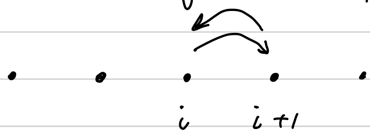
Nearest Neighbor Hopping Hamiltonian.



$$H = -t\sum_r a_{r+1}^\dagger a_r + a_r^\dagger a_{r+1}$$

Define fourier transform

$$a_k = \frac{1}{\sqrt{N}}\sum_r a_r e^{ikr} \qquad\qquad a_r = \frac{1}{\sqrt{N}}\sum_k a_k e^{ikr}$$

$$\Longleftrightarrow$$

$$a_k^\dagger = \frac{1}{\sqrt{N}}\sum_r a_r^\dagger e^{-ikr} \qquad\qquad a_r^\dagger = \frac{1}{\sqrt{N}}\sum_k a_k^\dagger e^{-ikr}$$

so that

$$H = -t\sum_r \left(\frac{1}{\sqrt{N}}\sum_k a_k^\dagger e^{-ik(r+1)}\right)\left(\frac{1}{\sqrt{N}}\sum_{k'} a_{k'} e^{ik'r}\right)$$

$$+ \left(\frac{1}{\sqrt{N}}\sum_k a_k^\dagger e^{-ikr}\right)\left(\frac{1}{\sqrt{N}}\sum_{k'} a_{k'} e^{ik'(r+1)}\right)$$

$$= -t\sum_r \left(\frac{1}{N}\sum_k\sum_{k'} a_k^\dagger a_{k'} e^{i(k'-k)r} e^{-ikr} + \frac{1}{N}\sum_k\sum_{k'} a_k^\dagger a_{k'} e^{i(k'-k)r} e^{ik'r}\right.$$

$$= \frac{-t}{N}\left(\sum_k\sum_{k'} a_k^\dagger a_{k'} e^{-ikr}\sum_r e^{i(k'-k)r}\right) + \sum_k\sum_{k'} a_k^\dagger a_{k'} e^{ik'r}\sum_r e^{i(k'-k)r}))$$

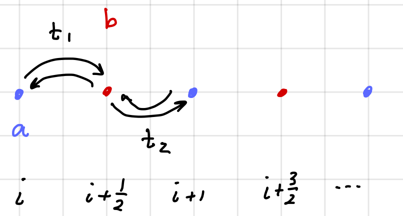
$$= -t\left(\sum_k\sum_{k'} a_k^\dagger a_{k'} e^{-ikr}\delta_{k'-k} + \sum_k\sum_{k'} a_k^\dagger a_{k'} e^{ik'r}\delta_{k'-k}\right)$$

$$= -t\left(\sum_k a_k^\dagger a_k e^{-ikr} + \sum_k a_k^\dagger a_k e^{ikr}\right)$$

$$= -t\sum_k a_k^\dagger a_k (2\cos kr)$$

$$= \sum_k a_k^\dagger \underbrace{(-2t\cos kr)}_{\varepsilon(k)} a_k$$

Bipartite Nearest Neighbor Hopping Hamiltonian



$$H = -t_1 \sum_i a_i^\dagger b_{i+1/2} + b_{i+1/2}^\dagger a_i \quad - t_2 \sum_i b_{i+1/2}^\dagger a_{i+1} + a_{i+1}^\dagger b_{i+1/2}$$

Defining Fourier Transforms

$$a_r = \frac{1}{\sqrt{N}} \sum_k e^{ikr} a_k \qquad\qquad b_{r+1/2} = \frac{1}{\sqrt{N}} \sum_k e^{ik(r+1/2)} b_k$$

$$a_r^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikr} a_k^\dagger \qquad\qquad b_{r+1/2}^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ik(r+1/2)} b_k^\dagger$$

We have

$$\sum_r a_r^\dagger b_{i+1/2} = \frac{1}{N} \sum_r \sum_k \sum_{k'} e^{-ikr} a_k^\dagger e^{ik'(r+1/2)} b_{k'}$$

$$= \frac{1}{N} \sum_k \sum_{k'} a_k^\dagger b_{k'} e^{+ik'r/2} \sum_r e^{i(k'-k)r}$$

$$= \sum_k a_k^\dagger b_k e^{+ikr/2}$$

$$\Rightarrow \sum_r b_{i+1/2}^\dagger a_r = \sum_k b_k^\dagger a_k e^{-ikr/2}$$

$$\& \quad \sum_r a_{r+1}^\dagger b_{i+1/2} = \frac{1}{N} \sum_r \sum_k \sum_{k'} e^{-ik(r+1)} a_k^\dagger e^{ik'(r+1/2)} b_{k'}$$

$$= \frac{1}{N} \sum_k \sum_{k'} a_k^\dagger b_{k'} e^{-ikr/2} \sum_r e^{i(k'-k)r}$$

$$= \sum_k a_k^\dagger b_k e^{-ikr/2}$$

$$\Rightarrow \sum_r b_{i+1/2}^\dagger a_{r+1} = \sum_k b_k^\dagger a_k e^{ikr/2}$$

