Estimation of Hurst Parameter of Mono Fractal Signals using Wavelets

Agrim Gupta, Saurabh Pinjani, Asim Ukaye

Main Course Assignment Presentation, EE-678

Overview of Presentation

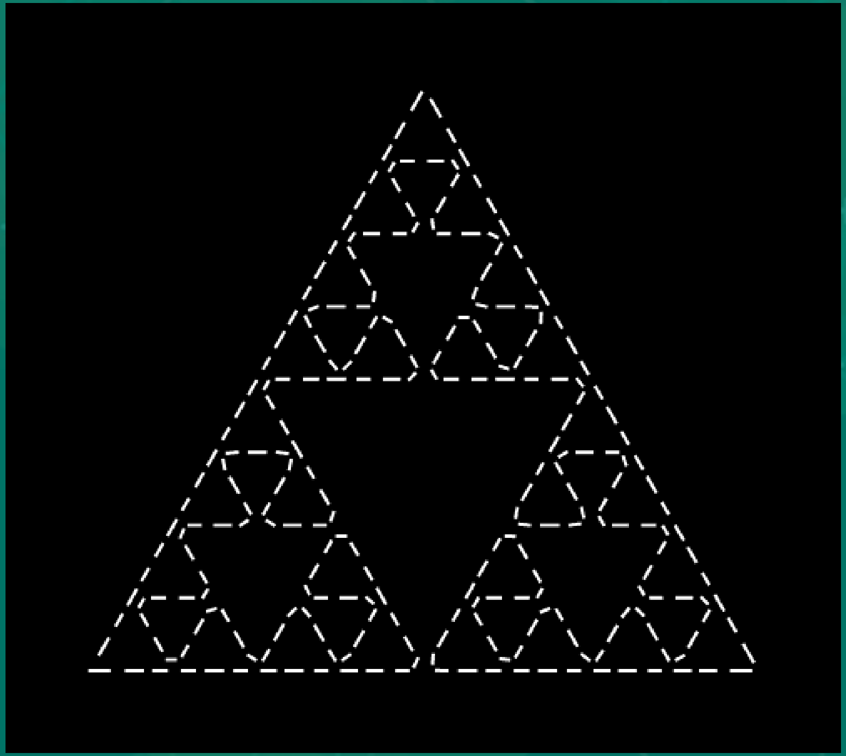
- Monofractal Processes Review
  - FBM Processes
  - 1/f processes

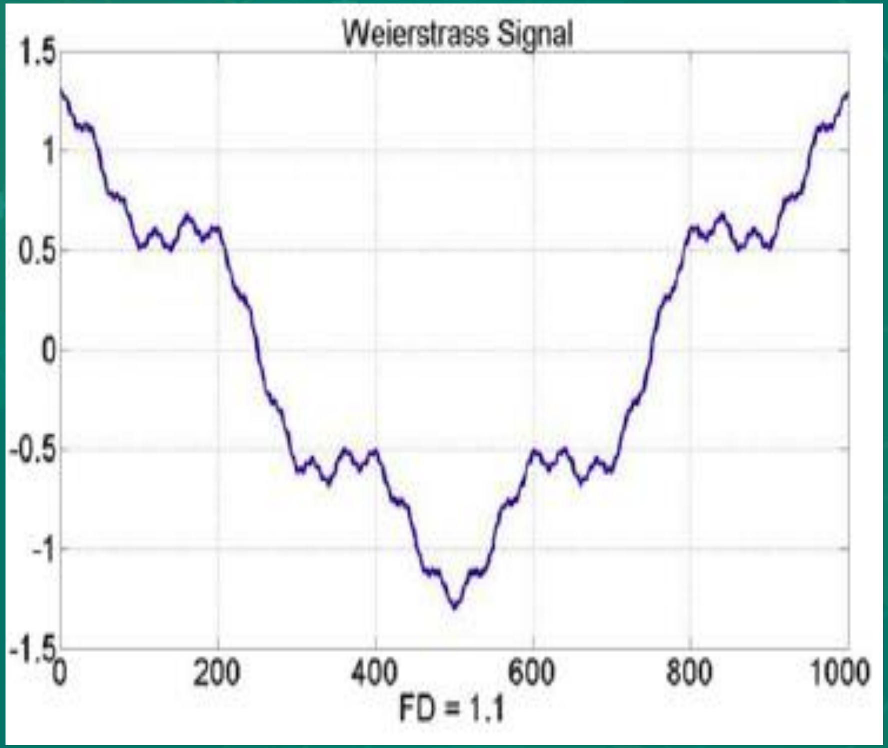
- Estimation of Hurst Parameter of FBM signal
  - PCA spectrum method
  - Proposed leading eigenvalue method

- Bayesian estimation of Hurst parameter for 1/f signal

- Future arears possible

Graphical Interpretation of Fractals





Introduction to Self-Similar Signals

A random process is said to be self-similar if its statistics are invariant to compressions and dilations of the waveform in time

