Report 3 - Financial Mathematics for Data Science

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The aim of this report is to introduce the Leisen-Reimer model and compare its rate of convergence to the Black-Scholes results in respect to the Cox-Ross-Rubinstein binomial model.

INTRODUCTION

It is a well known result that for increasingly large n the results from the Binomial Option Pricing Model converge to the Black-Scholes-Merton (BSM) pricing formula. The key to the success of the Binomial model stems from its simplicity, it is able to grasp the key option-pricing aspects while giving solid results with a relative low amount of mathematical effort. However since its discovery a few problems have risen, the main one being a slow convergence process to the BSM formula. In fact, the price given by the Binomial Option Pricing Model tends to have periodic oscillation around the BSM value. A solution to the slow convergence problem was proposed by D.Leisen and M.Reimer (LR) in 1995 and it is based around the construction of a different binomial tree having the strike price of the option at its center during each step of the algorithm. This assumption is justified since most of the time trading in options occurs in at-the-money and near-the-money options only, actually this requirement changes only minorly the tree's structure. The implementation of the LR model will allow us to significantly improve the convergence of the binomial model developed by Cox-Ross-Rubinstein (CRR) to the BSM formula. This results will be shown graphically applying both the CRR and LR approach to an example.

IMPLEMENTATION OF THE LR MODEL

The LR tree is generated using these parameters:

$$\begin{aligned} \bar{p} &= h^{-1}(d_1) \\ p &= h^{-1}(d_2) \\ u &= e^{r\Delta t}\frac{\bar{p}}{p} \\ d &= \frac{e^{r\Delta t}-pu}{1-p} \end{aligned}$$

where h^{-1} is a discrete approximation to the cumulative distribution function for a normal distribution. There are several ways this can be calculated. One suggested by Leisen and Reimer is to use the Peizer-Pratt Method-2-Inversion. The access to this type of inversion is allowed by the fact that the strike price is situated at the center of the tree at any step of the calculation:

$$h^{-1}(z) = \frac{1}{2} + \mathrm{sgn}(z)\cdot \cdot \sqrt{\left[\frac{1}{4} - \frac{1}{4}\exp\left\{-\left(\frac{z}{n+\frac{1}{3}+\frac{1}{10(n+1)}}\right)(n+\frac{1}{6})\right\}\right]}$$

where n is the number of time points in the model (including times 0 and T) and d1 and d2 are their usual definitions from the BSM formulation. This model, CRR and also the BSM formula are then implemented in Excel using VBA code. The code used for this work is reported in the Appendix section.

RESULTS

The different convergence properties of the LR and CRR models can be visualized with an example. The task is the pricing of a European Call option having the following parameters:

- Stock price S = 100
- Risk-free interest rate r = 0,1%
- Strike Price X = 100
- Maturity time T = 1 year
- Volatilty v = 20%

for various values of the time step n. I studied the convergence until $n = 100$ and the corresponding values of LR and CRR models are graphed in fig. 1. The result from the BSM formula is unique and equal to:

$$Price_{SBM} = 8,4333$$

It is evident from the graph the improvement in convergence speed and accuracy obtained by using the LR model in respect to the CRR one. The convergence is so precise that the LR model line is basically overlying on top of the ideal BSM formula's result already from the first values of n. The behaviour of the convergence is also different, as discussed above the CRR model oscillates around the BSM formula while the LR model has a monotone convergence, exactly as shown in fig.2, which is just a zoom of the fig.1 highlighting only the LR model with n from 2 to 100. From this work the improvement in efficiency to the binomial model given by LR is evident and cannot be overstated enough.

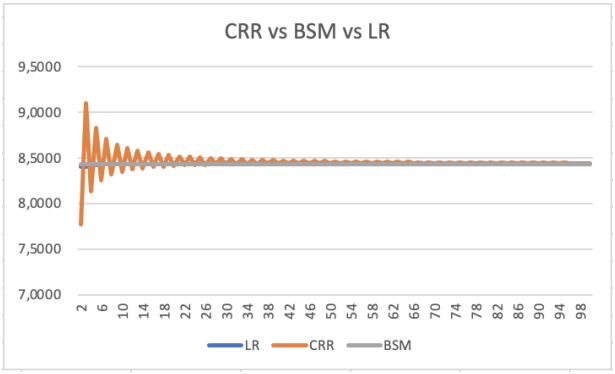

Fig. 1: Graph of the convergence of LR and CRR to BSM with $n < 100$

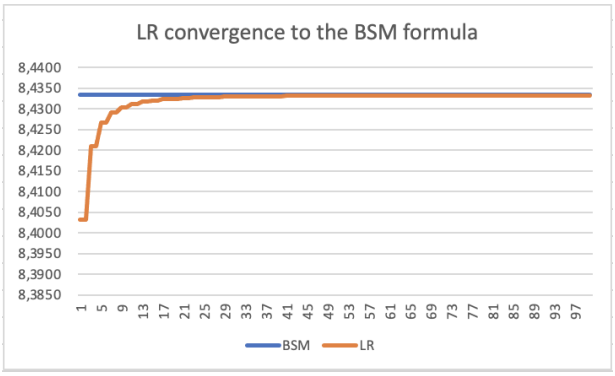

Fig. 2: Detail of the monotone convergence of LR to the BSM formula

APPENDIX

All the calculation necessary to obtain the results shown in this work are done in Excel using VBA formulas. What follows are the sources of the actual code implemented to write those formulas.

- CRR binomial model and SBM formula: http://www.anthony-vba.kefra.com/vba/vba7.htm

- Leisen Reimer: https://sites.google.com/view/vinegarhill-financelabs/binomial-lattice-framework/leisen-reimer