1.

(1) $f_1(x)=5,\ f_2(x)=\cos^2 x,\ f_3(x)=\sin^2 x$

$g(x)=C_1 5+C_2\cos^2 x+C_3\sin^2 x$

$=C_1 5+C_2\cos^2 x+C_3(1-\cos^2 x)$

$=C_1 5+C_3+(C_2-C_3)\cos^2 x$

$5C_1+C_3=0,\ C_2-C_3=0 \rightarrow g(x)=0$

$\therefore$ if $C_2=1,\ C_3=1,\ C_1=-\frac{1}{5}$

$g(x)=-\frac{1}{5}\cdot 5+1\cdot\cos^2 x+1\cdot\sin^2 x=-1+1=0$

$\therefore$ linearly dependent

(2) $f_1(x)=x,\ f_2(x)=x-1,\ f_3(x)=x+3$

$g(x)=C_1 x+C_2(x-1)+C_3(x+3)$

$=x(C_1+C_2+C_3)-C_2+3C_3$

$\begin{cases} C_1+C_2+C_3=0,\ C_2=3C_3 \rightarrow g(x)=0 \\ C_1+4C_3=0 \end{cases}$

$\therefore$ if $C_2=3,\ C_3=1,\ C_1=-4$

$g(x)=0$

$\therefore$ linearly dependent

(3) $f_1(x)=1+x,\ f_2(x)=x,\ f_3(x)=x^2$

$g(x)=C_1(1+x)+C_2(x)+C_3(x^2)$

$=C_3x^2+(C_1+C_2)x+C_1$

$C_3=0,\ C_1+C_2=0,\ C_1=0 \rightarrow g(x)=0$

$\therefore C_1=C_2=C_3=0$ is unique

$\therefore$ linearly independent

(4) $f_1(x)=x^2,\ f_2(x)=x^2\ln x,\ x>1$

$g(x)=C_1x^2+C_2x^2\ln x$

$C_1=0,\ C_2=0 \rightarrow g(x)=0$

$\therefore C_1=C_2=0$ is unique

$\therefore$ linearly independent

(5) $f_1(x)=\ln x,\ f_2(x)=\ln x^3$

$g(x)=C_1\ln x+C_2\ln x^3$

$=C_1\ln x+3C_2\ln x$

$=(C_1+3C_2)\ln x$

$C_1+3C_2=0 \rightarrow g(x)=0$

$\therefore$ if $C_1=3,\ C_2=-1$

$g(x)=0$

$\therefore$ linearly dependent

2. (1) Verify that $y_{p1} = 3e^{2x}$ and $y_{p2} = x^2 + 3x$ are, respectively, particular solutions of $y'' - 6y' + 5y = -9e^{2x}$ and $y'' - 6y' + 5y = 5x^2 + 3x - 16$.

$y'' - 6y' + 5y = -9e^{2x}$

$$\begin{array}{c|l} 5 & y_{p_1} = 3e^{2x} \\ -6 & y_{p_1}' = 6e^{2x} \\ 1 & y_{p_1}'' = 12e^{2x} \end{array}$$

$= 15e^{2x} - 36e^{2x} + 12e^{2x}$

$= \underline{-9e^{2x}}$

$y'' - 6y' + 5y = 5x^2 + 3x - 16$

$$\begin{array}{c|l} 5 & y_{p_2} = x^2 + 3x \\ -6 & y_{p_2}' = 2x + 3 \\ 1 & y_{p_2}'' = 2 \end{array}$$

$= 5(x^2 + 3x) - 6(2x + 3) + 2$

$= \underline{5x^2 + 3x - 16}$

(2) Use part (1) to find particular solutions of $y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$ and $y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}$.

$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$

$= (5x^2 + 3x - 16) + (-9e^{2x})$

$y_{p1}'' - 6y_{p1}' + 5y_{p1} = -9e^{2x}$

$y_{p2}'' - 6y_{p2}' + 5y_{p2} = 5x^2 + 3x - 16$

$(y_{p1}'' + y_{p2}'') - 6(y_{p1}' + y_{p2}') + 5(y_{p1} + y_{p2}) = (5x^2 + 3x - 16) + (-9e^{2x})$