$$x(t) \equiv a^{-H} x(at)$$

1/f processes

Self-similar signals that have a power spectrum obeying

$$S(\omega) = \frac{\sigma_x^2}{|\omega|^\gamma}$$

Where $\gamma = 2H + 1$

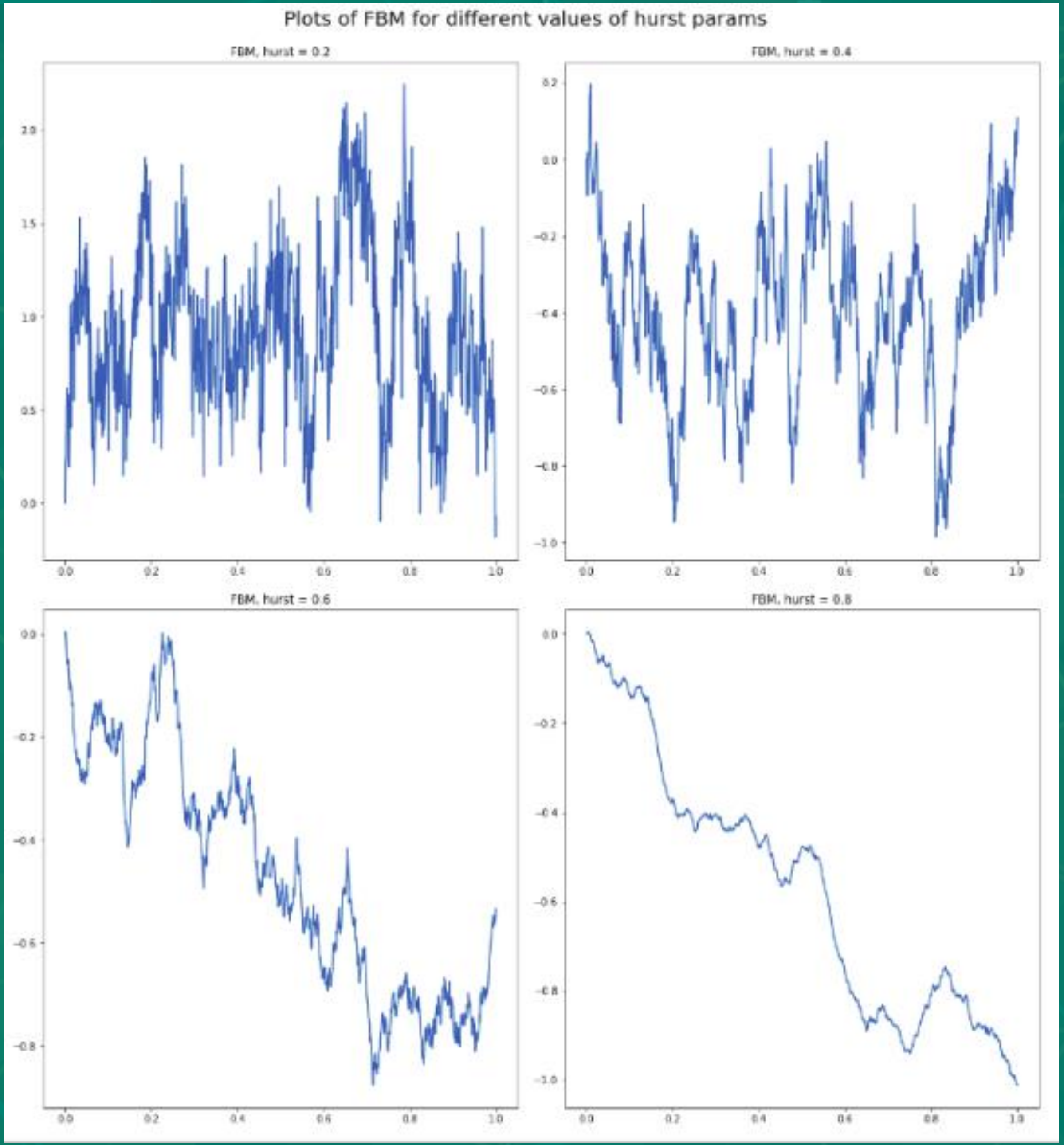
Fractional Brownian Motion

It is a generalization of Brownian motion.

- Unlike classical Brownian motion, the increments of fBm need not be independent.
- fBm is a continuous-time Gaussian process $B_H(t)$ on [0, *T*], which starts at zero, has expectation zero for all *t* in [0, *T*]
- It has the following covariance function:

$$E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

Graphical Interpretation of Hurst parameter



"Hurst Parameter fully characterizes the signal available at hand"

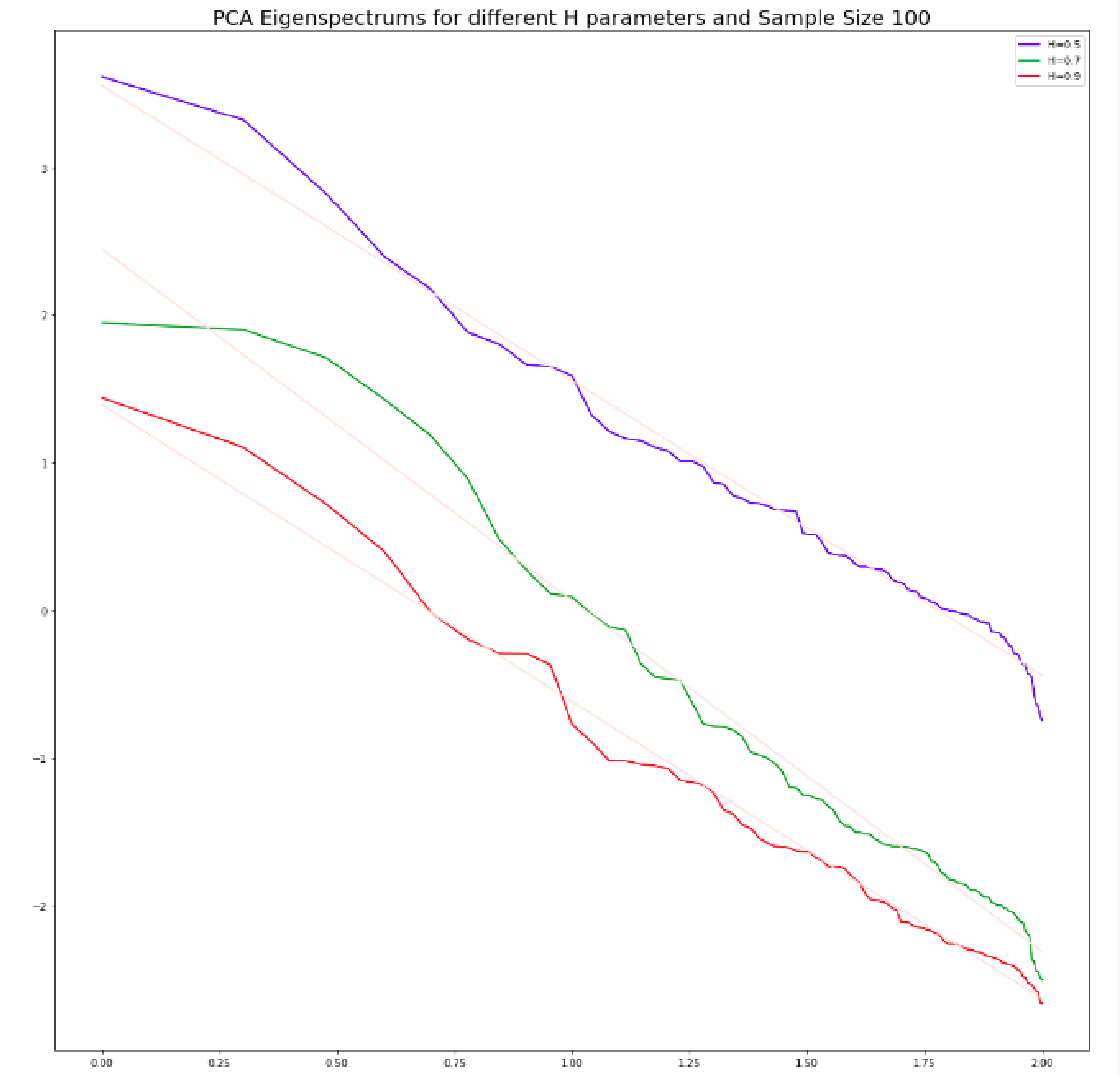
H parameter estimation using PCA Spectrum (Li et al. 2009)

Abstract

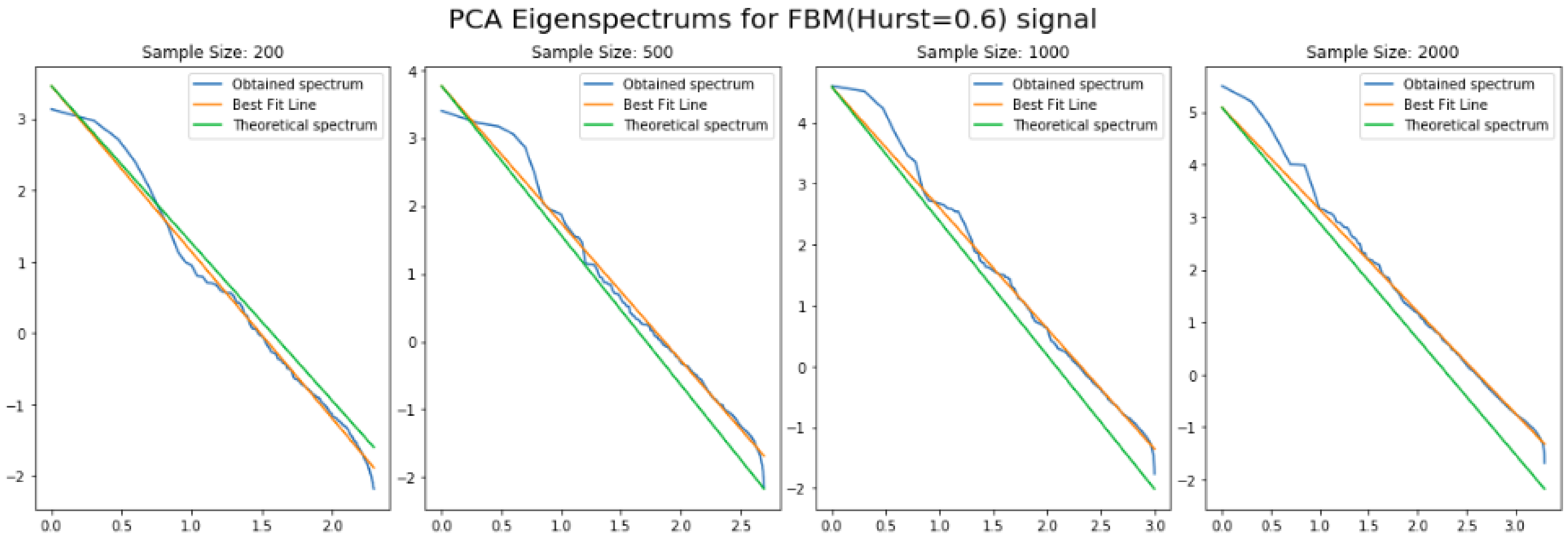
A PCA eigenspectrum method to estimate the hurst parameter of a FBM signal was proposed. PCA eigenspectrum of Autocov. matrix of FBM signal is linear, and the slope is indicative of the H parameter. The method is theoretically proved by showing equivalence of eigenvalues arising from discrete K-L expansion and Autocov. matrix

