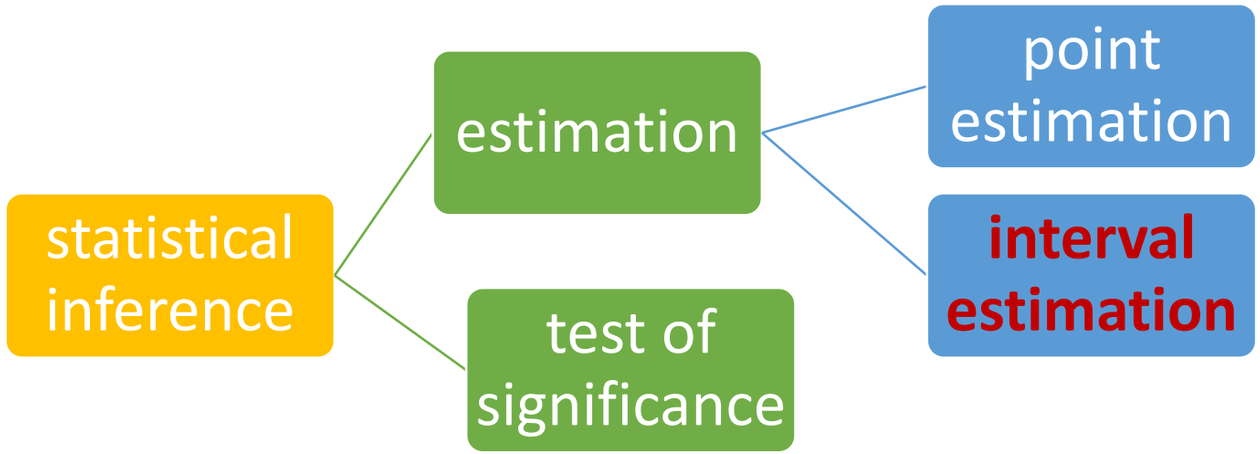
Statistical Inference

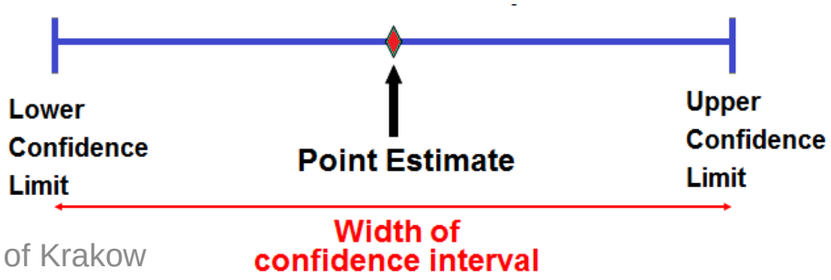


- The statistical inference consists in arriving at (quantitative) conclusions concerning a **population** where it is impossible or impractical to examine the entire set of observations that make up the population. Instead, we depend on a **subset** of observations - a **sample**.



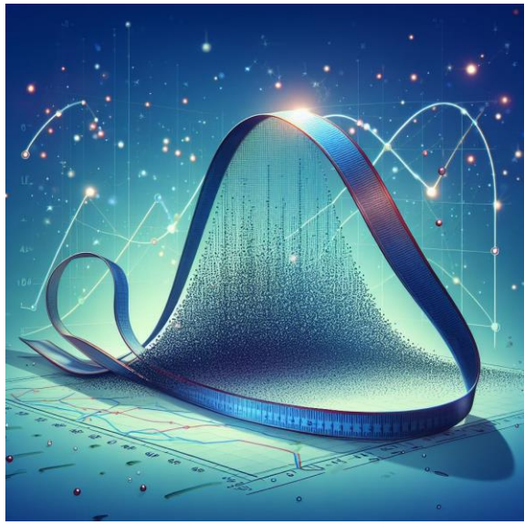
Example:
I checked 5 restaurants in Milano and claim that mean cost of pizza is 15 €.

In Milano pizza costs betweeen 10 and 20 €.

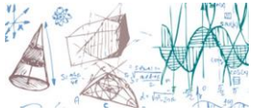


Estimation with confidence

- We discussed a number of approaches to study problem of parameter estimation – we called it the **point estimation**, since we were interested only in „a value” of some parameter
- If there is a special name, it means that there must be something more… **And it is**!
- The topic of today’s lecture will be the **interval estimation or estimation with confidence**
- We also make our first strides into **hypothesis testing**, for which defining the confidence if crucial

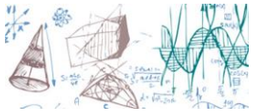


Estimation with confidence



- **Consider the following**: we performed an experiment and got estimate on a parameter using one of the methods we learned
- Not bad… **Next we repeat the experiment and got another estimate – what should we expect**? What kind of result should be treated as „**plausible**” and what „**unlikely**”?
- It is obvious that the ability of obtaining a plausible range of values for any unknown population parameter is a powerful tool
- Remember – we are talking about a range defined in the space of model parameters (the model must be itself of course reasonable)
- The min and max limit of this plausible parameter(s) range we call **confidence limits** and the corresponding range **confidence interval**

Example 1



As usual we start discussing a bunch of experiments and discuss along the topic at hand

