Introduction to Matrix
- matrix & vector
- addition & scalar multiplication
- product of matrices
- matrix as an operation
-

System of linear equations
- vectors & linear equations
- existence and uniqueness of solution
- elementary row operation
- Gauss elimination
- inverse matrices
-

Vector spaces & subspaces
- linear combinations, linear independence
- rank of matrix
- column & row spaces
-

Determinants
- determinant by cofactors
- properties for determinants
- Cramer's Rule

교재 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, 7.8

[1] matrix & vector

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$= [a_{ij}],\ 1 \le i \le m, 1 \le j \le n$

size of A 크기
entry of A
rectangular matrix (tall or fat matrix)
square matrix

$$v = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad w = [a_{11}\ a_{12}\ \cdots\ a_{1n}]$$

column vector, row vector

[2] addition & scalar multiplication
$A = B$, if $a_{ij} = b_{ij}, i = 1, \cdots, m,\ j = 1, \cdots, n$
$A + B =$
zero matrix O, 영행렬
cA, c는 상수, $cA =$
$A - B = A + (-1)B =$

Rules :
$m \times n$ matrices A, B, C
constant k, l

- $A + B = B + A$
- $(A+B) + C = A + (B+C)$
- $A + O = O + A = A$
- $A + (-A) = (-A) + A = O$
- $k(A+B) = kA + kB$
- $(k+l)A = kA + lA$
- $(kl)A = k(lA)$
- $1A = A$

[3] product of matrices

정의 : $m \times r$ matrix A and $r \times p$ matrix B

AB is defined by $m \times p$ matrix C whose entry

$c_{ij} = (a_{i1}, a_{i2}, \cdots, a_{ir}) \bullet (b_{1j}, b_{2j}, \cdots, b_{rj}) = \sum_{k=1}^{r} a_{ik} b_{kj}$

즉, $C = [c_{ij}],\ 1 \le i \le m, 1 \le j \le p$

Rules :
matrices A, B, C
constant k
- $k(AB) = (kA)B = A(kB)$
- $A(BC) = (AB)C$
- $(A+B)C = AC + BC$
- $C(A+B) = CA + CB$

[4] Transpose of $m \times n$ matrix $A = [a_{ij}]$
A^T, $n \times m$ matrix whose entry a_{ji}

Rules :
$m \times n$ matrices A, B and constant k

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

[5] 행렬의 곱셈에 대한 해석

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mr} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix}, \quad \vec{a}_i = [a_{i1}\ a_{i2}\ \cdots\ a_{ir}]$$

$$B=\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{r1} & \cdots & b_{rp} \end{bmatrix} = [\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_p], \quad \vec{b}_j = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{rj} \end{bmatrix}$$

1. 행벡터와 열벡터의 내적 (정의)

2. 행렬과 열벡터의 곱

$$C=AB=A[\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_p] = [A\vec{b}_1\ A\vec{b}_2\ \cdots\ A\vec{b}_p]$$

3. 행벡터와 행렬의 곱

$$C=AB=\begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix} B = \begin{bmatrix} \vec{a}_1 B \\ \vdots \\ \vec{a}_m B \end{bmatrix}$$

[6] matrix as an operation

permutation matrix P_{ij} (row exchanger)

$P_{12}A =$

elimination matrix E_{ij}

$E_{21}A =$

[6] Transposes

A^T : $n \times n$ matrix whose entry a_{ji}

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$

Def. A symmetric matrix has $A^T = A$.

A Skew symmetric has $A^T = -A$.

[7] 선형연립방정식과 행렬방정식

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

행렬방정식 표현 **계수행렬** **첨가행렬**

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -4 \\ -4 & 5 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$

첨가계수행렬 표현

$[A \mid \mathbf{b}]$

- coefficient matrix, augmented matrix

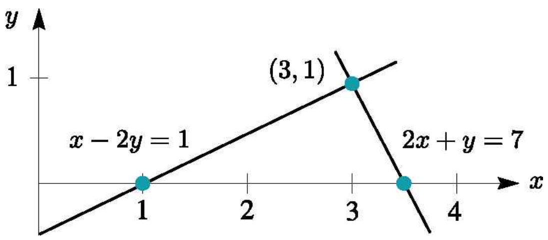
- Homogeneous equations
- Nonhomogeneous equations