Results and Shortcomings of the paper

1. Results obtained hold good for H parameter in range (0.5,1) & for sample sizes around 100
2. Many real life signals are of much larger length, and may show negative correlation, hence there is a lot of scope of progress
3. The theoretical proof which showed equivalence between discrete K-L expansion and Autocov. matrix is not robust and fails if KL expansion is done with a different basis

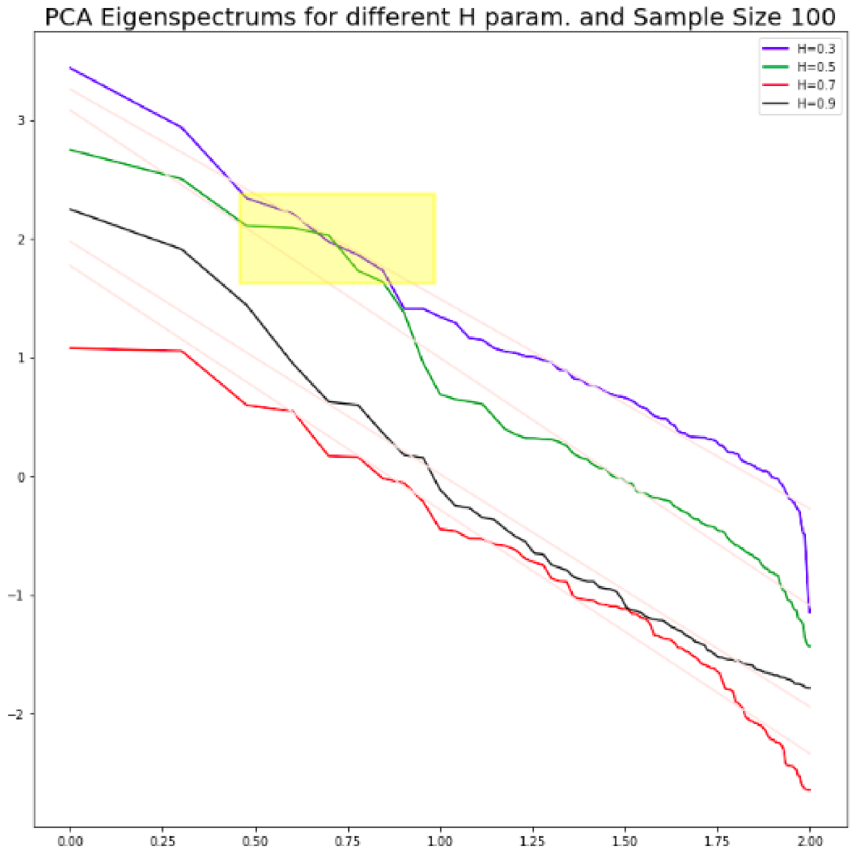


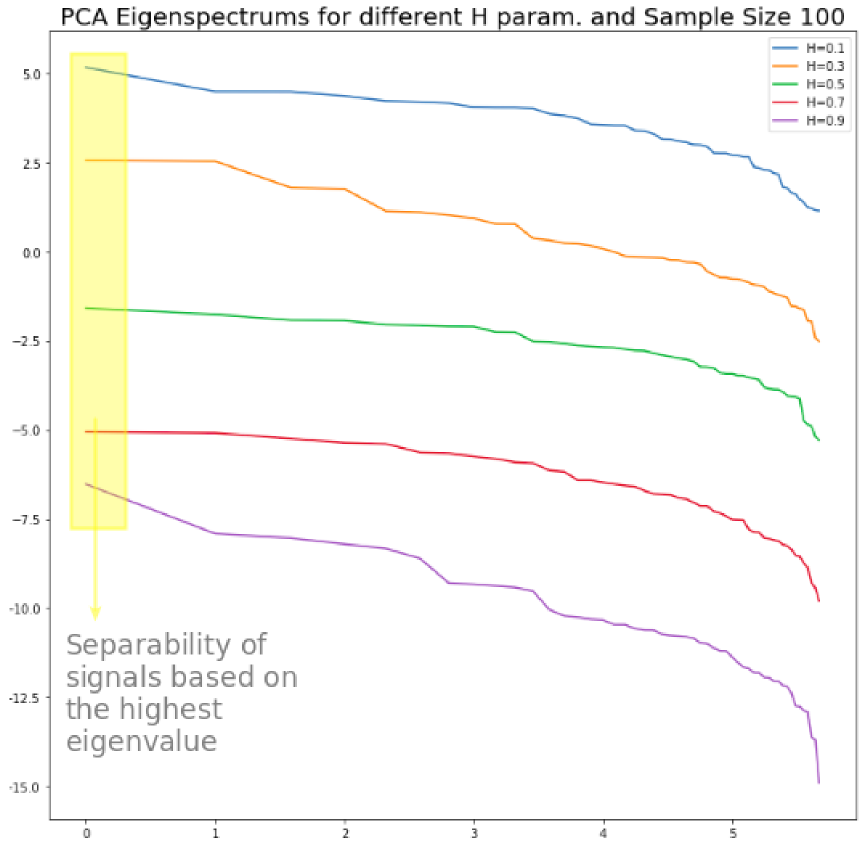
The method doesn't scale with sample size



