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컴퓨터 공학과/20171630/남주형

# 2.3절 기본행렬과 역행렬

#1 다음의 행렬이 기본 행렬인지 판단하여라. 만약 기본행렬이라면 역행렬을 구하여라.

(1) $\begin{bmatrix}1&2\\0&1\end{bmatrix} \xrightarrow{E_{12}(2)} \begin{bmatrix}1&0\\0&1\end{bmatrix} = I_2 \therefore \begin{bmatrix}1&2\\0&1\end{bmatrix} = E_{12}(2)$ $\quad E_{12}(2)^{-1} = E_{12}(-2) \quad \therefore E_{12}(2)^{-1} = \begin{bmatrix}1&-2\\0&1\end{bmatrix}$

(2) $\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \xrightarrow{C_{13}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I_3 \therefore \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} = C_{13}$ $\quad C_{13}^{-1} = C_{13} \quad \therefore C_{13}^{-1} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$

(3) $\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix} \xrightarrow{C_{23}} \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix} \xrightarrow{D_2(-1)} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I_3$ $I_3$에 행연산을 2번 시행해야 얻을수있는 행렬이므로 기본행렬이 아니다.

(4) $\begin{bmatrix}1&0&\frac{1}{2}\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{E_{13}(-\frac{1}{2})} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I_3 \therefore \begin{bmatrix}1&0&\frac{1}{2}\\0&1&0\\0&0&1\end{bmatrix} = E_{13}(\tfrac{1}{2})$ $\quad E_{13}(\tfrac{1}{2})^{-1} = E_{13}(-\tfrac{1}{2}) \quad \therefore E_{13}(\tfrac{1}{2})^{-1} = \begin{bmatrix}1&0&-\frac{1}{2}\\0&1&0\\0&0&1\end{bmatrix}$

(5) $\begin{bmatrix}1&3&0\\0&-1&0\\0&0&1\end{bmatrix} \xrightarrow{D_2(-1)} \begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{E_{12}(-3)} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I_3$ $I_3$에 행연산을 2번 시행해야 얻을수있는 행렬이므로 기본 행렬이 아니다.

(6) $\begin{bmatrix}-2&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{D_1(-\frac{1}{2})} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I_3 \therefore D_1(-2) = \begin{bmatrix}-2&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $\quad D_1(-2)^{-1} = D_1(-\tfrac{1}{2}) \quad \therefore D_1(-2)^{-1} = \begin{bmatrix}-\frac{1}{2}&0&0\\0&1&0\\0&0&1\end{bmatrix}$

(7) $\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&1\\0&0&0&1\end{bmatrix} \xrightarrow{C_{13}} \begin{bmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{E_{14}(-1)} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = I_4$ $I_4$에 행연산을 2번 시행해야 얻을수있는 행렬 이므로 기본 행렬이 아니다.

(8) $\begin{bmatrix}1&0&0&0\\0&1&0&-4\\0&0&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{E_{24}(4)} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = I_4$ $\quad \begin{bmatrix}1&0&0&0\\0&1&0&-4\\0&0&1&0\\0&0&0&1\end{bmatrix} = E_{24}(-4)$ $\quad E_{24}(-4)^{-1} = E_{24}(4)$

$\therefore E_{24}(-4)^{-1} = \begin{bmatrix}1&0&0&0\\0&1&0&4\\0&0&1&0\\0&0&0&1\end{bmatrix}$

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#2 다음의 m×n 행렬 A의 기약 행 사다리꼴 행렬 R을 구하여라 그리고 R=EA를 만족하는 m×m 정사각행렬 E를 구하여라.

(1)

$$A=\begin{bmatrix}2&6&6\\2&7&6\\2&7&7\end{bmatrix}\xrightarrow{D_1(\frac{1}{2})}\begin{bmatrix}1&3&3\\2&7&6\\2&7&7\end{bmatrix}\xrightarrow[E_{31}(-2)]{E_{21}(-2)}\begin{bmatrix}1&3&3\\0&1&0\\0&1&1\end{bmatrix}\xrightarrow[E_{32}(-1)]{E_{12}(-3)}\begin{bmatrix}1&0&3\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{E_{13}(-3)}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=R$$

R이 I 이므로 R=EA를 만족하는 E는 $A^{-1}$ 이다.