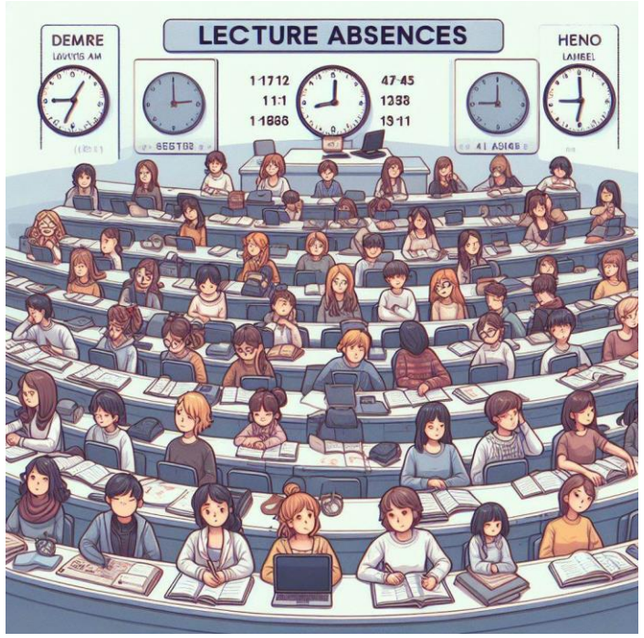
- **A psychology studies** were conducted to check correlation of the mental capabilities and proneness to injury among children. Total of 621 children were studied between ages of 4 and 11. The study period was divided into two intervals 4 – 7 and 8 – 11
- Say, one child experienced 3 accidents between ages 4 – 7. We could assume a Poisson model to describe the variability of the data sample. **With this single number we could still use the ML technique and obtain: $\hat{\mu} = 3$**
- We can than state that the accidents do happen but they are rather rare events
- However, it could be very useful to be able to provide some statement, with confidence 90%, that the mean lies in a range between this and that value (e.g., 1.0 – 5.0)

Example 2

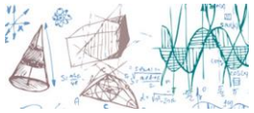
- **Mining disasters** (more than 10 victims) registered in a country in Europe were studied between 15 Mar 1851 and 22 Mar 1962
- Total of 191 accidents occurred – that is a lot…
- The first accident after the 15 Mar 1851 occurred 157 days after. Say, we stopped the study after the second accident. We would then obtain a single observation on a R.V. X. For this discussion we assume that it follows the exponential distribution
- **We have now an estimate on the mean time interval expected between accidents**
- Again – it would be really useful to be able to define an interval of confidence for this average time between disasters

Example 2

- **Lecture absences** of 113 student from a course A were noted over a period of two semesters (total of 24 lectures). In this case we could try to use a binomial model for the number of total missed lectures
- Say we divide the tested period into two parts – semester one and semester two. The corresponding number of lectures were 11 and 13
- The data showed that one student missed four (hmm… a lot) lectures, so the proportion of missed lectures would be $p = \frac{4}{11} = 0.364$
- One could question whether the binomial model is the best one to describe this data. It may so happen that some of the students are committed to the course…



Likely and un-likely



- Let's check out if we are able to work out some intuition about the confidence intervals by studying likely and not likely events
- We get back to the example 1 (mind that for all of the cases we are studying today we assume the models we picked are fine)
- Say, the true rate of accidents for young children is $\mu = 5$, which is higher than 3, but not by much. Taking that the true parent distribution of the R.V. X is $Poisson(\mu = 5)$, we can easily estimate the probability $P(X \leq 3; \mu = 5)$:

$$P(X \leq 3) = P(X = 0) + P(X = 1) + \cdots + P(X = 3) =$$

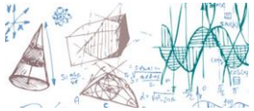
$$= e^{-5}\left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!}\right) = 0.265$$

- This is not that small, now if we test $\mu = 8$ and $\mu = 12$ we get

$$P(X \leq 3; \mu = 8) = P(X = 0) + \cdots + P(X = 3) = 0.042$$

$$P(X \leq 3; \mu = 12) = P(X = 0) + \cdots + P(X = 3) = 0.002$$

Likely and un-likely



- So, it seems unlikely that if the true mean number of accidents is 12 that we are going to observe $x = 3$
- We could also reverse the last argument and say that with the evidence (observation) $x = 3$ considering the true mean to be 12 is a serious overestimation
- We could now test the low values of true value of our model parameter? Let's try out $\mu = 1$ and $\mu = 0.5$