Proposed Algorithm

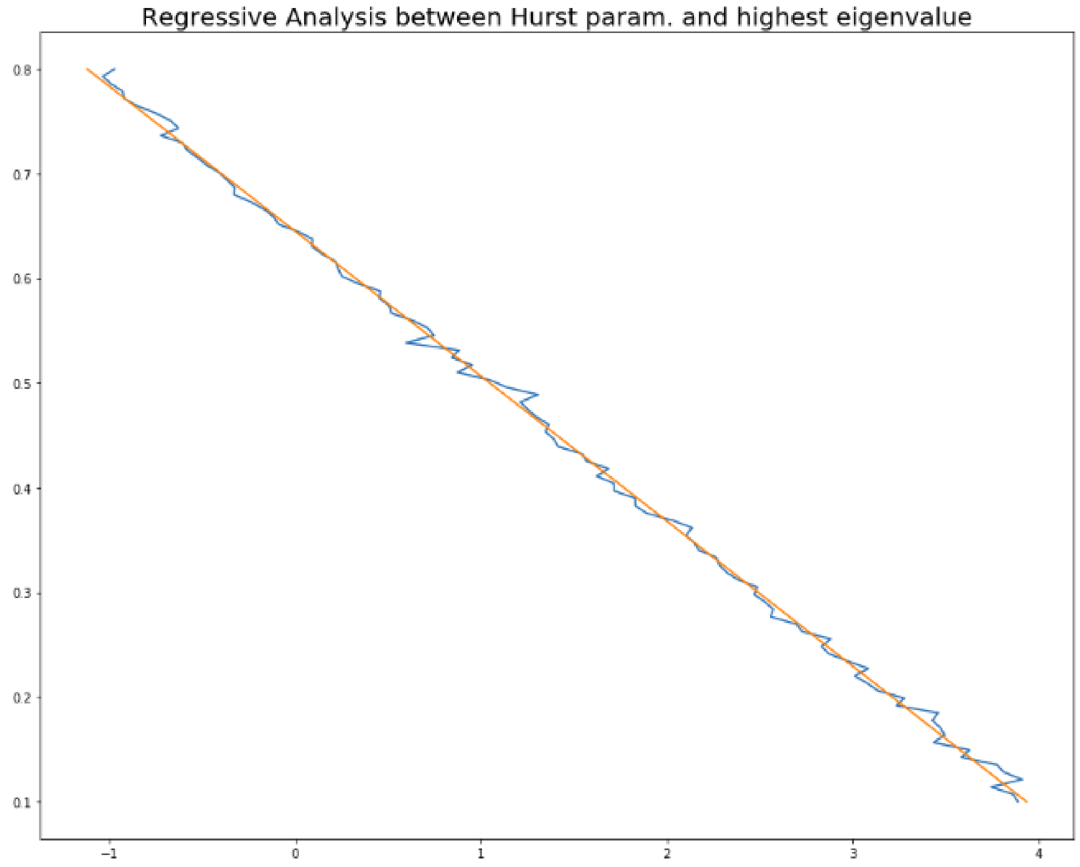
Instead of forming Autocov. matrix of the FBM signal, we first take DWT of the FBM signal, and then use the Detail Coefficients (high pass branch) to form the autocovariance. Plotting the Eigenspectrum of the modified autocov. matrix improves the separability of signals with different Hurst parameters.





Proposed Algorithm

Next, we apply regression over the observed highest eigen values as independent variable and the Hurst parameter as dependent variable. The Hurst parameter clearly shows a linear relationship with the observed highest eigen value, as shown in the below figure.



Dealing with Scalability with Sample Size

- The slope & intercept calculated after fitting remains approx. same as sample size increases
- However, if sample size is increased drastically, gradual changes in the slope & intercept are observed
- Applying regression with Slope, Intercept as dependent parameters, and sample size as independent variable can be done to correct for the gradual change
- Mathematical analysis can also be performed to bring out the weak dependence on sample size

Mathematical Analysis of the proposed claim

Using the fact that the highest eigenvalue(λ_1) of the Autocov. matrix with all +ve entries is $1+(M-1)*\Sigma(p*R(p))$, mathematical analysis of the results obtained was attempted. To make all the entries in the Autocov. matrix +ve, abs(min{R(p)}) can be added. Hence, the following formula was obtained, from which linearity when plotted in log log scale can be explained.

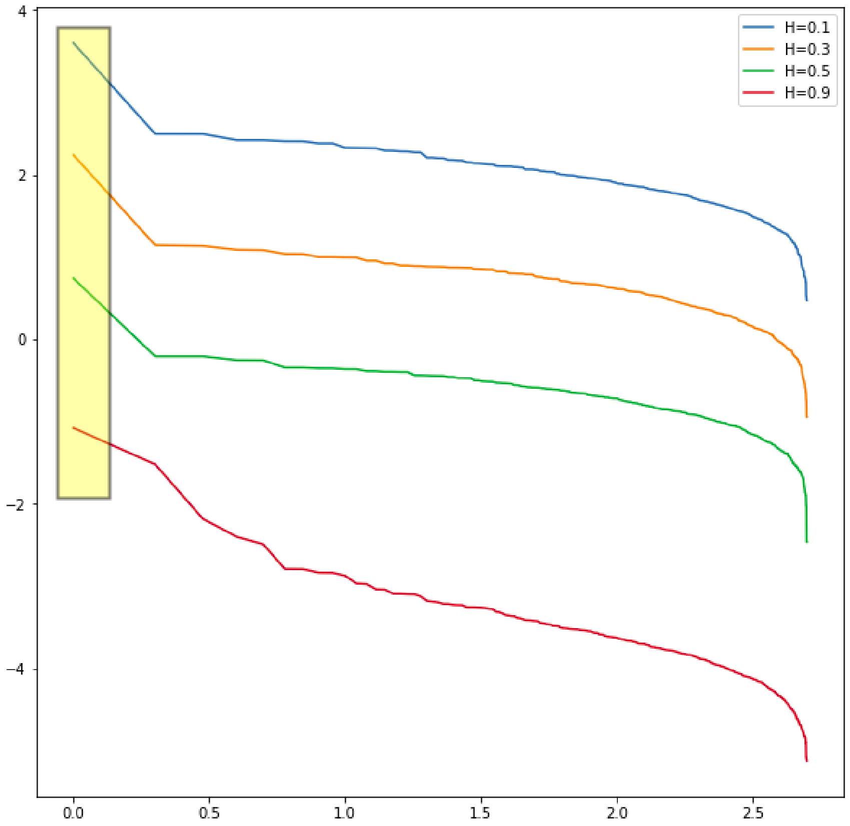
$$\lambda_1 = 1 + (M-1)\left(\frac{1}{\frac{M(M-1)}{2}} \sum_{p=1}^{M-1} pR[p] + |min_p(R[p])|\right)$$

However, to obtain the exact values of slope, we need to solve the minimization problem formally and account in for a more accurate expression for R[0], which has been left as a potential future work.

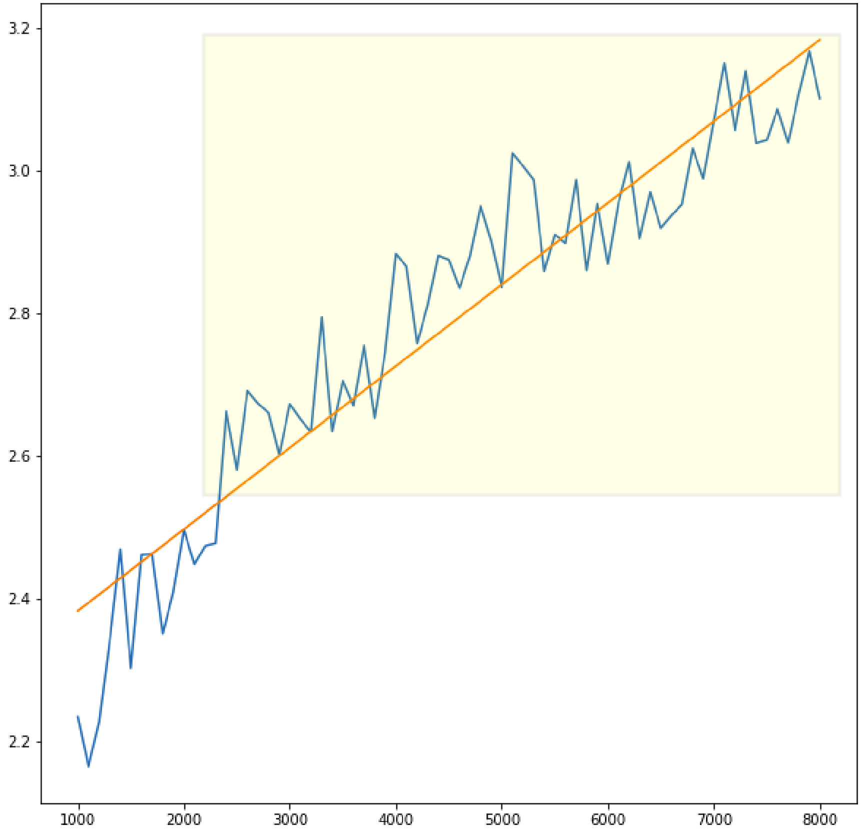
Explaining this via Discrete KL Expansion was also attempted, but it did not give fruitful result. Another issue faced was with various forms of R[k]. Using approximations to R[k] results in counterfeit results, and as a future work expression used for R[k] should be improved upon to explain the obtained results completely

Proposed Algorithm (contd.)

