visible degradation at a compression factor of 10. The artifacts introduced by our multiresolution coding scheme are nevertheless less annoying than those introduced by the DCT for higher compression ratios: this can be observed in Fig. 9 for a compression factor of 40. This is essentially due to the fact that a MT is better tuned to human vision than the DCT.

It is to be noticed that, for a low-correlated image like LENA which contains various types of stimuli, there is no significant difference between the use of a fixed filter bank designed by the ring algorithm on the basis of typical covariance lags and the use of a signal-adapted filter bank. A typical fixed filter bank is given by the 4 cell lattice whose successive angles are (in radians): 1.144826, -0.536006, 0.249848, -0.07327.

VI. Conclusion

We have shown that the optimization of the filter banks in a MT may lead to substantial gains (e.g., a factor of two) in the coding gain. The coding gain is an objective measure, involving the SNR of the reconstructed image. However, the ultimate quality of an image coder is influenced by other aspects than simply the SNR. For instance, specific artifacts introduced by a coding scheme contribute significantly to its subjective quality. We have shown that the typical ringing artifacts in a MT can be bounded since low order FIR filters usually allow to achieve most of the obtainable coding gain.

Some first principles considerations seem to indicate that the introduction of linear-phase constraints may lead to a reduction of artifacts. We are currently investigating signal-adapted MT's with the linear-phase constraint. Another issue that deserves further investigation is the local adaptation of the filter banks. Due to the nonstationary character of images one may indeed expect that space-varying filter banks may lead to further improved coding gains. There is a limit to this varying character though since at some point the coding of the filters will become predominant. A final issue concerns an investigation of the convergence properties of the ring algorithm or of any alternative optimization algorithm (involving homotopy methods?).

References

[1] M. Antonini, M. Barlaud, P. Mathieu, and I. Daubechies, "Image coding using vector quantization in the wavelet transform domain," in *Proc. Int. Conf. Acoust., Speech Signal Processing*, Albuquerque, NM, 1990, pp. 2297-2300.

[2] T. P. Barnwell, III and M. J. T. Smith, "Filter banks for anlysis-reconstruction systems: A tutorial," in *Proc. IEEE Int. Symp. Circuits Syst.*, New Orleans, LA, 1990, pp. 1999-2003.

[3] P. J. Burt and E. H. Adelson, "The Laplacian pyramid as a compact image code," *IEEE Trans. Commun.*, vol. COM-31, pp. 532-540, 1983.

[4] S. Comes and B. Macq, "Human visual quality criterion," in *Proc. Conf. Visual Commun. Image Processing*, vol. 1360, Lausanne, Switzerland: SPIE, 1990, pp. 2-13.

[5] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Commun. Pure Appl. Math.*, vol. 41, pp. 909-996, 1988.

[6] ——, "The wavelet transform, time-frequency localization and signal analysis," *IEEE Trans. Inform. Theory*, vol. 36, pp. 961-1005, Sept. 1990.

[7] P. Delsarte, B. Macq, and D. T. M. Slock, "Signal-adapted multiresolution transform for image coding," rep. no. M 376, Philips Res. Lab. Belgium, Feb. 1991.

[8] ——, "Efficient multiresolution signal coding via a signal-adapted perfect reconstruction filter pyramid," in *Proc. Int. Conf. Acoust. Speech Signal Processing*, Toronto, Canada, 1991.

[9] V. K. Jain and R. E. Crochiere, "Quadrarure mirror filter design in the time domain," *IEEE Trans. Acoust. Speech Signal Processing*, vol. ASSP-32, pp. 353-361, 1984.

[10] N. S. Jayant and P. Noll, *Digital Coding of Waveforms.* Englewood Cliffs, NJ: Prentice-Hall, 1984.

[11] G. Karlsson and M. Vetterli, "Extension of finite length signals for sub-band coding," *Signal Processing*, vol. 17, pp. 161-168, 1989.

[12] B. Macq, "Perceptual transforms and universal entropy coding for an integrated approach to picture coding," doctoral thesis, Univ. Louvain, 1989.

[13] S. G. Mallat, "A theory for multiresolution signal decomposition: The wavelet representation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 11, pp. 674-693, July 1989.

[14] ——, "Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$," *Trans. Amer. Math. Soc.*, vol. 315, pp. 69-87, 1989.

[15] ——, "Multifrequency channel decompositions of images and wavelet models," *IEEE Trans. Acoust. Speech Signal Processing*, vol. 37, pp. 2091-2110, 1989.

[16] J. B. O. S. Martens and G. M. M. Majoor, "The perceptual relevance of scale-space image coding," *Signal Processing*, vol. 17, pp. 353-364, 1989.

[17] Y. Meyer, *Ondelettes.* Paris, France: Hermann, 1990.

[18] V. P. Potapov, "The multiplicative structure of *J*-contractive matrix functions," in *American Mathematical Society Translations, Series 2, Vol. 15.* Providence, RI: Amer. Math. Soc., pp. 131-243, 1960.