$$P(X \geq 3; \mu = 1) = 1 - P(X \leq 2) = 0.08$$

$$P(X \geq 3; \mu = 0.5) = 1 - P(X \leq 2) = 0.014$$

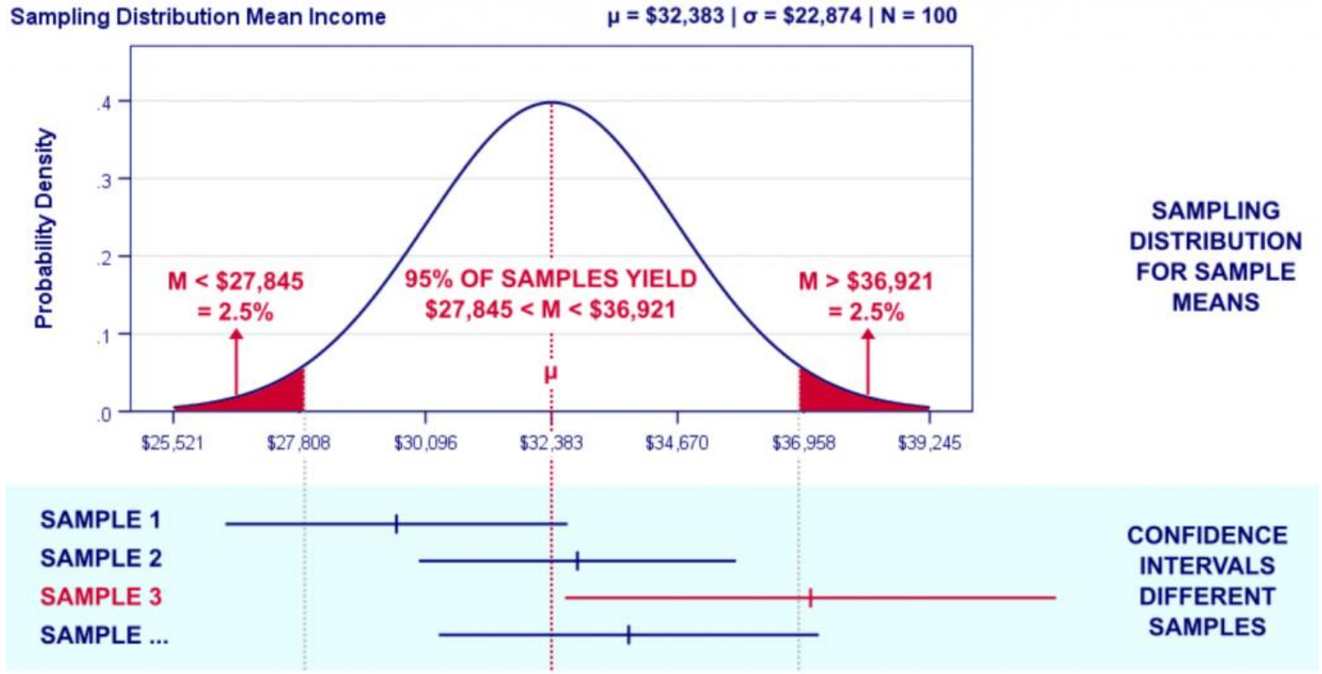
- The probabilities are low, sure, but we still should remember that they are indeed probable
- This exercise is actually vital for understanding confidence intervals and hypothesis testing – what we did was to propose a certain value of a parameter and evaluate chances to get a result as extreme as the one that was actually observed

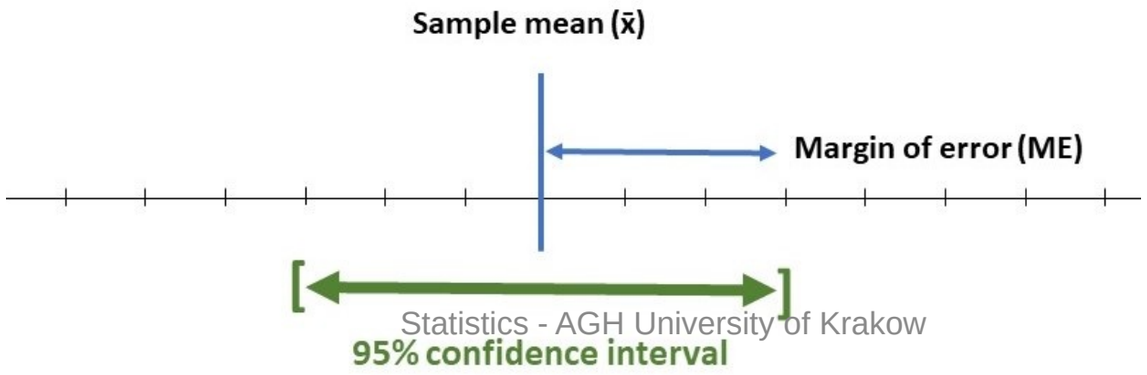
Confidence

- Statistical statements regarding R.Vs. and probability should always be interpreted in terms of model parameters and confidence
- **We express the confidence using fractional numbers** (%). So, we could say, for instance, a $\kappa\%$ confidence interval for parameter θ (based on an actual observation) is the interval from θ_- to θ_+, where $\kappa\% \rightarrow 99\%, 95\%, 90\%, \ldots$
- Its meaning is as follow: if we observe an event with the prob. of 95% we say it is reasonable, on the other hand if this is just 5% it should be considered **unlikely**
- So, what left now is to evaluate the confidence interval, we reserve for example **5%** of probability for „strange" events and consider both cases too-low-strange and too-high-strange
- This is, so called, **two tailed** or two sided confidence interval and we have reserved **2.5%** probability for very high and very low results

two tails…

Confidence intervals

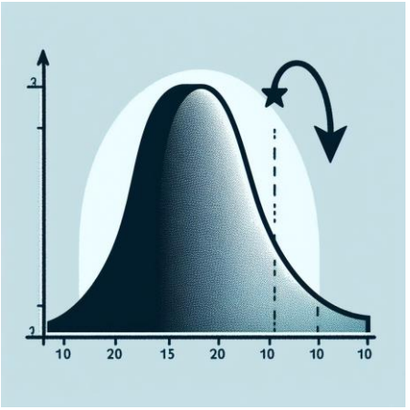




Confidence intervals

- We got the impression that reporting the value of an estimator (e.g. $\hat{X}$) tells us nothing about the magnitude of the discrepancy that may exist between the estimator and the estimated parameter $(E[X] = \mu)$.
- What would be the "confidence interval" of the estimated PARAMETER?
- WE MAY DEFINE the confidence interval in the following manner:
  - we start with choosing a value $\mathbf{1} - \boldsymbol{\alpha}$ of the **confidence level** as:

