MATH AT ONE GLANCE. CS 498 sl3 & sl4 Yinchen Xu.

0. introduction

Disclaimer: Not everything covered in lecture included in this file.

If conflict with lecture, consider me as wrong!!!

0.0. vector. An n-D vector has a general form of:

$\vec{v} = \sum_{i=1}^{n} a_i \hat{x_i}$ where $\hat{x_i}$ are unit vectors for an orthogonal basis coordinate $(\hat{x}, \hat{y}, \hat{z}; \hat{s}, \hat{\phi}, \hat{z}; \hat{r}, \hat{\phi}, \hat{\theta} \cdots)$.

a_i are scalars, representing the magnitude of the components.

$\hat{x_i}$ are vectors, representing the direction of components (all unit-length).

0.1. vector calculations.

Add, subtract, scaler-vector multiplication — Trivial.

0.1.0 dot product.

Define $\hat{x_i} \cdot \hat{x_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$ $\quad \vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i * v_i$.

Quick proof. $\vec{u} \cdot \vec{v} = (u_1 \hat{x}_1 + u_2 \hat{x}_2 + \cdots + u_n \hat{x}_n) \cdot (v_1 \hat{x}_1 + v_2 \hat{x}_2 + \cdots + v_n \hat{x}_n)$

$= \sum_{i,j|i=j}^{n} u_i v_j * \underbrace{(\hat{x_i} \cdot \hat{x_j})}_{1} + \sum_{i,j|i\neq j}^{n} u_i v_j * \underbrace{(\hat{x_i} \cdot \hat{x_j})}_{0}$.

$= \sum_{i}^{n} u_i v_i$. Also useful. $\vec{u} \cdot \vec{v} = |u||v| \cos\theta$

Quick proof 2: reconstruct the coord-system so that:

$\vec{u} = u\hat{u}$ $\quad \vec{v} = v_1 \hat{u} + v_2 \hat{u}^{\perp}$ (one unit vector is parallel to $\vec{u}$. another is $\perp$ to $\vec{u}$ and parallel to $\vec{u} - k\vec{v}$). trivial const.

($\theta \equiv$ angle between $\vec{u}$ and $\vec{v}$)

So $\vec{u} \cdot \vec{v} = u\hat{u} \cdot (v_1 \hat{u} + v_2 \hat{u}^{\perp}) = u\hat{u} \cdot v_1 \hat{u}$.

From graph we know that $v_1 \hat{u}$, $v_2 \hat{u}^{\perp}$ and $\vec{v}$ form a triangle

$v_1 \hat{u} \perp v_2 \hat{u}^{\perp}$ So $|v_1 \hat{u}| = \cos\theta |\vec{v}|$.

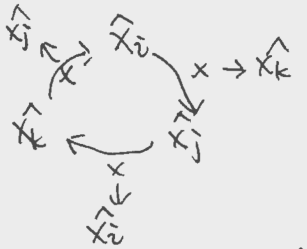
So $\vec{u} \cdot \vec{v} = u\hat{u} \cdot v_1 \hat{u} = u v_1 = |\vec{u}| \cdot \cos\theta |\vec{v}|$.

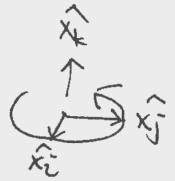
*. dot product result in scalar. useful in calculate projection, flux and judging directions of vectors.

0.1.1 cross product. (in 3-D space)

Define $\hat{x_i} \times \hat{x_j} = \begin{cases} \hat{x_k} & \text{if } \hat{x_i}, \hat{x_j}, \hat{x_k} \text{ are 3 coord satisfies right hand rule} \\ 0 & \\ -\hat{x_k} & \text{if } \hat{x_j}\ \hat{x_i}\ \hat{x_k} \text{ are } \ldots \end{cases}$

0.1.1 cross product (cont.) The order of right-hand coordinate is cyclic.





right hand rule

$$\vec{u} \times \vec{v} = (u_1\hat{x}_i + u_2\hat{x}_j + u_3\hat{x}_k) \times (v_1\hat{x}_i + v_2\hat{x}_j + v_3\hat{x}_k)$$

$$= u_1v_1(\hat{x}_i \times \hat{x}_i) + u_2v_2(\hat{x}_j \times \hat{x}_j) + u_3v_3(\hat{x}_k \times \hat{x}_k) \quad (\Longleftarrow \; 0+0+0)$$

$$+ u_1v_2(\hat{x}_i \times \hat{x}_j) + u_1v_3(\hat{x}_i \times \hat{x}_k) + u_2v_1(\hat{x}_j \times \hat{x}_i) + u_2v_3(\hat{x}_j \times \hat{x}_k)$$

$$+ u_3v_1(\hat{x}_k \times \hat{x}_i) + u_3v_2(\hat{x}_k \times \hat{x}_j)$$

$$= (u_2v_3 - u_3v_2)\hat{x}_i + (u_3v_1 - u_1v_3)\hat{x}_j + (u_1v_2 - u_2v_1)\hat{x}_k.$$

$\Uparrow$

some people prefer to memorize as

" $-(u_1v_3 - u_3v_1)\hat{x}_j$. depends on people.