[19] P. P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and applications: A tutorial," *Proc. IEEE*, vol. 78, pp. 56-93, 1990.

[20] P. P. Vaidyanathan and P.-Q. Hoang, "Lattice structures for optimal design and robust implementation of two-channel perfect-reconstruction QMF banks," *IEEE Trans. Acoust. Speech Signal Processing*, vol. 36, pp. 81-94, 1988.

Correlation Structure of the Discrete Wavelet Coefficients of Fractional Brownian Motion

A. H. Tewfik, *Member, IEEE*, and M. Kim

Abstract—**It is shown that the discrete wavelet coefficients of fractional Brownian motion at different scales are correlated and that their auto and cross-correlation functions decay hyperbolically fast at a rate much faster than that of the autocorrelation of the fractional Brownian motion itself. The rate of decay of the correlation function in the wavelet domain is primarily determined by the number of vanishing moments of the analyzing wavelet.**

Index Terms—**Wavelets, fractional Brownian motion, stochastic processes, multiscale analysis.**

I. Introduction

Fractional Brownian motion (fBm) [1] is a generalization of the usual Brownian motion. It was introduced to model processes that have long memory and/or a statistical self-similarity property. Although it is not stationary, its increments are *stationary* and *self-similar*. Its sample paths are fractal with probability one [2], i.e., the graph of fBm has a Hausdorff-Besicovitch dimension that is larger than its topological dimension with probability one. Fractional Brownian motion has been used in image generation and interpolation, texture classification and the modeling of burst errors in communication channels, $1/f$ noise in oscillators and current noise in metal films and semiconductor devices.

Manuscript received November 8, 1990; revised September 5, 1991. This work was supported in part by the National Science Foundation under Grant MIP-9010111.

The authors are with the Department of Electrical Engineering, University of Minnesota, Room 4-174 EE/CSci Building, Minneapolis, MN 55455.

IEEE Log Number 9105308.

0018-9448/92$03.00

Since fBm is nonstationary, it is best studied using time-frequency signal decomposition techniques. Among those techniques, wavelet decompositions [3]–[6] are particularly well suited to analyze fBm and study its statistical self-similarity since they lead to a multiresolution analysis of the underlying signal. Studies of the correlation structure of the continuous wavelet transform of fBm was accomplished in [7]–[8]. Here, we concentrate on stochastic *discrete wavelet transforms* of fBm. We show that, as with continuous wavelet transforms, the discrete wavelet coefficients corresponding to fBm are correlated from scale to scale. More importantly, we prove that the auto and cross-correlation functions of the discrete wavelet coefficients of fBm at different scales decay hyperbolically fast at a rate much faster than that of the fBm itself and that the rate of decay of the correlation function of the wavelet coefficients is primarily determined by the number of vanishing moments of the analyzing wavelet. This result indicates that if one wants to use a wavelet basis to provide an approximate Karhunen–Loeve expansion for fBm as in [9], it is highly advantageous to pick a wavelet with a large number of vanishing moments since such a wavelet will lead to a maximum decorrelation between the resulting wavelet coefficients.

It should also be pointed out that the results presented here are similar to those derived in [10] and used in [11] for the kernels of operators of Calderon–Zygmund type even though the covariance function of fBm need not have the decay properties of such kernels.

II. Fractional Brownian Motion

Fractional Brownian motion (fBm), $B_H(t)$, is a nonstationary zero-mean Gaussian random function. It is defined as follows [1]:

$$B_H(0) = 0$$

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+0.5)}\left\{\int_{-\infty}^{0}\left[(t-s)^{H-0.5} - (-s)^{H-0.5}\right]dB(s) + \int_0^t (t-s)^{H-0.5}dB(s)\right\}, \quad (2.1)$$

where $B(t)$ is an ordinary Brownian motion and $0 < H < 1$. Note that for $H = 0.5$, fBm is an ordinary Brownian motion. It is easily verified that the covariance function of $B_H(t)$ is given by

$$R_{B_H}(s,t) = E\left[B_H(s)B_H(t)\right] = \frac{V_H}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \quad s, t \in \mathbb{R} \quad (2.2)$$

where

$$V_H = \mathrm{var}\left[B_H(1)\right] = \frac{-\Gamma(2-2H)\cos(\pi H)}{\pi H(2H-1)} = \left[\Gamma(H+0.5)\right]^{-2} \cdot \left\{\int_{-\infty}^{0}\left[(t-s)^{H-0.5} - (-s)^{H-0.5}\right]^2 ds + \frac{1}{2H}\right\}. \quad (2.3)$$

The increments of fBm are zero-mean, stationary and self-similar [1]. However, they are not independent unless $H = 0.5$. In particular, it may be shown that

$$\mathrm{var}\left[B_H(t_2) - B_H(t_1)\right] = V_H(t_2 - t_1)^{2H}, \quad (2.4)$$

where V_H is defined as in (2.3). Equation (2.4) is often used to estimate the value of H from a sequence of observations corresponding to fBm.

