

$$1. A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{에 대하여.}$$

- (1) 열공간  $C(A)$ 을 구하시오.
- (2)  $C(A)$ 이  $\mathbb{R}^2$ 의 부분공간임을 보이고 평면상에 표시하시오.
- (3) 영공간  $N(A)$ 와 nullity를 구하시오.
- (4)  $N(A)$ 은  $\mathbb{R}^3$ 의 부분공간임을 설명하고 공간상에 표시하시오.
- (5)  $\text{rank}(A)$ 을 구하시오.

$$(1) C(A) = \left\{ Y \in \mathbb{R}^2 \mid Y = Ax, x \in \mathbb{R}^3 \right\} = \left\{ Y \in \mathbb{R}^2 \mid Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, x \in \mathbb{R}^3 \right\}$$

$$= \left\{ Y \in \mathbb{R}^2 \mid Y = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, x_1 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^2$$

(2)  $C(A)$ 는  $\mathbb{R}^2$ 의 부분집합이다.

$$\forall \lambda \in \mathbb{R} \quad u, v \in C(A)$$

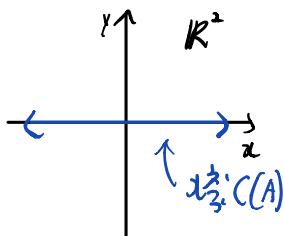
$$u = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

$$u+v = \begin{bmatrix} x_1+x_2 \\ 0 \end{bmatrix} \in C(A)$$

$\therefore C(A)$ 는  $\mathbb{R}^2$ 의 부분공간이다.

$$\begin{aligned} \lambda u &= \lambda \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda x_1 \\ 0 \end{bmatrix} \\ \lambda u &\in C(A) \end{aligned}$$



$$(3) \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = N(A) \subset \mathbb{R}^3$$

$$\text{nullity} = \dim(N(A)) = 2$$

(4)  $N(A) \subset \mathbb{R}^3$

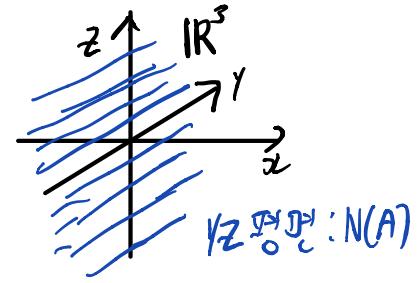
$$\forall \lambda \in \mathbb{R}, u, v \in N(A), u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_2 \end{bmatrix}$$

$$u+v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1+b_2 \end{bmatrix} \in N(A)$$

$$\lambda u = \lambda \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \lambda b_1 \end{bmatrix} \in N(A)$$

$\therefore N(A)$ 은  $\mathbb{R}^3$ 의 부분공간이다.

$$(5) \text{rank}(A) = \dim(C(A)) = 1$$



$$2. A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix} \text{에 대하여,}$$

(1) 열공간  $C(A)$ 을 구하시오.

(2) 영공간  $N(A)$ 와 nullity를 구하시오.

(3)  $\text{rank}(A)$ 을 구하시오.

(4) 선형연립방정식  $Ax = 0$ 의 해집합은 벡터공간인가? 그렇다면 기저와 차원을 구하시오.

(5) 선형연립방정식  $Ax = b$ ,  $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ 의 해집합을 구하고 벡터공간인지 아닌지 설명하시오.

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix} \xrightarrow{E_{31}(-2)} \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & -12 & -3 & 21 \end{bmatrix} \xrightarrow{E_{32}(3)} \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{D_2(\frac{1}{4})} \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_{12}(-5)} \begin{bmatrix} 1 & 0 & \frac{23}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(1) C(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$(2) N(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1 \begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \right\}$$

$$\text{nullity} = \dim(N(A)) = 2$$

$$(3) \text{rank}(A) = \dim(C(A)) = 2$$

(4)  $A_{11} = 0$ 의 해집합은  $N(A)$ 이다.

$$\forall \lambda \in \mathbb{R}, u, v \in N(A), u = a_1 \begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix} + b_1 \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix}, v = a_2 \begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

$$u + v = (a_1 + a_2) \begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix} + (b_1 + b_2) \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \in N(A), \lambda u = \lambda a_1 \begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix} + \lambda b_1 \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \in N(A)$$

$\therefore N(A)$ 는 벡터 공간이다.

