응용수학2

Introduction to ODE

Solution of First-Order ODEs
by Graphical and Numerical method

- To Solve First-Order ODEs
Separable
Exact
Linear

Existence & Uniqueness

교재 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7

[1] Ordinary Differential Equation 상미분방정식

$F(x, y, y', y'', \cdots, y^{(n)}) = 0$ 일반형(implicit form)
$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ 정규형(explicit form)

일반해(general solution)
특수해(particular solution)
특이해(singular solution)

IVP(Intial Value Problem) 초기값 문제
BVP(Boundary Value Problem) 경계값 문제

Order 계
Autonomous ($y' = f(y)$)
Linear 선형/ Nonlinear 비선형
Homogeneous 제차, 동차/ Inhomogeneous 비제차

Examples

(1) $y'' + 3xy + 72 = 0$
(2) $(y'')^3 - (y')^5 = \sin^2 x$
(3) $(\sin x) y'' + 2xy = 0$
(4) $xy''y' + y = x$
(5) $\begin{cases} y_1'(x) + 2y_2(x) + 3 = 0 \\ y_2'(x) + 2y_1'(x) + y_2(x) = 2 \end{cases}$
(6) $2\dfrac{\partial y}{\partial x}(x,z) + 3\dfrac{\partial y}{\partial z}(x,z) - 2x = 0$
(7) $y' = \cos x,\ y(0) = 1$
(8) $y'' + 16y = 0,\ y(\frac{\pi}{2}) = 2,\ y'(\frac{\pi}{2}) = 1$
(9) $y'' - 34y' - 2xy = x^2,\ y(0) = 1, y(2) = 4$

(10) $y' + 2y^{\frac{3}{2}} = 0$
일반해 $y = \dfrac{1}{(x+C)^2}$, C는 상수,
특수해 $y = \dfrac{1}{x^2}$
특이해 (singular solution) $y = 0$

(11) $(y')^2 - xy' + y = 0$
일반해 $y = Cx - C^2$, C는 상수,
특수해 $y = 0$
특이해 (singular solution) $y = \dfrac{x^2}{4}$

[2] First-Order ODEs

$$F(x, y, y') = 0 \text{ or } y' = f(x,y)$$

- 방향장(direction field)에 의한 해
(Integral Curve)

Graphical method
(1) pick (x, y), compute $f(x,y)$
(2) pick slope c, plot isocline $f(x,y) = c$
Draw an integral curve $y_1(x)$

Examples
(1) $y' = y$
(2) $y' = -\dfrac{x}{y}$

Numerical methods
Given IVP $y' = f(x, y),\ y(x_0) = y_0$.
Euler's method
$x_{n+1} = x_n + h,\ h:$ step size
$y_{n+1} = y_n + hA_n$
$A_n = f(x_n, y_n)$

Example
$y' = x^2 - y^2,\ y(0) = 1,\ h = 0.1$

n	x_n	y_n	A_n	hA_n
0	0	1	-1	-0.1
1				
⋮				

[How to solve the First-Order ODEs]

1. Separable(변수분리형): $y' = g(x)h(y)$꼴인 경우

$h(y) \neq 0$라 가정하고 $h_1(y)y' = g(x)$꼴의 1계 미분방정식을 생각하자.

$h_1(y(x))y'(x) = g(x)$

$\Rightarrow \int h_1(y(x))y'(x)\,dx = \int g(x)\,dx$

즉, $H_1(y) = G(x) + C$.

따라서, $y = H_1^{-1}(G(x)+C)$.

Examples

(1) $9yy' + 4x = 0$

(2) $y' = y$

(3) $y' = -\dfrac{y}{x}$

(4) $y' = -2xy$

(5) $y' = 1 + y^2$

(6) $2xyy' = y^2 - x^2$ [$y' = f(\dfrac{y}{x})$ 꼴이면...]

(7) $(2x-4y+5)y' + (x-2y+3) = 0$

[$ay+bx+c$를 치환하여 변형하면...]

2. Exact

Let $u(x,y)$ have continuous partials. Then total differential of u is defined as

$$du = u_x\,dx + u_y\,dy$$

and if $u(x,y) = c$(constant), then $du = 0$.