III. Discrete Wavelet Analysis

It has recently been shown [3]–[6] that there exists certain functions $\psi(t)$ called "wavelets" such that $\{\sqrt{2^j}\psi(2^jt - m)\}$, $j, m \in \mathbb{Z}$, form an orthonormal basis for $L^2(\mathbb{R})$. A wavelet function $\psi(t)$ is not arbitrary. Its construction is based on the solution $\phi(t)$ of a two scale difference equation (a dilation equation)

$$\phi(t) = \sum_k c_k \phi(2t - k). \quad (3.1)$$

The sequence $\{c_k\}$ is constrained to obey certain relationships. Those relationships guarantee that the wavelet decomposition of any function in $L^2(\mathbb{R})$ is numerically stable and that all translates and dilates of $\psi(t)$ are mutually orthogonal. We shall call any function $\phi(t)$ that obeys (3.1) with a valid coefficient sequence $\{c_k\}$ an "admissible scaling function." The wavelet $\psi(t)$ is then constructed from $\phi(t)$ as

$$\psi(t) = \sum_k (-1)^k c_{1-k}\phi(2t - k). \quad (3.2)$$

When the sequence $\{c_k\}$ is of finite length the resulting wavelet $\psi(t)$ has a finite support [3]. Furthermore, the constraints on the coefficient $\{c_k\}$ imply that $\psi(t)$ has at least one vanishing moment, i.e.,

$$\int_{-\infty}^{\infty} t^m \psi(t)\,dt = 0, \qquad m = 0, 1, \cdots, M-1, \quad (3.3)$$

where $M > 0$. Compactly supported wavelets with M vanishing moments and a support equal to the interval $[-(M-1), M]$ were provided in [3]. Since a wavelet function $\psi(t)$ can be shifted, we will assume here without loss of generality that the compactly supported wavelets to which we shall restrict our attention in this paper have a support given by the interval $[-K_1, K_2]$ where $K_1, K_2 > 0$. The corresponding scaling function $\phi(t)$ will then have the support $[-K_3, K_4]$, $K_3, K_4 > 0$ with $K_1 + K_2 = K_3 + K_4$ [3].

Using the previous construction of $\phi(t)$ and $\psi(t)$ and the properties of the sequence $\{c_k\}$ it can be proved that the span $\{\sqrt{2^J}\phi(2^Jt - m)\}$ = span $\{\sqrt{2^j}\psi(2^jt - m)\}_{j=-\infty}^{J-1}$ [3]–[5]. Now, let $f(t)$ be in $L^2(\mathbb{R})$. Define $b(j;m)$ the discrete "wavelet coefficients" of $f(t)$ at scale 2^j by

$$b(j;m) = \langle f(t), \sqrt{2^j}\psi(2^jt - m)\rangle = \sqrt{2^j}\int_{-\infty}^{\infty} f(t)\psi(2^jt - m)\,dt, \quad (3.4)$$

and $a(j;m)$ by

$$a(j;m) = \langle f(t), \sqrt{2^j}\phi(2^jt - m)\rangle = \sqrt{2^j}\int_{-\infty}^{\infty} f(t)\phi(2^jt - m)\,dt. \quad (3.5)$$

Since for any finite integer J, $\{\sqrt{2^J}\phi(2^J t - m)\} \cup \{\sqrt{2^j}\psi(2^j t - m)\}_{j=J}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$ $f(t)$ can be written as

$$f(t) = \sqrt{2^J} \sum_{m=-\infty}^{\infty} a(J; m)\phi(2^J t - m) + \sum_{j=J}^{\infty} \sum_{m=-\infty}^{\infty} \sqrt{2^j}\, b(j; m)\psi(2^j t - m). \quad (3.6)$$

Equation (3.6) provides a multiresolution decomposition of $f(t)$. Specifically, for fixed j, the sum $\{\sum_m \sqrt{2^j}\, b(j; m)\, \psi(2^j t - m)\}$ is the detail of $f(t)$ at scale 2^j whereas $\{\sum_m \sqrt{2^J}\, a(J; m)\phi(2^J t - m)\}$ yields an approximation to $f(t)$ up to scale 2^J. Wavelet decompositions are easily extended to higher dimensions [4]–[5].

IV. Correlation Structure of the Discrete Wavelet Coefficients of a Fractional Brownian Motion

As mentioned in the Introduction, the correlation structure of the continuous wavelet transform of fBm was studied earlier ([7]–[8]) and discrete orthogonal wavelets have been proposed as approximate Karhunen–Loeve expansions for such processes. Our aim is to investigate how the choice of a discrete orthogonal wavelet with compact support affects the correlation structure of the resulting wavelet coefficients of fBm. In particular, we are interested in relating the number of vanishing moments of a discrete orthogonal compactly supported analyzing wavelet to the correlation structure of the resulting wavelet coefficients of fBm. A by-product of this study will be to extend the results available for continuous wavelet transforms of fBm to the discrete case.

