1. $P(x,y)dx + Q(x,y)dy = 0$, $F(x,y) = F(y)$

↳ $\underbrace{FP}_{M=u_x}dx + \underbrace{FQ}_{N=u_y}dy = 0$

$F_y \cdot P + F \cdot P_y = FQ_x$

∴ $\frac{F_y}{F} = \frac{1}{P}(Q_x - P_y)$

$\int \frac{F_y}{F}dy = \int \frac{1}{P}(Q_x - P_y)dy$

$\ln|F| = \int \frac{1}{P}(Q_x - P_y)dy + C$

$F(y) = C \cdot e^{\int \frac{1}{P}(Q_x - P_y)dy}$

∴ $F(y) = e^{\int \frac{1}{P}(Q_x - P_y)dy}$

2.

(1) $xy' = y^2 + y$

↳ $\boxed{\frac{1}{y^2+y}dy = \frac{1}{x}dx}$

∴ Separable

↳ $\boxed{y' - \frac{1}{x}y = \frac{1}{x}y^2}$

∴ Bernoulli

(2) $\frac{dy}{dx} = y^2 - 4$

↳ $\boxed{\frac{1}{y^2-4}dy = 1dx}$

∴ Separable

(3) $x\frac{dy}{dx} + y = x^2y^2$

↳ $\boxed{y' + \frac{1}{x}y = xy^2}$

∴ Bernoulli

(4) $\frac{dy}{dx} = (-2x+y)^2 - 7$, $y(0) = 0$

↳ $-2x + y = u$

$y = u + 2x$

$y' = u' + 2$

∴ $u' + 2 = u^2 - 7$

$\frac{1}{u^2-9}u' = 1$

$\boxed{\frac{1}{u^2-9}du = 1dx}$

∴ Separable (치환)

(5) $\frac{dy}{dx} + y = x$, $y(0) = 4$

$(y-x)dx + 1dy = 0$

$M = u_x$ $N = u_y$

$M_y = 1$, $N_x = 0$

$M_y \neq N_x$ ∴ Not exact

$\frac{1}{Q}(P_y - Q_x) = \frac{1}{1}(1-0) = 1 = R(x)$

$F(x) = e^{\int 1dx} = e^x$

$\boxed{\underbrace{e^x(y-x)}_{M}dx + \underbrace{e^x}_{N}dy = 0}$

$M_y = e^x$, $N_x = e^x$

$M_y = N_x$

∴ exact

(6) $\frac{dy}{dx} - \frac{4}{x}y = x^6 e^x$

$\hookrightarrow \boxed{y' - \frac{4}{x}y = x^6 e^x}$

$\therefore$ linear

(7) $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$

$\boxed{\underbrace{(xy^2 - \cos x \sin x)}_{M} dx + \underbrace{y(x^2-1)}_{N} dy = 0}$

$M_y = 2xy$, $N_x = 2xy$

$M_y = N_x$

$\therefore$ exact

(8) $\underbrace{xy}_{M} dx + \underbrace{(2x^2 + 3y^2 - 20)}_{N} dy = 0$

$M_y = x$, $N_x = 4x$

$M_y \neq N_x$ $\therefore$ Not exact

$P = xy$, $Q = 2x^2 + 3y^2 - 20$

$P_y = x$, $Q_x = 4x$

$P_y - Q_x = -3x$

$\frac{1}{P}(Q_x - P_y) = \frac{3}{y} = R(y)$

$F(y) = e^{\int \frac{3}{y} dy} = e^{3\ln|y|} = y^3$

$\therefore \boxed{\underbrace{y^3 \cdot xy}_{M} dx + \underbrace{y^3 \cdot (2x^2 + 3y^2 - 20)}_{N} dy = 0}$

$M_y = 4xy^3$, $N_x = 4xy^3$

$M_y = N_x$

$\therefore$ exact

(9) $\frac{dy}{dx} = 1 + x + y + xy$, $y(0) = 0$

$\underbrace{(1 + x + y + xy)}_{M} dx + \underbrace{(-1)}_{N} dy = 0$

$M_y = 1 + x$, $N_x = 0$

$M_y \neq N_x$ $\therefore$ Not exact

$\frac{1}{Q}(P_y - Q_x) = -(1+x) = -1 - x = R(x)$

$F(x) = e^{-\int (1+x) dx} = e^{-(x + \frac{1}{2}x^2)}$

$\boxed{\underbrace{e^{-x-\frac{1}{2}x^2} \cdot (1 + x + y + xy)}_{M} dx + \underbrace{e^{-x-\frac{1}{2}x^2} \cdot (-1)}_{N} dy = 0}$

$M_y = e^{-x-\frac{1}{2}x^2} \cdot (1+x)$, $N_x = e^{-x-\frac{1}{2}x^2} \cdot (1+x)$

$M_y = N_x$

$\therefore$ exact