$$\Rightarrow H = -t_1 \sum_k a_k^\dagger b_k e^{+ikr/2} + b_k^\dagger a_k e^{-ikr/2}$$

$$- t_2 \sum_k a_k^\dagger b_k e^{-ikr/2} + b_k^\dagger a_k e^{ikr/2}$$

$$= \begin{bmatrix} a_k^\dagger & b_k^\dagger \end{bmatrix} \begin{bmatrix} 0 & -t_1 e^{+ikr/2} - t_2 e^{-ikr/2} \\ -t_1 e^{-ikr/2} - t_2 e^{ikr/2} & 0 \end{bmatrix} \begin{bmatrix} a_k \\ b_k \end{bmatrix} = \begin{bmatrix} a_k^\dagger & b_k^\dagger \end{bmatrix} M \begin{bmatrix} a_k \\ b_k \end{bmatrix}$$

We seek the transform U s.t.

$$H = [a_k^\dagger \; b_k^\dagger] U^\dagger U M U^\dagger U \begin{bmatrix} a_k \\ b_k \end{bmatrix}$$

$$= [u^\dagger \; v^\dagger] \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Computing the eigenvalues of M:

$$M\vec{v} = \lambda \vec{v}$$

$$\Rightarrow (M - \lambda I)\vec{v} = 0$$

$$\Rightarrow \det(M - \lambda I) = 0$$

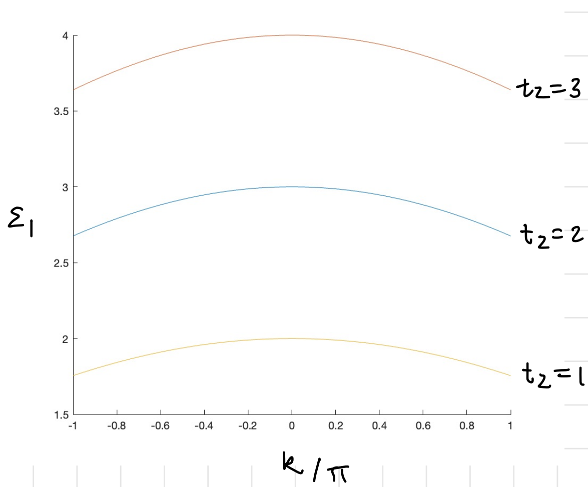
$$\Rightarrow \lambda^2 - \left| t_1 e^{ikr/2} + t_2 e^{-ikr/2} \right|^2 = 0$$

$$\Rightarrow \lambda = \pm \left| t_1 e^{ikr/2} + t_2 e^{-ikr/2} \right|$$

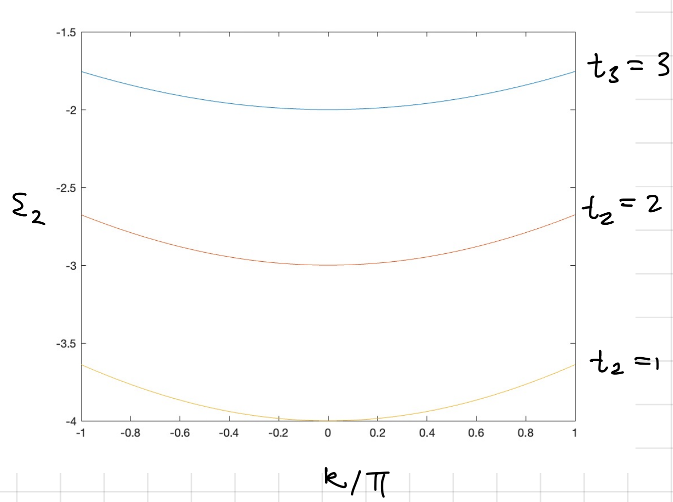
$$= \pm \left| t_1 e^{ikr} + t_2 \right|$$

$$\varepsilon_1 = + \left| t_1 e^{ikr} + t_2 \right| , \quad \varepsilon_2 = - \left| t_1 e^{ikr} + t_2 \right|$$

$t_1 = 1$



$t_1 = 1$



Solving directly in real space for a n-long chain w/ periodic b/cs we have

$$H = -t_1 \sum_i (a_i^\dagger b_{i+1/2} + b_{i+1/2}^\dagger a_i) - t_2 \sum_i (b_{i+1/2}^\dagger a_{i+1} + a_{i+1}^\dagger b_{i+1/2})$$

$$= \begin{bmatrix} a_1^\dagger & b_{3/2}^\dagger & a_2^\dagger & \cdots & a_n^\dagger & b_{n+1/2}^\dagger \end{bmatrix} \begin{bmatrix} 0 & -t_1 & & \cdots & & -t_2 \\ -t_1 & 0 & -t_2 & & & \\ & -t_2 & 0 & -t_1 & & \\ & & -t_1 & \ddots & -t_2 & \\ & & & & & -t_1 \\ -t_2 & & & & -t_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ b_{n+1/2} \end{bmatrix}$$