1. Solve the DEs

(1) $xy' = y^2 + y$

$\Rightarrow \frac{1}{y^2+y}\,dy = \frac{1}{x}\,dx$

$\frac{1}{y(y+1)}\,dy = \left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \frac{1}{x}\,dx$

$\int\left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \int \frac{1}{x}\,dx$

$\ln|y| - \ln|y+1| = \ln|x| + C$

$\ln\left|\frac{y}{y+1}\right| = \ln|x| + C$

$\frac{y}{y+1} = C\cdot x$

$1 + \frac{-1}{y+1} = C\cdot x$

$\frac{1}{y+1} = 1 - C\cdot x$

$y+1 = \frac{1}{1-C\cdot x}$

$y = \frac{1}{1-C\cdot x} - 1 = \frac{C\cdot x}{1-C\cdot x} = \frac{x}{\frac{1}{C} - x} = \frac{x}{C-x}$

$\therefore \boxed{y = \frac{x}{C-x}}$

(2) $\frac{dy}{dx} = y^2 - 4$

$\Rightarrow \frac{1}{y^2-4}\,dy = dx$

$\frac{1}{(y+2)(y-2)}\,dy = dx$

$\int \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = \int 1\,dx$

$\frac{1}{4}\left\{\ln|y-2| - \ln|y+2|\right\} = x + C$

$\ln\left|\frac{y-2}{y+2}\right| = 4x + C$

$\frac{y-2}{y+2} = C\cdot e^{4x}$

$1 + \frac{-4}{y+2} = C\cdot e^{4x}$

$\frac{4}{y+2} = 1 - C\cdot e^{4x}$

$y = \frac{4}{1-C\cdot e^{4x}} - 2$

$\boxed{y = \frac{2+2\cdot C\cdot e^{4x}}{1-Ce^{4x}}}$

(3) $x\frac{dy}{dx} + y = x^2y^2$

$\Rightarrow y' + \frac{1}{x}y = xy^2$

$u = y^{1-2} = y^{-1}$

$u' = -y^{-2}\cdot y'$

$y' = -\frac{u'}{u^2}$

$\therefore -\frac{u'}{u^2} + \frac{1}{x}\cdot\frac{1}{u} = \frac{x}{u^2}$

$u' - \frac{u}{x} = -x$

$\underbrace{\left(x - \frac{u}{x}\right)}_{P(x,y)}dx + \underbrace{1}_{Q(x,y)}\cdot du = 0$

$P_y = -\frac{1}{x} \qquad Q_x = 0$

$\frac{1}{Q}(P_y - Q_x) = R(x) = -\frac{1}{x}$

$\therefore F(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = e^{\ln|\frac{1}{x}|} = \frac{1}{x}$

$\frac{1}{x}u' - \frac{u}{x^2} = -1$

$\underbrace{\left(1 - \frac{u}{x^2}\right)}_{M}dx + \underbrace{\frac{1}{x}}_{N}du = 0$

$M_u = -\frac{1}{x^2} = N_x = -\frac{1}{x^2}$

$U(x,y) = \frac{1}{x}u + g(x) = 0$

$U_x(x,y) = -\frac{1}{x^2}u + g'(x) = -\frac{u}{x^2} + 1$

$\therefore g'(x) = 1$

$g(x) = x + C$

$U(x,y) = \frac{1}{x}u + x + C = \frac{1}{x}\frac{1}{y} + x + C$

$\therefore \boxed{y = \frac{1}{Cx - x^2}}$

(4) $\frac{dy}{dx} = (-2x+y)^2 - 7,\ y(0) = 0$

$u(x) = -2x + y$

$u' = -2 + y'$

$y' = u' + 2 = u^2 - 7$

$u' = u^2 - 9$

$dx = \frac{1}{u^2-9}\,du = \frac{1}{(u-3)(u+3)}\,du = \frac{1}{6}\left(\frac{1}{u-3} - \frac{1}{u+3}\right)du$

