Subiectul 1 a) Se considera multimea A = {1,2,4,8} si B $\subset$ A. Sa se determine toate submultimile B stiind ca numarul de functii de la A la B este egal cu numarul de functii de la B la A.
Rezolvare:

$Fie\ m = card(B).\ Cum\ B \subset A \implies m \leqslant 4$

$Numarul\ de\ functii\ de\ la\ A\ la\ B\ este\ egal\ cu\ m^4$

$$\implies m^4 = 4^m,\ m \leqslant 4 \implies m \in \{2,4\}$$

$Numarul\ de\ functii\ de\ la\ B\ la\ A\ este\ egal\ cu\ 4^m$

$\implies B\ este\ o\ submultime\ a\ lui\ A\ cu\ 2\ sau\ 4\ elemente \implies B\ poate\ fi\ una\ din\ multimile:$
$\{1,2\},\ \{1,4\},\ \{1,8\},\ \{2,4\},\ \{2,8\},\ \{4,8\},\ \{1,2,4,8\}$

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b) Dati exemplu, daca exista, de functie f: [23, 2024] $\rightarrow$ [1,2] $\cup$ [3,4] care este injectiva. Exista o astfel de functie surjectiva? Daca da, dati un astfel de exemplu.
Rezolvare:

$Cum\ [23,2024],\ [1,2] \cup [3,4] \subset \mathbb{R}\ (\ amandoua\ intervale\ ) \implies card([23,2024]) = card([1,2] \cup [3,4]) = card(\mathbb{R}).$ *De aici rezulta ca exista functie bijectiva de la* $[23,2024]$ *la* $[1,2] \cup [3,4]$*, deci exista si functie injectiva si functie surjectiva*

Exemplu de functie injectiva:

$$f: [23,\ 2024] \rightarrow [1,2] \cup [3,4]$$
$$f(x) = \frac{1}{x} + 1$$

Exemplu de functie surjectiva:

$$g: [23,2024] \rightarrow [1,2] \cup [3,4]$$

$$g(x) = \begin{cases} \{x\} + 1,\ x \in [23,24) \\ 2,\ x = 24 \\ \{x\} + 3,\ x \in (24,\ 2023) \\ 4,\ x = 2024 \end{cases}$$

Se poate arata usor ca f este injectiva si g este surjectiva. Orice functie strict monotona care are imaginea in intervalul $[1,2] \cup [3,\ 4]$ *este un exemplu bun, iar pentru surjectiva cel mai usor iti dai exemplu de o functie pe ramuri in care o ramura are imaginea* $[1,2]$ *si cealalta ramura are imaginea* $[3,4]$

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c) Fie a, b $\in \mathbb{R}$ si functia $f: \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = \begin{cases} x^2,\ x \leqslant -2 \\ ax + b,\ x \in (-2,\ 1) \\ -x^2 + 1,\ x \geqslant 1 \end{cases}$$

i) Pentru a = b = 2 calculati $f((-1,5]),\ f^{-1}([0,\ \infty))$

ii) Determinati a,b $\in \mathbb{R}$ astfel incat f sa fie bijectiva

Rezolvare:

i) $f((-1,\ 5]) = f((-1,1) \cup [1,5]) = f((-1,1)) \cup f([1,5]).$ *Deci trebuie sa calculam imaginea functiei pe intervalul* $(-1,\ 1)$ *(adica a doua ramura) si imaginea functiei pe intervalul* $[1,5]$ *(adica a treia ramura)*
$-1 < x < 1 \implies -2 < 2x < 2 \implies 0 < 2x + 2 < 4,\ \forall\, x \in (-1,\ 1) \implies f((-1,\ 1)) = (0,4)$

$1 \leqslant x \leqslant 5 \Longrightarrow 1 \leqslant x^2 \leqslant 25 \Longrightarrow -25 \leqslant -x^2 \leqslant -1 \Longrightarrow -24 \leqslant -x^2+1 \leqslant 0,\ \forall\, x \in \mathbb{R} \Longrightarrow$
$f([1,5]) = [-24,\ 0]$

$\Longrightarrow f((-1,\ 5]) = f((-1,1)) \cup f([1,5]) = (0,4) \cup [-24,\ 0] = [-24,\ 4)$

