응용수학

Introduction

Homogeneous Linear ODEs of 2nd order
Reduction of order
Homogeneous Linear DE
(with constant coefficients)
Euler Cauchy equation
Wronskian and linear independence of solutions

Nonhomogeneous Linear ODEs
Undetermined coefficient method
Variation-of-Parameter formula

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How to Solve 2nd Order Linear ODE

$$y'' + P(x)y' + Q(x)y = r(x) \qquad (N)$$
$$y'' + P(x)y' + Q(x)y = 0 \qquad (H)$$

Facts:
1. (H) has two "linearly independent" solutions $y_1(x)$ and $y_2(x)$.

2, $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$, c_1, c_2: constants, is again a solution for (H).
(superposition principle, 중첩원리)

이 때, 모든 $y_h(x)$의 집합은 벡터공간을 이루며, Fundamental Set of Solution(FSS)라 한다.

3. If you get a particular solution $y_p(x)$ for (N), the general solution of (N) is
$y(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$.

Also we can determine c_1 and c_2 with the initial condition.

[Find FSS y_h for (H)]

- To find the second solution $y_2(x)$ given a solution $y_1(x)$ of (H) such that y_1 and y_2 are linearly independent.

(i) $y_1(x)$: known solution to (H)

(ii) Assume that $y_2(x) = u(x) y_1(x)$ is another solution.

(iii) reduction of order

$y_2{}'(x) =$
$y_2{}''(x) =$

Let $w = u'$. Then we get
$$y_1 w' + (2y_1{}' + Py_1)w = 0$$

$$u(x) = C_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx + C_2$$ and so

(iv) $y_2 = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx$

(v) In fact, $y_1(x)$ and $y_2(x)$ are linearly independent.

Example $(x^2 - x)y'' - xy' + y = 0$

Consider n-th Order Linear ODE

$$y^{(n)}(x)+\ \cdots+a_1(x)y'(x)+a_0(x)y(x)=f(x)\ (N)$$
$$y^{(n)}(x)+\ \cdots\ +a_1(x)y'(x)+a_0(x)y(x)=0\ (H)$$

(1) The set of all solutions to (H) forms the **vector space** with addition and scalar multiplication.

- $y\equiv 0$ is always a solution to (H), it is called the trivial solution.

(2) basis, linearly independent

: A set of functions $f_1(x), f_2(x),\cdots,f_n(x)$ is said to be **linearly dependent** if there exist constants $c_1, c_2, \cdots, c_n$, not all zero, such that

$$c_1f_1(x)+c_2f_2(x)+\ \cdots\ +c_nf_n(x)=0$$

: A set $f_1(x), f_2(x)$ is linearly dependent

$\Leftrightarrow\ f_1(x)=kf_2(x)$, for some $k\neq 0$

Examples (linearity for functions)

(i) $f_1=x,\ f_2=x\ln x,\ f_3=x^2\ln x$

(ii) $f_1=\sqrt{x}+5, f_2=\sqrt{x}+5x, f_3=x-1, f_4=x^4$

(3) Fundamental set of Solutions (FSS)

: Any set $y_1,\ y_2,\ \cdots,\ y_n$ of linearly independent solutions of (H) is said to be a fundamental set of solutions.

Example

$$(*)\ \ x^3y''' - 2xy' + 4y = 0$$

$y=c_1x^2+c_2x\ln x+c_3x^2\ln x$ is a FSS of $(*)$

In fact, $x^2,\ x\ln x,\ x^2\ln x$ are solutions to $(*)$ and linearly independent.

(4) The general solution of (N) is

Homogeneous linear ODE with constant coefficient

$$y''+P(x)y'+Q(x)y=\ 0\qquad (H)$$

Let $P(x)=P,\ Q(x)=Q$.

Assume $y=e^{mx}$ is a solution (H).

$e^{mx}(m^2+Pm+Q)=0$ and so

$m^2+Pm+Q=0$ (characteristic equation)

case1: distinct real roots $m_1,\ m_2$

$y_1=e^{m_1x},\ \ y_2=e^{m_2x}$: FSS

case2: repeated real roots m

$y_1=e^{mx},\ \ y_2=xe^{mx}$: FSS

case3: conjugate complex roots $\alpha\pm i\beta$

$y_1=e^{\alpha x}\cos(\beta x),\ \ y_2=e^{\alpha x}\sin(\beta x)$: FSS

Remark Eular formula(Taylor series)

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ \cdots$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ \cdots$$
$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ \cdots$$
$$e^{ix}=\cos x+i\sin x,\quad e^{-ix}=\cos x-i\sin x$$
$$\cos x=\frac{e^{ix}+e^{-ix}}{2},\quad \sin x=\frac{e^{ix}-e^{-ix}}{2i}$$

Examples

(1) $y''-y=0$

(2) $y''+y'-2y=0,\ y(0)=4,\ y'(0)=-5$

(3) $y''+y=0$

(4) $y''+0.2y'+4.01y=0$

[Cauchy-Euler Equation]

A linear differential equation of the form
$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = f(x)$
where $a_n, a_{n-1}, \cdots, a_0$ are constants, is known as a Cauchy-Euler Equation.
(일반적으로 해를 $(0,\infty)$, 또는 $(-\infty,0)$에서 구한다.)

$$x^2 y'' + ax y' + by = 0$$

Try a solution of the form $y = x^m$!!!