[8] 선형 연립방정식의 해에 대한 두 가지 해석

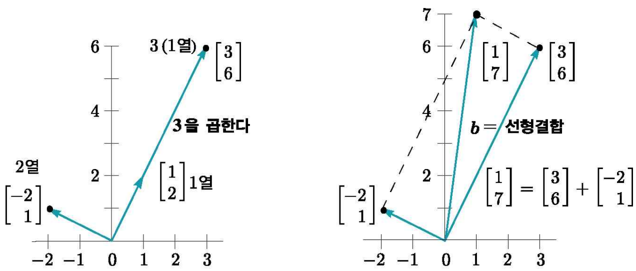
[2차 선형 연립방정식]

$$\begin{cases} x - 2y = 1 \\ 2x + y = 7 \end{cases}, \quad x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

ROWS : 두 직선의 교집합



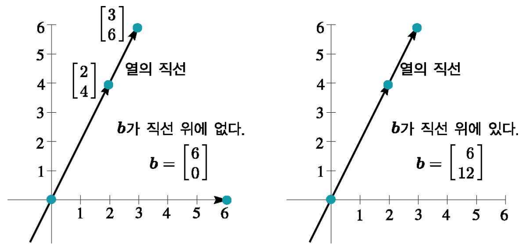
COLUMNS : 상수벡터와 열벡터들의 선형결합



(a) (b)

해의 기하학적 해석

$$\begin{cases} 2x+3y=b_1 \\ 4x+6y=b_2 \end{cases}, \qquad x\begin{bmatrix}2\\4\end{bmatrix}+y\begin{bmatrix}3\\6\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

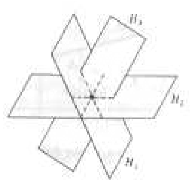


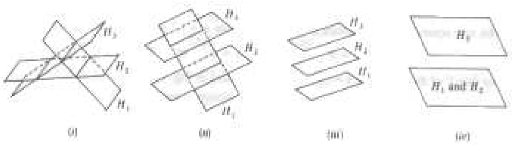
[3차 선형 연립방정식]

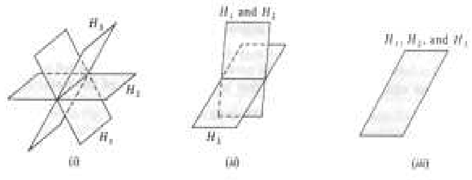
$$\begin{cases} x+2y+3z=6 \\ 2x+5y+2z=4 \\ 6x-3y+z=2 \end{cases},$$

$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$$

ROWS :
COLUMNS :
기하학적 해석







[9] Gauss elimination

How to find the solutions?
- pivoting
- forward elimination
- upper triangular matrix
- back substitution

: elementary row operation을 유한번 적용하여 echelon form(사다리꼴)으로 변환해가는 과정

$$P_{12}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad E_{21}=\begin{bmatrix}1&0&0\\k&1&0\\0&0&1\end{bmatrix},\quad E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&k&1\end{bmatrix},$$

$\cdots$

Example : $C=\begin{bmatrix}2&1&1&1\\4&-6&0&2\\-2&7&2&3\end{bmatrix}$

$$E_{32}E_{31}E_{21}C=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&1&1&1\\4&-6&0&2\\-2&7&2&3\end{bmatrix}$$

$$=\begin{bmatrix}2&1&1&1\\0&-8&-2&0\\0&0&1&4\end{bmatrix}$$

(1) Gauss elimination (unique solution)

$$\begin{cases} 2x+y+z=1 \\ 4x-6y=2 \\ -2x+7y+2z=3 \end{cases} \qquad \begin{cases} x-y+z=0 \\ 2x-2y+2z=0 \\ 10y+25z=90 \\ 20x+10y=80 \end{cases}$$

(2) Gauss elimination (no solution)

$$\begin{cases} 3x+2y+z=3 \\ 2x+y+z=0 \\ 6x+2y+4z=6 \end{cases}$$

(3) Gauss elimination (infinitely many solutions)

$$\begin{cases} 3x+2y+2z+5w=1 \\ 6x+4y+z+2w=2 \\ 3z+w=2 \end{cases}$$

[10] Inverse matrices

Def. A matrix A is invertible if there exists B such that $AB=BA=I$, we call B by the inverse of A and write it by A^{-1}.