Let $\psi(t)$ be an orthogonal wavelet with compact support $[-K_1, K_2]$, $K_1, K_2 > 0$, and with M vanishing moments. We define the discrete random processes $b(j; m)$ and $a(j; m)$ corresponding to fBm $B_H(t)$ as in (3.4) and (3.5) by replacing $f(t)$ with $B_H(t)$ and letting the integral hold in the mean-square sense. We shall call $b(j; m)$ the (j, m)th wavelet coefficient of fBm. The autocorrelation function of the process $b(j; m)$ is given by

$$\begin{aligned} R_b(j; k; m, n) &= E[b(j; m)b(k; n)] \\ &= \sqrt{2^{j+k}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{B_H}(s, t)\psi(2^j t - m) \cdot \psi(2^k s - n)\, dt\, ds, \\ &= \frac{-V_H}{2}\frac{1}{\sqrt{2^{j+k}}} \int_{-K_1}^{K_2} dt \int_{-K_1}^{K_2} ds \cdot |2^{-j}t - 2^{-k}s + 2^{-j}m - 2^{-k}n|^{2H}\psi(t)\psi(s) \\ &= \frac{-V_H}{2}\frac{2^{-2Hj}}{\sqrt{2^{j+k}}} \int dt \int ds \cdot |t + 2^{j-k}n - m|^{2H}\psi(2^{j-k}s - t)\psi(s). \quad (4.1) \end{aligned}$$

Using the facts that $\psi(t)$ has compact support and $0 < H < 1$, it may be shown that $R_b(j, k; m, n)$ is finite for all j and $k \geq 0$ and all finite m and n. Note that at a single scale (for $j = k$) $R_b(j, k; m, n)$ is a function of $|m - n|$, i.e., $b(j; m)$ is a *stationary* process at fixed j as observed for continuous wavelet transforms in [7].

If the wavelet basis corresponding to any given $\psi(t)$ provides a good approximation to a Karhunen–Loeve expansion for fBm, we would expect $R_b(j, k; m, n)$ to be negligible for $j \neq k$ and $m \neq n$. To get an intuitive feeling for how the regularity of $\psi(t)$ can affect $R_b(j, k; m, n)$ let us define the function $\Lambda(2^{j-k}; t)$ as

$$\Lambda(2^{j-k}; t) = \int ds\psi(2^{j-k}s - t)\psi(s). \quad (4.2)$$

Then $R_b(j, k; m, n)$ can be written as

$$R_b(j, k; m, n) = -\frac{V_H}{2}\frac{2^{-2Hj}}{\sqrt{2^{j+k}}} \int_S dt\, |t + 2^{j-k}n - m|^{2H}\Lambda(2^{j-k}; t). \quad (4.3)$$

Observe that if $\psi(t)$ has M vanishing moments then $\Lambda(2^{j-k}; t)$ will have $2M$ vanishing moments since

$$\begin{aligned} &\int dt\, t^m \int ds\psi(2^{j-k}s - t)\psi(s) \\ &= -\int dt \int ds(2^{j-k}s - t)^m \psi(t)\psi(s) \\ &= -\int dt \int ds \sum_n \binom{m}{n}(2^{j-k}s)^{m-n}(-t)^n\psi(t)\psi(s) \\ &= 0, \qquad \text{for } m < 2M. \end{aligned}$$

Note also that $\Lambda(2^{j-k}; t)$ will have compact support if $\psi(t)$ has compact support. Now consider the integral

$$\int dt f(t + \alpha)\Lambda(2^{j-k}; t)$$

where α is a parameter. If $\Lambda(2^{j-k}; t)$ has $2M$ vanishing moments and $f(t + \alpha)$ is $2M$ times differentiable over the finite support S of $\Lambda(2^{j-k}; t)$ then

$$\left|\int dt f(t + \alpha)\Lambda(2^{j-k}; t)\right| < C \sup_{t \in S}\left|\frac{d^{2M}f(t + \alpha)}{dt^{2M}}\right|,$$

for some finite constant C that depends on $\psi(t)$. Hence, if $|d^{2M}f(t + \alpha)/dt^{2M}|$ decreases rapidly as the parameter α increases then $|\int dt f(t + \alpha)\Lambda(2^{j-k}; t)|$ will also decrease rapidly as α increases.

In the case that is of interest to us here $f(t + \alpha) = |t + \alpha|^{2H}$. This function is *not* differentiable everywhere with respect to t. However, as long as $-\alpha \notin S$ then $|t + \alpha|^{2H}$ will have $2M$ continuous derivatives in the variable t for $t \in S$. Furthermore, its $2M$th derivative with respect to t decreases as $\alpha^{2(H-M)}$ as α tends to infinity. Thus, we should expect that a wavelet $\psi(t)$ with a larger number of vanishing moments will lead to greater decorrelation of the wavelet coefficients of fBm. The above observation is confirmed by the following theorem that is the main result of this correspondence.