Pentru a calcula $f^{-1}([0,\ \infty))$ *determinam mai intai imaginea pe fiecare ramura a functiei pentru a stabili cum impartim intervalul* $[0,\ \infty)$ *in functie de imaginile ramurilor.*

$x \leqslant -2 \Longrightarrow x^2 \geqslant 4,\ \forall\, x \in (-\infty,\ -2] \Longrightarrow f((-\infty,\ -2]) = [4,\ \infty)$

$-2 < x < 1 \Longrightarrow -4 < 2x < 2 \Longrightarrow -2 < 2x+2 < 4,\ \forall\, x \in (-2,1) \Longrightarrow f((-2,1)) = (-2,4)$

$x \geqslant 1 \Longrightarrow -x^2 \leqslant -1 \Longrightarrow -x^2+1 \leqslant 0,\ \forall\, x \in [1,\ \infty) \Longrightarrow f([1,\ \infty)) = (-\infty,\ 0]$

Deci ramurile functiei au imaginile, $[4,\infty)$, $(-2,\ 4)$ *respectiv* $(-\infty,\ 0]$. *Atunci:*

$f^{-1}([0,\ \infty)) = f^{-1}(0) \cup f^{-1}((0,4)) \cup f^{-1}([4,\infty))$

Ramura III : $-x^2+1 = 0 \Longrightarrow x^2 = 1 \Longrightarrow x = 1 \Longrightarrow f^{-1}(0) = 1$

Ramura II: $0 < 2x+2 < 4 \Longrightarrow -1 < x < 1 \Longrightarrow f^{-1}((0,4)) = (-1,1)$

Ramura I: $x^2 \geqslant 4 \Longrightarrow x \leqslant -2 \Longrightarrow f^{-1}([4,\ \infty)) = (-\infty,-2]$

$\Longrightarrow f^{-1}([0,\infty)) = \{1\} \cup (-1,\ 1) \cup (-\infty,-2] = (-\infty,-2] \cup (-1,\ 1]$

ii) Cum f este bijectiva $\Longrightarrow$ *f este surjectiva* $\Longrightarrow$ *Im(f)* $= \mathbb{R}$. *Vom pleca de la faptul ca imaginea functiei este* $\mathbb{R}$ *deci reuniunea imaginilor pentru fiecare ramura trebuie sa fie* $\mathbb{R}$. *Din punctul i) stim ca imaginea primei ramuri este* $[4,\infty)$ *iar imaginea celei de a treia ramura este* $(-\infty,\ 0]$. *Cum f este si injectiva* $\Longrightarrow$ *imaginea celei de a doua ramura trebuie sa fie* $(0,\ 4)$. *Deci* $0 < ax+b < 4,\ \forall\, x \in (-2,\ 1)$.
Sunt doua cazuri care trebuie verificate:

Cazul 1: $a \geqslant 0$

$-2 < x < 1 \Longrightarrow -2a < ax < a \Longrightarrow -2a + b < ax+b < a+b,\ \forall\, x \in (-2,1)$

$$\Longrightarrow \begin{cases} -2a+b = 0 \\ a+b = 4 \end{cases} \Longrightarrow 3a = 4 \Longrightarrow a = \frac{4}{3} \geqslant 0 \Longrightarrow b = \frac{8}{3}$$

Cazul 2: $a < 0$

$-2 < x < 1 \Longrightarrow a < ax < -2a \Longrightarrow a+b < ax+b < -2a+b,\ \forall\, x \in (-2,1)$

$$\Longrightarrow \begin{cases} a+b = 0 \\ -2a+b = 4 \end{cases} \Longrightarrow 3a = -4 \Longrightarrow a = -\frac{4}{3} < 0 \Longrightarrow b = \frac{4}{3}$$

Se verifica dupa pentru fiecare caz ca f este si injectiva (fiecare ramura este injectiva in plus ramurile nu au puncte comune)

Subiectul 2

a) Se considera permutarea $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & 8 & 10 & 9 & 7 & 3 & 2 & 5 & 6 & 4 & 1 \end{pmatrix} \in S_{11}$. Descompuneti σ in produs de cicilii disjuncti si in produs de transpozitii. Aflati signatura si ordinul lui σ si apoi calculati σ^{-2024}.

b) Determinati toate permutarile $\tau \in S_9$ cu proprietatea $\tau^3 = (1\,5)(3\,4\,8\,9)$.

c) Determinati cel mai mic numar natural nenul k cu proprietatea ca oricum alegem k transpozitii distincte din S_{2024} , exista printre acestea doua care nu comuta.