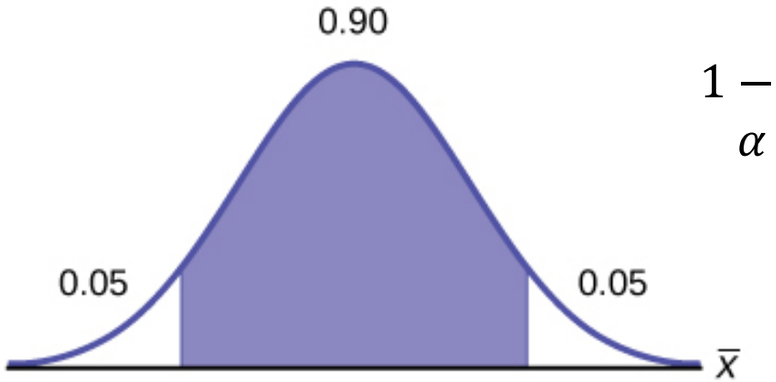
$$1 - \alpha, \qquad 0 < \alpha < 1$$



  - usually $\alpha = 0.01;\ \mathbf{0.05};\ 0.1$
  - the confidence interval Δ is chosen in such a way that the probability for Δ to cover the unknown parameter (like μ or σ^2) is $1 - \alpha$

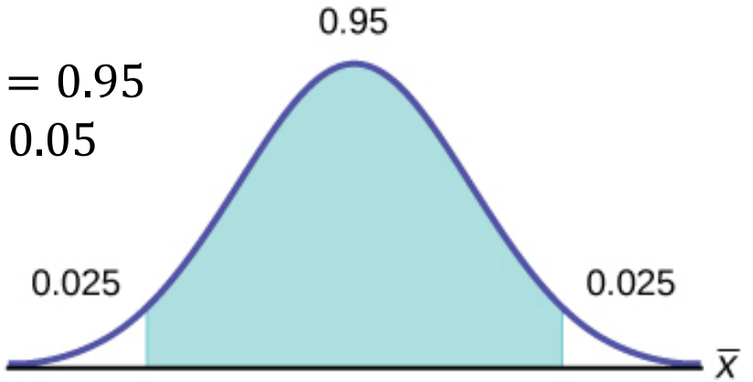
$C.I.$ for the normal distribution

- We already know a lot about evaluating probabilities using the normal distribution



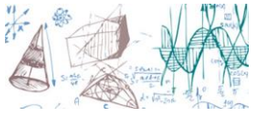
$1 - \alpha = 0.95$

$\alpha = 0.05$



Confidence Level	Alpha	Alpha/2	z alpha/2
90%	10%	5.0%	1.645
95%	5%	2.5%	1.96
98%	2%	1.0%	2.326
99%	1%	0.5%	2.576

$C.I.$ for the normal distribution



- Using the plot or the table from the previous slide we write for the critical values $z_c = \pm 1.96$, which corresponds to the confidence level of 95%:

$$P\left(-1.96 \leq \frac{\bar{X} - \boldsymbol{\mu}}{\sigma/\sqrt{n}} \leq 1.96\right) = 0.95$$

- As usual, there are some tricks… For instance if we knew the distribution variance σ (remember the normal model has two parameters!) we could immediately solve these inequalities

$$P\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \boldsymbol{\mu} \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

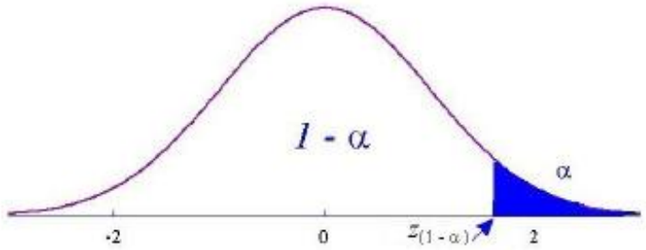
- This is a random interval, defined around the sample mean, which contains the unknown population mean with the probability of 95%. So, the 95% C.I. for μ is given by

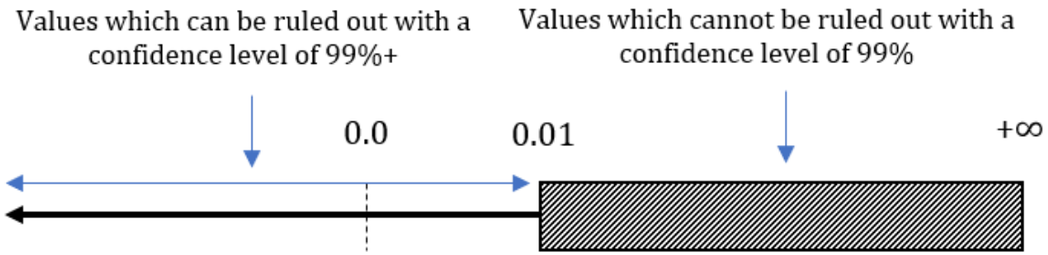
$$C.I._{95\%}^{\mathcal{N}} = \left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

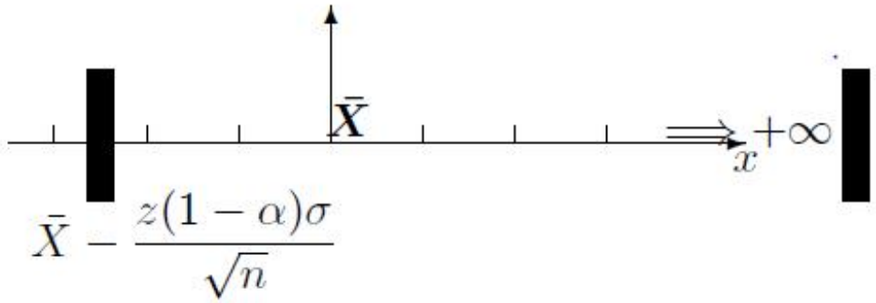
One-sided CI

1. LOWER one-sided confidence interval: $\alpha_1 = 0 \quad z(\alpha_1) = -\infty$ $z(\alpha_2) = z(1-\alpha)$; the interval is:

