20171630, 남주형, 컴퓨터공학과

1.2절 대각행렬, 삼각행렬, 대칭행렬

#1 정리 1.2.1의 (1)을 증명하여라. (Hint: 수학적 귀납법을 이용한다.)

(i) $k=1$ 일때

$$D^1 = \begin{bmatrix} (d_{11}) & 0 & \cdots & 0 \\ 0 & (d_{22}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn}) \end{bmatrix}$$

(ii) $k=a$ 일때, 정리 1.2.1-(1)이 성립한다고 가정하면

$$D^a = \begin{bmatrix} (d_{11})^a & 0 & \cdots & 0 \\ 0 & (d_{22})^a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^a \end{bmatrix}$$

위의 등식에 D^1을 곱해주면

$$D^{a+1} = \begin{bmatrix} (d_{11})^a & 0 & \cdots & 0 \\ 0 & (d_{22})^a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^a \end{bmatrix} \begin{bmatrix} (d_{11}) & 0 & \cdots & 0 \\ 0 & (d_{22}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn}) \end{bmatrix} = \begin{bmatrix} (d_{11})^{a+1} & 0 & \cdots & 0 \\ 0 & (d_{22})^{a+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^{a+1} \end{bmatrix}$$

따라서 $k=a+1$ 일 때도 정리 1.2.1-(1)이 성립한다.

(i),(ii)에 의해 모든 자연수 k에 대하여 정리 1.2.1-(1)이 성립한다.

#2 정리 1.2.1의 (2)를 증명하여라. (Hint: 역행렬의 정의를 만족함을 보인다.)

$$D \cdot D^{-1} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} 1/d_{11} & 0 & \cdots & 0 \\ 0 & 1/d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$D^{-1} \cdot D = \begin{bmatrix} 1/d_{11} & 0 & \cdots & 0 \\ 0 & 1/d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_{nn} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\therefore$ $D \cdot D^{-1} = I_n = D^{-1} D$ 를 만족한다.

따라서 D의 역행렬은 D^{-1} 이다.

20171630, 남주형, 컴퓨터공학과

\# 3 다음의 삼각행렬이 가역행렬 일 때 역행렬을 구하여라.

(1) $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$, $U^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

$$UU^{-1} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} ax_{11}+bx_{21}+cx_{31}, & ax_{12}+bx_{22}+cx_{32}, & ax_{13}+bx_{23}+cx_{33} \\ dx_{21}+ex_{31}, & dx_{22}+ex_{32}, & dx_{23}+ex_{33} \\ fx_{31}, & fx_{32}, & fx_{33} \end{bmatrix}$$

$$\cancel{\begin{bmatrix} \frac{1}{a} & -\frac{b}{ad} & \\ 0 & \frac{1}{d} & -\frac{e}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}} \qquad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow fx_{31} = 0 \Rightarrow x_{31} = 0$

$fx_{32} = 0 \Rightarrow x_{32} = 0$

$fx_{33} = 1 \Rightarrow x_{33} = \frac{1}{f}$

$dx_{21} + e\cancel{x_{31}}^{0} = 0 \Rightarrow x_{21} = 0$

$dx_{22} + e\cancel{x_{32}}^{0} = 1 \Rightarrow x_{22} = \frac{1}{d}$

$dx_{23} + ex_{33} = 0$

$= dx_{23} + \frac{e}{f} = 0 \Rightarrow x_{23} = -\frac{e}{df}$

$ax_{11} + b\cancel{x_{21}}^{0} + c\cancel{x_{31}}^{0} = 1 \Rightarrow x_{11} = \frac{1}{a}$

$\cancel{ax_{12} + bx_{22} + cx_{32} = 0}$

$\cancel{= ax_{12} + \frac{b}{d} - \frac{ce}{df} = 0 \Rightarrow x_{12} = \frac{ce}{adf} - \frac{b}{ad}}$