Rezolvare:

$a)\ \sigma = (1\,11)(2\,8\,5\,7)(3\,10\,4\,9\,6)\ -\ produs\ de\ ciclii\ disjuncti$

$\sigma = (1\,11)(2\,8)(8\,5)(5\,7)(3\,10)(10\,4)(4\,9)(9\,6)\ -\ produs\ de\ transpozitii \implies 8\ transpozitii$

$sgn(\sigma) = (-1)^8 = 1$

$ord(\sigma) = [2,4,5] = 4 \cdot 5 = 20$ (*ordinul lui* σ *este cel mai mic multiplu comun al ordinilor ciclilor din descompunerea lui* σ *in ciclii*

$\sigma^{-2024} = \sigma^{-4} \cdot \left(\sigma^{20}\right)^{-101} = \sigma^{-4} \cdot e^{-101} = \sigma^{-4}$

$\sigma^{-1} = (11\,1)(7\,5\,8\,2)(6\,9\,4\,10\,3) \implies \sigma^{-4} = (6\,3\,10\,4\,9)$ (*ciclii* $(11\ 1)$ *si* $(7\,5\,8\,2)$ *dispar deoarece au ordinii* 2 *respectiv* 4 *deci ridicati la puterea a* $4-a$ *dau amandoi* e)

$b)$ *Fie* $\tau = \delta_1\delta_2 ... \delta_n\ -$ *descompunerea in n ciclii disjuncti a lui* τ*, avand lungimile* $l_1,\ l_2, ...,\ l_n. \implies$

$\tau^3 = \delta_1^3\delta_2^3 ... \delta_n^3 = (1\,5)(3\,4\,8\,9)$

$$Stim\ ca\ \delta_k^3 = \begin{cases} 3\ ciclii\ de\ lungimile\ \dfrac{l_k}{3}\ daca\ 3|\ l_k \\ 1\ ciclu\ de\ lungime\ l_k\ daca\ 3 \nmid l_k \end{cases} \implies Cum\ 3 \nmid 2\ si\ 3 \nmid 4 \implies$$

Un ciclu de lungime 2 *poate proveni doar dintr – un alt ciclu de lungime* 2 *ridicat la a* $3-a$ *si un ciclu de lungime* 4 *poate proveni doat dintr – un alt ciclu de lungime* 4 *ridicat la a* $3-a$.

Deci τ *are in descompunerea sa un ciclu de lungime* 2 *si un ciclu de lungime* $4 \implies l_1 = 2,\ l_2 = 4$

Insa cum τ^3 *are* 3 *puncte fixe, aceste* 3 *puncte pot proveni dintr – un ciclcu de lungime* 3 *ridicat la* $3-a$ *sau din aceleasi* 3 *puncte fixe ridicate la puterea a* $3-a$*, deci avem* 2 *cazuri* :

Cazul 1 :

$n = 3,\ l_1 = 2,\ l_2 = 4,\ l_3 = 3$

$\implies \delta_1^3 = (1\,5) \implies \delta_1 = (1\,5)$

$\delta_2^3 = (3\,4\,8\,9) \implies \left(\delta_2^3\right)^3 = (3\,4\,8\,9)^3 \implies \delta_2 \cdot \left(\delta_2^4\right)^2 = (3\,9\,8\,4) \implies \delta_2 = (3\,9\,8\,4)$

$\delta_3^3 = (2)(6)(7) \implies \delta_3 = (2\,6\,7)\ sau\ \delta = (2\,7\,6)$

$\implies \tau = (1\,5)(3\,9\,8\,4)(2\,6\,7)\ sau\ \tau = (1\,5)(3\,9\,8\,4)(2\,7\,6)$

Cazul 2 :

$n = 2,\ l_1 = 2,\ l_2 = 4$

$\implies$ (*din cazul* 1) $\tau = (1\,5)(3\,9\,8\,4)$

Deci avem 3 *permutari* τ *care verifica egalitatea* : $(1\,5)(3\,9\,8\,4)(2\,6\,7),\ (1\,5)(3\,9\,8\,4)(2\,7\,6),\ (1\,5)(3\,9\,8\,4)$

c) Stim faptul ca 2 transpozitii disjuncte comuta oricum am alege aceste transpozitii din S_n. Deci pentru a il determina pe k vom determina mai intai cate transpozitii sunt disjuncte din S_{2024}. Acestea sunt :