$$\left(\bar{X} - z(1-\alpha)\frac{\sigma}{\sqrt{n}},\ +\infty\right)$$





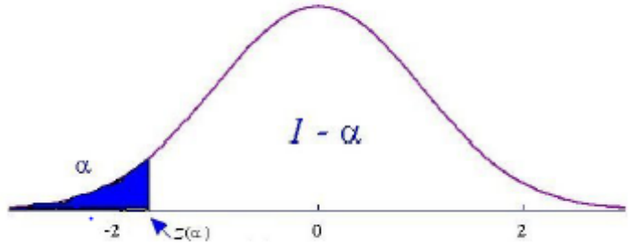


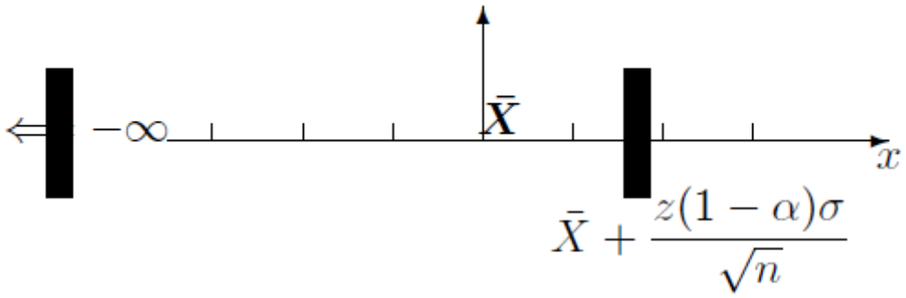
we may be $1-\alpha$ certain that μ is **no less** than $\bar{X} - \frac{z(1-\alpha)\sigma}{\sqrt{n}}$

One-sided CI

2. UPPER one-sided confidence interval $\alpha_2 = 0 \quad z(1-\alpha_2) = \infty$ the interval is:

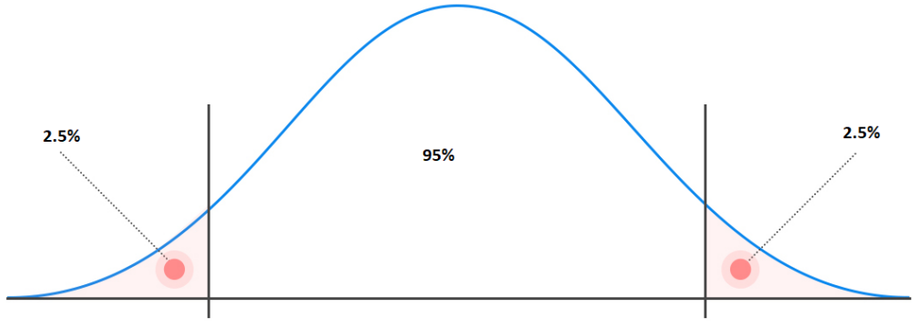
$$\left(-\infty,\ \bar{X} - z(\alpha)\frac{\sigma}{\sqrt{n}}\right) \equiv \left(-\infty,\ \bar{X} + z(1-\alpha)\frac{\sigma}{\sqrt{n}}\right)$$



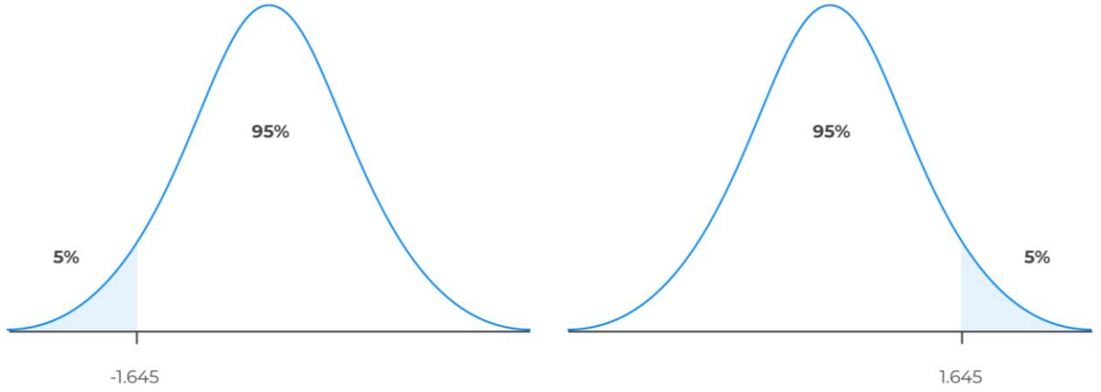


we may be $1-\alpha$ certain that μ is **not greater** than $\bar{X} + \frac{z(1-\alpha)\sigma}{\sqrt{n}}$