Theorem 1: Let $B_H(t)$ be fBm with a parameter H, $0 < H < 1$. Furthermore, let $\psi(t)$ be a discrete orthogonal wavelet with M vanishing moments and supported over the interval $[-K_1, K_2]$, $K_1, K_2 > 0$. Define the stochastic process $b(j; m)$ as

$$b(j; m) = \sqrt{2^j}\int_{-\infty}^{\infty} B_H(t)\psi(2^j t - m)\, dt,$$

where the integral holds in the mean-square sense and where the cross-correlation $R_b(j, k; m, n)$ between $b(j; m)$ and $b(k; n)$ is given by (4.1). Then,

a) for fixed j the 1-D processes $2^{j(H+0.5)} b(j; m)$ are stationary and have the same statistics;
b) for fixed j and k $R_b(j, k; m, n)$ decays as $O(|2^{-j}m - 2^{-k}n|^{2(H-M)})$ for all m, n such that $|2^{-j}m - 2^{-k}n| > \max(2^{-j}K_1 + 2^{-k}K_2, 2^{-k}K_1 + 2^{-j}K_2)$.

Statement a) in Theorem 1 follows directly from (4.1). The proof of Statement b) is given in the Appendix. It is based on the fact that if $|2^{-j}m - 2^{-k}n| > \max(2^{-j}K_1 + 2^{-k}K_2, 2^{-k}K_1 + 2^{-j}K_2)$ then $-(2^{j-k}n - m) \notin S$ and the function $|t + 2^{j-k}n - m|^{2H}$ admits a Taylor series expansion for all $t \in S$, where S is the support of the function $\Lambda(2^{j-k}; t)$ defined in (4.2).

V. Discussion

To illustrate Theorem 1, we computed the correlation matrix of the wavelet coefficients of a 128×1 vector of samples of fBm with $H = 0.2$. In particular, to study the effect of M on the correlation structure of the resulting wavelet coefficients, two wavelet decompositions were computed using wavelets with 4 and 10 vanishing moments.

Following [5] and [11], we considered the given 128×1 vector to be an approximation of the fBm at a highest scale of 128. Note that with a finite set of data, it is not possible to compute all of $a(j; m)$'s and $b(j; m)$'s exactly, particularly those at scale-shift pairs that lead to the support of the correspondingly scaled and shifted wavelet or scaling function straddling the boundary of the interval. As the scale becomes coarser, the number of $a(j; m)$'s and $b(j; m)$'s which can be computed exactly becomes fewer unless we decrease the support of the wavelet. This effectively implies that at coarser scales one is restricted to use wavelets with a smaller number of vanishing moments.

Using the algorithm [5] one can compute a wavelet decomposition of the process from a finite set of data by implicitly assuming a periodic extension of the data. The algorithm of [5] computes 2^{j-1} samples of $a(j-1; m)$ and $b(j-1; m)$ at scale 2^{j-1} from 2^j samples of $a(j; m)$ at scale 2^j by filtering $a(j; m)$ with two filters with impulse responses $\{c_{-k}/\sqrt{2}\}$ and $\{(-1)^k c_{k-1}/\sqrt{2}\}$, where the length of the sequence $\{c_k\}$ is less than or equal to 2^j. Since a sequence $\{c_k\}$ of length at least $2M$ is required to produce a wavelet with M vanishing moments, we were restricted to use a wavelet with M number of vanishing moments such that $2M \leq 2^j$ at scale 2^j, i.e., we used wavelets with a smaller number of vanishing moments at coarser scales.

In the first decomposition that we computed, a wavelet with four vanishing moments was used from scale 128 down to scale 4 whereas a wavelet with ten vanishing moments was used in the second decomposition from scale 128 down to scale 32. At lower resolution scales wavelets with a number of vanishing moments equal to the number of coefficients being computed at that scale were used, e.g., a wavelet with two vanishing moments was used to compute the two wavelet coefficients at scale 2 from the 4 coefficients of the approximation of the 128×1 vector of fBm samples at scale 4. Fig. 1 shows the magnitudes of the entries of the 128×128 autocorrelation matrix of the 128×1 vector of samples of fBm while Figs. 2 and 3 show those of the entries of the autocorrelation matrices corresponding to the wavelet coefficients computed starting with a wavelet with 4 and 10 vanishing moments at high scales, respectively. The rapid hyperbolic decay of the magnitudes of the entries of the autocorrelation matrices corresponding to the computed wavelet coefficients away from the main diagonal in each subblock in those matrices is clear in both Figs. 2 and 3. The scaling behavior of the main diagonal entries of these correlation matrices is also evident.

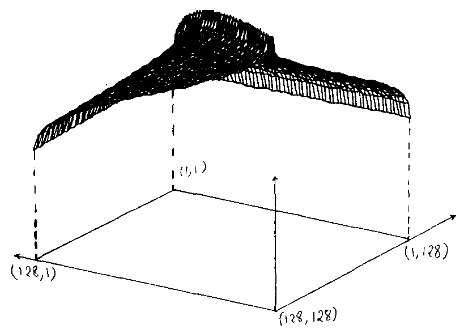

Fig. 1. Magnitudes the entries of the covariance matrix of a fBm with $H = 0.2$.

