1. (1) A skew-symmetric matrix satisfies $A^T = -A$. In 3 by 3 case, show that the determinant must be zero.

(2) How about a 4 by 4 skew-symmetric matrix? Give a 4 by 4 skew-symmetric matrix with $\det A \neq 0$.

(1) $A^T = -A$

$\det(A^T) = \det(-A) = (-1)^3 \det(A)$ $\qquad$ ($\because \det(cA) = c^n \det(A)$)

$\det(A^T) = \det(A)$ $\qquad$ ($\because \det(A^T) = \det(A)$)

$\therefore -\det(A) = \det(A)$

$\det(A) = 0$

$\therefore \det(A) = \det(A^T) = \det(-A) = 0$

(2) $\det(-A) = (-1)^4 \det(A)$

$\therefore \det(A^T) = \det(-A) = \det(A)$

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}, \quad -A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{0} \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = -1 \cdot C_{13} = -1 \cdot \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = -1(-1) = 1$$

2. Let $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.

Find the determinant of A^{-1} and $A - \lambda I$. For which values of λ is $A - \lambda I$ a singular matrix?

$$\left[\begin{array}{cc|cc} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow[E_{21}(-4)]{P_{12}} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -10 & 1 & -4 \end{array}\right] \xrightarrow[E_{12}(-3)]{D_2(-\frac{1}{10})} \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{10} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{10} & \frac{2}{5} \end{array}\right]$$

$\therefore A^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{2}{5} \end{bmatrix}$, $\quad \det(A^{-1}) = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$

$A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$, $\quad \det(A - \lambda I) = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10$

$\det(A - \lambda I) = 0$ 이면 $A - \lambda I$는 singular matrix

$\therefore \lambda = 2, 5$ 일때 $A - \lambda I$는 singular matrix

3. By applying row operations to produce an upper triangular matrix, compute

$$① \det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \quad \text{and} \quad ② \det\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

①

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \xrightarrow[E_{31}(1)]{E_{21}(-2)} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \xrightarrow[E_{42}(-1)]{E_{12}(-1),\ E_{32}(-1)} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow{E_{13}(-1)} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} \xrightarrow[E_{34}(-2),\ E_{24}(-1),\ E_{14}(3)]{D_4(\frac{1}{6})} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B$$

$$\det B = \frac{1}{6}\det A = 6$$

$$\therefore \det A = 36$$

②

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{P_{14}} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[E_{31}(-1),\ E_{41}(-2)]{E_{21}(-1)} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow[E_{42}(1)]{E_{12}(-1)} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -4 \end{bmatrix} \xrightarrow[E_{43}(1)]{E_{13}(-1)} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \xrightarrow[E_{24}(-\frac{1}{5}),\ E_{34}(-\frac{1}{5})]{E_{14}(\frac{4}{5})} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} = B$$

$$\det B = -\det A = -5$$

$$\therefore \det A = 5$$

4. Find the cofactor matrix C of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and compare AC^T with A^{-1}.

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$AC^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} = \det A \cdot I$$

$\therefore$ $AC^T = \det A \cdot I$

$\therefore$ $A^{-1} = \dfrac{1}{\det A} \cdot C^T$