Two- and one-sided CI

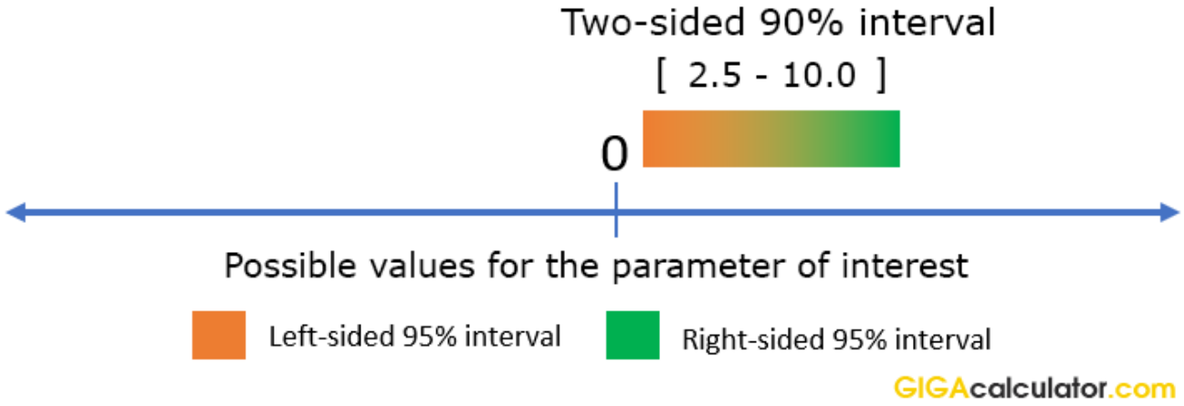


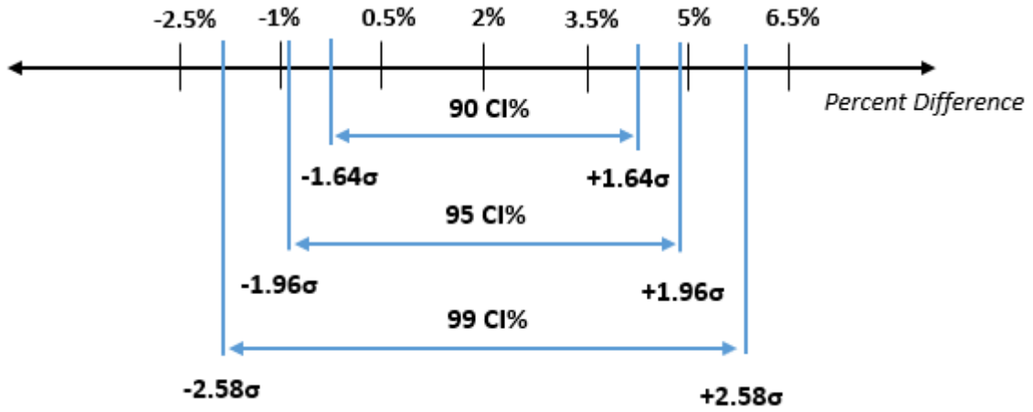
Confidence Interval = (point estimate) +/- (critical value)*(standard error)



Two- and one-sided CI

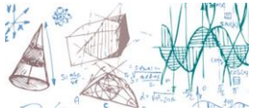
Confidence Interval = (point estimate) +/- (critical value)*(standard error)





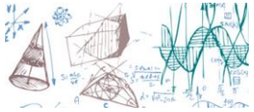
Each point within the confidence interval is equally likely to be the true value

Interpretation of $C.I.$



- Observe, we are able to define a $C.I.$ using the formula describing the model and „probability points” that follow from definition of the confidence level
- If we obtain a single measurement then we will get an $C.I.$ spanning $0.27X$ to $39.5X$
- The proper interpretation is, that this interval **contains** the unknown parameter with probability **0.95**
- In other words: if we repeat an experiment 100 times, and calculate each time the $C.I.$ (random interval) then we should expect that about 95 times the unknown parameter will be inside these intervals
- The parameter is a number and the confidence statement is made based on properties of the random interval – it may or may not contain the parameter!

$C.I.$ for the normal distribution



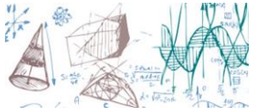
- Imagine that we want to test the accuracy of some timer device using a more accurate one (like an digital stop-watch)
- It could go, for instance, like that – we set the tested timer to 5 minutes and we measure the actual time interval
- Assume that the observed data variation is a consequence of the scale precision (you may not be able to set the actual time) and the precision of the time mechanism - $\mathcal{N}(\mu, \sigma^2)$
- Say we made n observations and obtained sample mean and sample variance $\bar{x} = 294.8$, $s^2 = 3.12$ respectively
- We know, that if one draws a sample from a normal distribution the sampling distribution of $\bar{X}$ is also normal

$$\bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \rightarrow Z = \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right) \sim \mathcal{N}(0,1)$$

- And in general we can write:

$$P\left(-z_c \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_c\right) = 1 - \alpha$$

$C.I.$ for the normal distribution



- A general formula that can be applied for the normal distribution for its mean is then

$$C.I._{100\cdot(1-\alpha)\%}^{\mathcal{N}} = \left(\bar{X} - z_c\frac{\sigma}{\sqrt{n}}, \bar{X} + z_c\frac{\sigma}{\sqrt{n}}\right)$$

- Nice, but… what **if we do not know the distribution** variance (and we usually do not)? The most sensible approach would be to use the sample variance to estimate σ^2