$\cancel{ax_{13} + bx_{23} + cx_{33} = 0}$

$ax_{12} + bx_{22} + c\cancel{x_{32}}^{0} = 0$

$= ax_{12} + \frac{b}{d} = 0 \Rightarrow x_{12} = -\frac{b}{ad}$

$ax_{13} + bx_{23} + cx_{33} = 0$

$ax_{13} - \frac{be}{df} + \frac{c}{f} = 0 \Rightarrow x_{13} = \frac{be}{adf} - \frac{c}{af}$

$$\Rightarrow U^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{b}{ad} & \frac{be}{adf} - \frac{c}{af} \\ 0 & \frac{1}{d} & -\frac{e}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$$

20171630, 남주혁, 컴퓨터 공학과

#3 다음 삼각행렬이 가역행렬일 때 역행렬을 구하여라.

(2)

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}, \quad L^{-1} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$LL^{-1} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} ax_{11}, & ax_{12}, & ax_{13} \\ bx_{11}+cx_{21}, & bx_{12}+cx_{22}, & bx_{13}+cx_{23} \\ dx_{11}+ex_{21}+fx_{31}, & dx_{12}+ex_{22}+fx_{32}, & dx_{13}+ex_{23}+fx_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow ax_{11} = 1 \rightarrow x_{11} = \frac{1}{a}$

$ax_{12} = 0 \rightarrow x_{12} = 0$

$ax_{13} = 0 \rightarrow x_{13} = 0$

$bx_{11} + cx_{21} = 0$

$= \frac{b}{a} + cx_{21} = 0 \rightarrow x_{21} = -\frac{b}{ac}$

$b\cancel{x_{12}}^{0} + cx_{22} = 1 \rightarrow x_{22} = \frac{1}{c}$

$b\cancel{x_{13}}^{0} + cx_{23} = 0 \rightarrow x_{23} = 0$

$dx_{11} + ex_{21} + fx_{31} = 0$

$= \frac{d}{a} - \frac{be}{ac} + fx_{31} = 0 \rightarrow x_{31} = \frac{be}{acf} - \frac{d}{af}$

$\cancel{dx_{12}}^{0} + ex_{22} + fx_{32} = 0$

$= \frac{e}{c} + fx_{32} = 0 \rightarrow x_{32} = -\frac{e}{cf}$

$\cancel{dx_{13}}^{0} + \cancel{ex_{23}}^{0} + fx_{33} = 1 \rightarrow x_{33} = \frac{1}{f}$

$$\Rightarrow L^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b}{ac} & \frac{1}{c} & 0 \\ \frac{be}{acf} - \frac{d}{af} & -\frac{e}{cf} & \frac{1}{f} \end{bmatrix}$$

20171630, 남주형, 컴퓨터공학과

#4 정리 1.2.3의 (4)를 증명하여라. (Hint: $A=[a_{ij}]_{m\times r}$, $B=[b_{ij}]_{r\times n}$ 이라고 하고 $(AB)^T$의 (i,j)성분과 $(B^T)(A^T)$의 (i,j)성분이 같음을 보인다.)

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \quad B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}$$

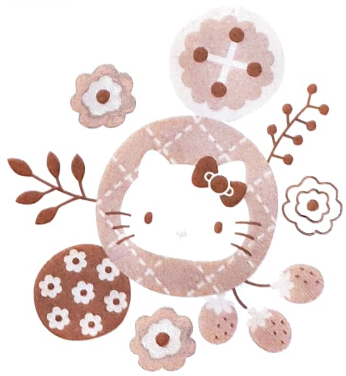
$$AB=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1r}b_{r1}, & a_{11}b_{12}+a_{12}b_{22}+\cdots+a_{1r}b_{r2}, & \cdots, & a_{11}b_{1n}+a_{12}b_{2n}+\cdots+a_{1r}b_{rn} \\ a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2r}b_{r1}, & a_{21}b_{12}+a_{22}b_{22}+\cdots+a_{2r}b_{r2}, & \cdots, & a_{21}b_{1n}+a_{22}b_{2n}+\cdots+a_{2r}b_{rn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11}+a_{m2}b_{21}+\cdots+a_{m2}b_{r1}, & a_{m1}b_{12}+a_{m2}b_{22}+\cdots+a_{mr}b_{r2}, & \cdots, & a_{m1}b_{1n}+a_{m2}b_{2n}+\cdots+a_{mr}b_{rn} \end{bmatrix}$$