$(1\ 2),\ (3\ 4),\ (5\ 6),\ldots,\ (2021\ 2022),\ (2023\ 2024)$. *In total* $\dfrac{2023-1}{2}+1 = 1011+1 = 1012.$

Deci orice alta transpozitie am alege din S_{2024} va exista o transpozitie in sirul de mai sus cu care nu va fi disjuncta deci cu care nu va comuta. Deci $k = 1012+1 = 1013$

Subiectul 3

a) Fie functia $f:\ \mathbb{Z}_{23} \to \mathbb{Z}_{23}$ data de $f(\widehat{a}) = \widehat{a}^{21}$. Determinati $f^{-1}(\widehat{1})$ si $\mathrm{Im}(f)$.

b) Determinati ultimele 3 cifre ale lui $23^{2024^{14}}$.

c) Determinati numarul elementelor de ordin 4 din grupul produs direct $(\mathbb{Z}_{10} \times \mathbb{Z}_{24},\ +)$.

d) Este grupul $(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8,\ +)$ izomorf cu un subgrup al lui S_{16}? Daca da, precizati un astfel de izomorfism.

Rezolvare:

*a) Pentru a determina $f^{-1}(\widehat{1})$ trebuie sa rezolvam ecuatia $f(\widehat{x}) = \widehat{1}. \implies \widehat{x}^{21} = \widehat{1} \Leftrightarrow x^{21} \equiv 1 \ (mod\ 23).$
Cum 23 este prim $\implies \forall\, k \in \mathbb{Z}^*,\ (k,23) = 1,$ deci si $(x,23) = 1.$ Putem aplica Teorema lui Euler $\implies$
$x^{\phi(23)} \equiv 1\ (mod\ 23) \Leftrightarrow x^{22} \equiv 1\ (mod\ 23) \Leftrightarrow x\cdot x^{21} \equiv 1\ (mod\ 23) \Leftrightarrow x \equiv 1\ (mod\ 23),\ \widehat{x} \in \mathbb{Z}_{23}$
$\implies \widehat{x} = \widehat{1}.$*

Demonstram ca functia este surjectiva.

*Fie $f(\widehat{a}) = \widehat{k} \implies a^{21} \equiv k\ (mod\ 23).$ Asemantor mai sus, din Teorema lui Euler $\implies a \equiv k\ (mod\ 23)$
Cum $(k,23) = 1,\ \forall \widehat{k} \in \mathbb{Z}_{23} \setminus \{\widehat{0}\} \implies \exists \widehat{k^{-1}} \in \mathbb{Z}_{23} \setminus \{\widehat{0}\}$ astfel incat $\widehat{k}\cdot\widehat{k^{-1}} = \widehat{1} \implies k^{-1}a \equiv 1\ (mod\ 23).$
Ecuatia aceasta are solutia unica in $\mathbb{Z}_{23} \setminus \{\widehat{0}\}$ $\left(deoarece\ \ (k^{-1},\ 23) = 1,\ \forall \widehat{k^{-1}} \in \mathbb{Z}_{23}\right)$ deci pentru orice*

$\widehat{k} \in \mathbb{Z}_{23} \setminus \{\widehat{0}\} \implies \exists\, \widehat{a} \in \mathbb{Z}_{23}$ astfel incat $f(\widehat{a}) = \widehat{k}.$ Si cum $f(\widehat{0}) = \widehat{0}^{21} = \widehat{0}, \implies f$ este surjectiva, deci $Im(f) = \mathbb{Z}_{23}.$

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b) Trebuie sa calculam $23^{2024^{14}}(mod\ 1000).\ 1000 = 2^3\cdot 5^3.$ Folosind Lema Chineza a Resturilor, pentru a determina $(mod\ 1000)$ este suficient sa calculam $\left(mod\ 2^3\right)$ si $\left(mod\ 5^3\right)$.

