

1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

Check that $\lambda_1\lambda_2\lambda_3 =$ the determinant.

$$|\lambda I - A| = \begin{vmatrix} \lambda-3 & -4 & -2 \\ 0 & \lambda-1 & -2 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3 - 4\lambda^2 + 3\lambda, \quad \lambda=3, 1, 0$$

$$\lambda_1=3, \quad \lambda I - A = \begin{bmatrix} 0 & -4 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\substack{P_{12} \\ E_{21}(2) \\ D_1(\frac{1}{2})}} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -6 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\substack{E_{32}(\frac{1}{6}) \\ D_2(-\frac{1}{6})}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_{12}(1)}$$

$$d = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$$\lambda_2=1, \quad \lambda I - A = \begin{bmatrix} -2 & -4 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{D_1(-\frac{1}{2}) \\ P_{13} \\ E_{32}(2)}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$d = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0 \xrightarrow{\substack{E_{32}(2) \\ E_{12}(-1)}}$$

$$\lambda_3=0, \quad \lambda I - A = \begin{bmatrix} -3 & -4 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{E_{12}(4) \\ D_1(-\frac{1}{3}) \\ D_2(-1)}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$d = t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$\det A = 0 = \lambda_1\lambda_2\lambda_3 = 3 \cdot 1 \cdot 0$$

$$|\lambda I - B| = \begin{vmatrix} \lambda & 0 & -2 \\ 0 & \lambda-2 & 0 \\ -2 & 0 & \lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 - 4\lambda + 8 \\ = (\lambda-2)^2(\lambda+2), \quad \lambda=2, -2$$

$$\lambda_1=\lambda_2=2, \quad \lambda I - B = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} \xrightarrow{\substack{E_{31}(1) \\ D_1(\frac{1}{2})}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$d = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\lambda_3 = -2 \quad \lambda_2 I - B = \begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix} \xrightarrow{\begin{array}{l} E_{31}(-1) \\ D(-\frac{1}{2}) \\ D_2(-\frac{1}{4}) \end{array}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$\lambda_1 \lambda_2 \lambda_3 = 2 \cdot 2 \cdot -2 = -8 = \det B$$

2. Suppose that λ is an eigenvalue of A , and \vec{x} is its eigenvector.

(1) If s is a scalar, show that this same \vec{x} is an eigenvector of $sI - A$, and find the eigenvalue.

(2) Assuming $\lambda \neq 0$, show that \vec{x} is also an eigenvector of A^{-1} and find the eigenvalue.

(3) Let $A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$. Use (1) and (2) to find the eigenvalues and eigenbases for

the eigenspaces of A^{-1} and $3I - A$.

$$(1) \quad A \vec{x} = \lambda \vec{x}$$

$$(sI - A) \vec{x} = (s - \lambda) \vec{x}$$

$\therefore A$ 와 $sI - A$ 는 같은 고유벡터를 가지고

$sI - A$ 의 고유값은 $s - \lambda$ 이다.

$$(2) \quad A \vec{x} = \lambda \vec{x}$$

$$\lambda \vec{x} = A^{-1} \vec{x}$$

$\therefore A$ 와 A^{-1} 은 같은 고유벡터를 가지고

A^{-1} 의 고유값은 $\frac{1}{\lambda}$ 이다.

$$(3) \quad |\lambda I - A| = \begin{vmatrix} \lambda + 2 & -2 & -3 \\ 2 & \lambda - 3 & -2 \\ 4 & -2 & \lambda - 5 \end{vmatrix} = (\lambda^3 - 6\lambda^2 - \lambda + 30) + (16) + (12) \\ + (12\lambda - 36) + (-4\lambda - 8) + (4\lambda - 20) \\ = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3) \\ \therefore \lambda = 1, 2, 3$$

$$\lambda_1 = 1, \quad \left[\begin{array}{ccc} 3 & -2 & -3 \\ 2 & -2 & -2 \\ 4 & -2 & -4 \end{array} \right] \xrightarrow{E_{12}(-1)} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array} \right] \xrightarrow{E_{32}(-2)} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\alpha_1 = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_2 = 2, \quad \left[\begin{array}{ccc} 4 & -2 & -3 \\ 2 & -1 & -2 \\ 4 & -2 & -3 \end{array} \right] \xrightarrow{E_{31}(-1)} \left[\begin{array}{ccc} 2 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{D_2(\frac{1}{2})} \left[\begin{array}{ccc} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

$$\alpha_2 = t \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, t \neq 0$$

$$\lambda_3 = 3, \quad \left[\begin{array}{ccc} 5 & -2 & -3 \\ 2 & 0 & -2 \\ 4 & -2 & -2 \end{array} \right] \xrightarrow{E_{12}(2)} \left[\begin{array}{ccc} 1 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & -2 & 2 \end{array} \right] \xrightarrow{E_{21}(-1)} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

$$\alpha_3 = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, t \neq 0$$

① A^{-1}

$$\lambda = 1, \frac{1}{2}, \frac{1}{3}$$

$$\beta_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\beta_2 = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\beta_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

② $3I - A$

$$\lambda = 2, 1, 0$$

$$\beta_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\beta_2 = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\beta_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\beta_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\beta_2 = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\beta_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$