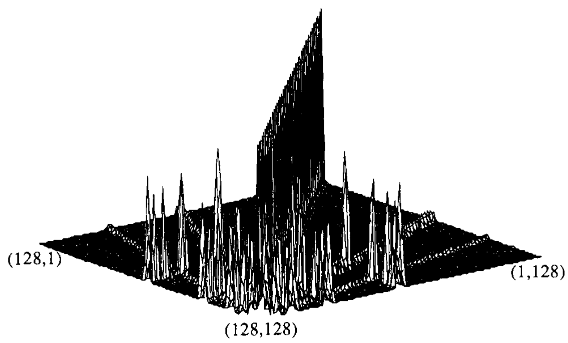

Fig. 2. Magnitudes the entries of the covariance matrix of the discrete wavelet coefficients of a fBm with $H = 0.2$ with $M = 4$ at higher scales.

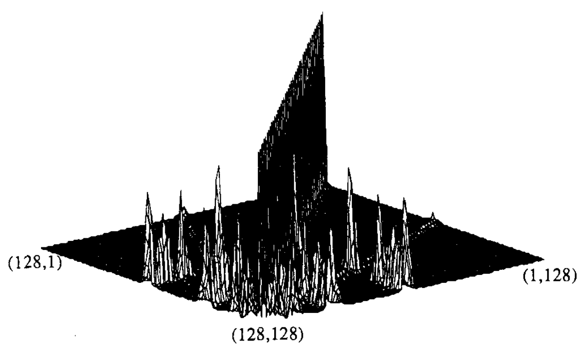

Fig. 3. Magnitudes the entries of the covariance matrix of the discrete wavelet coefficients of a fBm with $H = 0.2$ with $M = 10$ at higher scales.

Observe also the presence of extraneous peaks off the main diagonal in each subblock. These peaks are not predicted by the theory presented in this correspondence and are due primarily to periodic wrap-around effects associated with the algorithm of [5]. The artifacts corresponding to the implicit periodic assumption made by the algorithm of [5] are particularly large at coarser scales (see the discussion in [12]).

More importantly, note that the width of the bands containing entries of nonnegligible magnitude around the main diagonal in each subblock of the autocorrelation matrices of the wavelet coefficients *does not seem to have increased* in Fig. 3 when compared with Fig. 2 even though a wavelet with a larger number of vanishing moments was used. This observation may be surprising at first. The smallest possible support for a wavelet $\psi(t)$ that has M vanishing moments is $2M-1$ [3]. Hence, as M increases we would expect the band in which the decay rate given in Theorem 1 does not hold to increase in width. Fortunately, the effective support of the wavelet $\psi_0(t)$ with smallest compact support and M vanishing moments is much smaller than its total support [3]. Specifically, $\psi_0(t)$ is essentially nonzero over an interval $[-K_1', K_2']$ that is much smaller than $[-K_1, K_2]$ and $|\psi_0(t)| \le \epsilon \|\psi_0(t)\|_\infty$ for some small ϵ and for all $t \in [-K_1, K_1'] U [K_2', K_2]$ (see, for example [3, Fig. 7]). In particular, the effective support of $\psi_0(t)$ increases very slowly with M and hence, the width of the band in which the decay rate of Theorem 1 does not hold also *grows slowly* as M increases provided that $\psi_0(t)$ is used to analyze the fBm.

Second, Theorem 1 indicates that from a computational point of view, processing the wavelet coefficients of a fBm is *more efficient* than processing samples of the fBm itself. In particular, it may be shown that over any finite interval $[-T_1, T_2]$, $T_1, T_2 > 0$, fractional Brownian motion $B_H(t)$ admits the decomposition

$$B_H(t) = \sum_{m=-N_1}^{N_2} a(m)\phi(t-m) + \sum_{j\ge 0} \sum_{m=-M_1(j)}^{M_2(j)} \sqrt{2}\, b(j;m)\psi(2^j t - m), \quad t \in [-T_1, T_2], \tag{5.1}$$

where the equality holds in the mean-square sense and $M_1(j) = \lfloor 2^j T_1 + K_2 \rfloor$,[1] $M_2(j) = \lfloor 2^j T_2 + K_1 \rfloor$, $N_1 = \lfloor T_1 + K_4 \rfloor$ and $N_2 = \lfloor T_2 + K_3 \rfloor$. In the above equation $a(m)$ is defined as the discrete time random process

$$a(m) = \int_{-\infty}^{\infty} dt B_H(t)\phi(t-m), \tag{5.2}$$

where the integral is to be understood in the mean square sense and $b(j;m)$ is given in Theorem 1. Equation (5.1) may be established using standard techniques and the fact that over any finite region in the (s,t) plane $R_{B_H}(s,t)$ has a finite energy and hence, it may be expanded over any finite region $I = \{(s,t): -T_1 \le s, t \le T_2; T_1, T_2 > 0\}$ as