(1) If A is invertible, A is square and B is unique.

(2) $E_{21}=\begin{bmatrix}1 & & \\ k & 1 & \\ & & 1\end{bmatrix}$, $E_{21}^{-1}=$

$D=\begin{bmatrix}d_1 & & \\ & d_2 & \\ & & d_3\end{bmatrix}$, $D^{-1}=$

(3) If there exists nonzero vector $\vec{x}$ such that $A\vec{x}=\vec{0}$ then A is not invertible.

(4) $(A^{-1})^{-1}=A$

(5) $(AB)^{-1}=B^{-1}A^{-1}$

(6) $(ABCD)^{-1}=D^{-1}C^{-1}B^{-1}A^{-1}$

How to find the inverse of A?

$$AB=I,\quad B=\begin{bmatrix} \\ \overrightarrow{c_1} & \overrightarrow{c_2} & \overrightarrow{c_3} \\ \\ \end{bmatrix},\quad I=\begin{bmatrix} \\ \overrightarrow{e_1} & \overrightarrow{e_2} & \overrightarrow{e_3} \\ \\ \end{bmatrix}$$

$$\begin{cases} A\overrightarrow{c_1} = \overrightarrow{e_1} \\ A\overrightarrow{c_2} = \overrightarrow{e_2} \\ \quad\vdots \qquad \vdots \\ A\overrightarrow{c_n} = \overrightarrow{e_n} \end{cases}$$

Example (Gauss-Jordan elimination)

$$A=\begin{bmatrix}2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2\end{bmatrix},$$

$$[\,A : I\,]=\begin{bmatrix}2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1\end{bmatrix}$$

[11] Vector spaces and Subspaces

Def. A <u>vector space</u> V is the set of vectors together with rules for vector addition and scalar multiplication by numbers. The set is closed under the vector addition and the scalar multiplication and the 8 conditions must be satisfied.

Def. A <u>subspace</u> W is the subset of a vector space V which is itself a vector space.

Theorem.
Let V be a vector space and $\varnothing \neq W \subset V$.
Then the followings are equivalent.
(1) W is a subspace of V.
(2) $\forall v_1, v_2 \in W,\ k \in R,\ \ v_1+v_2 \in W$ and $kv_1 \in W$.

Examples for vector spaces

(1) Euclidean vector space R^n

(2) $V=\{\vec{0}\}$ zero vector space

(3) $M=$ $m\times n$ matrices

(4) $P_n=$ polynomials of degree $\le n$

Subspaces in the vector spaces

(1) lines through the origin in R^2 or R^3

(2) planes through the origin in R^3

(3) symmetric matrices

(4) (upper or lower) triangular matrices

(5) diagonal matrices

Remark. not subspaces!!
quarter-plane, half-line, $\cdots$

[Basic definitions]

1. Linear combination of $v_1, v_2, \cdots, v_n$
& Linear independence of $\{v_1, v_2, \cdots, v_n\}$

(1) If $c_1, c_2, \cdots, c_n$ are constants, we call
$c_1v_1+c_2v_2+\cdots+c_nv_n$
the linear combination of vectors $v_1, v_2, \cdots, v_n$.

(1) If the equation

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

holds only with $c_1 = c_2 = \cdots = c_n = 0$, then we call $\{v_1, v_2, \cdots, v_n\}$is linearly independent.

(If not, linearly dependent.)

2. Basis and Dimension

(1) 'basis' of a vector space V is the set of vectors $\{v_1, v_2, \cdots, v_r\}$ in a vector space V such that

(i) they are independent

(ii) $V = span\{v_1, v_2, \cdots, v_r\}$

Remark.