$x^m (m^2 + (a-1)m + b) = 0$, $m^2 + (a-1)m + b = 0$

case1 : distinct real roots
case2 : double root
case3 : conjugate complex roots

Examples
(1) $x^2 y'' - 2x y' - 4y = 0$
(2) $4x^2 y'' + 8x y' + y = 0$

<u>Existence & Uniqueness Theorem for IVP</u>

Consider a IVP
$y^{(n)}(x) + p_{n-1}(x) y^{(n-1)}(x) + \cdots + p_0(x) y(x) = 0 \quad (H)$
$y(x_0) = K_0, y'(x_0) = K_1, \cdots, y^{(n-1)}(x_0) = K_{n-1} \quad (I.C.)$

If the coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x_0)$ are continuous near x_0, then IVP has a unique solution.

Example Let $y'' + 16y = 0 \quad (H)$.

$y(x) = C_1 \cos(4x) + C_2 \sin(4x)$ is a general solution to (H)

(0) $y'' + 16y = 0$, $y(0) = 0$, $y'(0) = 0$

(1) $y'' + 16y = 0$, $y(0) = 0$, $y(\frac{\pi}{2}) = 0$

(2) $y'' + 16y = 0$, $y(0) = 0$, $y(\frac{\pi}{2}) = 1$

[Wronskian
and linear independence of solutions]

With $y_1(x)$ and $y_2(x)$ being the solutions of
$$y'' + P(x)y' + Q(x)y = 0 \qquad (H),$$

Wronski determinant(Wronskian) of $y_1(x)$ and $y_2(x)$ is defined by
$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$
Then
(1) two solutions y_1, y_2 are linearly dependent on an interval I
$\Leftrightarrow$ (2) $W(y_1(x), y_2(x)) = 0$, for some $x^* \in I$.

Proof
(1) $\Rightarrow$(2)
(2) $\Rightarrow$(1)

[참고] Let $f_1, f_2, \cdots, f_n$ be solutions of (H).
$y^{(n)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0 \quad (H)$
$$W(f_1, f_2, \cdots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n)} & f_2^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$$

If $y_1, \cdots, y_n$ are solutions to (H), then the followings are equivalent.
(1) W is not zero function
(2) $y_1, \cdots, y_n$ are linearly independent
[If $y_1, \cdots, y_n$ are solutions to (H). Then $W(y_1, \cdots, y_n)(x) \equiv 0$ or $W(y_1, \cdots, y_n)(x) \neq 0, \forall x$]

Example $y''' - 6y'' + 11y' - 6y = 3 \ (N)$
$y''' - 6y'' + 11y' - 6y = 0 \ (H)$
1. $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ satisfy (H)
2. $W(y_1, y_2, y_3) \neq 0$
3. General solution of (H)
4. $y = 1$ is a particular solution of (N)
5. General solution of (N)
$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + 1$$

How to Solve 2nd Order Linear ODE (N)

$$y'' + P(x)y' + Q(x)y = r(x) \qquad (N)$$
$$y'' + P(x)y' + Q(x)y = 0 \qquad (H)$$

A General Solution $y(x)$ to (N) as

$$y(x) = y_h(x) + y_p(x)$$

Find a particular solution y_p to (N)

(1) undetermined coefficients

Trial Particular Solutions

(Select a suitable type of particular solution with undetermined coefficients)

$r(x)$	$y_p(x)$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + D$
$\sin(2x)$ or $\cos(2x)$	$A\sin(2x) + B\cos(2x)$
e^{2x}	Ae^{2x}
$x e^{3x}$	$(Ax+B)e^{3x}$
$(x+1)\sin(2x)$	$(Ax+B)\sin(2x)$
$e^{3x}\sin(2x)$	$e^{3x}(A\sin(2x) + B\cos(2x))$

(2) variation of parameter

y_1, y_2: solutions to (H)

y_p : particular solution to (N)

assume $y_p = u_1 y_1 + u_2 y_2$ with additional assumption $u_1' y_1 + u_2' y_2 = 0$.

Then we have

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = r \end{cases}$$

and so

$$u_1'(x) = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_1'(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$y_p = -y_1 \int \frac{y_2 r}{W(y_1, y_2)}\,dx + y_2 \int \frac{y_1 r}{W(y_1, y_2)}\,dx$$

Examples

1. $y'' + 4y' - 2y = 2x^2 - 3x + 6$
2. $y'' - y' + y = 2\sin(3x)$
3. $y'' - 2y' - 3y = 4x - 5 + 6x e^{2x}$
4. $y'' - 5y' + 4y = 8e^{x}$
5. $y'' - 2y' + y = e^{x}$
6. $4y'' + 36y = \operatorname{cosec}(3x)$
7. $y'' + y = \sec x$