Report 6 - Financial Mathematics for Data Science

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The aim of this report is to build an Excel pricer for various types of options (European, Asian and Lookback) using Monte Carlo techniques and the Euler's scheme discretization.

FIRST PART - BRIEF STUDY OF A GBM

An option on equity may be modelled with one source of uncertainty: the price of the underlying stock in question. The price of the underlying instrument is usually modelled such that it follows a Geometric Brownian Motion (GBM) with constant drift μ and volatility σ. So:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where dW_t is found via a random sampling from a normal distribution. In this report we will study options from a stock having the following parameters:

- Spot price $S = 100$
- Strike price $K = 99$
- Maturity $T = 1$ year
- Volatility $\sigma = 20\%$
- Risk free interest rate $R = 1\%$

In Fig.1 is represented the graph from 100 paths of a GBM having the parameters reported in the table to give an idea of the stochastic behaviour of Eqn.1 (In absence of dividend yield the drift is equal to the risk-free interest rate) [1].

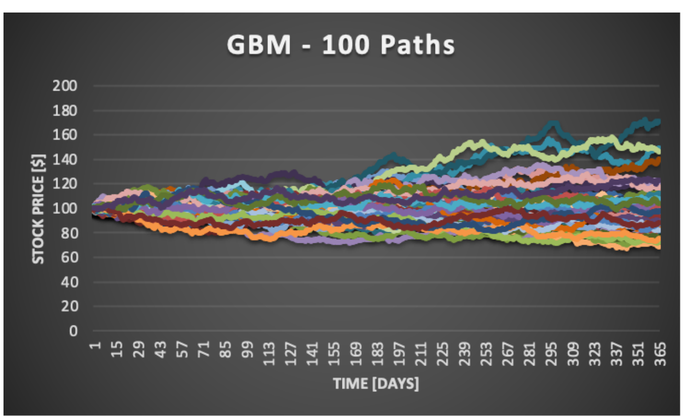

Fig. 1: Graph for 100 paths of the GBM defined by the parameters in the first part

SECOND PART - PRICER FOR EUROPEAN OPTIONS

The first option studied is the European type. It is possible to implement an option pricer in VBA based on Monte Carlo simulations using this reference [2]. The user interface is the shown in fig.2.

Using this code is possible to price European vanilla options using as parameters the ones listed in the first part. The additional parameters necessary for the Monte Carlo simulation are: $N = 1000$ iterations and Number of Steps $= 1$. Results are shown in the upper left corner of the final results summary table tab.I, in the "Standard" row.

THIRD PART - EULER SCHEME DISCRETIZATION

Monte Carlo simulation in the context of option pricing refers to a set of techniques to generate underlying values over time. Typically the dynamics of these stock prices and interest rates are assumed to be driven by Eqn.1 however, is done at discrete time steps. Hence, the first step in any simulation is to find a way to "discretize" a continuous-time process into a discrete time process. The Euler scheme discretization solves this problem starting from the simulation of S_t over the time interval [0; T], which we assume to be discretized as $0 = t_1 < t_2 < ... < t_m =$ T, where the time increments are equally spaced with width dt. Integrating Eqn.1 from t to $t+dt$ produce:

$$S_{t+dt} = S_t + \int_t^{t+dt} \mu\left(S_u, u\right) du + \int_t^{t+dt} \sigma\left(S_u, u\right) dW_u \tag{2}$$

by approximating the integrals using the left point rule and solving them we obtain:

$$S_{t+dt} = S_t + \mu\left(S_t, t\right) dt + \sigma\left(S_t, t\right) \sqrt{dt} Z \tag{3}$$

and by applying this result to the Black-Scholes model we get an implementable discratization form of the equation [3]:

$$S_{t+dt} = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right) dt + \sigma\sqrt{dt} Z\right) \tag{4}$$

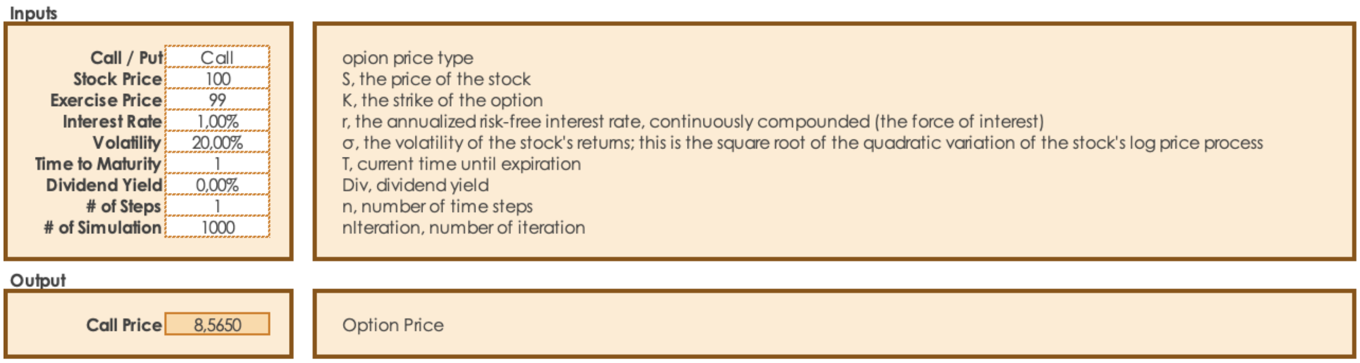

Fig. 2: User Interface for the European options pricer implemented in Excel

it is now possible to implement Eqn.4 in VBA and price European options by simulating $N = 1000$ scenarios of 252 steps each (in such way that $dt = 1$ day). Results are shown in the lower left corner in the previously cited Tab.I, in the "Euler Scheme" row.

