3(5). Considerăm $\varphi: \mathbb{Q}[X] \to \frac{\mathbb{Q}[X]}{(X^2-506)}$,

$\varphi(f) = \widehat{f(2X)}$.

Fie $f, g \in \mathbb{Q}[X]$.

$\varphi(f+g) = \widehat{(f+g)(2x)} = \widehat{f(2x)+g(2x)} = \widehat{f(2x)} + \widehat{g(2x)} = \varphi(f)+\varphi(g)$

$\varphi(fg) = \widehat{(fg)(2x)} = \widehat{f(2x)g(2x)} = \widehat{f(2x)}\,\widehat{g(2x)} = \varphi(f)\varphi(g)$

Evident, $\varphi(1) = \hat{1}$.

Ca urmare, φ e morfism unitar de inele.

Fie $w = \hat{v} \in \frac{\mathbb{Q}[X]}{(X^2-506)} \overset{not}{=} A$.

Atunci $w = \varphi\left(v\left(\frac{1}{2}X\right)\right)$

Ca urmare, φ e surjectivă, deci $\operatorname{Im}\varphi = A$.

Fie $f \in \mathbb{Q}[X]$.

$f \in \ker\varphi \iff \varphi(f) = \hat{0} \iff \widehat{f(2x)} = \hat{0} \iff$

$f(2x) \vdots X^2-506 \iff \exists g \in \mathbb{Q}[X]\ \ f(2x) = (x^2-506)g(x)$

~~$\iff \exists g \in \mathbb{Q}[X]$~~ $f(x) = \left(\left(\frac{x}{2}\right)^2-506\right)g\left(\frac{x}{2}\right) = (x^2-2024)\boxed{\frac{1}{4}g\left(\frac{x}{2}\right)}$

$\iff \exists h \in \mathbb{Q}[X]\ \ f(x) = (x^2-2024)h(x)$ $\mathbb{Q}[X]$

$\iff f \in (x^2-2024)$.

Ca urmare, $\ker\varphi = (x^2-2024)$

Dar $\ker\varphi \trianglelefteq S_4$ ②

$\boxed{T}$ (Cauchy): Dacă G e grup finit, $p \in \mathbb{N}$ e prim și $p \mid |G|$, atunci există $g \in G$ cu $\mathrm{ord}(g) = p$

Dacă $\ker\varphi \ni \{(a,b,c),(a,b)\}$:
Atunci $\forall \sigma \in \ker\varphi$ $\sigma(d) = d$.
Dar $(a,d)\underline{(a,b)}(a,d)^{-1} = (a,d)(a,b)(a,d) = (b,d) \notin \ker\varphi$
Deci $\ker\varphi \ntrianglelefteq S_4$, $\cancel{}$.

Presupunem că S_3 are un subgrup $H \xrightarrow{\sim} \mathbb{Z}_2 \times \mathbb{Z}_2$ (grup factor)
Atunci $4 = |H| \mid |S_3| = 6$, $\cancel{}$.
Rezultă deci că S_3 n-are subgrupuri izomorfe (grupuri factor) cu $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$H \overset{not}{=} \{e, (1,2), (3,4), (1,2)(3,4)\} \leq S_4$ și are toate elementele $\neq e$ de ordin 2, deci $H \xrightarrow{\sim} \mathbb{Z}_2 \times \mathbb{Z}_2$

Orice subgrup al lui $\mathbb{Z}_{48}$ e de forma $\hat{d}\mathbb{Z}_{48}$ $d \mid 48$, deci e ciclic, deci nu e izomorf cu $\mathbb{Z}_2 \times \mathbb{Z}_2$ (care, având doar elemente de ordin ≤ 2, nu e ciclic)
Orice grup factor al lui $\mathbb{Z}_{48}$ e de forma $\dfrac{\mathbb{Z}_{48}}{\hat{d}\mathbb{Z}_{48}} \xrightarrow{\sim} \mathbb{Z}_d$
deci e ciclic . . . . .