$$[A\ \ I_3]=\begin{bmatrix}2&6&6&1&0&0\\2&7&6&0&1&0\\2&7&7&0&0&1\end{bmatrix}\xrightarrow{D_1(\frac{1}{2})}\begin{bmatrix}1&3&3&\frac{1}{2}&0&0\\2&7&6&0&1&0\\2&7&7&0&0&1\end{bmatrix}\xrightarrow[E_{31}(-2)]{E_{21}(-2)}\begin{bmatrix}1&3&3&\frac{1}{2}&0&0\\0&1&0&-1&1&0\\0&1&1&-1&0&1\end{bmatrix}$$

$$\xrightarrow[E_{32}(-1)]{E_{12}(-3)}\begin{bmatrix}1&0&3&\frac{7}{2}&-3&0\\0&1&0&-1&1&0\\0&0&1&0&-1&1\end{bmatrix}\xrightarrow{E_{13}(-3)}\begin{bmatrix}1&0&0&\frac{7}{2}&0&-3\\0&1&0&-1&1&0\\0&0&1&0&-1&1\end{bmatrix}$$

$$\therefore A^{-1}=\begin{bmatrix}\frac{7}{2}&0&-3\\-1&1&0\\0&-1&1\end{bmatrix}=E$$

(2)

$$A=\begin{bmatrix}1&6&4\\2&4&-1\\-1&2&5\end{bmatrix}\xrightarrow[E_{31}(1)]{E_{21}(-2)}\begin{bmatrix}1&6&4\\0&-8&-9\\0&8&9\end{bmatrix}\xrightarrow{E_{23}(1)}\begin{bmatrix}1&6&4\\0&-8&-9\\0&0&0\end{bmatrix}\xrightarrow{D_2(-\frac{1}{8})}\begin{bmatrix}1&6&4\\0&1&\frac{9}{8}\\0&0&0\end{bmatrix}$$

$$\xrightarrow{E_{12}(-6)}\begin{bmatrix}1&0&-\frac{11}{4}\\0&1&\frac{9}{8}\\0&0&0\end{bmatrix}=R$$

$$R=EA\qquad\begin{bmatrix}1&0&-\frac{11}{4}\\0&1&\frac{9}{8}\\0&0&0\end{bmatrix}=\begin{bmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{bmatrix}\begin{bmatrix}1&6&4\\2&4&-1\\-1&2&5\end{bmatrix}$$

$$x_1+2y_1-z_1=1\qquad x_2+2y_2-z_2=0\qquad x_3+2y_3-z_3=0$$

$$6x_1+4y_1+2z_1=0\qquad 6x_2+4y_2+2z_2=1\qquad 6x_3+4y_3+2z_3=0$$

$$4x_1-y_1+5z_1=-\frac{11}{4}\qquad 4x_2-y_2+5z_2=\frac{9}{8}\qquad 4x_3-y_3+5z_3=0$$

$$\therefore\begin{bmatrix}1&2&-1&1&0&0\\6&4&2&0&1&0\\4&-1&5&-\frac{11}{4}&\frac{9}{8}&0\end{bmatrix}\xrightarrow[E_{31}(-4)]{E_{21}(-6)}\begin{bmatrix}1&2&-1&1&0&0\\0&-8&8&-6&1&0\\0&-9&9&-\frac{27}{4}&\frac{9}{8}&0\end{bmatrix}$$

$$\xrightarrow{E_{32}(-\frac{9}{8})}\begin{bmatrix}1&2&-1&1&0&0\\0&-8&8&-6&1&0\\0&0&0&-\frac{63}{8}&\frac{9}{8}&0\end{bmatrix}$$

∴ E의 해가 존재하지 않는다.

(3)

$$A=\begin{bmatrix}1&2&3&4&1\\2&5&3&5&6\\1&0&8&9&-6\end{bmatrix}\xrightarrow[E_{31}(-1)]{E_{21}(-2)}\begin{bmatrix}1&2&3&4&1\\0&1&-3&-3&4\\0&-2&5&5&-7\end{bmatrix}\xrightarrow[E_{32}(2)]{E_{12}(-2)}\begin{bmatrix}1&0&9&10&-7\\0&1&-3&-3&4\\0&0&-1&-1&1\end{bmatrix}$$