$$S^2 = \frac{1}{n-1}\sum_i (X_i - \bar{X})^2 \rightarrow E[S^2] = \sigma^2$$

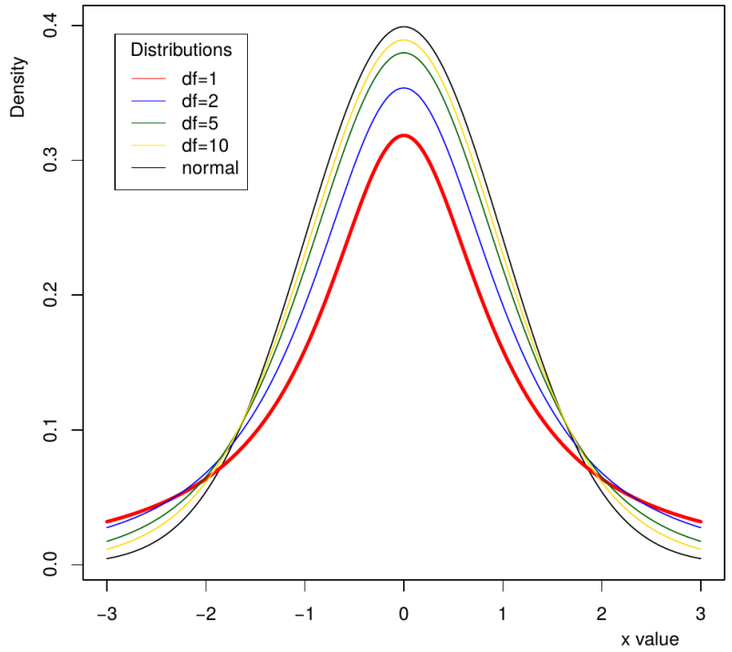
- We define a new R. V. T

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \rightarrow P\left(-t \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t\right) = 1 - \alpha$$

- The R.V. T follows the **Student's t-distribution** (actually there is a whole family of distribution) $T \sim t(\nu)$

t-distribution

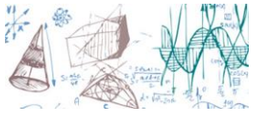
- t-distribution is similar to the normal on (obviously!)



	50%	90%	95%	99%	99.9%
DF=5	0.73	2.02	2.57	4.03	6.87
DF=10	0.70	1.81	2.23	3.17	4.59
DF=20	0.69	1.72	2.09	2.85	3.85
DF=30	0.68	1.70	2.04	2.75	3.65
DF=50	0.68	1.68	2.01	2.68	3.50
(Normal)	0.67	1.64	1.96	2.58	3.29

- The larger the ν the more resemblance to the normal curve
- We use tables to evaluate the critical values t_c for a given confidence levels, let's continue on the next slide…

$C.I.$ for t-distribution



- Start with some formalities… If we draw a sample of size n from a normal distribution with the mean μ, the $R.V.\ T$

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t \quad (\nu = n - 1)$$

- Where $\bar{X}$ is the sample mean and S its standard deviation

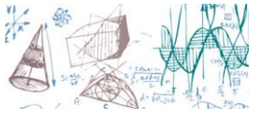
$$P\left(-t_c \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t_c\right) = 1 - \alpha$$

$$P\left(\bar{X} - t_c \frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_c \frac{S}{\sqrt{n}}\right) = 1 - \alpha$$

- And the $C.I.$ is centred about the sample mean, which contains the true unknown population parameter μ with probability $1 - \alpha$

$$C.I._{100\cdot(1-\alpha)}^{t} = \left(\bar{x} - t_c \frac{S}{\sqrt{n}}, \bar{x} + t_c \frac{S}{\sqrt{n}}\right)$$

$C.I.$ for t-distribution



- For our timer example, let's pick up the confidence level to be 90% and assume that collected data sample in $n = 11$

$$\frac{1}{2}\alpha = 0.05 \rightarrow t_c = \pm 1.81$$

These guys will set the unit!

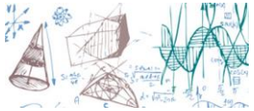
$$C.I._{90\%}^{t(10)} = \left(\bar{x} - t_c \frac{S}{\sqrt{n}}, \bar{x} + t_c \frac{S}{\sqrt{n}}\right) =$$

$$= \left(294.8 - 1.81\frac{1.77}{\sqrt{11}}, 294.8 + 1.81\frac{1.77}{\sqrt{11}}\right) =$$

$$= (293.8, 295.8)$$

- With larger data sample, our C.I. is now nicely narrow (so, we add some predictability actually)
- Also, note that 300 second (this was the setting on the timer) is not included inside the interval
- Is it an indication that the device goes consistently early?

Exponential distribution



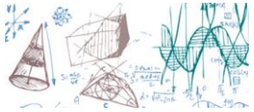
- Let's evaluate the C.I. for the mining accidents example
- We assumed that the R.V. T follows the exponential model, we have a single observation of $t = 157$ days
- We ask for $C.L. = 100(1-\alpha)\% = 90\%$, $\alpha = \frac{1}{2}\alpha = 0.05$

$$P(T \leq t) = 1 - e^{-\frac{t}{\mu}} = 0.05 \rightarrow \frac{t}{\mu} = -\ln(0.95) \rightarrow \mu_+ = 3060\ (days)$$

$$P(T \geq t) = e^{-\frac{t}{\mu}} = 0.05 \rightarrow \frac{t}{\mu} = -\ln(0.05) \rightarrow \mu_- = 52.4\ (days)$$

- As another summary we should stress, that evaluation of the C.I. requires: **data sample, a model** (to evaluate probabilities) and the **parameter** we want to evaluate
- Using these two examples, try to come up with 90% C.I. for the absence case

Confidence



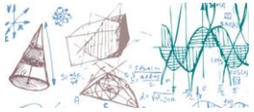
- In order to find the confidence interval (C.I.) we solve

$$P(X \leq 3; \mu) = e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!}\right) = 0.025 \rightarrow \mu_{-} = 0.62$$

$$P(X \geq 3; \mu) = 1 - e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!}\right) = 0.025 \rightarrow \mu_{+} = 8.8$$

- And our statistical statement would be: **a 95% confidence interval for the parameter μ of the Poisson model, evaluated using a single observation is** $(\mu_{-}, \mu_{+}) = (0.62, 8.8)$
- The obtained confidence interval is very wide – we can make it better by collecting more data!
- The grand summary of what we did: with an observation(s) on the R.V. X, assuming X follows some specified model with an unknown parameter θ, we may evaluate a 95% confidence interval for θ by solving $P(X \leq x) = \frac{1}{2}\alpha$ and $P(X \geq x) = \frac{1}{2}\alpha$. We define the confidence level as $C.L. = (1 - \alpha)\%$