$\widehat{3x+2} \cdot \widehat{ax+b} = \hat{1} \iff \widehat{3ax^2 + (2a+3b)x + 2b} = \hat{1} \iff$

$\widehat{3a(x^2-2024)} + \widehat{6072a + (2a+3b)x + 2b} = \hat{1} \iff$

$\widehat{(2a+3b)x + 6072a + 2b} = \hat{1} \iff \begin{cases} 2a+3b = 0 \\ 6072a + 2b = 1 \end{cases}$

$\forall n \in \mathbb{N}^* \ \forall d \in \mathbb{N} \quad d/n \Rightarrow \dfrac{\mathbb{Z}_n}{d\mathbb{Z}_n} \simeq \mathbb{Z}_d$ ↳ grupuri, inele

$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \mathbb{Z}_2 \times \hat{2}\mathbb{Z}_4 \leq \mathbb{Z}_2 \times \mathbb{Z}_4$

$\dfrac{\mathbb{Z}_2 \times \mathbb{Z}_4}{\{0\} \times \hat{2}\mathbb{Z}_4} \simeq \dfrac{\mathbb{Z}_2}{\{0\}} \times \dfrac{\mathbb{Z}_4}{\hat{2}\mathbb{Z}_4} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

$\operatorname{ord}_{G_1 \times G_2}((x_1, x_2)) = [\operatorname{ord}_{G_1}(x_1), \operatorname{ord}_{G_2}(x_2)]$

$\operatorname{ord} \hat{x} = 2 \iff \operatorname{ord}_{48} x \cdot \hat{1} = 2 \iff \dfrac{\operatorname{ord} \hat{1}}{(x, \operatorname{ord} \hat{1})} = 2 \iff \dfrac{48}{(x, \operatorname{ord} \hat{1})} = 2 \iff$

$\iff (x, \overset{48}{\operatorname{ord} \hat{1}}) = 24.$

Dar $x \in \{0, 1, \dots, 47\}$ $\Big\} \iff x = 24.$

$|G| = |H| \cdot |G : H|$; Dacă $H \trianglelefteq G$, $|G : H| = |G/H|$

Presupun că $\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \dfrac{S_4}{H}$ pt un anumit subgrup ⊴ normal H al lui S_4

Considerăm $\varphi : S_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$, $\varphi = \alpha \circ \pi$.

Atunci $\dfrac{S_4}{\ker \varphi} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, $\Big\}$ $S_4 \xrightarrow{\pi} \dfrac{S_4}{H} \xrightarrow[\sim]{\alpha} \mathbb{Z}_2 \times \mathbb{Z}_2$

deci $4 = |\mathbb{Z}_2 \times \mathbb{Z}_2| = \left|\dfrac{S_4}{\ker \varphi}\right| = |S_4 : \ker \varphi| = |S_4| : |\ker \varphi| =$

$= 24 : |\ker \varphi| \Rightarrow |\ker \varphi| = 6.$

Cf T.F. Izom. pt. inele,

$$\frac{\mathbb{Q}[x]}{(x^2-2024)} \simeq A = \frac{\mathbb{Q}[x]}{(x^2-506)}$$

1.(1) Fie $x \in \mathbb{R}$.

Evident, $x-x=0 \in n\mathbb{Z}$, deci $x \rho_n x$.

Ca urmare, ρ_n e reflexivă (R)

Fie $x, y \in \mathbb{R}$ astfel ca $x \rho_n y$. Atunci

$x-y \in n\mathbb{Z}$, deci $y-x \in n\mathbb{Z}$, deci $y \rho_n x$.

Ca urmare, ρ_n e simetrică (S)

Fie $x, y, z \in \mathbb{R}$ astfel încât $x \rho_n y$ și $y \rho_n z$

Atunci $x-y \in n\mathbb{Z}$ și $y-z \in n\mathbb{Z}$

Ca urmare, $x-z=(x-y)+(y-z) \in n\mathbb{Z}$, deci $x \rho_n z$.