One can also understand this as calculating determinant for:

$$\det \begin{vmatrix} \hat{x}_i & \hat{x}_j & \hat{x}_k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{u} \times \vec{v}.$$

* Both types of product have distributive ~~ta~~ rules:

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \;;\; (\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}.$$

*. Dot product is commute, cross product is not, but always satisfies another rule:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad \vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

*. They have associative rule <u>with scalars</u>:

$$(2\vec{a}) \cdot \vec{b} = 2(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (2\vec{b}) \;;\; (2\vec{a}) \times \vec{b} = 2(\vec{a} \times \vec{b}) = \vec{a} \times (2\vec{b}).$$

0.2. Matrix. Matrix can be considered as an Operator, can be operated on inputs like vectors and another matrix, resulting in another vector or matrix.

0.2.0 matrix vector multiplication. $M\vec{v} = \vec{v}'$ $\Leftarrow$ result.

operator $\uparrow$ input $\uparrow$

M's column should be equal to $\vec{v}$'s dimension.

M's row dot $\vec{v}$, Nothing special.

0.2.1 Matrix - Matrix Multiplication. $M_1 M_2 = M'$.

~~Understand: Find an individual matrix that~~ M_1: operator M_2: input M': result.

M_1's column # should be the same as M_2's row #.

M_1's ith ~~column~~ row dot M_2's jth column $\Rightarrow$ M''s ith row jth column's entry.

$$\begin{bmatrix} \text{---} \\ \\ \end{bmatrix}\begin{bmatrix} | & \\ | & \end{bmatrix} = \begin{bmatrix} a & \\ & \end{bmatrix}$$

1. Transformation.: <operator> <input> $\Rightarrow$ <result>.

I've mentioned matrix as a form of operator.
This can be useful to understand calculation order of M-v multiplication and M-M multiplication.

Consider $M_1 M_2 v = v'$. way 1: $(M_1 M_2) v = v'$

$\Rightarrow$ Calculate the matrix represent the net ~~transform eq~~, T', equivalent to first T_2 then T_1.

(or, applying T_1 on T_2)

way 2: $M_1 (M_2 v) = v'$ $\Rightarrow$ Calculate the result of apply T_2 on v. then calculate the result of apply T_1 to the previously temp result.

These 2 ways are equivalent. $\Rightarrow$ Associateive rule for M multiplication is valid

$\Rightarrow$ M multiplication ~~is~~ mostly does not commute ($M_1 M_2 \neq M_2 M_1$).

because order of transform matters!

*. FYI, Transform represented by matrix with non-zero determinant is a linear transformation.

Common Transformation.

1.0. translate. $\begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & j \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Nothing special.

1.1. rotational $\begin{bmatrix} \mathbb{R} & & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ where $\mathbb{R}$ is a 3x3 matrix. called rotational matrix.

Determine whether a matrix is rotational matrix:

If $M^{-1} = M^T$. Then it is. Proof omitted.

* M^{-1}. inverse matrix $\Rightarrow$ $M M^{-1} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} (= I)$

M^T. transpose matrix. $\Rightarrow$ M^T's ith row is M's ith column.

2. Quaternion. Can be considered both as transformation and configuration (like vector).

$$Q = \langle a, b, c, d \rangle = \langle \cos\frac{\theta}{2}, v_1 \sin\frac{\theta}{2}, v_2 \sin\frac{\theta}{2}, v_3 \sin\frac{\theta}{2} \rangle$$

$$= \langle \cos\frac{\theta}{2}, \vec{v} \sin\frac{\theta}{2} \rangle.$$

It represents a rotation transformation with $\vec{v}$ as center axis and θ as magnitude.

$\Rightarrow$ So ratio of b, c, d matters rather than the whole magnitude.

$\Rightarrow$ a can determine θ because for $0 \le \theta \le 2\pi$. θ and $\cos\frac{\theta}{2}$ have bijection.

Why add $\sin\frac{\theta}{2}$? To make the statement below valid:

Q is unit ~~if and only~~ if $\vec{v}$ is unit. Quick proof. $|Q|^2 = a^2+b^2+c^2+d^2$

$= \cos^2\frac{\theta}{2} + (v_1^2+v_2^2+v_3^2)\sin^2\frac{\theta}{2}$ if $\vec{v}$ is unit $\Rightarrow$ $= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$. also unit.

Why unit quaternion is good? ①. check validity. of $Q_1 \circ Q_2 = Q_3$

if Q_1, Q_2 are unit quaternion. Then Q_3 should also be unit quaternion.

proof too long, will be added separately.

If Q_1 and Q_2 are unit, but you get Q_3 non-unit $\Rightarrow$ you do something wrong!

②. Accuracy. Knowing unit $Q \circ$ unit Q = unit Q.

when computer trying calculate Q multiplication. If we don't use unit Q.

it may either:
- expand to infinity, causing out-of-domain problem
- shrink to small floats, causing accuracy problem

(FYI, if a float shrinks to a number smaller than $\varepsilon_{machine}$, computer will claim $1+\varepsilon_{machine} == 1$, normally $\varepsilon_{machine} \sim 2^{-24}$ for float).

Comment: NOT ALL contents above are required to master the class.
NOT ALL contents needed for the class is included above.

Considering this is not a math class, I am not completely strict with some of the definitions.

Considering maybe some students have not taken MATH 241, 225, 415 or 416.
I have not gone deep to these concepts.

Hope this file can help you on your understanding about the referred concepts!!!