컴퓨터 공학과 / 20171630 / 남주형

# 4장 4.5.6절 연습문제

#1 실수체 $\mathbb{R}$ 위의 벡터공간 $\mathbb{R}^4$에서 다음과 같이 주어진 벡터들의 선형독립, 선형종속 여부를 판단하여라. 만약, 선형종속이라면 이들 중 하나의 벡터를 다른 벡터들의 선형결합으로 나타내어라.

(1) $(4,-3,6,2), (1,8,3,1), (3,-2,-1,0)$

$\rightarrow c_1(4,-3,6,2)+c_2(1,8,3,1)+c_3(3,-2,-1,0)=(0,0,0,0)$

$$\begin{cases} 4c_1+c_2+3c_3=0 \\ -3c_1+8c_2-2c_3=0 \\ 6c_1+3c_2-c_3=0 \\ 2c_1+c_2=0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 4 & 1 & 3 & 0 \\ -3 & 8 & -2 & 0 \\ 6 & 3 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow[E_{34}(-3)]{E_{12}(1)} \left[\begin{array}{ccc|c} 1 & 9 & 1 & 0 \\ -3 & 8 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow[E_{41}(-2),\ D_2(-1)]{E_{21}(3)}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 1 & 0 \\ 0 & 35 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -17 & -2 & 0 \end{array}\right] \xrightarrow{E_{24}(2)} \left[\begin{array}{ccc|c} 1 & 9 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -17 & -2 & 0 \end{array}\right] \xrightarrow[E_{23}(3),\ E_{43}(2)]{E_{13}(-1)} \left[\begin{array}{ccc|c} 1 & 9 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -17 & 0 & 0 \end{array}\right] \xrightarrow[E_{42}(17)]{E_{12}(-9)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore c_1=c_2=c_3=0$을 유일하게 가진다.

$\therefore$ 선형 독립이다.

(2) $(1,0,-1,0), (-2,-2,0,-4), (-1,1,0,-6), (0,3,1,-2)$

$\rightarrow c_1(1,0,-1,0)+c_2(-2,-2,0,-4)+c_3(-1,1,0,-6)+c_4(0,3,1,-2)=(0,0,0,0)$

$$\begin{cases} c_1-2c_2-c_3=0 \\ -2c_2+c_3+3c_4=0 \\ -c_1+c_4=0 \\ -4c_2-6c_3-2c_4=0 \end{cases} \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 0 & 0 \\ 0 & -2 & 1 & 3 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -4 & -6 & -2 & 0 \end{array}\right] \xrightarrow[D_3(-1),\ C_{31}]{E_{31}(1)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & -2 & 1 & 3 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & -4 & -6 & -2 & 0 \end{array}\right] \xrightarrow[D_2(-\frac{1}{2})]{E_{32}(-1),\ E_{42}(-2)}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -8 & -8 & 0 \end{array}\right] \xrightarrow[D_3(-\frac{1}{2}),\ E_{23}(-\frac{1}{2})]{E_{43}(-4)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore c_1, c_2, c_3, c_4$는 무수히 많은 해를 가진다.

$\therefore$ 선형 종속이다.

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 0 & 1 \\ 0 & -2 & 1 & 3 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & -4 & -6 & -2 & 0 \end{array}\right] \xrightarrow{E_{31}(1)\ D_3(-1)\ E_{32}(-1)\ E_{42}(-2)\ D_2(-\frac{1}{2})\ E_{43}(-4)\ D_3(-\frac{1}{2})\ E_{23}(-\frac{1}{2})} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} c_1=t+1 \\ c_2=t \\ c_3=-t \\ c_4=t \end{cases}$$

$\rightarrow (0,0,0,0)$이 아니라 $(1,0,-1,0)$에 대해 위 계산을 반복한다. $\therefore$ $\boxed{(t+1)(1,0,-1,0)+t(-2,-2,0,-4)-t(-1,1,0,-6)+t(0,3,1,-2) = (1,0,-1,0)}$