$$\xrightarrow[E_{23}(-3)]{E_{13}(+9)}\begin{bmatrix}1&0&0&1&2\\0&1&0&0&1\\0&0&-1&-1&1\end{bmatrix}\xrightarrow{D_3(-1)}\begin{bmatrix}1&0&0&1&2\\0&1&0&0&1\\0&0&1&1&-1\end{bmatrix}=R$$

$$R=EA\qquad\begin{bmatrix}1&0&0&1&2\\0&1&0&0&1\\0&0&1&1&-1\end{bmatrix}=\begin{bmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{bmatrix}\begin{bmatrix}1&2&3&4&1\\2&5&3&5&6\\1&0&8&9&-6\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\2&5&0&0&1&0\\3&3&8&0&0&1\\4&5&9&1&0&1\\1&6&-6&2&1&-1\end{array}\right]\xrightarrow[E_{41}(-4),\ E_{51}(-1)]{E_{21}(-2),\ E_{31}(-3)}\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&-2&-2&1&0\\0&-3&5&-3&0&1\\0&-3&5&-3&0&1\\0&4&-7&1&1&-1\end{array}\right]$$

$$\xrightarrow{E_{43}(-1)}\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&-2&-2&1&0\\0&-3&5&-3&0&1\\0&0&0&0&0&0\\0&4&-7&1&1&-1\end{array}\right]\xrightarrow{C_{45}}\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&-2&-2&1&0\\0&-3&5&-3&0&1\\0&4&-7&1&1&-1\\0&0&0&0&0&0\end{array}\right]\xrightarrow[E_{42}(-4)]{E_{12}(-2),\ E_{32}(3)}\left[\begin{array}{ccc|ccc}1&0&5&5&-2&0\\0&1&-2&-2&1&0\\0&0&-1&-9&3&1\\0&0&1&9&-3&-1\\0&0&0&0&0&0\end{array}\right]$$

$$\xrightarrow[D_3(-1)]{E_{43}(1)}\left[\begin{array}{ccc|ccc}1&0&5&5&-2&0\\0&1&-2&-2&1&0\\0&0&1&9&-3&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right]\xrightarrow[E_{23}(2)]{E_{13}(-5)}\left[\begin{array}{ccc|ccc}1&0&0&-40&13&5\\0&1&0&16&-5&-2\\0&0&1&9&-3&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right]\Rightarrow E=\begin{bmatrix}-40&13&5\\16&-5&-2\\9&-3&-1\end{bmatrix}$$

(4)

$$A=\begin{bmatrix}3&-1&0\\-2&3&1\\0&1&-2\\2&1&5\end{bmatrix}\xrightarrow{E_{14}(-1)}\begin{bmatrix}1&-2&-5\\-2&3&1\\0&1&-2\\2&1&5\end{bmatrix}\xrightarrow[E_{41}(-2)]{E_{21}(2)}\begin{bmatrix}1&-2&-5\\0&1&9\\0&1&-2\\0&5&15\end{bmatrix}\xrightarrow[E_{32}(-1),\ E_{12}(2)]{E_{43}(-5)}\begin{bmatrix}1&0&13\\0&1&9\\0&0&-11\\0&0&25\end{bmatrix}$$

$$\xrightarrow{D_3(-\frac{1}{11})}\begin{bmatrix}1&0&13\\0&1&9\\0&0&1\\0&0&25\end{bmatrix}\xrightarrow[E_{23}(-9),\ E_{43}(-25)]{E_{13}(-13)}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}=R$$

$$R=EA\qquad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}&&&\\&4\times4&&\\&&&\end{bmatrix}\begin{bmatrix}3&-1&0\\-2&3&1\\0&1&-2\\2&1&5\end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc}3&-2&0&2&1&0&0&0\\-1&3&1&1&0&1&0&0\\0&1&-2&5&0&0&1&0\end{array}\right]\xrightarrow{C_{12}}\left[\begin{array}{cccc|cccc}-1&3&1&1&0&1&0&0\\3&-2&0&2&1&0&0&0\\0&1&-2&5&0&0&1&0\end{array}\right]\xrightarrow[C_{23}]{D_1(-1)}\left[\begin{array}{cccc|cccc}1&-3&-1&-1&0&-1&0&0\\0&1&-2&5&0&0&1&0\\3&-2&0&2&1&0&0&0\end{array}\right]$$