Comparing the prices obtained from the standard Monte Carlo method in the second part of this report to the ones obtained by implementing the Euler scheme tells us that the latter are on average 10% higher that the first ones

	European		Asian		Lookback	
	Call	Put	Call	Put	Call	Put
Standard	8,57	6,60	5,17	4,04	14,65	15,34
Euler Scheme	9,16	7,45	5,56	4,05	15,37	16,47

Tab. I: Prices (in \$) of various types of options obtained via Monte Carlo techniques using as input parameters the ones listed in the first part. The first row are results obtained with the standard Monte Carlo method, the second row is calculated using the Euler Scheme discretization for the computation of the stochastic integrals

FOURTH PART - PRICER FOR ASIAN OPTIONS

The payoff of an Asian option is based on the difference between an asset's average price over a given time period, and a fixed price called the strike price. Asian options are popular because they tend to have lower volatility than options whose payoffs are based purely on a single price point. It is also harder for big traders to manipulate an average price over an extended period than a single price, so Asian options offer further protection against risk. One disadvantage of Asian options is that their prices are very hard to compute using standard techniques, but this is a more achievable using Monte Carlo simulations. We can modify the VBA code shown in Fig.2 to include Asian options just by keeping track of the average stock price until maturity and by implementing the corresponding payoff, which in our case is the one for fixed strike type of options. The payoffs for Call and Put are:

$$C(T) = \max(A(0,T) - K, 0 \tag{5}$$

$$P(T) = \max(K - A(0,T), 0) \tag{6}$$

where $A(0,T)$ is the arithmetic average of the stock price between time 0 and T. It is also possibly to estimate the Asian option prices using the Euler scheme discretization just by modifying that code too in the same way as we have done for the standard Monte Carlo pricer. After simulating $N = 1000$ paths we obtain an estimation of the Asian option price having the parameters listed in the first part of the report. These results are shown in the "Asian" column of Tab.I for both methods. We can see that this type of option is cheaper than the European one, as expected, because the averaging done in the payoff calculation reduces its volatility. We can also notice that even for this type of option the prices obtained using the Euler scheme discretization are on average higher than by using the standard method, in this particular case by 4%.

FIFTH PART - PRICER FOR LOOKBACK OPTIONS

In the final part of this report we want to show that we are able to adapt our code also to different types of exotic options, for example the Lookback option. A Lookback option is a type of exotic option that gives the holder the opportunity of knowing its history when deciding on the appropriate time to exercise it. This option has a benefit of reducing the uncertainties arising when timing the market entry. The strike price is derived from the optimal value of the price of the underlying asset during the lifespan of the option. It allows the holder to review the historical prices of the underlying asset over the life of the option. There are two types of Lookback options which are the fixed strike price and the floating strike price. The payoff from a floating Lookback call is the amount that the final asset price exceeds the minimum asset price achieved during the life of the option. The payoff from a floating Lookback put is the amount by which the maximum asset price achieved during the life of the option exceeds the final asset price [1].

We can build a Lookback option pricer by modifying the original VBA code for the European options in the same way as we have done for the Asian options. Just by modifying the calculation of the payoff using the Lookback one and by keeping track of the absolute Max and Min value of the stock. Doing the same on the code for the Euler scheme we obtain once again an estimation of the price of the option, this time the Lookback option, for the usual Monte Carlo method and the one implementing the discretization. After doing the usual simulation with $N = 1000$ paths we get the final estimation of the prices, they can be seen in the same summary table as before Tab.I in the column "Lookback".

From the table we immediately notice that the Lookback option is much more expensive than the European one, costing around double the money. This is expected because is much more expensive for the option writer of a Call to buy the underlying at the maximum price occurred during a period of time rather than the spot price, which will always be equal or less than the maximum price. We can also notice that even in this case the price obtained from the Euler scheme is greater than the standard method, this time by 6%.

CONCLUSIONS AND COMPARISON BETWEEN STANDARD AND EULER METHOD

In this report we managed to build a very flexible option pricer based on Monte Carlo techniques which can be easily adapted to the pricing of exotic option just by modifying the payoff calculation inside the VBA code to the one corresponding to the option of interest. We have also noticed that the Euler scheme discretization techniques consistently gives an higher option price in respect to the standard Monte Carlo method, on average by around 7%. The cause of this could be the type of approximation done in the discretization. We wrote that we are solving the integrals by applying the left point rule but if our time steps are not small enough in respect to the total interval this could mean that a significant error could arise from the approximation of the rectangles area in the integration process. In our case we only have 252 time steps in our interval and the average error of 7% noticed in this report could be an alarm signal that the time step $dt = 1$ day chosen is too small. We suggest that by applying a much smaller time step, for example by a factor of 10, the error should be much smaller than the one observed with the given parameters.

[1] J. C. Hull, "Futures and other derivatives," (2015).
[2] "https://www.spreadsheetweb.com/how-to-calculate-option-pricing-using-monte-carlo-simulations-in-excel/," .
[3] "https://frouah.com/finance%20notes/euler%20and," .