

1. Find the eigenvalues and eigenvectors of  $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

Check that  $\lambda_1 \lambda_2 \lambda_3 = \text{the determinant}$ .

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -4 & -2 \\ 0 & \lambda - 1 & -2 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3 - 4\lambda^2 + 3\lambda, \quad \lambda = 3, 1, 0$$

$$\lambda_1 = 3, \quad \lambda_1 I - A = \begin{bmatrix} 0 & -4 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow[\substack{D_1(\frac{1}{2}) \\ E_{21}(2)}]{\substack{P_{12} \\ E_{21}(2)}} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -6 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow[\substack{D_2(-\frac{1}{6}) \\ E_{12}(1)}]{E_{32}(\frac{1}{2})} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$$\lambda_2 = 1, \quad \lambda_2 I - A = \begin{bmatrix} -2 & -4 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{E_{32}(2) \\ E_{12}(-1)}]{\substack{D_1(-\frac{1}{2}) \\ P_{23}}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$$\lambda_3 = 0, \quad \lambda_3 I - A = \begin{bmatrix} -3 & -4 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{D_1(-\frac{1}{3}) \\ D_2(-1)}]{E_{12}(-4)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$\det A = 0 = \lambda_1 \lambda_2 \lambda_3 = 3 \cdot 1 \cdot 0$$

$$|\lambda I - B| = \begin{vmatrix} \lambda & 0 & -2 \\ 0 & \lambda - 2 & 0 \\ -2 & 0 & \lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 - 4\lambda + 8 = (\lambda - 2)^2(\lambda + 2), \quad \lambda = 2, -2$$

$$\lambda_1 = \lambda_2 = 2, \quad \lambda_1 I - B = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} \xrightarrow[\substack{D_1(\frac{1}{2})}]{E_{31}(1)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\lambda_3 = -2 \quad \lambda_2 I - B = \begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix} \xrightarrow{\substack{E_3(+1) \\ R_1(-\frac{1}{2}) \\ R_2(-\frac{1}{4})}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$\lambda_1 \lambda_2 \lambda_3 = 2 \cdot 2 \cdot -2 = -8 = \det B$$

2. Suppose that  $\lambda$  is an eigenvalue of  $A$ , and  $\vec{x}$  is its eigenvector.

(1) If  $s$  is a scalar, show that this same  $\vec{x}$  is an eigenvector of  $sI - A$ , and find the eigenvalue.

(2) Assuming  $\lambda \neq 0$ , show that  $\vec{x}$  is also an eigenvector of  $A^{-1}$  and find the eigenvalue.

(3) Let  $A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$ . Use (1) and (2) to find the eigenvalues and eigenbases for

the eigenspaces of  $A^{-1}$  and  $3I - A$ .

$$(1) \quad A \vec{x} = \lambda \vec{x}$$

$$(sI - A) \vec{x} = (s - \lambda) \vec{x}$$

$\therefore A$  와  $sI - A$  는 같은 고유벡터를 가지고

$sI - A$  의 고유값은  $s - \lambda$  이다.

$$(2) \quad A \vec{x} = \lambda \vec{x}$$

$$\frac{1}{\lambda} \vec{x} = A^{-1} \vec{x}$$

$\therefore A$  와  $A^{-1}$  은 같은 고유벡터를 가지고

$A^{-1}$  의 고유값은  $\frac{1}{\lambda}$  이다.

$$(3) \quad |\lambda I - A| = \begin{vmatrix} \lambda+2 & -2 & -3 \\ 2 & \lambda-3 & -2 \\ 4 & -2 & \lambda-5 \end{vmatrix} = (\lambda^3 - 6\lambda^2 - \lambda + 30) + (16) + (12) \\ + (12\lambda - 36) + (-4\lambda - 8) + (4\lambda - 20) \\ = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda-2)(\lambda-3)$$

$$\therefore \lambda = 1, 2, 3$$

$$\lambda_1 = 1, \begin{bmatrix} 3 & 2 & -3 \\ 2 & 2 & 2 \\ 4 & 2 & -4 \end{bmatrix} \xrightarrow{\substack{E_{12}(-1) \\ E_{32}(-2) \\ E_{21}(-2) \\ D_2(-\frac{1}{2})}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{E_{32}(-2)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_2 = 2, \begin{bmatrix} 4 & 2 & -3 \\ 2 & 1 & -2 \\ 4 & 2 & -3 \end{bmatrix} \xrightarrow{\substack{E_{31}(-1) \\ E_{12}(-2) \\ P_{12}}} \begin{bmatrix} 2 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{D_1(\frac{1}{2}) \\ E_{12}(1)}} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = t \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, t \neq 0$$

$$\lambda_3 = 3, \begin{bmatrix} 5 & 2 & -3 \\ 2 & 0 & -2 \\ 4 & 2 & 2 \end{bmatrix} \xrightarrow{\substack{E_{12}(-2) \\ E_{32}(-2) \\ D_2(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow{\substack{E_{21}(-1) \\ E_{32}(-1) \\ D_2(\frac{1}{2}) \\ E_{12}(2)}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, t \neq 0$$

①  $A^{-1}$

$$\lambda = 1, \frac{1}{2}, \frac{1}{3}$$

$$\lambda_1 = 1$$

$$\beta_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_2 = \frac{1}{2}$$

$$\beta_2 = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\lambda_3 = \frac{1}{3}$$

$$\beta_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

②  $3I - A$

$$\lambda = 2, 1, 0$$

$$\lambda_1 = 2$$

$$\beta_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_2 = 1$$

$$\beta_2 = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\lambda_3 = 0$$

$$\beta_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$