$$\xrightarrow{E_{31}(-3)}\left[\begin{array}{cccc|cccc}1&-3&-1&-1&0&-1&0&0\\0&1&-2&5&0&0&1&0\\0&7&3&5&1&3&0&0\end{array}\right]\xrightarrow[E_{32}(-7)]{E_{12}(3)}\left[\begin{array}{cccc|cccc}1&0&-7&14&0&-1&3&0\\0&1&-2&5&0&0&1&0\\0&0&17&-30&1&3&-7&0\end{array}\right]$$

$$\xrightarrow{D_3(\frac{1}{17})}\left[\begin{array}{cccc|cccc}1&0&-7&14&0&-1&3&0\\0&1&-2&5&0&0&1&0\\0&0&1&-\frac{30}{17}&\frac{1}{17}&\frac{3}{17}&-\frac{7}{17}&0\end{array}\right]\xrightarrow[E_{23}(2)]{E_{13}(7)}\left[\begin{array}{cccc|cccc}1&0&0&\frac{28}{17}&\frac{7}{17}&\frac{4}{17}&-\frac{18}{17}&0\\0&1&0&\frac{25}{17}&\frac{2}{17}&\frac{6}{17}&\frac{3}{17}&0\\0&0&1&-\frac{30}{17}&\frac{1}{17}&\frac{3}{17}&-\frac{7}{17}&0\end{array}\right]$$

∴ 무수히 많은 E가 존재한다.

#3 다음의 행렬이 가역행렬인지 조사하여라. 만약 가역행렬이라면 기본행렬의 곱으로 나타내어라. 또한 역행렬도 구하여라.

(1) $A=\begin{bmatrix}-3 & 6\\ 4 & 5\end{bmatrix} \xrightarrow{D_1(-\frac{1}{3})} \begin{bmatrix}1 & -2\\ 4 & 5\end{bmatrix} \xrightarrow{E_{21}(-4)} \begin{bmatrix}1 & -2\\ 0 & 13\end{bmatrix} \xrightarrow{D_2(\frac{1}{13})} \begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix} \xrightarrow{E_{12}(2)} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I$ ∴ 가역행렬이다.

$$E_{12}(2)\,D_2(\tfrac{1}{13})\,E_{21}(-4)\,D_1(-\tfrac{1}{3})\,A = I$$

$$A = D_1(-\tfrac{1}{3})^{-1}\,E_{21}(-4)^{-1}\,D_2(\tfrac{1}{13})^{-1}\,E_{12}(2)^{-1}$$

$$= D_1(-3)\,E_{21}(4)\,D_2(13)\,E_{12}(-2)$$

$$A = \begin{bmatrix}-3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 13\end{bmatrix}\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-3 & 6\\ 4 & 5\end{bmatrix}$$

$$\begin{bmatrix}-3 & 6 & 1 & 0\\ 4 & 5 & 0 & 1\end{bmatrix} \xrightarrow{D_1(-\frac{1}{3})} \begin{bmatrix}1 & -2 & -\frac{1}{3} & 0\\ 4 & 5 & 0 & 1\end{bmatrix} \xrightarrow{E_{21}(-4)} \begin{bmatrix}1 & -2 & -\frac{1}{3} & 0\\ 0 & 13 & \frac{4}{3} & 1\end{bmatrix} \xrightarrow{D_2(\frac{1}{13})} \begin{bmatrix}1 & -2 & -\frac{1}{3} & 0\\ 0 & 1 & \frac{4}{39} & \frac{1}{13}\end{bmatrix}$$

$$\xrightarrow{E_{12}(2)} \begin{bmatrix}1 & 0 & -\frac{5}{39} & \frac{2}{13}\\ 0 & 1 & \frac{4}{39} & \frac{1}{13}\end{bmatrix} \quad \therefore\ A^{-1} = \begin{bmatrix}-\frac{5}{39} & \frac{2}{13}\\ \frac{4}{39} & \frac{1}{13}\end{bmatrix}$$

(2) $A=\begin{bmatrix}6 & -4\\ -3 & 2\end{bmatrix}$ $\quad \begin{bmatrix}6 & -4 & 1 & 0\\ -3 & 2 & 0 & 1\end{bmatrix} \xrightarrow{E_{21}(\frac{1}{2})} \begin{bmatrix}6 & -4 & 1 & 0\\ \boxed{0 \quad 0} & & 0 & \frac{1}{2}\end{bmatrix}$ ∴ 가역행렬이 아니다.