$$(AB)^T=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1r}b_{r1}, & a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2r}b_{r1}, & \cdots, & a_{m1}b_{11}+a_{m2}b_{21}+\cdots+a_{m2}b_{r1} \\ a_{11}b_{12}+a_{12}b_{22}+\cdots+a_{1r}b_{r2}, & a_{21}b_{12}+a_{21}b_{22}+\cdots+a_{2r}b_{r2}, & \cdots, & a_{m1}b_{12}+a_{m2}b_{22}+\cdots+a_{mr}b_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{1n}+a_{12}b_{2n}+\cdots+a_{1r}b_{rn}, & a_{21}b_{1n}+a_{22}b_{2n}+\cdots+a_{2r}b_{rn}, & \cdots, & a_{m1}b_{1n}+a_{m2}b_{2n}+\cdots+a_{mr}b_{rn} \end{bmatrix}$$

$$A^T=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1r} & a_{2r} & \cdots & a_{mr} \end{bmatrix}, \quad B^T=\begin{bmatrix} b_{11} & b_{21} & \cdots & b_{r1} \\ b_{12} & b_{22} & \cdots & b_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{rn} \end{bmatrix}$$

$$(B^T)(A^T)=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1r}b_{r1}, & a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2r}b_{r1}, & \cdots, & a_{m1}b_{11}+a_{m2}b_{21}+\cdots+a_{m2}b_{r1} \\ a_{11}b_{12}+a_{12}b_{22}+\cdots+a_{12}b_{r2}, & a_{21}b_{12}+a_{21}b_{22}+\cdots+a_{2r}b_{r2}, & \cdots, & a_{m1}b_{1}+a_{m2}b_{22}+\cdots+a_{mr}b_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{1n}+a_{12}b_{2n}+\cdots+a_{1r}b_{rn}, & a_{21}b_{1n}+a_{22}b_{2n}+\cdots+a_{2r}b_{rn}, & \cdots, & a_{m1}b_{1n}+a_{m2}b_{2n}+\cdots+a_{mr}b_{rn} \end{bmatrix}$$

$\therefore (AB)^T=(B^T)(A^T)$



2017 1630, 남주형, 컴퓨터공학과

#5 정리 1.2.3의 (5)를 증명하여라. (Hint: 정리 1.1.3의 (5)와 정리 1.2.3의 (4)를 이용한다.)

$(AA^{-1})^T = \{(A^{-1})^T\}(A)^T$

$I^T = (A^{-1})^T (A)^T$

$(A^{-1}A)^T = (A)^T\{(A^{-1})^T\}$

$I = (A)^T\{(A^{-1})^T\}$

~~∴ $(A^{-1}A)^T = I = (AA^{-1})^T$~~

∴ $(A^{-1})^T A^T = A^T (A^{-1})^T = I$

∴ A가 가역행렬이면 A^T도 가역행렬이다.

$((A^{-1})^T \cdot A^T)^{-1} = (A^T)^{-1}((A^{-1})^T)^{-1}$

$I = (A^T)^{-1}((A^{-1})^T)^{-1}$

$(A^{-1})^T = (A^T)^{-1}\cancel{((A^{-1})^T)^{-1}}(A^{-1})^T$

∴ $(A^{-1})^T = (A^T)^{-1}$

#6 정리 1.2.4의 (4)를 증명하여라. (Hint: 정리 1.2.3의 (4)를 이용한다.)

$(A\cdot A^{-1})^T = (A^{-1})^T A^T = I$

$= (A^{-1})^T A$ ∵ A가 대칭행렬이기 때문에 $A = A^T$

$I = (A^{-1})^T A$

$A^{-1} = (A^{-1})^T A A^{-1}$

∴ $A^{-1} = (A^{-1})^T$

∴ A가 가역행렬이면 A^{-1}도 대칭행렬이다.