$\therefore A_{11} = 0$ 의 해집합은 벡터 공간이다.

$$(5) \left[ \begin{array}{cccc|c} 1 & 5 & 7 & 9 & 1 \\ 0 & 4 & 1 & 1 & 0 \\ 2 & -2 & 11 & -3 & 1 \end{array} \right] \xrightarrow{E_{31}(-2)} \left[ \begin{array}{cccc|c} 1 & 5 & 7 & 9 & 1 \\ 0 & 4 & 1 & 1 & 0 \\ 0 & -12 & -3 & -21 & 0 \end{array} \right] \xrightarrow{E_{32}(3)} \left[ \begin{array}{cccc|c} 1 & 5 & 7 & 9 & 1 \\ 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{E_{12}(-5)} \left[ \begin{array}{cccc|c} 1 & 0 & \frac{23}{4} & \frac{1}{4} & 1 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} \frac{23}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{bmatrix} \mid t_1, t_2 \in \mathbb{R} \right\}$$

$$\forall \lambda \in \mathbb{R}, u, v \in S, \quad u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} \frac{23}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + b_1 \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} \frac{23}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$$

$$u+v = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (a_1+a_2) \begin{bmatrix} \frac{23}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + (b_1+b_2) \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{bmatrix} \notin S$$

$\therefore S$ 는 벡터 공간이 아니다.

3. 행렬  $A$ 의 크기가  $5 \times 3$ 일 때  $\text{rank}(A)=3$  이라 한다.  $N(A)$ 를 구하시오.

$$N(A) = \{x \in \mathbb{R}^5 \mid Ax = 0\}, \quad A = [a_1 \ a_2 \ a_3]$$

$Ax = x_1a_1 + x_2a_2 + x_3a_3, \text{rank}(A)=3$  이므로  $a_1, a_2, a_3 \rightarrow \text{lin. indep.}$

$\therefore x_1 = x_2 = x_3 = 0$  일때만  $Ax = 0$  성립

$$\therefore N(A) = \{0\}$$

4. 다음 집합이 벡터공간을 이루는지 판단하고 벡터공간인 경우 기저와 차원을 구하시오.

(1) the set of upper triangular matrices with size 3 by 3

(2) the set of matrices with size 2 by 3

(3)  $V = \{v = (x, y) \mid x \geq 0, y \geq 0\}$

$$(1) \quad S = \left\{ A \mid A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, a, b, c, d, e, f \in \mathbb{R} \right\}$$

$$\forall \lambda \in \mathbb{R}, u, v \in S, \quad u = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix}, \quad v = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{bmatrix}$$

$$u+v = \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ 0 & a_4+b_4 & a_5+b_5 \\ 0 & 0 & a_6+b_6 \end{bmatrix} \in S$$

$$\lambda u = \begin{bmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ 0 & \lambda a_4 & \lambda a_5 \\ 0 & 0 & \lambda a_6 \end{bmatrix} \in S$$

$\therefore S$  는 벡터 공간을 이루다.

$$B_S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

$$\dim(S) = 6$$

$$(2) S = \left\{ A \mid A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, a, b, c, d, e, f \in \mathbb{R} \right\}$$

$$\forall \lambda \in \mathbb{R}, u, v \in S, u = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, v = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix}$$

$$u+v = \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_4+b_4 & a_5+b_5 & a_6+b_6 \end{bmatrix} \in S$$

$$\lambda u = \begin{bmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ \lambda a_4 & \lambda a_5 & \lambda a_6 \end{bmatrix} \in S$$

$\therefore S$  는 벡터 공간을 이루다.

$$B_S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\dim(S) = 6$$

$$(3) V = \{v = (x, y) \mid x \geq 0, y \geq 0\}$$

$$\forall \lambda \in \mathbb{R}, u, v \in V, u = (x_1, y_1), v = (x_2, y_2)$$

$$u+v = (x_1+x_2, y_1+y_2) \in V$$

$$\lambda u = (\lambda x_1, \lambda y_2)$$

$$\Leftrightarrow \text{if } \lambda = -1, \lambda u \notin V$$

$\therefore V$  는 벡터 공간이 아니다.