

equation technique of dynamic programming. Some reductions which are useful from the computational viewpoint are indicated, and several applications to radar and communication system theory are sketched.

¹ Wald, A., *Statistical Decision Functions* (New York: John Wiley & Sons, 1950).

² Wald, A., *Sequential Analysis* (New York: John Wiley & Sons, 1947).

³ Bussgang, J. J., and D. Middleton, "Optimum sequential detection of signals in noise," *IRE Trans. on Inform. Theory*, IT-1, 5 (December) 1955.

⁴ Middleton, D., *An Introduction to Statistical Communication Theory* (New York: McGraw-Hill, 1960), Chapter 20, esp. refs. 32-40.

⁵ *Ibid.*, Chapter 21.

⁶ Bellman, R., *Dynamic Programming* (Princeton University Press, 1957).

⁷ Bellman, R., *Adaptive Control: A Guided Tour* (Princeton University Press, 1961).

⁸ Bellman, R., and R. Kalaba, "On communication processes involving learning and random duration," *1958 IRE National Convention Record*, part 4, pp. 16-21.

THE GEODESICS IN GÖDEL'S UNIVERSE

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1. In 1949, Gödel¹ discovered a solution of Einstein's field equations which, unlike the Friedmann-Lemaître solutions of relativistic cosmology, is incompatible with Weyl's postulate² and does not allow the possibility of defining a universal cosmic time. On these accounts, Gödel's universe has many unusual properties, which Gödel³ has enumerated and discussed. It does not, however, seem to have been noticed that the equations for the geodesics with Gödel's metric can be explicitly integrated and that in terms of the explicit solution for the geodesics, many of the properties of Gödel's universe can be more readily understood.

2. Gödel's metric is

$$ds^2 = a^2(dx_0^2 - dx_1^2 + \frac{1}{2}e^{2x_1}dx_2^2 - dx_3^2 + 2e^{x_1}dx_0dx_2), \quad (1)$$

where $a^{-1} = R$ is the "radius" of the universe. With R chosen as the unit of "distance," the metric can be written alternatively in the form

$$ds^2 = (dx_0 + e^{x_1}dx_2)^2 - dx_1^2 - \frac{1}{2}e^{2x_1}dx_2^2 - dx_3^2. \quad (2)$$

The Christoffel symbols for the metric (1) have been listed by Gödel¹ and using them, the equations governing the geodesics can be readily written down. They are:

$$\frac{du^0}{ds} + 2u^0u^1 + e^{x_1}u^1u^2 = 0, \quad (3)$$

$$\frac{du^1}{ds} + e^{x_1}u^0u^2 + \frac{1}{2}e^{2x_1}(u^2)^2 = 0, \quad (4)$$

$$\frac{du^2}{ds} - 2e^{-x_1}u^1u^0 = 0, \quad (5)$$

and
$$\frac{du^3}{ds} = 0, \quad (6)$$

where u^μ is the four-velocity:

$$u^\mu = \frac{dx^\mu}{ds}. \quad (7)$$

For the metric (1), this four-velocity is time-like.

3. Equation (6) immediately integrates to give

$$u^3 = C \quad \text{and} \quad x_3 = Cs + c_3, \quad (8)$$

where C and c_3 are constants. Equations (3)–(6) allow two additional integrals. The first of these is the metric rewritten in terms of u^μ ; thus (cf. equation (2)),

$$(u^0 + e^{x_1}u^2)^2 - (u^1)^2 - \frac{1}{2}e^{2x_1}(u^2)^2 - (u^3)^2 = 1. \quad (9)$$

The second follows from multiplying equations (4) and (5) by $u^1(=dx_1/ds)$ and $e^{2x_1}u^2/2$, respectively, and adding; we then obtain

$$u^1 \frac{du^1}{ds} + \frac{1}{2} e^{2x_1} \frac{dx_1}{ds} (u^2)^2 + \frac{1}{2} e^{2x_1} u^2 \frac{du^2}{ds} = 0; \quad (10)$$

and the integral of this equation is

$$(u^1)^2 + \frac{1}{2} e^{2x_1} (u^2)^2 = B^2 = \text{constant}. \quad (11)$$

Using equations (8) and (11), we can rewrite equation (9) in the form

$$(u^0 + e^{x_1}u^2)^2 = 1 + B^2 + C^2 = \frac{1}{2}D^2 \quad (\text{say}). \quad (12)$$