$23^{2024^{14}}\left(mod\ 2^3\right) \equiv (-1)^{2024^{14}}\left(mod\ 2^3\right) \equiv 1\ \left(mod\ 2^3\right)$

$\left(23, 5^3\right) = 1 \implies$ *Din Teorema lui Euler* $\implies 23^{2024^{14}}\left(mod\ 5^3\right) \equiv 23^{2024^{14}\left(mod\ \phi(5^3)\right)}\left(mod\ 5^3\right)$

$\phi\left(5^3\right) = 125\cdot\left(1-\dfrac{1}{5}\right) = 25\cdot 4 = 100$

Calculam $2024^{14}\ (mod\ 100) \equiv 24^{14}(mod\ 100).\ 100 = 2^2\cdot 5^2.$ Folosind Lema Chineza a Resturilor, pentru a determina $(mod\ 100)$ este suficient sa calculam $\left(mod\ 2^2\right)$ si $\left(mod\ 5^2\right)$

$24^{14}\ \left(mod\ 2^2\right) \equiv 0\ \left(mod\ 2^2\right) \implies 24^{14} = 4k$

$24^{14}\ \left(mod\ 5^2\right) \equiv (-1)^{14}\ \left(mod\ 5^2\right) \equiv 1\ \left(mod\ 5^2\right) \implies 24^{14} = 25l+1$

$\implies 4k = 25l+1 \implies 25l \equiv -1\ (mod\ 4) \implies l \equiv 3\ (mod\ 4) \implies l = 4l_1+3 \implies 24^{14} = 25(4l_1+3)+1 =$
$= 100l_1+76 \implies 24^{14}\ (mod\ 100) \equiv 76\ (mod\ 100)$ *(Din L. C. R)* $\implies$

$23^{2024^{14}}(mod\ 5^3) \equiv 23^{76}(mod\ 5^3) \equiv 529^{38}(mod\ 5^3) \equiv 29^{38}(mod\ 5^3) \equiv 91^{19}(mod\ 5^3) \equiv 91\cdot 31^9(mod\ 5^3)$
$= 71\cdot 86^4(mod\ 125) \equiv 71\cdot 21^2(mod\ 125) \equiv 71\cdot 66\ (mod\ 125) \equiv 61\ (mod\ 125) \implies$
$23^{2024^{14}} = 125p + 61$

Din $23^{2024^{14}}(mod\ 2^3) \equiv 1\ (mod\ 2^3) \implies 23^{2024^{14}} = 8q + 1$
$\implies 125p + 61 = 8q + 1 \implies 125p + 60 = 8q \implies 125p \equiv 60\ (mod\ 8) \implies 5p \equiv 4\ (mod\ 8) \implies$
$p \equiv 20\ (mod\ 8) \implies p \equiv 4\ (mod\ 8) \implies p = 8p_1 + 4 \implies 23^{2024^{14}} = 125(8p_1 + 4) + 61 = 1000p_1 + 561 \implies$
(*Din L. C. R*) $23^{2024^{14}}(mod\ 1000) \equiv 561\ (mod\ 1000)$. *Deci ultimele 3 cifre sunt* 561

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c) *Fie* $(\widehat{x},\ \overline{y})$ *un element de ordin* 4 *din* $\mathbb{Z}_{10} \times \mathbb{Z}_{24}$.
$ord((\widehat{x},\ \overline{y})) = [ord(\widehat{x}),\ ord(\overline{y})] = 4 = 2^2$

Din Teorema lui Lagrange: $ord(\widehat{x}) \mid \mathbb{Z}_{10} \implies ord(\widehat{x}) \mid 10 \implies$
$ord(\widehat{x}) \in \{1, 2, 5, 10\}$, *dar* $[ord(\widehat{x}),\ ord(\overline{y})] = 4 = 2^2$
$\implies ord(\widehat{x}) \in \{1, 2\}$

Cazul 1: $ord(\widehat{x}) = 1 \implies ord(\overline{y}) = 4$
$$ord(\widehat{x}) = \frac{10}{(x, 10)} = 1 \implies (x, 10) = 10,\ \widehat{x} \in \mathbb{Z}_{10} \implies \widehat{x} = \widehat{0}$$

$$ord(\overline{y}) = \frac{24}{(y, 24)} = 4 \implies (y, 24) = 6,\ \overline{y} \in \mathbb{Z}_{24} \implies \overline{y} = \overline{6}\ sau\ \overline{y} = \overline{18}$$
$\implies$ *perechile* $(\widehat{0},\ \overline{6}), (\widehat{0},\ \overline{18})$