After adding the abs(min(R[p])) term, the plots do change slightly, however the separability and linearity continue to hold



Regressing the maximum eigen value with sample size shows a linear trend as well



Estimation of Fractal Signals from Noisy Measurements using Wavelets (G.Wornell and A. Oppenheim)

What is this research about?

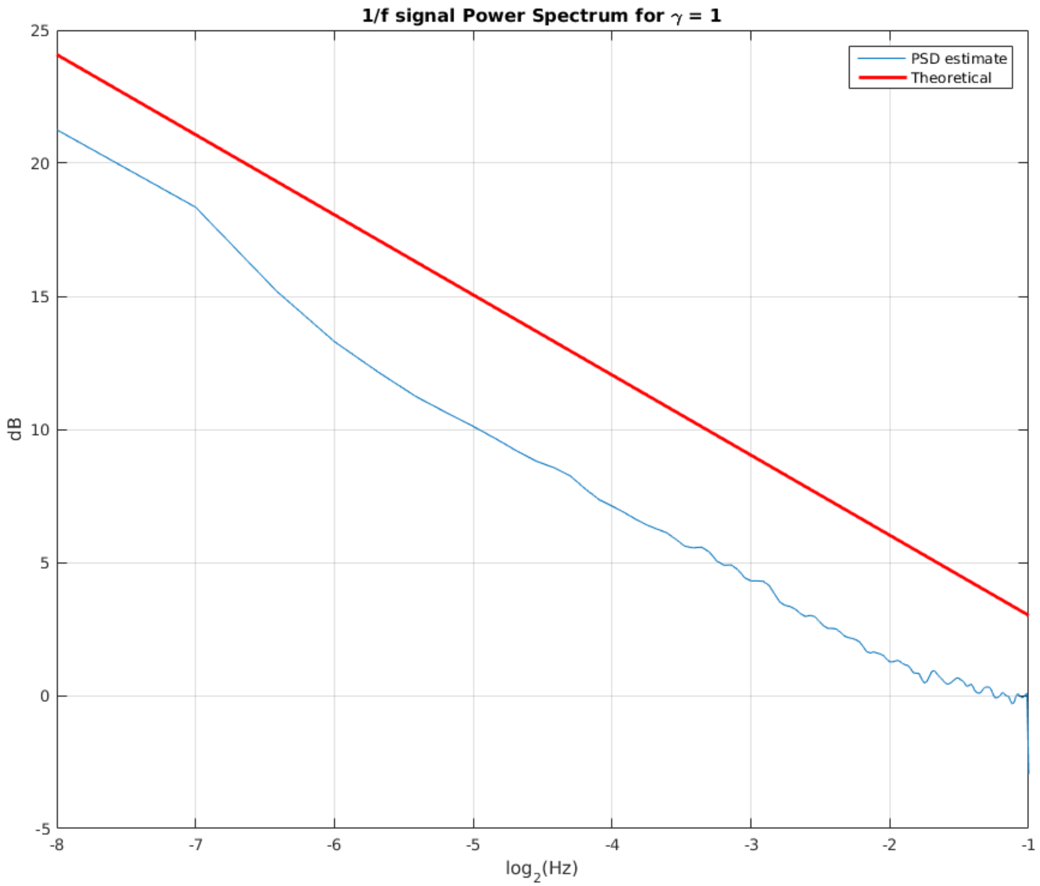
This paper shows a Robust, Computationally efficient and consistent Iterative algorithm to estimate **Fractality Parameters (H),** and variance estimates of **1/f processes** using wavelet expansion.

How this fits into our project?

- Understanding the wavelet domain analysis of fractals through implementation.
- Effect of choice of wavelet basis on estimates of H

Scope for innovation:

- Researching the possibility of extending the framework of this paper to f-ARIMA processes



$$\frac{k_1}{|\omega|^\gamma} \leq S(\omega) \leq \frac{k_2}{|\omega|^\gamma}$$

$$\mathrm{var}\, x_n^m = \sigma^2 2^{-\gamma m}$$

Estimation of Fractal Signals from Noisy Measurements using Wavelets (G.Wornell and A. Oppenheim)

Methodology in Implementation:

1. Generate a 1/f process x(t), with known parameters (γ, N)
2. Establish signal parameters (β , σ^2) ; $\beta = 2^\gamma$
3. Add Gaussian White Noise of known SNR
   $$r(t) = x(t) + w(t)$$
4. Obtain wavelet coefficients for noisy signal (r_n^m)
5. Feed the variance of coefficients (σ^2_m) of noisy signal in the E-M algorithm
6. Obtain estimated signal parameters
   $$\Theta = [\beta_{ML}, \sigma^2_{ML}, \sigma^2_{w,\ ML}]$$
7. Establish effect on estimation error with input parameters:
   a) Resolution
   b) Choice of Wavelet basis
   c) SNR
8. Reconstruct the original signal using estimated parameters.

γβσ

E-M algorithm:

Estimate step: To find out the Maximum Likelihood function for required parameters. Likelihood function:

$$L(\Theta) = -\frac{1}{2}\sum_{m\in\mathfrak{M}} N(m)\left\{\frac{\hat{\sigma}_m^2}{\sigma_m^2} + \ln\left(2\pi\sigma_m^2\right)\right\}$$

$$\Theta = (\beta, \sigma^2, \sigma_w^2)$$

Maximize step: Maximize said function (using calculus) independently for each parameter

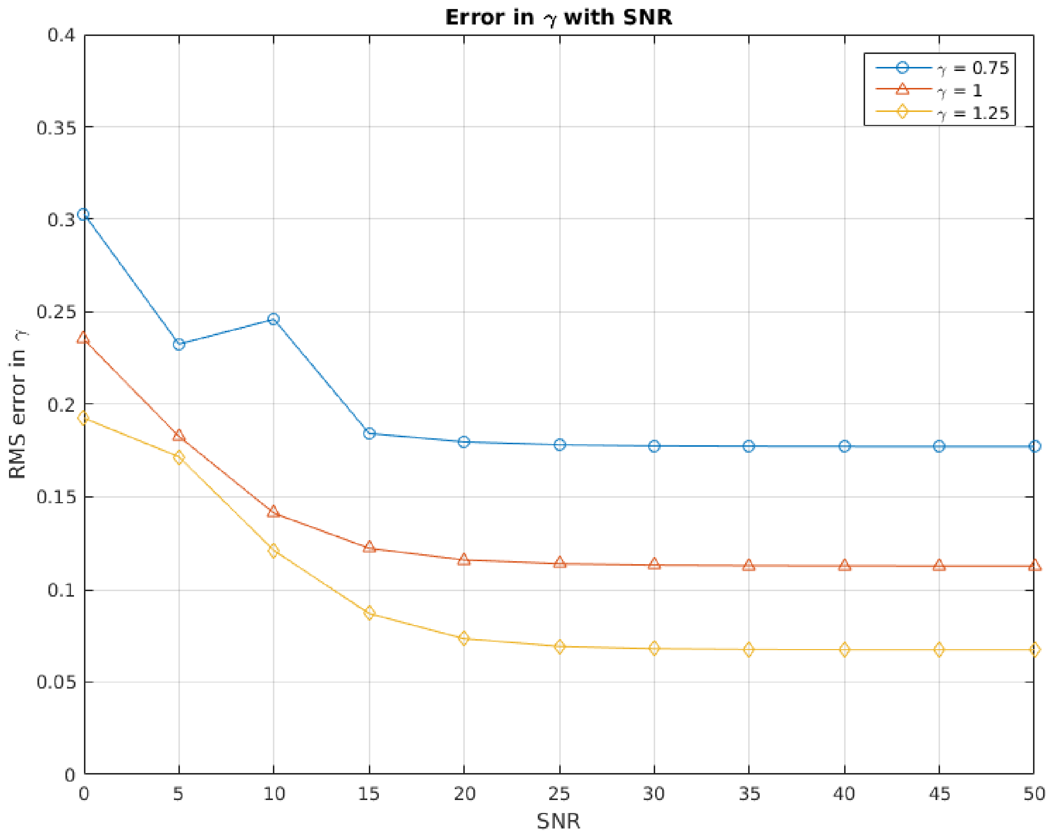
$$\hat{\beta}^{(l+1)} \leftarrow \sum_{m\in\mathfrak{M}} C_m N(m) S_m^x(\hat{\Theta}^{(l)})\beta^m = 0$$