Let $S = \{v_1, v_2, \cdots, v_r\} \subset V$. A subspace SS of V can be obtained by

$SS = all\ c_1v_1 + c_2v_2 + \cdots + c_rv_r$

; the subspace of V spanned by $\{v_1, v_2, \cdots, v_r\}$

; the smallest subspace containing$\{v_1, v_2, \cdots, v_r\}$

(2) For a vector space, 'basis' is not unique but its number(cardinality) is unique. In this case the cardinality is called thes'dimension' of a vector space V ($\dim(V)$).

Remark. If V is zero vector space, then $\dim(V) = 0$.

If not,

$\dim(V) =$ cardinality of a basis of

= V에 속한 선형독립인 벡터들의 최대 수

= V를 생성할 수 있는 벡터들의 최소 수

Examples

(1) $\dim(R^n) = n$

(2) $\dim(P_n) = n+1$

(3) $\dim(M_{m\times n}) = mn$

(4) $\dim(Sym_n) = \dfrac{n(n+1)}{2}$,

Sym_n : $n \times n$ symmetric matrices

(5) $\dim(span\{v_1, v_2, \cdots, v_r\}) \le r$

[12] Rank of a matrix A

Def. rank$A$= 독립인 행벡터의 최대 수

Example.

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Properties.

1. 'rank(계수)' is invariant to elementary row operation.

2. rank$A$= 행렬 A에서 독립인 열벡터의 최대 수.
즉, rank$A$ = rankA^T

3. For a $m \times n$ matrix A, rank$A \le \min\{m, n\}$

4. For $v_1, v_2, \cdots, v_p \in R^n$, if $n < p$ then they are linearly dependent.

5. Let $A = \{v_1, v_2, \cdots, v_p\}$, $v_i \in R^n$.
If rank$A = p$, they are linearly independent.
If rank$A < p$, they are linearly dependent.

Def. Column space & Row space of A

$C(A)$=span of column vectors of A

$R(A)$=span of row vectors of A

Remark

$$\dim(C(A)) = \dim(R(A)) = rank(A)$$

Def. Null space & nullity of A

For a given matrix A, the solution set of the homogeneous system $AX = 0$ is a vector space, called the null space of A, and its dimension is called the nullity of A.

Remark

$rank(A)$+nullity(A)=number of columns of A

[13] cofactor expansion for determinant

Def. Let $A = [a_{ij}]_{n\times n}$.

(1) cofactor C_{ij} of a_{ij} $=(-1)^{i+j}|M_{ij}|$
(2) minor M_{ij}: A의 i행과 j열을 제외한 행렬

Then we define the determinant of A as

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in},\ \ 1\le i\le n$$

Remark 모든 행에 관한 여인수 전개와 모든 열에 관한 여인수 전개는 그 값이 같고, 이를 행렬의 행렬식(determinant)이라 한다.

Theorem.

1. Interchange of two rows multiplies the value of determinant by -1.

$$d(P_{ij}A) = |P_{ij}A| = -|A|$$

2. $d(E_{ij}A) = |E_{ij}A| = |A|$

3. $d(D_i(c)A) = |D_i(c)A| = c|A|$

Properties for determinant.

1. 두 행이 같은 행렬의 행렬식은 0

2. 모두 0인 행이 존재하는 행렬의 행렬식은 0

3. 삼각행렬의 행렬식은 대각 성분들의 곱

4. 가역행렬의 행렬식은 0이 아니고 행렬식이 0이 아니면 가역행렬

5. $d(AB) = |AB| = |A||B|$

6. $d(A^T) = |A^T| = |A|$

7. $d(kA) = |kA| =$

[14] Cramer' rule

Solving $Ax=b$ if A is invertible.

$$x = A^{-1}b = \frac{1}{|A|}C^Tb,\ \ C=[C_{ij}]_{n\times n}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix},\ \ x_j = \frac{|B_j|}{|A|},\ j=1,\cdots.n$$

B_j: A의 j열을 b로 대체한 행렬

Inverse of A: $A^{-1} = \frac{1}{|A|}C^T,\ \ C=[C_{ij}]_{n\times n}$

$$\begin{bmatrix} a\ b \\ c\ d \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a\ b \\ c\ d \end{vmatrix}}\begin{bmatrix} C_{11}\ C_{12} \\ C_{21}\ C_{22} \end{bmatrix}^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$