(3) $(1,2,3,4),(1,-6,-5,-4),(1,4,5,6)$

$c_1(1,2,3,4)+c_2(1,-6,-5,-4)+c_3(1,4,5,6)=(0,0,0,0)$

$$\begin{cases} c_1+c_2+c_3=0 \\ 2c_1-6c_2+4c_3=0 \\ 3c_1-5c_2+5c_3=0 \\ 4c_1-4c_2+6c_3=0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -6 & 4 & 0 \\ 3 & -5 & 5 & 0 \\ 4 & -4 & 6 & 0 \end{array}\right] \xrightarrow[E_{41}(-4)]{E_{21}(-2),\ E_{31}(-3)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -8 & 2 & 0 \\ 0 & -8 & 2 & 0 \\ 0 & -8 & 2 & 0 \end{array}\right] \xrightarrow[D_2(-\frac{1}{8}),\ E_{21}(-1)]{E_{32}(-1),\ E_{42}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∴ $c_1, c_2, c_3$는 무수히 많은 해가 존재한다.

∴ 선형종속이다.

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & 1 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

※ 첫번째 벡터를 선형결합으로 나타내보자.

$$\begin{pmatrix} c_1=-\frac{5}{4}t+1 \\ c_2=\frac{1}{4}t \\ c_3=t \end{pmatrix} \quad \left(-\frac{5}{4}t+1\right)(1,2,3,4)+\left(\frac{1}{4}t\right)(1,-6,-5,-4)+(t)(1,4,5,6)=(1,2,3,4)$$

#2 다음과 같이 주어진 벡터들이 선형독립이 되도록 실수 $t$의 값을 결정하여라.

(1) $(t,1,1),(1,t,1),(1,1,t) \in \mathbb{R}^3$

$c_1(t,1,1)+c_2(1,t,1)+c_3(1,1,t)=(0,0,0)$

$$\begin{cases} tc_1+c_2+c_3=0 \\ c_1+tc_2+c_3=0 \\ c_1+c_2+tc_3=0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 1 & t & 1 & 0 \\ 1 & 1 & t & 0 \end{array}\right] \xrightarrow{C_{13}} \left[\begin{array}{ccc|c} 1 & 1 & t & 0 \\ 1 & t & 1 & 0 \\ t & 1 & 1 & 0 \end{array}\right] \xrightarrow[E_{31}(-t)]{E_{21}(-1)} \left[\begin{array}{ccc|c} 1 & 1 & t & 0 \\ 0 & t-1 & 1-t & 0 \\ 0 & 1-t & 1-t^2 & 0 \end{array}\right] \xrightarrow[(단,\ t\neq1)]{D_2\left(\boxed{\frac{1}{t-1}}\right)}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & t & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1-t & 1-t^2 & 0 \end{array}\right] \xrightarrow[(단,\ t\neq1)]{D_3\left(\boxed{\frac{1}{1-t}}\right)} \left[\begin{array}{ccc|c} 1 & 1 & t & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1+t & 0 \end{array}\right] \xrightarrow[E_{32}(-1)]{E_{12}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & t+1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2+t & 0 \end{array}\right] \xrightarrow[(단,\ t\neq-2)]{D_3\left(\boxed{\frac{1}{2+t}}\right)} \left[\begin{array}{ccc|c} 1 & 0 & t+1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[E_{23}(1)]{E_{13}(-(t+1))} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

∴ $c_1=c_2=c_3=0$ 에서 유일한 해를 가진다.

∴ 1, -2를 제외한 모든 실수 $t$에서 선형독립이다.

i) $t=1$ 일때 $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[E_{31}(-1)]{E_{21}(-1)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ∴ $c_1, c_2, c_3$는 무수한 해를 가지므로 선형종속이다.

ii) $t=-2$ 일때 $\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{C_{13}} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right] \xrightarrow[E_{31}(2)]{E_{21}(-1)} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \xrightarrow[D_2(-\frac{1}{3})]{E_{32}(1)} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_{12}(-1)}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ ∴ $c_1, c_2, c_3$는 무수한해를 가지므로 선형 종속이다.