Ca urmare, ρ_n e tranzitivă (T)

Din (R), (S), (T) obținem faptul că ρ_n e relație de echivalență.

(2) $\frac{1}{\rho_n} = \frac{2024}{\rho_n} \iff 2024 \rho_n 1 \iff 2024-1 \in n\mathbb{Z} \iff$

$2023 \in n\mathbb{Z} \iff \exists k \in \mathbb{Z}\ 2023 = nk \iff n \mid 2023 \iff$

$\xLeftrightarrow{n \in \mathbb{N}^* \setminus \{1\}} n \in \{7, 17, 119, 289, 2023\}$.

(3) Cf sol. de la (2), $n_0=7$.

Fie $x, y \in [0, n_0) = [0, 7)$ cu $x \neq y$.

Presupunem că $\frac{x}{\rho_7} = \frac{y}{\rho_7}$. Atunci $x-y \in 7\mathbb{Z} \iff$

$\exists k \in \mathbb{Z}\quad x-y=7k \overset{x\neq y}{\Longrightarrow} |x-y| \geq 7.$ (5)

Dar $0 \leq x < 7$

$0 \leq y < 7 \Rightarrow -7 < -y \leq 0 \Big\} \Rightarrow -7 < x-y < 7 \Rightarrow |x-y| < 7.$

Rămâne, deci că $\frac{x}{\rho_7} \neq \frac{y}{\rho_7}$.

$k \in \mathbb{Z}$: $7k \leq x < 7k+7$ (2)

$k \leq \frac{x}{7} < k+1$

(4). Fie $x \in \mathbb{R}$

luăm $x' = x - 7\left[\frac{x}{7}\right] = 7\left\{\frac{x}{7}\right\}$ (1)

$\frac{x}{7} - 1 < \left[\frac{x}{7}\right] \leq \frac{x}{7} \Rightarrow -\frac{x}{7} \leq -\left[\frac{x}{7}\right] < 1 - \frac{x}{7} \Rightarrow -x \leq -7\left[\frac{x}{7}\right] < 7 - x \overset{(1)}{\Rightarrow}$

$x' \in [0,7)$. În plus, $x - x' = x - \left(x - 7\left[\frac{x}{7}\right]\right) = 7\left[\frac{x}{7}\right] \in 7\mathbb{Z}$,

deci $x' \rho_7 x$

Deci, $[0,7)$ e un sistem complet de reprezentanți pt ρ_7. colectivă pt

Cf. (3), el e și independent.

Ca urmare, $[0,7)$ e un exemplu de SCIR pt ρ_7.

1.(5) Consider $g : \mathbb{R} \to [0,7)$, $g(x) = 7\left\{\frac{x}{7}\right\}$ ⑥

din ultima de la (4), g e corect definită.

Fie $x, y \in \mathbb{R}$ $x \rho_g y \Leftrightarrow g(x) = g(y) \Leftrightarrow x - 7\left[\frac{x}{7}\right] = y - 7\left[\frac{y}{7}\right]$ (10)

(10) $\Rightarrow x - y = 7\left(\left[\frac{x}{7}\right] - \left[\frac{y}{7}\right]\right) \in 7\mathbb{Z} \Rightarrow x \rho_7 y$

Reciproc $x \rho_7 y \Rightarrow x - y \in 7\mathbb{Z} \Leftrightarrow \exists k \in \mathbb{Z}\ x - y = 7k \Leftrightarrow \exists k \in \mathbb{Z}\ \frac{x}{7} = \frac{y}{7} + k$

$\Rightarrow \left\{\frac{x}{7}\right\} = \left\{\frac{y}{7}\right\} \Rightarrow g(x) = 7\left\{\frac{x}{7}\right\} = 7\left\{\frac{y}{7}\right\} = g(y) \Rightarrow x \rho_g y$.

Ca urmare, $\rho_g = \rho_7$.