Cazul 2: $ord(\widehat{x}) = 2 \implies ord(\overline{y}) = 4$
$$ord(\widehat{x}) = \frac{10}{(x, 10)} = 2 \implies (x, 10) = 5,\ \widehat{x} \in \mathbb{Z}_{10} \implies \widehat{x} = \widehat{5}$$

$$ord(\overline{y}) = \frac{24}{(y, 24)} = 4 \implies (y, 24) = 6,\ \overline{y} \in \mathbb{Z}_{24} \implies \overline{y} = \overline{6}\ sau\ \overline{y} = \overline{18}$$
$\implies$ *perechile* $(\widehat{5},\ \overline{6}), (\widehat{5},\ \overline{18})$

$\implies$ *Avem* 4 *elemente in* $\mathbb{Z}_{10} \times \mathbb{Z}_{24}$ *de ordin* 4.

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d) *Cum grupul* $(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8,\ +)$ *nu este ciclic (un element are ordinul maxim* 8 *si grupul are* 64 *de elemente) ca sa fie izomorf cu un subgrup al lui* S_{16} *trebuie sa gasim un subgrup astfel incat toate elementele sa aiba ordinul* 1 *sau* 2 *sau* 4 *sau* 8. *Practic trebuie sa vedem daca exista in* S_{16} *un subgrup generat de un ciclu de lungime* 2, *de unul de lungime* 4 *si de unul de lungime* 8. *Si intr – adevar daca alegem subgrupul generat de ciclii*: $(1\ 2),\ (3\ 4\ 5\ 6),\ (7\ 8\ 9\ 10\ 11\ 12\ 13\ 14)$ *acesta va fi un grup cu* 64 *de elemente in care fiecare element are ordinul* 1 *sau* 2 *sau* 4 *sau* 8. *Izomorfismul propriu – zis are forma urmatoare*: (*notam cu G subgrupul generat de ciclii anteriori*)
$$\varphi: \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \to G$$
$$\varphi\left(\left(\widehat{x}, \overline{y}, \tilde{z}\right)\right) = (1\ 2)^x(3\ 4\ 5\ 6)^y(7\ 8\ 9\ 10\ 11\ 12\ 13\ 14)^z$$
Se poate arata cu tabelul functiei φ *ca* φ *este izomorfism de grupuri.*

Subiectul 4

a) Fie submultimea I a lui $\mathbb{Q}[X]$ care contine toate polinoamele care dau restul 0 prin impartirea X^2-23.

1) Dati exemplu de un polinom care nu apartine submultimii I si are gradul 4. Demonstrati ca I este un ideal al lui $\mathbb{Q}[X]$.

2) Demonstrati ca inelul factor $\mathbb{Q}[X]/I$ este izomorf cu inelul $\mathbb{Q}\left[\sqrt{23}, +, \cdot\right]$, unde $\mathbb{Q}\left[\sqrt{23}\right] = \left\{a+b\sqrt{23} \mid a,\ b \in \mathbb{Q}\right\}$.

3) Determinati $U(\mathbb{Q}[X]/I)$, elementele inversabile ale inelului factor $\mathbb{Q}[X]/I$.

b) Fie polinomul $P(X) = X^{444}+X^{333}+X^{222}+X^{111}+1 \in \mathbb{C}[X]$. Aratati ca restul impartirii lui $P(X)$ la polinomul $X^4+X^3+X^2+X+1$ este 0.

Rezolvare:

1) *Fie* $P(X) \in I \implies$ *Restul impartirii lui* $P(X)$ *la* X^2-23 *este* $0 \implies P(X) = (X^2-23)G(X)$, *unde* $G(X) \in \mathbb{Q}[X]$

Un exemplu de polinom care nu apartine submultimii I si are gradul 4 este $(X^2-23)(X^2+1)+1$

Demonstram ca I este ideal al lui $\mathbb{Q}[X]$:

Fie $P(X), G(X) \in I \Leftrightarrow P(X) = (X^2-23)p(X),\ G(X) = (X^2-23)g(X), p(X), g(X) \in \mathbb{Q}[X] \implies$ $P(X)-G(X) = (X^2-23)(p(X)-g(X)) \in I.$ *Cum* $P(X)$ *si* $G(X)$ *au fost alese aleator* $\implies (I,+)$ *subgrup a lui* $(\mathbb{Q}[X], +)$