(2) $(-3,1,t,5),\ (2t,-2,0,22),\ (5,-3,2,-t) \in \mathbb{R}^4$

$c_1(-3,1,t,5)+c_2(2t,-2,0,22)+c_3(5,-3,2,-t)=(0,0,0,0)$

$$\begin{cases} -3c_1 + 2tc_2 + 5c_3 = 0 \\ c_1 - 2c_2 - 3c_3 = 0 \\ tc_1 + 2c_3 = 0 \\ 5c_1 + 22c_2 - tc_3 = 0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} -3 & 2t & 5 & 0 \\ 1 & -2 & -3 & 0 \\ t & 0 & 2 & 0 \\ 5 & 22 & -t & 0 \end{array}\right] \xrightarrow[E_{31}(-t),\ E_{41}(-5)]{C_{12},\ E_{21}(3)} \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2t-6 & -4 & 0 \\ 0 & 2t & 2+3t & 0 \\ 0 & 32 & 15-t & 0 \end{array}\right] \xrightarrow[D_2(-\frac{1}{6})]{E_{23}(-1)}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1+\frac{1}{2}t & 0 \\ 0 & 2t & 2+3t & 0 \\ 0 & 32 & 15-t & 0 \end{array}\right] \xrightarrow[E_{32}(-2t),\ E_{42}(-32)]{E_{12}(2)} \left[\begin{array}{ccc|c} 1 & 0 & t-1 & 0 \\ 0 & 1 & 1+\frac{1}{2}t & 0 \\ 0 & 0 & -t^2+t+2 & 0 \\ 0 & 0 & -17-17t & 0 \end{array}\right] \xrightarrow[(\text{단, } t\neq -1,\ t\neq 2)]{D_3\left(-\frac{1}{(t+1)(t-2)}\right)} \left[\begin{array}{ccc|c} 1 & 0 & t-1 & 0 \\ 0 & 1 & 1+\frac{1}{2}t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -17-17t & 0 \end{array}\right]$$

$$\xrightarrow[E_{23}(-(1+\frac{1}{2}t)),\ E_{43}(17+17t)]{E_{13}(-(t-1))} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∴ $t\neq -1,\ t\neq 2$ 일때 $c_1=c_2=c_3=0$ 에서 유일한 해를 가진다.

∴ $-1, 2$를 제외한 모든 실수 $t$에서 선형독립이다.

i) $t=-1$ 일때

$$\left[\begin{array}{ccc|c} -3 & -2 & 5 & 0 \\ 1 & -2 & -3 & 0 \\ -1 & 0 & 2 & 0 \\ 5 & 22 & 1 & 0 \end{array}\right] \xrightarrow[C_{13}]{D_3(-1)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 1 & -2 & -3 & 0 \\ -3 & -2 & 5 & 0 \\ 5 & 22 & 1 & 0 \end{array}\right] \xrightarrow[E_{31}(3),\ E_{41}(-5)]{E_{21}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 22 & 11 & 0 \end{array}\right] \xrightarrow[E_{41}(11),\ D_2(-\frac{1}{2})]{E_{31}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∴ $t=-1$일때 무수히 많은 해를 가지므로 선형종속이다.

ii) $t=2$일때

$$\left[\begin{array}{ccc|c} -3 & 4 & 5 & 0 \\ 1 & -2 & -3 & 0 \\ 2 & 0 & 2 & 0 \\ 5 & 22 & -2 & 0 \end{array}\right] \xrightarrow[C_{13}]{D_3(\frac{1}{2})} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -2 & -3 & 0 \\ -3 & 4 & 5 & 0 \\ 5 & 22 & -2 & 0 \end{array}\right] \xrightarrow[E_{31}(3),\ E_{41}(-5)]{E_{21}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 22 & -7 & 0 \end{array}\right] \xrightarrow[E_{42}(11),\ D_2(-\frac{1}{2})]{E_{32}(2)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -51 & 0 \end{array}\right] \xrightarrow[E_{24}(-2),\ C_{34}]{D_4(-\frac{1}{51}),\ E_{14}(-1)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∴ $t=2$일때 $c_1=c_2=c_3=0$ 에서 유일한 해를 가진다.

∴ $t=2$일때 선형 독립이다.

∴ 최종적으로 $t=-1$을 제외한 모든 실수 $t$에서 선형독립이다.