$\therefore y_p(x) = (x^2 + 3x) + (3e^{2x})$

$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}$

$= -2(5x^2 + 3x - 16) - \frac{1}{9}(-9e^{2x})$

$\therefore y_p(x) = -2(x^2 + 3x) - \frac{1}{9}(3e^{2x})$

$= -2x^2 - 6x - \frac{1}{3}e^{2x}$

3. The indicated function $y_1(x)$ is a solution of the given differential equation. Find a second solution $y_2(x)$.

(1) $y''-4y'+4y=0$; $y_1(x)=e^{2x}$

$\lambda^2-4\lambda+4=0$, $\lambda=2$, $y_1(x)=e^{2x}$

$y_2(x)=xe^{2x}$

(2) $y''+16y=0$; $y_1(x)=\cos(4x)$

$\lambda^2+16=0$, $\lambda=\pm 4i$, $y_a(x)=e^{4xi}$, $y_b(x)=e^{-4xi}$

$\begin{cases} y_a(x)=\cos 4x+i\sin 4x \\ y_b(x)=\cos 4x-i\sin 4x \end{cases}$

$\therefore y_1(x)=\cos 4x$, $y_2(x)=\sin 4x$

(3) $y''-y=0$; $y_1(x)=\cosh x=\dfrac{e^x+e^{-x}}{2}$

$\lambda^2-1=0$, $\lambda=\pm 1$, $y_a(x)=e^x$, $y_b(x)=e^{-x}$

$\therefore y_1(x)=\dfrac{e^x+e^{-x}}{2}$, $y_2(x)=\dfrac{e^x-e^{-x}}{2}=\sinh x$

$\therefore y_2(x)=\sinh x$

(4) $xy''+y'=0$; $y_1(x)=\ln x$

$y=x^m$, $y'=mx^{m-1}$, $y''=m(m-1)x^{m-2}$

$x^{m-1}(m^2-m)+x^{m-1}(m)=0$

$x^{m-1}(m^2)=0$

$m^2=0$, $m=0$

$y_a(x)=x^0=1$, $y_b(x)=\ln x\cdot y_a(x)=\ln x$

$\therefore y_1(x)=\ln x$, $y_2(x)=1$

$\therefore y_2(x)=1$

(5) $x^2y'' - 3xy' + 4y = 0$; $y_1(x) = x^2$

$y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$\cancel{x^m}(m^2 - m - 3m + 4) = 0$

$m^2 - 4m + 4 = 0$, $m = 2$

$y_a(x) = x^2$, $y_b(x) = \ln x \cdot x^2$

$\therefore y_1(x) = x^2$, $y_2(x) = x^2 \cdot \ln x$

$\therefore y_2(x) = x^2 \cdot \ln x$

(6) $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$; $y_1(x) = x+1$

$$y'' + \frac{2(1+x)}{(1-2x-x^2)}y' - \frac{-2}{(1-2x-x^2)}y = 0$$

$p = \frac{2(1+x)}{(1-2x-x^2)}$, $y_2(x) = y_1 \int \frac{1}{y_1^2} e^{-\int p\,dx} dx$, $-\int p\,dx = \int \frac{2(1+x)}{x^2+2x-1} dx = \ln|x^2+2x-1|$

$$y_2(x) = (x+1) \int \frac{1}{(x+1)^2}(x^2+2x-1)\,dx$$

$$= (x+1) \int \frac{(x+1)^2 - 2}{(x+1)^2} dx = (x+1) \int \left(1 - \frac{2}{(x+1)^2}\right) dx$$

$$= (x+1)\left(x + \frac{2}{x+1}\right) = x^2 + x + 2$$

$\therefore y_2(x) = x^2 + x + 2$

4. Homogeneous인 이계선형미분방정식 $y''+p(x)y'+q(x)y=0\,(p(x), q(x)$는 연속)의 모든 해는 독립인 두 해의 선형결합의 형태이다. 즉, 특이해(singular solution)는 존재하지 않는다. 이에 대한 이유를 설명하시오. (교재 78쪽의 정리4를 참고해도 좋음)

$y''+p(x)y'+q(x)y=0$

$y(x)=C_1y_1(x)+C_2y_2(x)$ ... general sol

C_1, C_2가 항상 존재함을 보이면 된다.

C_1, C_2를 구하기위해 임의의 x_0를 넣어보자.

$y(x_0)=k_0$, $y'(x_0)=k_1$

$C_1y_1(x_0)+C_2y_2(x_0)=k_0$

$C_1y_1'(x_0)+C_2y_2'(x_0)=k_1$

$$\underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{A}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} k_0 \\ k_1 \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\frac{1}{\det A}\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}\begin{bmatrix} k_0 \\ k_1 \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\frac{1}{W(y_1(x_0), y_2(x_0))}\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}\begin{bmatrix} k_0 \\ k_1 \end{bmatrix}$$

$y_1(x)$ & $y_2(x)$ linearly Independent

$\therefore\ W(y_1(x_0), y_2(x_0))\neq 0$

$\therefore\ C_1, C_2$가 항상 존재한다.

$\therefore$ 특이해(singular solution)는 존재하지 않는다.