Interpretation of $C.I.$



- Since, we need data to construct a $C.I.$ it follows that **for different sample we obtain a different interval** – we call it random interval (θ_-, θ_+ are $R.Vs.$ themselves)
- Consider again a single observation of $R.V.X$ that follows an exponential distribution (the math is very easy). The parameter is μ. Now concentrate! We can express the respective limits of the $C.I.$ using the value of that parameter

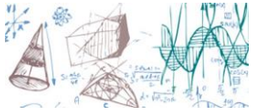
$$x_{0.025} \to 1 - e^{-\frac{x_{0.025}}{\mu}} = 0.025 \to x_{0.025} = -\ln(0.975) = 0.025\mu$$

$$x_{0.975} \to 1 - e^{-\frac{x_{0.975}}{\mu}} = 0.975 \to x_{0.975} = -\ln(0.05) = 3.69\mu$$

- Thus, we have $P(0.025\mu \leq X \leq 3.69\mu) = 0.95.$ We can also rewrite this in terms of the unknown parameter

$$P\left(\frac{X}{3.69} \leq \mu \leq \frac{X}{0.025}\right) = P(0.27X \leq \mu \leq 39.5X) = 0.95$$

$C.I.$ for the variance



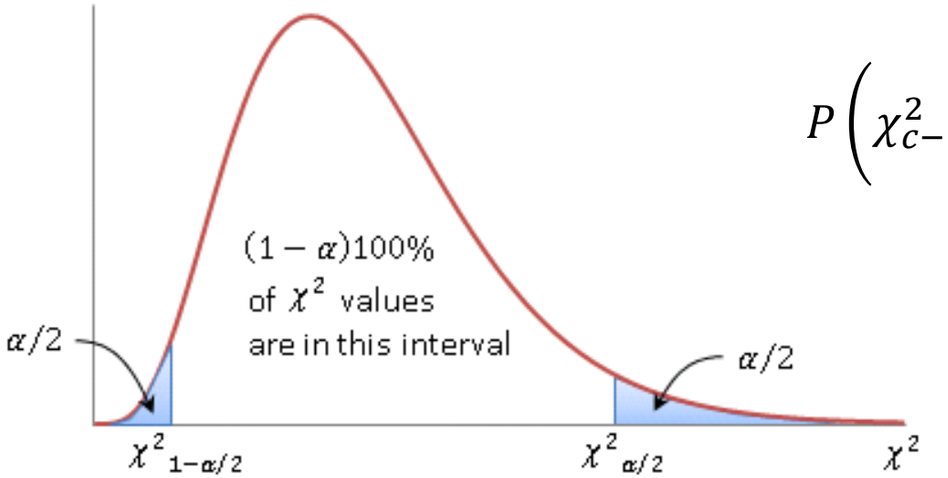
- Imagine that a company is delivering composite fibres for aircraft wings. In that case a great care should be taken to produce fibres that do not vary too much in tensile strength (expressed in kg)
- A sample of 8 fibres were taken and tested, the results were as follow $\bar{x} = 150.72\ kg$ and $s^2 = 37.75\ kg^2$. **Our mission is to find a confidence interval for the variance**
- We assume that the parent distribution of the fibre strength is normal, thus the sampling distribution of variance should follow the $\chi^2(\nu = n - 1)$ distribution

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(\nu = n - 1)$$

- The χ^2 is a family of curves and for increasing number of degrees of freedom it is getting more and more symmetric

$C.I.$ for the variance

- Again, the game is to find critical points using a given model (in this case the chi-squared)

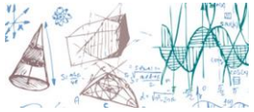


$$P\left(\chi^2_{c-} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{c+}\right) = 1-\alpha \rightarrow \chi^2_{c-} = q_{\frac{1}{2}\alpha}, \chi^2_{c+} = q_{1-\frac{1}{2}\alpha}$$

$$P\left(\frac{(n-1)S^2}{\chi^2_{c+}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{c-}}\right) = 1-\alpha$$

$$C.I._{100\cdot(1-\alpha)\%}^{\chi^2(v)} = \left(\frac{(n-1)S^2}{\chi^2_{c+}}, \frac{(n-1)S^2}{\chi^2_{c-}}\right)$$

$C.I.$ for the variance



- Getting back to the fibre strength example, we are searching for $C.I._{90\%}^{\chi^2(9)}$, the critical points (from tables) $\chi^2_{5\%} = 3.325$ and $\chi^2_{95\%} = 16.919$ for $\chi^2(\nu = 9)$ distribution
- Our probability statement then is

$$P\left(3.325 \leq \frac{9s^2}{\sigma^2} \leq 16.919\right) = 0.9$$

$$C.I._{90\%}^{\chi^2(9)} = \left(\frac{(n-1)S^2}{\chi^2_{c+}}, \frac{(n-1)S^2}{\chi^2_{c-}}\right) = \left(\frac{9s^2}{16.919}, \frac{9s^2}{3.325}\right)$$
$$= (0.53s^2, 2.71s^2) = \cdots$$

- For the timer example, this would give us

$$C.I._{90\%}^{\chi^2(10)} = \left(\frac{10 \cdot 3.12}{18.307}, \frac{10 \cdot 3.12}{3.247}\right) = (1.70, 9.38)$$

- Try to work out the C.I. for the normal standard deviation