$\int \frac{1}{6}\left(\frac{1}{u-3} - \frac{1}{u+3}\right)du = \int 1 \cdot dx$

$\frac{1}{6}\left(\ln|u-3| - \ln|u+3|\right) = x + C$

$\ln\left|\frac{u-3}{u+3}\right| = 6x + C$

$\frac{u-3}{u+3} = C \cdot e^{6x}$

$1 + \frac{-6}{u+3} = C \cdot e^{6x}$

$\frac{-6}{u+3} = C \cdot e^{6x} - 1$

$\frac{u+3}{\cancel{\pm}6} = \frac{1}{1 - C \cdot e^{6x}}$

$u = \frac{6}{1 - C \cdot e^{6x}} - 3 = \frac{3 + 3C \cdot e^{6x}}{1 - C \cdot e^{6x}}$

$\underset{-2x+y}{\overset{\shortparallel}{u}}$

$y = 2x + \frac{3(1 + C \cdot e^{6x})}{1 - C \cdot e^{6x}}$

$y(0) = 0, \quad 0 = 0 + \frac{3(1+C)}{1-C}, \quad \therefore C = -1$

$\therefore \boxed{y = 2x + \frac{3(1 - e^{6x})}{1 + e^{6x}}}$

(5) $\frac{dy}{dx} + y = x,\ y(0) = 4$

$y' + y = x$

$\underbrace{(y-x)}_{P}dx + \underbrace{1}_{Q} \cdot dy = 0$

$P_y = 1 \qquad Q_x = 0$

$\frac{1}{Q}\left(P_y - Q_x\right) = 1 = R(x)$

$F(x) = e^{\int 1\,dx} = e^{x}$

$\underbrace{e^{x}(y-x)}_{M}dx + \underbrace{e^{x}}_{N}dy = 0$

$M_y = e^{x} = N_x = e^{x}$

$u(x,y) = \int e^{x}\,dy = e^{x}y + g(x)$

$u_x = e^{x} \cdot y + g'(x) = e^{x} \cdot y - xe^{x}$

$\therefore g'(x) = -xe^{x}$

$g(x) = -\left(xe^{x} - \int e^{x}\,dx\right)$

$= -xe^{x} + e^{x} + C$

$\therefore u(x,y) = e^{x}y - xe^{x} + e^{x} + C = 0$

$e^{x}y - xe^{x} + e^{x} = C$

$y(0) = 4,\ 4 + 1 = C = 5$

$e^{x}y - xe^{x} + e^{x} = 5$

$(y - x + 1) = 5/e^{x}$

$\boxed{y = \frac{5}{e^{x}} + x - 1}$

(6) $\frac{dy}{dx} - \frac{4}{x}y = x^6 e^x$

$y' - \frac{4}{x}y = x^6 e^x$

$\underbrace{-\left(\frac{4}{x}y + x^6 e^x\right)}_{P} dx + \underbrace{1}_{Q} \cdot dy = 0$

$P_y = -\left(\frac{4}{x}\right) \quad Q_x = 0$

$\frac{1}{Q}(P_y - Q_x) = -\frac{4}{x} = R(x)$

$F(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = x^{-4}$

$\underbrace{-\left(\frac{4}{x^5}y + x^2 e^x\right)}_{M} dx + \underbrace{x^{-4}}_{N} dy = 0$

$M_y = -\left(\frac{4}{x^5}\right) = N_x = -\frac{4}{x^5}$

$U(x,y) = \int x^{-4} dy = \frac{y}{x^4} + g(x)$

$U_x = -\frac{4y}{x^5} + g'(x) = -\frac{4y}{x^5} - x^2 e^x$

$g'(x) = -x^2 e^x$

$g(x) = -\left(x^2 e^x - 2\left(xe^x - \int e^x dx\right)\right)$

$= -x^2 e^x + 2xe^x - 2e^x + C$

$\therefore U(x,y) = y \cdot x^{-4} - x^2 e^x + 2xe^x - 2e^x + C$

$y \cdot x^{-4} - x^2 e^x + 2xe^x - 2e^x = C$

$\boxed{y = x^6 e^x - 2x^5 e^x + 2x^4 e^x + C \cdot x^4}$

(7) $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}, \ y(0) = 2$

$\underbrace{(xy^2 - \cos x \sin x)}_{P} dx + \underbrace{(y(x^2-1))}_{Q} dy = 0$