(3) $A=\begin{bmatrix}3 & 4 & 1\\ 1 & 0 & 3\\ 2 & 5 & -4\end{bmatrix}$ $\quad \begin{bmatrix}3 & 4 & 1 & 1 & 0 & 0\\ 1 & 0 & 3 & 0 & 1 & 0\\ 2 & 5 & -4 & 0 & 0 & 1\end{bmatrix} \xrightarrow{L_{12}} \begin{bmatrix}1 & 0 & 3 & 0 & 1 & 0\\ 3 & 4 & 1 & 1 & 0 & 0\\ 2 & 5 & -4 & 0 & 0 & 1\end{bmatrix} \xrightarrow[E_{31}(-2)]{E_{21}(-3)} \begin{bmatrix}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 4 & -8 & 1 & -3 & 0\\ 0 & 5 & -10 & 0 & -2 & 1\end{bmatrix}$

$\xrightarrow{E_{23}(-1)} \begin{bmatrix}1 & 0 & 3 & 0 & 1 & 0\\ 0 & -1 & 2 & 1 & -1 & -1\\ 0 & 5 & -10 & 0 & -2 & 1\end{bmatrix} \xrightarrow{E_{32}(5)} \begin{bmatrix}1 & 0 & 3 & 0 & 1 & 0\\ 0 & -1 & 2 & 1 & -1 & -1\\ \boxed{0 \quad 0 \quad 0} & & & 5 & -7 & -4\end{bmatrix}$ ∴ 가역행렬이 아니다.

(4) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_{21}(1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[E_{32}(-1)]{C_{23}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[E_{13}(-1)]{D_3(\frac{1}{2})} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

∴ A는 가역행렬이다. $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$E_{13}(-1)\, D_3(\tfrac{1}{2})\, E_{32}(-1)\, C_{23}\, E_{21}(1)\, A = I$

$$A = E_{21}(1)^{-1}\, C_{23}^{-1}\, E_{32}(-1)^{-1}\, D_3(\tfrac{1}{2})^{-1}\, E_{13}(-1)^{-1}$$

$$A = E_{21}(-1)\, C_{23}\, E_{32}(1)\, D_3(2)\, E_{13}(1)$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) $A = \begin{bmatrix} 4 & 8 & -7 & 14 \\ 2 & 5 & -4 & 6 \\ 0 & 2 & 1 & -7 \\ 3 & 6 & -5 & 10 \end{bmatrix}$

$$\left[\begin{array}{cccc|cccc} 4 & 8 & -7 & 14 & 1 & 0 & 0 & 0 \\ 2 & 5 & -4 & 6 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -7 & 0 & 0 & 1 & 0 \\ 3 & 6 & -5 & 10 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[E_{42}(-1)]{E_{12}(-2)} \left[\begin{array}{cccc|cccc} 0 & -2 & 1 & 2 & 1 & -2 & 0 & 0 \\ 2 & 5 & -4 & 6 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -7 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 4 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[E_{21}(-2)]{C_{14}} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 4 & 0 & -1 & 0 & 1 \\ 0 & 3 & -2 & -2 & 0 & 3 & 0 & -2 \\ 0 & 2 & 1 & -7 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 2 & 1 & -2 & 0 & 0 \end{array}\right] \xrightarrow[E_{43}(1)]{E_{23}(-1)} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 4 & 0 & -1 & 0 & 1 \\ 0 & 1 & -3 & 5 & 0 & 3 & -1 & -2 \\ 0 & 2 & 1 & -7 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -5 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[E_{32}(-2)]{E_{12}(-1)} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & -1 & 0 & -4 & 1 & 3 \\ 0 & 1 & -3 & 5 & 0 & 3 & -1 & -2 \\ 0 & 0 & 7 & -17 & 0 & -6 & 3 & 4 \\ 0 & 0 & 2 & -5 & 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[E_{34}(-3)]{E_{14}(-1)} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & -1 & -2 & 0 & 3 \\ 0 & 1 & -3 & 5 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -2 & -3 & 0 & 0 & 4 \\ 0 & 0 & 2 & -5 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[E_{43}(-2)]{E_{23}(3)} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & -1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 & -9 & 3 & -1 & 4 \\ 0 & 0 & 1 & -2 & -3 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 & 7 & -2 & 1 & -8 \end{array}\right] \xrightarrow[\substack{E_{14}(-4) \\ E_{24}(1) \\ E_{34}(2)}]{D_4(-1)} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 27 & -10 & 4 & -29 \\ 0 & 1 & 0 & 0 & -16 & 5 & -2 & 18 \\ 0 & 0 & 1 & 0 & -17 & 4 & -2 & 20 \\ 0 & 0 & 0 & 1 & -7 & 2 & -1 & -8 \end{array}\right]$$