$$\begin{aligned} R_{B_H}(s,t) = &\sum_{n=-N_1}^{N_2} \sum_{m=-N_1}^{N_2} R_a(m,n)\phi(t-m)\phi(s-n) \\ &+ \sum_{n=-N_1}^{N_1} \sum_{j\ge 0} \sum_{m=-M_1(j)}^{M_2(j)} \sqrt{2^j} R_{ab}(j;m,n) \\ &\cdot [\phi(s-n)\psi(2^j t - m) + \phi(t-n)\psi(2^j s - m)] \\ &+ \sum_{j\ge 0} \sum_{m=-M_1(j)}^{M_2(j)} \sum_{k\ge 0} \sum_{m=-M_1(k)}^{M_2(k)} \\ &\sqrt{2^{j+k}} R_b(j,k;m,n)\psi(2^j t - m)\psi(2^k s - n), \\ &-T_1 \le s, t \le T_2 \end{aligned} \tag{5.3}$$

by first computing its wavelet transform in the s variable and then in the t variable over a suitable large finite interval that contains I. In (5.3), $R_a(m,n)$ and $R_{ab}(j;m,n)$ are given by

$$R_a(m,n) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds R_{B_H}(s,t)\phi(t-m)\phi(s-n), \tag{5.4}$$

$$R_{ab}(j;m,n) = \sqrt{2^j} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds R_{B_H}(s,t)\phi(s-n)\psi(2^j t - m), \tag{5.5}$$

and $R_b(j,k;m,n)$ is given by (4.1). Hence, processing fBm over any finite interval is equivalent to processing the processes $a(m)$ and $b(j;m)$. The advantage of processing the processes resulting from an exact wavelet decomposition or those obtained via the algorithm of [5] is that the correlation matrix of the process $b(j;m)$ or its approximation computed via the algorithm of [5] is very *sparse* and has a *nice structure* as demonstrated by Theorem 1 and Figs. 2 and 3. It is then possible to exploit this structure to develop fast algorithms for solving detection and estimation problems involving fBm (see [13] for details.)

VI. Conclusion

In this correspondence, we have shown that the wavelet coefficients of fBm are correlated form scale to scale and that their auto and cross-correlation decay hyperbolically fast at a rate much faster than that of the autocorrelation of the fBm. The rate of decay is determined by the parameter H of fBm and the regularity of the analyzing wavelet. The results of this correspondence can be extended in a straightforward manner to cover 2-D fBm.

Appendix

Proof of Theorem 1b):

Let $\alpha = 2^{j-k}n - m$ and note that we have from (4.1)

$$R_b(j,k;m,n) = -\frac{V_H}{2} \frac{2^{-2Hj}}{\sqrt{2^{j+k}}} \int_S dt\, |t+\alpha|^{2H} \Lambda(2^{j-k};t), \tag{A.1}$$

where Λ is defined as in (4.2) and S is the support of $\Lambda(2^{j-k};t)$. Let $f(t+\alpha) = |t+\alpha|^{2H}$. Clearly, if $|\alpha| > \max(K_1 + 2^{j-k}K_2, 2^{j-k}K_1 + K_2)$, then $-\alpha \notin S$ and $f(t+\alpha)$ is differentiable of all order for all $t \in S$. Write $f(t+\alpha)$ as

$$f(t+\alpha) = |\alpha|^{2H}\left(1 + \frac{t}{\alpha}\right)^{2H}. \tag{A.2}$$

Since $|\alpha| > \max(K_1 + 2^{j-k}K_2, 2^{j-k}K_1 + K_2)$, $|t/\alpha| < 1$ and the Taylor series expansion of $(1 + t/\alpha)^{2H}$ is given by the well

[1] $\lfloor a \rfloor$ denotes the largest integer that is smaller than or equal to a.

known power series expansion of $(1+x)^{\beta}$, $|x|<1$. In particular, we have

$$f(t+\alpha) = |\alpha|^{2H} \cdot \left\{1 + \sum_{p=1}^{\infty} \frac{2H(2H-1)\cdots(2H-p+1)}{p!}\left(\frac{t}{\alpha}\right)^{p}\right\}. \quad (A.3)$$

The power series in (A.3) converges uniformly and absolutely for all $t \in S$. Hence, we can substitute (A.3) into (A.1) and interchange the order of integration and summation to obtain

$$R_b(j,k;m,n) = -\frac{V_H}{2}\frac{2^{-2Hj}}{\sqrt{2^{j+k}}}|\alpha|^{2H} \cdot \left\{\int_S dt\Lambda(2^{j-k};t) + \sum_{p=1}^{\infty}\frac{2H(2H-1)\cdots(2H-p+1)}{p!} \int_S dt\left(\frac{t}{\alpha}\right)^{p}\Lambda(2^{j-k};t)\right\}. \quad (A.4)$$