$$\hat{\sigma}^{2(l+1)} = \frac{\sum_{m\in\mathfrak{M}} N(m) S_m^x(\hat{\Theta}^{(l)})\,[\hat{\beta}^{(l+1)}]^m}{\sum_{m\in\mathfrak{M}} N(m)}$$

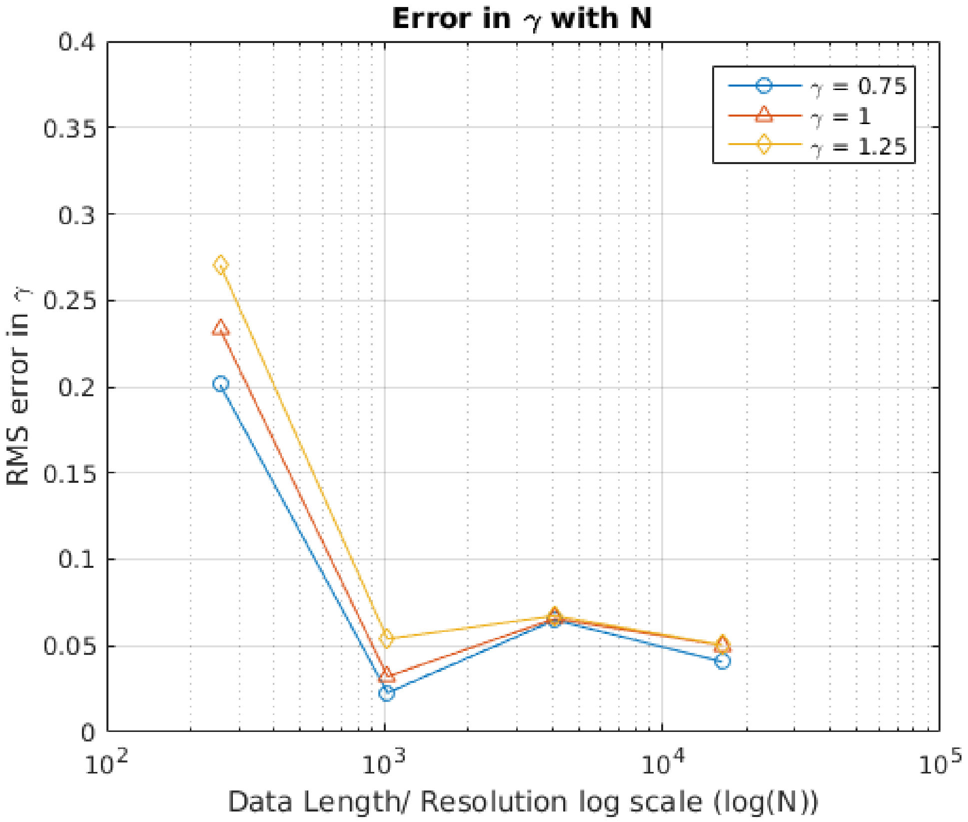
$$\hat{\sigma}_w^{2(l+1)} = \frac{\sum_{m\in\mathfrak{M}} N(m) S_m^w(\hat{\Theta}^{(l)})}{\sum_{m\in\mathfrak{M}} N(m)}$$

$$C_m \triangleq \frac{m}{\sum_{m\in\mathfrak{M}} mN(m)} - \frac{1}{\sum_{m\in\mathfrak{M}} N(m)}.$$

Estimation of Fractal Signals from Noisy Measurements using Wavelets (G.Wornell and A. Oppenheim)