∴ 가역행렬이다.

$$A^{-1} = \begin{bmatrix} 27 & -10 & 4 & -29 \\ -16 & 5 & -2 & 18 \\ -17 & 4 & -2 & 20 \\ -7 & 2 & -1 & -8 \end{bmatrix}$$

$E_{34}(2)\cdot E_{24}(1)\cdot E_{14}(-4)\, D_4(-1)\, E_{43}(-2)\, E_{23}(3)\, E_{34}(-3)\, E_{14}(-1)\, E_{32}(-2)\, E_{12}(-1)\, E_{43}(1)\, E_{23}(-1) \rightarrow$

$\rightarrow E_{21}(-2)\, C_{14}\, E_{42}(-1)\, E_{12}(-2)\, A = I$

$$A = E_{12}(-2)^{-1} E_{42}(-1)^{-1} C_{14}^{-1} E_{21}(-2)^{-1} E_{23}(-1)^{-1} E_{43}(1)^{-1} E_{12}(-1)^{-1} E_{32}(-2)^{-1} E_{14}(-1)^{-1} E_{34}(-3)^{-1} E_{23}(3)^{-1} E_{43}(-2)^{-1} D_4(-1)^{-1} E_{14}(-4)^{-1} E_{24}(1)^{-1} E_{34}(2)^{-1}$$

$$A = E_{12}(2)\, E_{42}(1)\, C_{14}\, E_{21}(2)\, E_{23}(1)\, E_{43}(-1)\, E_{12}(1)\, E_{32}(2)\, E_{14}(1)\, E_{34}(3)\, E_{23}(-3)\, E_{43}(2)\, D_4(-1)\, E_{14}(4)\, E_{24}(-1)\, E_{34}(-2)$$

(6) $A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 1 & -2 & 5 & 3 \\ 2 & -6 & 2 & 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 3 & 5 & 1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -2 & 5 & 3 & 0 & 0 & 1 & 0 \\ 2 & -6 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{C_{13}} \left[\begin{array}{cccc|cccc} 1 & -2 & 5 & 3 & 0 & 0 & 1 & 0 \\ 1 & -3 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 5 & 1 & 0 & 0 & 0 \\ 2 & -6 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{E_{21}(-1),\ E_{41}(-2)} \left[\begin{array}{cccc|cccc} 1 & -2 & 5 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & -5 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & -2 & -8 & -6 & 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow{C_{23},\ E_{32}(1),\ E_{42}(2),\ E_{12}(2)} \left[\begin{array}{cccc|cccc} 1 & 0 & 11 & 13 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & 1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 & 2 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{E_{43}(-1)} \left[\begin{array}{cccc|cccc} 1 & 0 & 11 & 13 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & 1 & 1 & -1 & 0 \\ \boxed{0 & 0 & 0 & 0} & 1 & -1 & -1 & 1 \end{array}\right]$$

∴ 가역행렬이 아니다.

## 3.1절 행렬식의 계산

#1 행렬 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 에서 1행, 2행, 1열, 2열 에 대하여 각각 행렬식을 전개하고, 그 값을 구해서 같은 값임을 확인하여라.

행렬식

1행: $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = aC_{11} + bC_{12} = ad - bc$

2행: $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = cC_{21} + dC_{22} = -cb + da = ad - bc$

1열: $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = aC_{11} + cC_{21} = ad - cb = ad - bc$

2열: $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = bC_{12} + dC_{22} = -bc + da = ad - bc$

같은 값이다.

#2 다음의 행렬에서 서로 다른 2개의 행 또는 열을 선택해서 선택된 행 또는 열에 대하여 각각 행렬식을 전개하고, 그 값을 구하여라.