Since $\psi(t)$ has M vanishing moments the first $2M$ moments of $\Lambda(2^{j-k};t)$ are zero, and it follows that

$$R_b(j,k;m,n) = C_1|\alpha|^{2(H-M)} + R_{2M+1}, \quad (A.5)$$

where

$$C_1 = (-1)^{M+1}\frac{V_H}{2}2^{-(M+0.5)(j+k)}2^{-2j(H-M)} \cdot \frac{2H(2H-1)\cdots(2H-2M+1)}{2M!}\binom{2M}{M} \cdot \left(\int_{-K_1}^{K_2} t^M\psi(t)\,dt\right)^2, \quad (A.6)$$

and

$$R_{2M+1} = \frac{V_H}{2}\frac{2^{-2Hj}}{\sqrt{2^{j+k}}}|\alpha|^{2H} \cdot \sum_{p=2M+1}^{\infty}\frac{2H(2H-1)\cdots(2H-p+1)}{p!} \cdot \int_{-K_1}^{K_2}\int_{-K_1}^{K_2}\left(\frac{2^{j-k}s-t}{\alpha}\right)^{p}\psi(t)\psi(s)\,dt\,ds\Bigg\}. \quad (A.7)$$

Note that since $\psi(t) \in L^1(\mathbb{R})$, C_1 is finite. Observe also that since $M \geq 1$ and $0 < H < 1$,

$$|R_{2M+1}| \leq C_2|\alpha|^{2H}\sum_{i=1}^{\infty}\beta^{2M+i}, \quad (A.8)$$

where

$$C_2 = \frac{V_H}{2}\frac{2^{-2Hj}}{\sqrt{2^{j+k}}}\left|\frac{2H(2H-1)\cdots(2H-2M+1)}{2M!}\right| \cdot \left(\int_{-K_1}^{K_2}|\psi(t)|\,dt\right)^2 \quad (A.9)$$

$$\beta = \sup_{(t,s)\in\Omega}\left|\frac{2^{j-k}s-t}{\alpha}\right| < 1, \quad (A.10)$$

and $\Omega = \{(t,s) \mid -K_1 \leq t,\ s \leq K_2\}$. Since $0<\beta<1$, it then follows that

$$|R_{2M+1}| \leq C_3|\alpha|^{2(H-M)-1}, \quad (A.11)$$

where C_3 is a finite constant. Combining (A.5) and (A.11) we obtain

$$R_b(j,k;m,n) = O\left(|2^{-j}m - 2^{-k}n|^{2(H-M)}\right),$$

for all m, n such that $|2^{-j}m - 2^{-k}n| > \max(2^{-j}K_1 + 2^{-k}K_2,\ 2^{-k}K_1 + 2^{-j}K_2)$.

References

[1] B. B. Mandelbrot and J. W. Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Rev.*, vol. 10, pp. 422-437, 1968.

[2] R. J. Adler, *The Geometry of Random Fields.* New York: Wiley, 1981.

[3] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Comm. Pure and Appl. Math.*, vol. 41, pp. 909-996, 1988.

[4] S. G. Mallat, "A theory for multiresolution signal decomposition: The wavelet representation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 11, pp. 674-693, 1989.

[5] —, "Multifrequency channel decompositions of images and wavelet models," *IEEE Trans. Acoust., Speech, and Signal Processing*, vol. 37, pp. 2091-2110, 1989.

[6] G. Strang, "Wavelets and dilation equations: A brief introduction," *SIAM Rev.*, vol. 31, pp. 614-627, 1989.

[7] P. Flandrin, "On the spectrum of fractional Brownian motions," *IEEE Trans. Inform. Theory*, vol. 35, pp. 197-199, Jan. 1989.

[8] M. Vergassola and U. Frisch, "Scaling exponents for global and local self-similar processes," preprint, Aug. 1990.

[9] G. W. Wornell, "A Karhunen-Loeve-like expansion for $1/f$ processes via wavelets," *IEEE Trans. Inform. Theory*, vol. 36, pp. 859-861, July 1990.

[10] Y. Meyer, "Wavelets and operators," in *Analysis at Urbana*, E. Berkson et al., Eds. Cambridge, UK: Cambridge Univ. Press, 1989.

[11] G. Beylkin, R. Coifman, and V. Rokhlin, "Fast wavelet transforms and numerical algorithms I," *Comm. Pure and Appl. Math.*, vol. 44, pp. 141-183, Mar. 1991.

[12] S. A. Golden, "Identifying multiscale statistical models using the wavelet transform," S. M. thesis, Dept. of Elect. Eng. and Comp. Sci., Mass. Inst. of Technol., Cambridge, MA.

[13] A. H. Tewfik and M. Kim, "Fast multiscale statistical signal processing algorithms," submitted to the *IEEE Trans. Signal Processing*, May 1991; see also "Fast multiscale detection in the presence of fractional Brownian motions," *Proc. SPIE Conf. Advanced Algorithms and Architectures for Signal Processing V*, San Diego, CA, July 1990.