#3 실수 체 $\mathbb{R}$ 위의 벡터 공간 $Mat_{2\times2}(\mathbb{R})$에서 다음과 같이 주어진 네 벡터가 선형 종속이 되도록 a, b, c 의 값을 결정하여라.

$$\begin{bmatrix} 3 & -1 & a \\ -5 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 9 & 8 \\ 2 & b & 5 \end{bmatrix}, \begin{bmatrix} 2 & 4 & -5 \\ -6 & 2 & 2 \end{bmatrix}, \begin{bmatrix} c & 30 & 26 \\ -5 & 10 & 19 \end{bmatrix}$$

$$c_1\begin{bmatrix} 3 & -1 & a \\ -5 & 0 & 2 \end{bmatrix} + c_2\begin{bmatrix} -1 & 9 & 8 \\ 2 & b & 5 \end{bmatrix} + c_3\begin{bmatrix} 2 & 4 & -5 \\ -6 & 2 & 2 \end{bmatrix} + c_4\begin{bmatrix} c & 30 & 26 \\ -5 & 10 & 19 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} 3c_1 - c_2 + 2c_3 + c\,c_4 = 0 \\ -c_1 + 9c_2 + 4c_3 + 30c_4 = 0 \\ ac_1 + 8c_2 - 5c_3 + 26c_4 = 0 \\ -5c_1 + 2c_2 - 6c_3 - 5c_4 = 0 \\ bc_2 + 2c_3 + 10c_4 = 0 \\ 2c_1 + 5c_2 + 2c_3 + 19c_4 = 0 \end{cases} \rightarrow \left[\begin{array}{cccc|c} 3 & -1 & 2 & c & 0 \\ -1 & 9 & 4 & 30 & 0 \\ a & 8 & -5 & 26 & 0 \\ -5 & 2 & -6 & -5 & 0 \\ 0 & b & 2 & 10 & 0 \\ 2 & 5 & 2 & 19 & 0 \end{array}\right]$$

선형종속이 되기 위해선 무수히 많은 해를 가져야 한다. ∴ 영행이 3개이상 존재해야 한다.

$$\xrightarrow{C_{12},\ C_{24},\ C_{36}} \left[\begin{array}{cccc|c} -1 & 9 & 4 & 30 & 0 \\ -5 & 2 & -6 & -5 & 0 \\ 2 & 5 & 2 & 19 & 0 \\ 3 & -1 & 2 & c & 0 \\ 0 & b & 2 & 10 & 0 \\ a & 8 & -5 & 26 & 0 \end{array}\right] \xrightarrow{E_{21}(-5),\ E_{31}(2),\ D_1(-1)} \left[\begin{array}{cccc|c} 1 & -9 & -4 & -30 & 0 \\ 0 & -43 & -26 & -195 & 0 \\ 0 & 23 & 10 & 79 & 0 \\ 3 & -1 & 2 & c & 0 \\ 0 & b & 2 & 10 & 0 \\ a & 8 & -5 & 26 & 0 \end{array}\right] \xrightarrow{E_{23}(2)} \left[\begin{array}{cccc|c} 1 & -9 & -4 & -30 & 0 \\ 0 & 3 & -6 & 3 & 0 \\ 0 & 23 & 10 & 79 & 0 \\ 3 & -1 & 2 & c & 0 \\ 0 & b & 2 & 10 & 0 \\ a & 8 & -5 & 26 & 0 \end{array}\right]$$

$$\xrightarrow{D_2(\frac{1}{3}),\ E_{12}(9),\ E_{32}(-23)} \left[\begin{array}{cccc|c} 1 & 0 & -22 & -21 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 56 & 56 & 0 \\ 3 & -1 & 2 & c & 0 \\ 0 & b & 2 & 10 & 0 \\ a & 8 & -5 & 26 & 0 \end{array}\right] \xrightarrow{D_3(\frac{1}{56}),\ E_{13}(22),\ E_{23}(2)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 3 & -1 & 2 & c & 0 \\ 0 & b & 2 & 10 & 0 \\ a & 8 & -5 & 26 & 0 \end{array}\right] \xrightarrow{E_{41}(-3),\ E_{61}(-a)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 2 & c-3 & 0 \\ 0 & b & 2 & 10 & 0 \\ 0 & 8 & -5 & 26-a & 0 \end{array}\right]$$

$$\xrightarrow{E_{42}(1),\ E_{52}(-b),\ E_{62}(-8)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & c & 0 \\ 0 & 0 & 2 & 10-3b & 0 \\ 0 & 0 & -5 & 2-a & 0 \end{array}\right] \xrightarrow{E_{43}(-2),\ E_{53}(-2),\ E_{63}(5)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & c-2 & 0 \\ 0 & 0 & 0 & 8-3b & 0 \\ 0 & 0 & 0 & 7-a & 0 \end{array}\right]$$