(1) $A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

1행: $\det(A) = 1C_{11} + 4C_{12} - 2C_{13} = 1\begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix}$

$= 1(6) - 4(9) - 2(12+2) = -58$

2행: $\det(A) = 3C_{21} + 2C_{22} + 0C_{23} = -3\begin{vmatrix} 4 & -2 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix}$

$= -3(12+8) + 2(3-2) = -58$

(2) $A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 6 & 0 \\ -3 & 7 & 2 \end{bmatrix}$

1행 : $\det(A) = -2C_{11} + 0C_{12} + 0C_{13} = -2\begin{vmatrix} 6 & 0 \\ 7 & 2 \end{vmatrix} = -2(12) = -24$

3열 : $\det(A) = 0C_{31} + 0C_{32} + 2C_{33} = 2\begin{vmatrix} -2 & 0 \\ 4 & 6 \end{vmatrix} = 2(-12) = -24$

(3) $A = \begin{bmatrix} 3 & 0 & 7 & 0 \\ 2 & 6 & 11 & 12 \\ 4 & 1 & -1 & 2 \\ 1 & 5 & 2 & 10 \end{bmatrix}$

1행 : $\det(A) = 3C_{11} + 0C_{12} + 7C_{13} + 0C_{14}$

$= 3\begin{vmatrix} 6 & 11 & 12 \\ 1 & -1 & 2 \\ 5 & 2 & 10 \end{vmatrix} + 7\begin{vmatrix} 2 & 6 & 12 \\ 4 & 1 & 2 \\ 1 & 5 & 10 \end{vmatrix}$

$= 3(1C_{21} - 1C_{22} + 2C_{23}) + 7(4C_{21} + 1C_{22} + 2C_{23})$

$= -3\begin{vmatrix} 11 & 12 \\ 2 & 10 \end{vmatrix} - 3\begin{vmatrix} 6 & 12 \\ 5 & 10 \end{vmatrix} - 6\begin{vmatrix} 6 & 11 \\ 5 & 2 \end{vmatrix} - 28\begin{vmatrix} 6 & 12 \\ 5 & 10 \end{vmatrix} + 7\begin{vmatrix} 2 & 12 \\ 1 & 10 \end{vmatrix} - 14\begin{vmatrix} 2 & 6 \\ 1 & 5 \end{vmatrix}$

$= -3(110-24) - 3(60-60) - 6(12-55) - 28(60-60) + 7(20-12) - 14(10-6)$

$= 0$

2열 : $\det(A) = 0C_{12} + 6C_{22} + 1C_{32} + 5C_{42}$

$= 6\begin{vmatrix} 3 & 7 & 0 \\ 4 & -1 & 2 \\ 1 & 2 & 10 \end{vmatrix} - 1\begin{vmatrix} 3 & 7 & 0 \\ 2 & 11 & 12 \\ 1 & 2 & 10 \end{vmatrix} + 5\begin{vmatrix} 3 & 7 & 0 \\ 2 & 11 & 12 \\ 4 & -1 & 2 \end{vmatrix}$

$= 6(3C_{11} + 7C_{12} + 0C_{13}) - (3C_{11} + 7C_{12} + 0C_{13}) + 5(3C_{11} + 7C_{12} + 0C_{13})$

$= 6\left(3\begin{vmatrix} -1 & 2 \\ 2 & 10 \end{vmatrix} - 7\begin{vmatrix} 4 & 2 \\ 1 & 10 \end{vmatrix}\right) - \left(3\begin{vmatrix} 11 & 12 \\ 2 & 10 \end{vmatrix} - 7\begin{vmatrix} 2 & 12 \\ 1 & 10 \end{vmatrix}\right) + 5\left(3\begin{vmatrix} 11 & 12 \\ -1 & 2 \end{vmatrix} - 7\begin{vmatrix} 2 & 12 \\ 4 & 2 \end{vmatrix}\right)$

$= 18(-10-4) - 7(40-2) - 3(110-24) + 7(20-12) + 15(22+12) - 35(4-48)$

$= 0$

(4) $A = \begin{bmatrix} 5 & 8 & -4 & 2 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

2행 : $\det(A) = 0C_{21} + 0C_{22} + 6C_{23} + 0C_{24}$

$= -6 \begin{vmatrix} 5 & 8 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{vmatrix}$

$= -6\left(0C_{21} + 0C_{22} + 2C_{23}\right)$

$= -6\left(-2\begin{vmatrix} 5 & 8 \\ 0 & 0 \end{vmatrix}\right)$

$= 12(0) = 0$

4행 : $\det(A) = 0C_{41} + 0C_{42} + 0C_{43} - 1C_{44}$

$= -\begin{vmatrix} 5 & 8 & -4 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{vmatrix}$

$= -\left(0C_{21} + 0C_{22} + 6C_{23}\right)$

$= 6\begin{vmatrix} 5 & 8 \\ 0 & 0 \end{vmatrix}$

$= 6(0) = 0$

#3 다음을 만족하는 실수 λ를 구하여라.