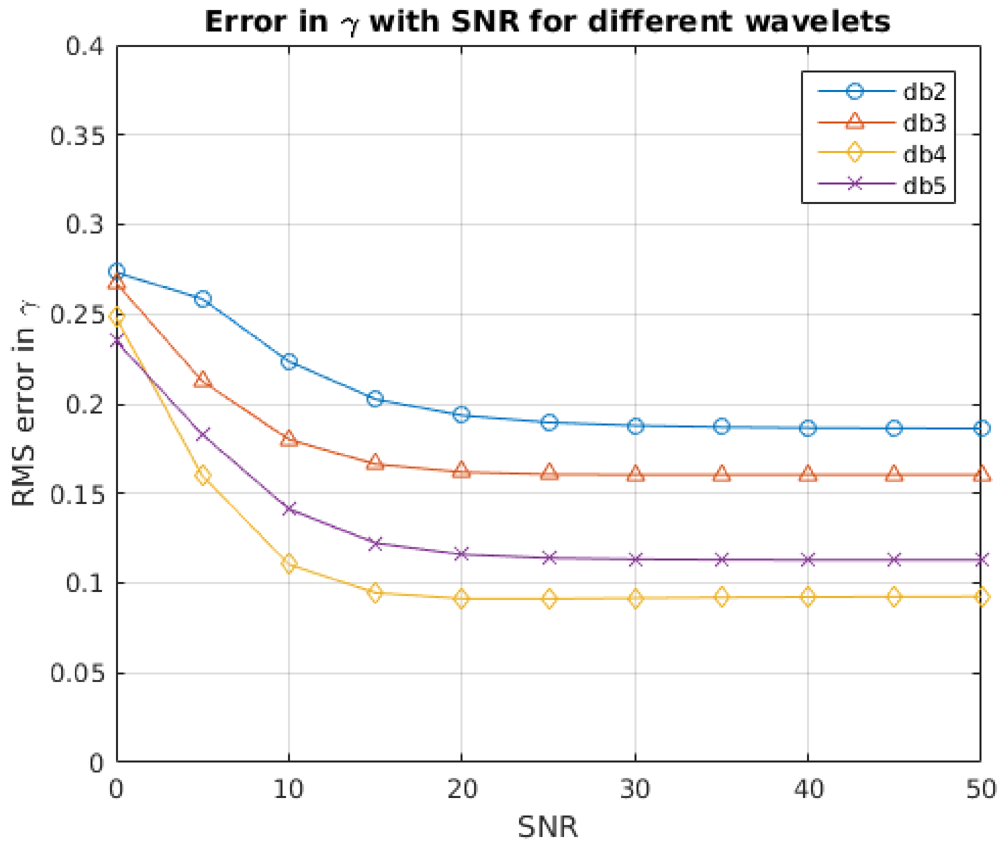


Plot for variation of absolute error in γ with SNR for N = 2048

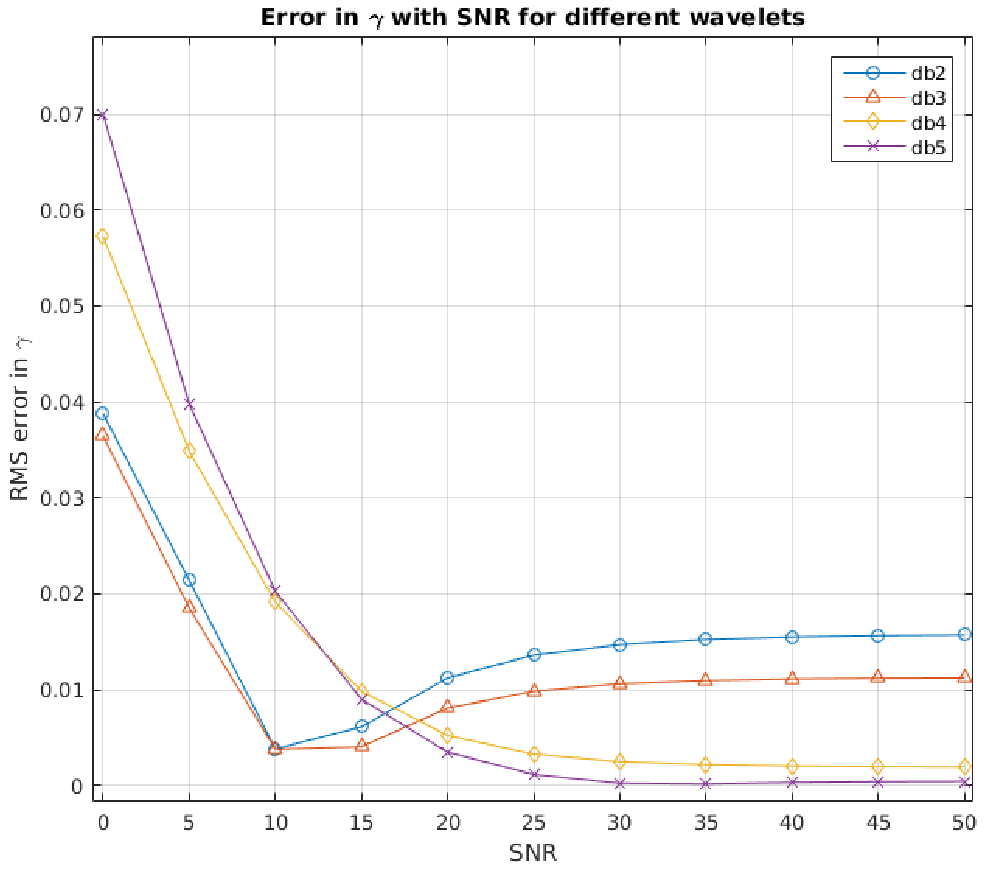


Plot for variation of absolute error in γ with N for SNR = 20

Estimation of Fractal Signals from Noisy Measurements using Wavelets (G.Wornell and A. Oppenheim)

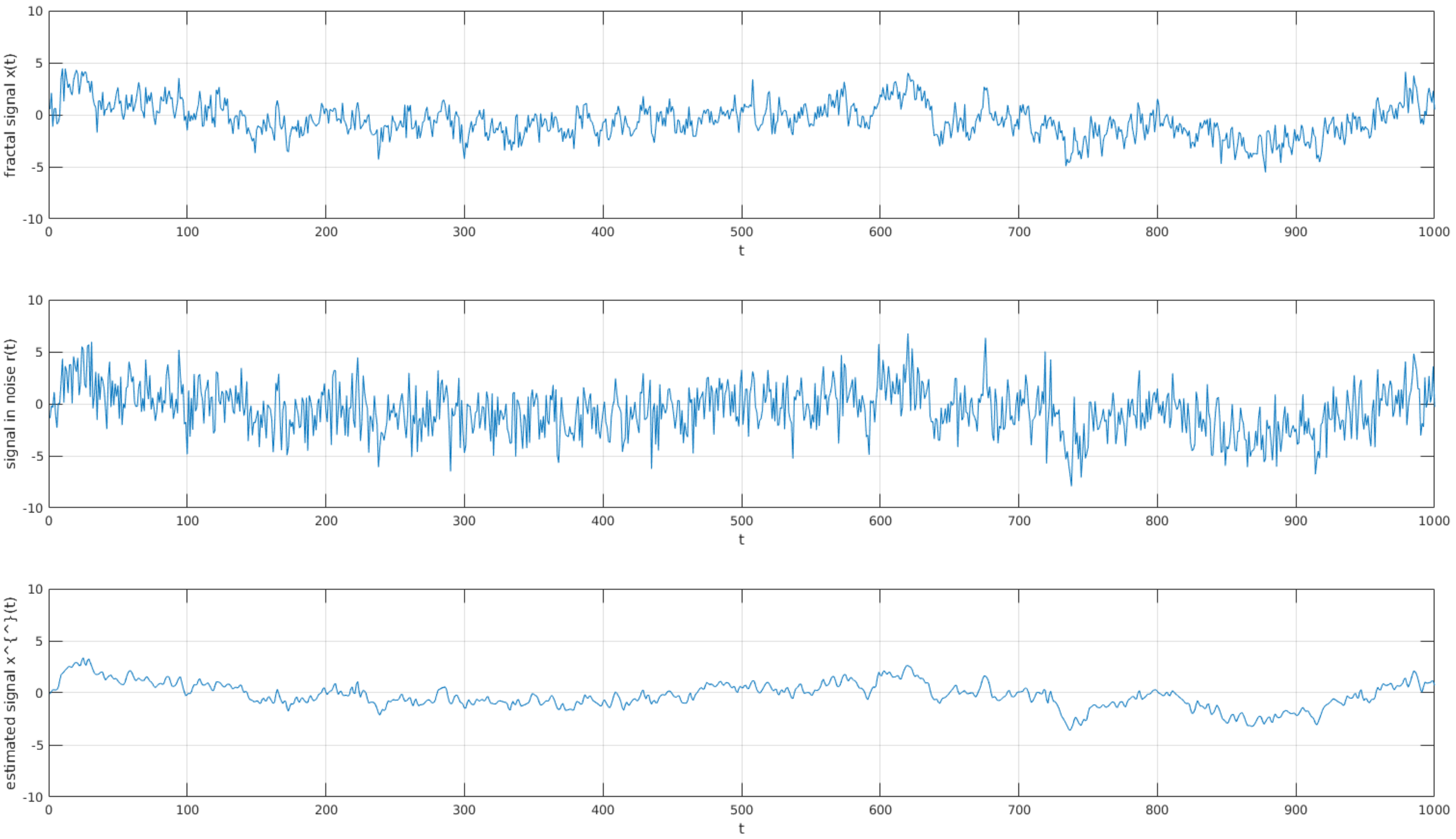


Plot for variation of absolute error in γ for different Daubechies family wavelet basis with SNR
(N = 2048)



Plot for variation of absolute error in γ for different Daubechies family wavelet basis with SNR
(N = 65536)

Estimation of Fractal Signals from Noisy Measurements using Wavelets (G.Wornell and A. Oppenheim)