영행이 3개이상 존재하여야 무수히 많은 해를 가지므로, $c=2$, $b=\frac{8}{3}$, $a=7$ 일때 선형종속이 된다.

$\therefore a=7,\ b=\frac{8}{3},\ c=2$

# 4 다음 벡터공간 W의 기저와 차원을 구하여라. 또한 주어진 벡터 $\mathbb{w} \in W$를 앞에서 구한 기저 벡터들의 선형결합으로 나타내어라

(1) $W = \{(2t-s, s, t) \mid s, t \in \mathbb{R}\}$, $\mathbb{w} = (9, -5, 2)$

$$W = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad W = \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

W는 $A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$의 열공간이다. $\therefore A^T = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow[E_{21}(-2)]{D_1(-1)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{D_2(\frac{1}{2})} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$

$\xrightarrow{E_{12}(1)} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$

$\therefore B = \{(1, 0, \frac{1}{2}), (0, 1, \frac{1}{2})\}$

$\therefore$ 2차원이다.

$c_1(1, 0, \frac{1}{2}) + c_2(0, 1, \frac{1}{2}) = (9, -5, 2)$

$\begin{bmatrix} 1 & 0 & | & 9 \\ 0 & 1 & | & -5 \\ \frac{1}{2} & \frac{1}{2} & | & 2 \end{bmatrix} \xrightarrow[E_{32}(-\frac{1}{2})]{E_{31}(-\frac{1}{2})} \begin{bmatrix} 1 & 0 & | & 9 \\ 0 & 1 & | & -5 \\ 0 & 0 & | & 0 \end{bmatrix}$ $\therefore c_1 = 9,\ c_2 = -5$

$\therefore 9(1, 0, \frac{1}{2}) - 5(0, 1, \frac{1}{2}) = (9, -5, 2) = \mathbb{w}$

(2) $W = \{(-a+b+c, a-b+c, a+b-c) \mid a, b, c \in \mathbb{R}\}$, $\mathbb{w} = (1, 2, 3)$

$$W = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad W = \left\langle \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\rangle$$

W는 $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$의 열공간. $\therefore A^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[C_{23}]{C_{12}} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow[E_{31}(1)]{E_{21}(-1)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[D_3(\frac{1}{2})]{D_2(\frac{1}{2})} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow[E_{23}(1)]{E_{12}(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$\therefore$ 3차원이다.

$1(1, 0, 0) + 2(0, 1, 0) + 3(0, 0, 1) = (1, 2, 3) = \mathbb{w}$

(3) $W = \{(a+2b+c,\ -3a-b+2c,\ a-2b-3c) \mid a,b,c \in \mathbb{R}\}$, $w = (-5,10,-1)$

$$W = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} + c\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \qquad W = \left\langle \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\rangle$$

W는 $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix}$ 의 열공간이다.

$$A^T = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -1 & -2 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow[E_{31}(-1)]{E_{21}(-2)} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 5 & -4 \\ 0 & 5 & -4 \end{bmatrix} \xrightarrow[D_2(\frac{1}{5})]{E_{32}(-1)} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -\frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_{12}(3)} \begin{bmatrix} 1 & 0 & -\frac{7}{5} \\ 0 & 1 & -\frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore B = \{(1,0,-\frac{7}{5}),\ (0,1,-\frac{4}{5})\}$

$\therefore$ 2차원이다.