(1) $\begin{vmatrix} 1-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = 0$

$= (1-\lambda)(3-\lambda) - 4 = 0$

$\lambda^2 - 4\lambda + 3 - 4 = 0$

$\lambda^2 - 4\lambda - 1 = 0$

$\lambda = 2 \pm \sqrt{5}$

(2) $\begin{vmatrix} 2-\lambda & 4 & 2 \\ 1 & -\lambda & 1 \\ 1 & -4 & 5-\lambda \end{vmatrix} = 0 = 1C_{21} - \lambda C_{22} + 1C_{23} = -\begin{vmatrix} 4 & 2 \\ -4 & 5-\lambda \end{vmatrix} - \lambda\begin{vmatrix} 2-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} - \begin{vmatrix} 2-\lambda & 4 \\ 1 & -4 \end{vmatrix}$

$= -(20 - 4\lambda + 8) - \lambda(\lambda^2 - 7\lambda + 10 - 2) - (4\lambda - 8 - 4)$

$= -\lambda^3 + 7\lambda^2 - 8\lambda - 16 = 0$

$\lambda^3 - 7\lambda^2 + 8\lambda + 16 = 0$

$(\lambda+1)(\lambda^2 - 8\lambda + 16) = 0 \quad (\lambda+1)(\lambda-4)^2 = 0 \quad \therefore \lambda = -1, 4$

Date. / /

#4 정리 3.1.3이 성립하는 이유를 설명하여라

정리 3.1.3 행렬 $A=[a_{ij}]_{n\times n}$, $B=[b_{ij}]_{n\times m}$, $C=[c_{ij}]_{n\times m}$과 영행렬 $O$에 대하여 다음이 성립한다.

$$\begin{vmatrix} A & B \\ O & I_m \end{vmatrix} = \begin{vmatrix} I_m & O \\ C & A \end{vmatrix} = |A|$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_{n1} & \cdots & b_{nm} \\ 0_{11} & \cdots & 0_{1n} & 1_{11} & 0 \cdots & 0 \\ & & & 0 & 1_{22} \cdots & 0 \\ 0_{m1} & \cdots & 0_{mn} & 0 & 0 \cdots & 1_{mm} \end{vmatrix} = 1\cdot C_{n+m\,n+m} = \begin{vmatrix} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1,m-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_{n1} & \cdots & b_{n,m-1} \\ 0_{11} & \cdots & 0_{1n} & 1_{11} & 0 \cdots & 0 \\ & & & 0 & 1_{22} \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{m-1,1} & \cdots & 0_{m-1,n} & 0 & 0 \cdots & 1_{m-1,m-1} \end{vmatrix} = \cdots = \begin{vmatrix} a_{11} & \cdots & a_{1n} & b_{11} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_{n1} \\ 0_{11} & \cdots & 0_{1n} & 1_{11} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = |A|$$

$$\begin{vmatrix} 1_{11} & 0 \cdots & 0 & 0_{11} & \cdots & 0_{1n} \\ 0 & 1_{22} \cdots & 0 & & & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \cdots & 1_{mm} & 0_{m1} & \cdots & 0_{mn} \\ c_{11} & \cdots & c_{1m} & a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nm} & a_{n1} & \cdots & a_{nn} \end{vmatrix} = 1\cdot C_{11} = \begin{vmatrix} 1_{22} & 0 \cdots & 0 & 0_{21} & \cdots & 0_{2n} \\ 0 & 1_{33} \cdots & 0 & & & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \cdots & 1_{mm} & 0_{m1} & \cdots & 0_{mn} \\ c_{12} & \cdots & c_{1m} & a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n2} & \cdots & c_{nm} & a_{n1} & \cdots & a_{nn} \end{vmatrix} = \cdots = \begin{vmatrix} 1_{mm} & 0_{m1} & \cdots & 0_{mn} \\ c_{1m} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{nm} & a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = |A|$$

$$\therefore \begin{vmatrix} A & B \\ O & I_m \end{vmatrix} = \begin{vmatrix} I_m & O \\ C & A \end{vmatrix} = |A|$$