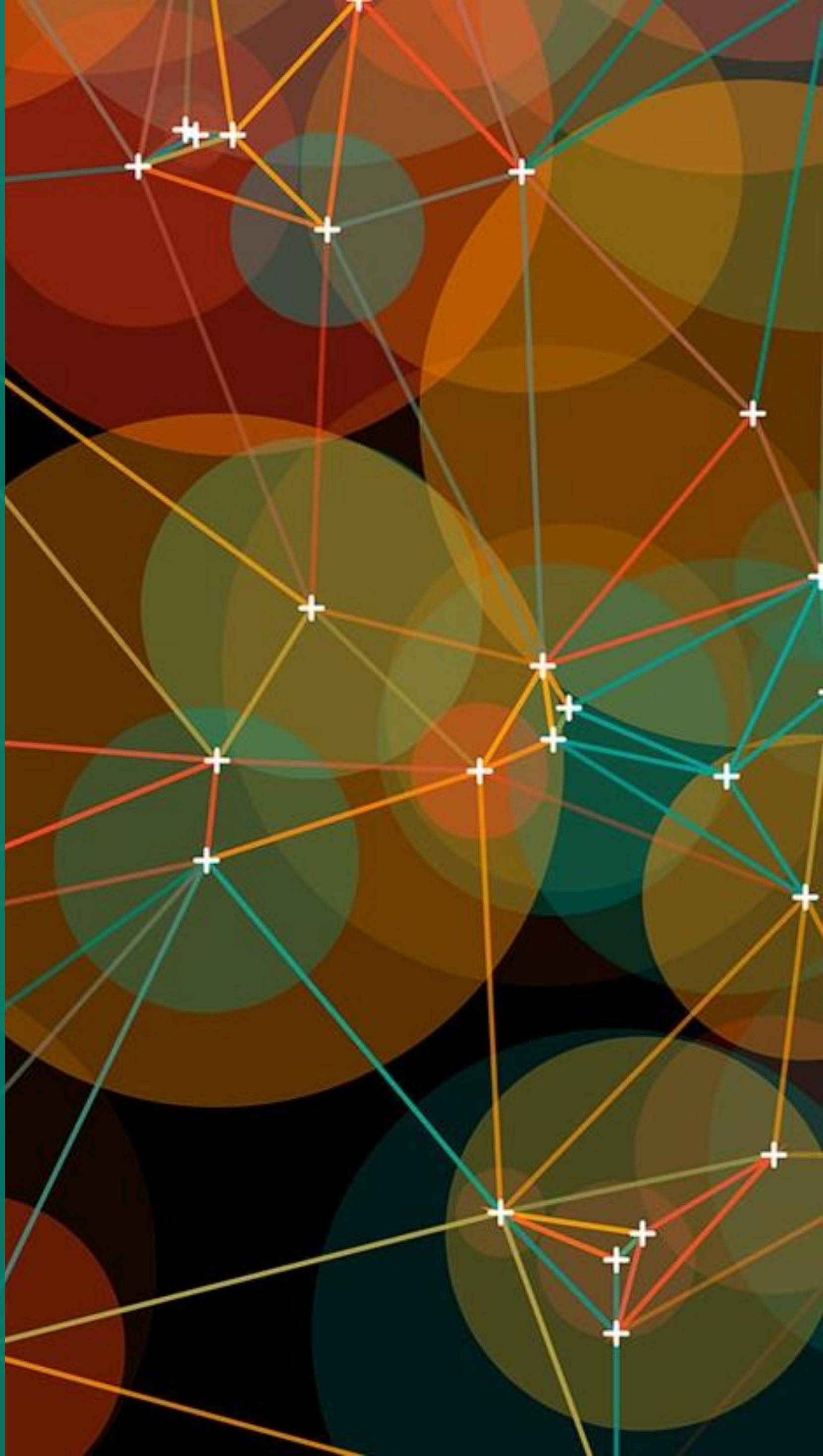
Signal Reconstruction for a signal with $\gamma = 1$, $N = 65536$ and SNR = 0 dB

Applications

- Finance
  - Fractional Black Scholes model
  - Volatility modelling
- Long Range Dependence in DNA
- Solid State Physics
  - Band Gaps due to Bloch like eigen states

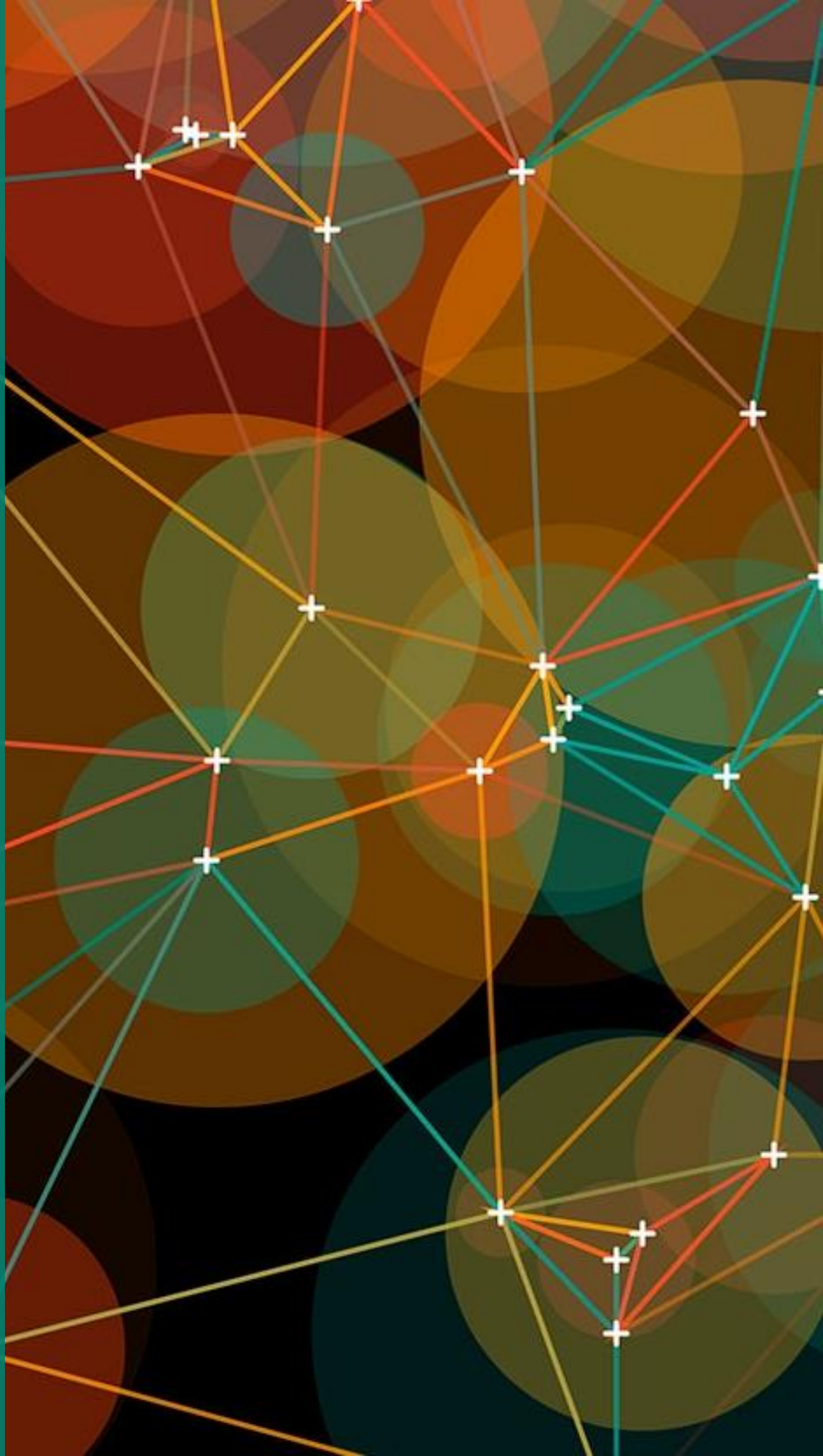
Future Work Possible

- Completing the theoretical analysis of the proposed method
- Characterizing the obtained eigenspectrum completely, not just the leading eigenvalue
- Utilizing different wavelets and extending the analysis to wave packets
- Extending the analysis using wavelet leaders to multi fractal signals



Key Takeaways !

- Self-Similar signals and their properties
- Incremental Analysis of non-stationary processes like FBM helps in scaling up the algorithms to large sample sizes
- Interpretation of the leading eigenvalue of a Autocovariance matrix
- Bayesian Estimation for Hurst parameter of 1/f processes
- Avenues of interest associated with this



Questions ?

Self Similar signals be like