For example,

$$u(x,y) = x + x^2y^3 = c$$

$$du = (1+2xy^3)\,dx + 3x^2y^2\,dy = 0$$

$$\frac{dy}{dx} = -\frac{1+2xy^3}{3x^2y^2} \quad (*)$$

If a DE is given (*), then a solution is

$$x + x^2y^3 = c.$$

Definition: A first order ODE

$M(x,y) + N(x,y)\,y' = 0$ or $M(x,y)\,dx + N(x,y)\,dy = 0$

is exact if the differential form

$M(x,y)\,dx + N(x,y)\,dy$ is the total differential of some function $u(x,y)$.

Theorem1: If there exists C^2−function $u(x,y)$ such that $u_x = M$, $u_y = N$, then

$M_y(x,y) = N_x(x,y), \forall\, x, y.$

Theorem2:

If $M_y(x,y) = N_x(x,y)$ on $(x,y) \in D \subset R^2$ where M, N: C^1−functions, D: simply connected region, then there exists $u(x,y)$ defined on D such that $u_x = M$, $u_y = N$.

참고) simply connected region

Solve Exact DE :

0. check $M_y(x,y) = N_x(x,y)$

1. find $u(x,y)$ such that $u_x = M$, $u_y = N$

2. $u(x,y) = C$, (C: constant) general solution!!

Examples

(1) $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$

(2) $\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0$

(3) $\sin x \cosh(y)\,dx - \cos x \sinh(y)\,dy = 0$

(4) $-y\,dx + x\,dy = 0$

[Reduction to exact form]

(*) $-y\,dx + x\,dy = 0$ is not exact!!

Multiply it by $\dfrac{1}{x^2}$, we get an exact ODE

(**) $-\dfrac{y}{x^2}dx + \dfrac{1}{x}dy = 0.$

In fact, $d(\dfrac{y}{x}) = -\dfrac{y}{x^2}dx + \dfrac{1}{x}dy = 0.$

To find Integrating Factor(IF) in particular case:

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

$$\text{multiply } F(x,y) \Rightarrow FP\,dx + FQ\,dy = 0$$

$$(FP)_y = (FQ)_x, \quad F_y P + F Q_y = F_x Q + F Q_x$$

(i) $F(x,y) = F(x)$

$\dfrac{F_x}{F} = \dfrac{1}{Q}(P_y - Q_x) \equiv R(x)$ (depending on x only)

$$\vdots$$

$\Rightarrow \; F(x) = e^{\int R(x)\,dx}$

(ii) $F(x,y) = F(y)$ 인 경우

$\dfrac{F_y}{F} = \dfrac{1}{P}(Q_x - P_y) \equiv R(y)$(depending on y only)

$\Rightarrow \; F(y) = e^{\int R(y)\,dy}$

Example

$(e^{x+y} + y e^y)\,dx + (x e^y - 1)\,dy = 0$

3. First order Linear ODEs

$$y' + p(x)\,y = q(x) \quad \text{(standard linear form)}$$

(1) If $q(x) = 0$, then (homogeneous case)

$$y' + p(x)\,y = q(x) \;\Rightarrow\; \frac{y'}{y} = -\,p(x)$$

(2) If $q(x) \neq 0$, then (nonhomogeneous case)

$$y' + p(x)\,y = q(x) \;\Rightarrow\; (p(x)\,y - q(x))\,dx + 1\,dy = 0$$

Linear DE for nonhomogeneous case):

0. standard linear form
1. calculate IF
2. multiply both sides by IF
3. integrating!!

Examples

(1) $y' - y = e^{2x}$

(2) $y' + (\tan x)\,y = \sin 2x,\; y(0) = 1$

4. Reduction to linear form (Bernoulli)

$$y' + p(x)\,y = q(x)\,y^{\alpha}, \; \alpha \neq 0, 1$$

Substitution $u(x) = (y(x))^{1-a}$

$\Rightarrow \; u' + (1-a)\,p\,u = (1-a)\,q$ (linear)

Examples

(1) $y' = \dfrac{y}{x} - y^2$

(2) Logistic equation $y' = Ay - By^2$

[Existence and Uniqueness]

Consider the IVP

$$(*) \;\; y' = f(x,y), \;\; y(x_0) = y_0$$

(1) Suppose that two-variable function $f(x,y)$ is continuous near (x_0, y_0) and bounded then $(*)$ has at least solution $y(x)$.

(2) Let f and f_y be continuous near (x_0, y_0) and both are bounded then $(*)$ has at most one solution $y(x)$.

Examples

(1) $y' = 1 + y^2,\; y(0) = 0$

(2) $y' = x\sqrt{y},\; y(0) = 0$

(3) $y' = x\sqrt{y},\; y(0) = 1$