$c_1(1,0,-\frac{7}{5}) + c_2(0,1,-\frac{4}{5}) = (-5,10,-1)$

$$\left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 10 \\ -\frac{7}{5} & -\frac{4}{5} & 1 \end{array}\right] \xrightarrow[E_{32}(\frac{4}{5})]{E_{31}(\frac{7}{5})} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 10 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore c_1 = -5$, $c_2 = 10$

$\therefore -5(1,0,-\frac{7}{5}) + 10(0,1,-\frac{4}{5}) = (-5,10,-1) = w$

(4) $W = \{(x_1,x_2,x_3,x_4,x_5) \mid x_1 - 2x_3 + 3x_5 = 0\}$, $w = (1,-1,5,0,3)$

$$W = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad W = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

W는 $A = \begin{bmatrix} 0 & 2 & 0 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 의 열공간이다.

$$A^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[D_1(-\frac{1}{3})]{C_{14}} \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{3} \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[C_{34}]{C_{24}} \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{E_{31}(-2)} \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore B = \{(1,0,0,0,-\frac{1}{3}),\ (0,1,0,0,0),\ (0,0,1,0,\frac{2}{3}),\ (0,0,0,1,0)\}$

$\therefore$ 4차원이다.

$c_1(1,0,0,0,-\frac{1}{3}) + c_2(0,1,0,0,0) + c_3(0,0,1,0,\frac{2}{3}) + c_4(0,0,0,1,0) = (1,-1,5,0,3) = w$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{2}{3} & 0 & 3 \end{array}\right] \xrightarrow[E_{53}(-\frac{2}{3})]{E_{51}(\frac{1}{3})} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore c_1 = 1$, $c_2 = -1$, $c_3 = 5$, $c_4 = 0$

$\therefore 1\cdot(1,0,0,0,-\frac{1}{3}) - 1\cdot(0,1,0,0,0) + 5(0,0,1,0,\frac{2}{3}) = (1,-1,5,0,3) = w$

(5) $W = \left\{ \begin{bmatrix} a-b & b-c \\ a+b & b+c \end{bmatrix} \middle| a,b,c \in \mathbb{R} \right\}$, $\mathbb{W} = \begin{bmatrix} 2 & -7 \\ 4 & 9 \end{bmatrix}$

$$W = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad W = \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

W 는 $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ 의 열공간이다. $A^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{21}(1)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{32}(1)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

$\xrightarrow[E_{13}(-1),\ E_{23}(-2)]{D_3(\frac{1}{2})} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\therefore B = \{(1,0,0,-1), (0,1,0,-1), (0,0,1,1)\}$

$\therefore$ 3차원이다.

$\therefore B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

$c_1(1,0,0,-1) + c_2(0,1,0,-1) + c_3(0,0,1,1) = (2,-7,4,9)$

$2(1,0,0,-1) + (-7)(0,1,0,-1) + 4(0,0,1,1) = (2,-7,4,9)$

$\therefore\ 2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - 7\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ 4 & 9 \end{bmatrix} = \mathbb{W}$

(6) $A = \begin{bmatrix} 3 & 2 & 3 & -2 \\ 1 & 1 & 1 & 0 \end{bmatrix} \in Mat_{2\times4}(\mathbb{R})$ 에 대하여 W는 제차 연립선형 방정식 $AX=0$의 해공간.

$\mathbb{W} = (1,2,-3,-1)$

$\begin{bmatrix} 3 & 2 & 3 & -2 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[E_{21}(-3)]{C_{12}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \xrightarrow[E_{12}(-1)]{D_2(-1)} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ $x_3 = s,\ x_4 = t$ 라하자.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \qquad X = \left\langle \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

X는 $A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ 의 열공간이다. $A^T = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix} \xrightarrow[E_{21}(-2)]{D_1(-1)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -2 & 2 & 1 \end{bmatrix} \xrightarrow{D_2(-\frac{1}{2})} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -\frac{1}{2} \end{bmatrix}$

$\therefore B = \{(1,0,-1,0), (0,1,-1,-\frac{1}{2})\}$

$\therefore$ 2차원이다.

$c_1(1,0,-1,0) + c_2(0,1,-1,-\frac{1}{2}) = (1,2,-3,-1) = \mathbb{W}$, $c_1 = 1, c_2 = 2$

$1\cdot(1,0,-1,0) + 2\cdot(0,1,-1,-\frac{1}{2}) = (1,2,-3,-1) = \mathbb{W}$