Fie $P(X) \in I$ *si* $G(X) \in \mathbb{Q}[X] \implies P(X)\cdot G(X) = (X^2-23)p(X)G(X) = (X^2-23)(p(X)G(X)) \in I.$ *Cum* $P(X)$ *si* $G(X)$ *au fost alese aleator* $\implies \forall P(X) \in I$ *si* $\forall G(X) \in \mathbb{Q}[X] \implies P(X)G(X) \in I$

$\implies$ *I este ideal, mai precis* $I = (X^2-23)$

2) *Pentru a demonstra izomorfismul vom folosi Teorema Fundamentala de Izomorfism (de la Inele). Trebuie sa ne construim o functie* $\varphi : \mathbb{Q}[X] \to \mathbb{Q}\left[\sqrt{23}\right]$ *care sa fie morfism de inele surjectiv cu nucleul* $Ker\ \varphi = I.$

Fie $\varphi : \mathbb{Q}[X] \to \mathbb{Q}\left[\sqrt{23}\right],\ \varphi(P(X)) = P\left(\sqrt{23}\right)$

Pasul 1: φ *este morfism de inele*

$\varphi(P(X)+G(X)) = \varphi(H(X)) = H\left(\sqrt{23}\right) = P\left(\sqrt{23}\right)+G\left(\sqrt{23}\right) = \varphi(P(X))+\varphi(G(X)),\ \forall P(X), G(X) \in \mathbb{Q}[X]$

$\varphi(P(X)\cdot G(X)) = \varphi(H(X)) = H\left(\sqrt{23}\right) = P\left(\sqrt{23}\right)\cdot G\left(\sqrt{23}\right) = \varphi(P(X))\cdot\varphi(G(X)),\ \forall P(X), G(X) \in \mathbb{Q}[X]$

$\varphi(0) = 0$

Din toate 3 rezulta ca φ *este morfism de inele*

Pasul 2: $Im(\varphi) = \mathbb{Q}\left[\sqrt{23}\right]$

$\varphi(bX+a) = a+b\sqrt{23},\ \forall a,b \in \mathbb{Q} \implies Im(\varphi) = \mathbb{Q}\left[\sqrt{23}\right]$

Pasul 3: *Demonstram ca* $Ker(\varphi) = I$ *printr-o dubla incluziune*

"$\supseteq$". *Fie* $P(X) \in I \implies P(X) = (X^2-23)G(X),\ G(X) \in \mathbb{Q}[X] \implies \varphi(P(X)) = \varphi\left((X^2-23)G(X)\right) =$ (*din* φ *morfism*) $= \varphi(X^2-23)\cdot\varphi(G(X)) = (23-23)\cdot\varphi(G(X)) = 0 \implies Ker(\varphi) \supseteq I$

" $\subseteq$ ". *Fie* $P(X) \in Ker(\varphi) \implies \varphi(P(X)) = 0$

Folosind Teorema Impartirii cu Rest $\implies \exists q(X),\ r(X) \in \mathbb{Q}[X]$ *astfel incat* :

$$P(X) = (X^2 - 23)q(X) + r(X),\ \text{unde } rang(r(X)) < 2 \implies r(X) = aX + b,\ a, b \in \mathbb{Q} \implies$$

$$\varphi(P(X)) = \varphi\left((X^2-23)q(X) + r(X)\right) = (\text{din } \varphi \text{ morfism}) = \varphi\left((X^2-23)q(X)\right) + \varphi(aX+b) = \varphi\left(X^2-23\right) \cdot$$

$$\varphi(q(X)) + a\sqrt{23} + b = 0 + a\sqrt{23} + b = a\sqrt{23} + b = 0$$

$$\implies a\sqrt{23} = -b \implies \text{Pentru } a \neq 0,\ \sqrt{23} = \frac{-b}{a} \in \mathbb{Q} - \text{fals} \implies a = 0 \implies b = 0 \implies r(X) = 0 \implies$$

$$P(X) = (X^2-23)q(X) \implies Ker(\varphi) \subseteq I$$

Din dubla incluziune $\implies Ker(\varphi) = I.$

Din cei 3 pasi de mai sus $\implies$ (*conform Teoremei Fundamentale de Izomorfism*) $\mathbb{Q}[X]/I \approx \mathbb{Q}\left[\sqrt{23}\right].$