$P_y = 2xy = Q_x = 2xy$

$U(x,y) = \frac{1}{2}(x^2-1)y^2 + g(x)$

$U_x = y^2 x + g'(x) = xy^2 - \cos x \sin x$

$g'(x) = -\cos x \sin x$

$g(x) = -\int \cos x \sin x \, dx = \int u \, du = \frac{1}{2}u^2 + C$

$(\cos x = u, \ -\sin x \, dx = du)$

$\therefore g(x) = \frac{1}{2}\cos^2 x + C$

$U(x,y) = \frac{1}{2}(x^2-1)y^2 + \frac{1}{2}\cos^2 x + C$

$(x^2-1)y^2 + \cos^2 x = C$

$y(0) = 2, \ -1 \cdot 4 + 1 = -3 \quad \therefore C = -3$

$\boxed{(x^2-1)y^2 + \cos^2 x = -3}$

(8) $xy\,dx + (2x^2 + 3y^2 - 20)dy = 0$

$\underbrace{xy}_{P} dx + \underbrace{(2x^2 + 3y^2 - 20)}_{Q} dy = 0$

$P_y = x, \ Q_x = 4x$

$\frac{1}{P}(Q_x - P_y) = \frac{1}{xy}(4x - x) = \frac{3}{y} = R(y)$

$F(y) = e^{\int \frac{3}{y}dy} = e^{3\ln|y|} = y^3$

$\underbrace{y^3 \cdot xy}_{M} dx + \underbrace{y^3 \cdot (2x^2 + 3y^2 - 20)}_{N} dy = 0$

$M_y = 4xy^3 = N_x = 4xy^3$

$U(x,y) = \frac{1}{2}x^2 y^4 + g(y)$

$U_y = 2x^2 y^3 + g'(y) = 2x^2 y^3 + 3y^5 - 20y^3$

$g'(y) = 3y^5 - 20y^3$

$g(y) = \frac{1}{2}y^6 - 5y^4 + C$

$U(x,y) = \frac{1}{2}x^2 y^4 + \frac{1}{2}y^6 - 5y^4 + C$

$\boxed{\frac{1}{2}x^2 y^4 + \frac{1}{2}y^6 - 5y^4 = C}$

(9) $\frac{dy}{dx} = 1 + x + y + xy, \ y(0) = 0$

$\frac{dy}{dx} = (1+x)(1+y)$

$\left(\frac{1}{1+y}\right) dy = (1+x)dx$

$\int \frac{1}{1+y} dy = \int (1+x)dx$

$\ln|1+y| = x + \frac{1}{2}x^2 + C$

$y + 1 = Ce^{x + \frac{1}{2}x^2}$

$y = Ce^{x + \frac{1}{2}x^2} - 1$

$y(0) = 0, \ 0 = C - 1, \ C = 1$

$\boxed{y = -1 + e^{x + \frac{1}{2}x^2}}$

2. Given the family of all curves $xy=c$. Find the orthogonal family

$xy=c$

$y+xy'=0$

$y'=-\frac{y}{x}$

$\left(-\frac{y}{x}\right)\cdot\left(\frac{x}{y}\right)=-1$

$\therefore\ y'=\frac{x}{y}$

$y\,dy=x\,dx$

$\int y\,dy=\int x\,dx$

$\frac{1}{2}y^2=\frac{1}{2}x^2+C$

$\therefore\ \boxed{x^2-y^2=C}$