3) $\mathbb{Q}[X]/I = \{\widehat{aX+b} \mid a, b \in \mathbb{Q}\}$

$$\widehat{aX+b} \in U(\mathbb{Q}[X]/I) \Leftrightarrow \exists\, \widehat{cX+d},\ c,d \in \mathbb{Q} \text{ astfel incat } (\widehat{aX+b})(\widehat{cX+d}) = \hat{1} \Leftrightarrow \overline{acX^2 + adX + bcX + bd} = \hat{1}$$

$$\Leftrightarrow \overline{(ad+bc)X} + \widehat{bd} = \hat{1} \Leftrightarrow ad + bc = 0 \text{ si } bd = 1$$

$$\text{Din } bd = 1 \implies b \neq 0 \text{ si } d = \frac{1}{b} \in \mathbb{Q} \implies c = -\frac{a}{b^2} \in \mathbb{Q}$$

$$\implies (\widehat{aX+b})\left(\widehat{-\frac{a}{b^2}X + \frac{1}{b}}\right) = \overline{-\frac{a^2}{b^2}X^2 + \frac{a}{b}X - \frac{a}{b}X + 1} = \hat{1}$$

$$\implies U(\mathbb{Q}[X]/I) = \left\{\widehat{aX+b} \mid a \in \mathbb{Q} \text{ si } b \in \mathbb{Q}^*\right\}$$

b) Pornim de la ecuatia $X^5 = 1 \Leftrightarrow X^5 - 1 = 0 \Leftrightarrow (X-1)\left(X^4 + X^3 + X^2 + X + 1\right) = 0.$ *Deci radacinile polinomului* $X^4 + X^3 + X^2 + X + 1$ *in* $\mathbb{C}[X]$ *sunt toate radacinile ecuatiei* $X^5 = 1$ *diferite de* 1.

Aceste radacini sunt chiar $\epsilon,\ \epsilon^2, \epsilon^3, \epsilon^4$, *unde* $\epsilon = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}.$

$$\text{Demonstram ca } P(\epsilon) = P(\epsilon^2) = P(\epsilon^3) = P(\epsilon^4) = 0$$

Trebuie sa precizam inainte cateva aspecte. Cum $\epsilon,\ \epsilon^2, \epsilon^3, \epsilon^4$ *sunt si radacini a ecuatiei* $X^5 = 1 \implies \epsilon^5 = 1.$ *si* $\epsilon,\ \epsilon^2, \epsilon^3, \epsilon^4$ *radacini ai polinomului* $X^4 + X^3 + X^2 + X + 1 \implies \epsilon^4 + \epsilon^3 + \epsilon^2 + \epsilon + 1 = 0.$ *Vom folosi aceste egalitati pentru a demonstra* $P(\epsilon) = P(\epsilon^2) = P(\epsilon^3) = P(\epsilon^4) = 0.$

$$P(\epsilon) = \epsilon^{444} + \epsilon^{333} + \epsilon^{222} + \epsilon^{111} + 1 = (\epsilon^5)^{88} \cdot \epsilon^4 + (\epsilon^5)^{66} \cdot \epsilon^3 + (\epsilon^5)^{44} \cdot \epsilon^2 + (\epsilon^5)^{22} \cdot \epsilon + 1 = \epsilon^4 + \epsilon^3 + \epsilon^2 + \epsilon + 1 = 0$$

$$P(\epsilon^2) = \epsilon^{888} + \epsilon^{666} + \epsilon^{444} + \epsilon^{222} + 1 = \epsilon^3 + \epsilon + \epsilon^4 + \epsilon^2 + 1 = 0$$

$$P(\epsilon^3) = \epsilon^{1332} + \epsilon^{999} + \epsilon^{666} + \epsilon^{333} + 1 = \epsilon^2 + \epsilon^4 + \epsilon + \epsilon^3 + 1 = 0$$

$$P(\epsilon^4) = \epsilon^{1776} + \epsilon^{1332} + \epsilon^{888} + \epsilon^{444} + 1 = \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + 1 = 0$$

Deci toate radacinile lui $X^4 + X^3 + X^2 + X + 1$ *sunt si radacini ale lui* $P(X) \implies X^4 + X^3 + X^2 + X + 1 | P(X)$

$\implies$ *Restul impartirii lui* $P(X)$ *la* $X^4 + X^3 + X^2 + X + 1$ *este* 0.