It is clear from the manner in which the constants B , C , and D have been introduced that

$$\frac{1}{2}D^2 \geq B^2 + 1; \quad (13)$$

and further that B , C , and D are real constants. From equations (11) and (12), we obtain

$$u^2 = \sqrt{2}e^{-x_1}[B^2 - (u^1)^2]^{1/2} \quad (14)$$

and
$$u^0 = \frac{D}{\sqrt{2}} - e^{x_1}u^2 = \frac{1}{\sqrt{2}} \{D - 2[B^2 - (u^1)^2]^{1/2}\}. \quad (15)$$

The ambiguities in the signs which arise from extracting the square roots of equations (11) and (12) can be resolved (without loss of generality) by requiring that D be always positive and allowing B to be positive or negative. However, (13) requires that

$$D > \sqrt{2}|B|. \quad (16)$$

4. With u^0 and u^2 given by equations (14) and (15), equation (4) becomes

$$\frac{du^1}{ds} = B^2 - (u^1)^2 - D[B^2 - (u^1)^2]^{1/2}. \quad (17)$$

Letting $u^1 = B \sin \theta$, (18)

(which is a permissible transformation, since (11) requires that $|u^1| < |B|$), equation (17) becomes

$$\frac{d\theta}{ds} = -(D - B \cos \theta). \quad (19)$$

Remembering that in any case $D > |B|$, we can integrate equation (19) in the form

$$s - s_0 = - \frac{2}{(D^2 - B^2)^{1/2}} \tan^{-1} \left\{ \left(\frac{D+B}{D-B} \right)^{1/2} \tan \frac{1}{2} \theta \right\}, \quad (20)$$

where s_0 is a constant. With the definition

$$s - s_0 = \frac{2}{(D^2 - B^2)^{1/2}} \sigma, \quad (21)$$

we can rewrite equation (20) more conveniently in the form

$$\tan \frac{1}{2} \theta = -\sqrt{\alpha} \tan \sigma, \quad \text{where } \alpha = \frac{D-B}{D+B}. \quad (22)$$

Elementary identities which follow from equation (22) are

$$\cos \theta = \frac{1 - \alpha \tan^2 \sigma}{1 + \alpha \tan^2 \sigma}, \quad \text{and} \quad \sin \theta = -2\sqrt{\alpha} \frac{\tan \sigma}{1 + \alpha \tan^2 \sigma}. \quad (23)$$

5. By making use of equations (21), (22), and (23), equation (18) becomes

$$\frac{dx_1}{d\sigma} = u^1 \frac{ds}{d\sigma} = - \frac{4B}{D+B} \frac{\tan \sigma}{1 + \alpha \tan^2 \sigma}. \quad (24)$$

This equation readily integrates to give

$$x_1 = - \frac{2B}{(\alpha - 1)(D+B)} \log \left(\frac{1 + \alpha \tan^2 \sigma}{1 + \tan^2 \sigma} \right) + c_1, \quad (25)$$

where c_1 is a constant. We find that in virtue of the definition of α , the coefficient in front of the logarithm in equation (25) is unity; we are thus left with

$$x_1 = \log \left(\frac{1 + \alpha \tan^2 \sigma}{1 + \tan^2 \sigma} \right) + c_1. \quad (26)$$

An alternative form of this last integral is

$$e^{x_1} = e^{c_1} (\cos^2 \sigma + \alpha \sin^2 \sigma). \quad (27)$$

6. With u^1 given by equation (18), equation (15) can be rewritten as

$$\frac{dx_0}{d\sigma} = \left(\frac{2}{D^2 - B^2} \right)^{1/2} (D - 2B \cos \theta), \quad (28)$$

or making use of the relation given in (23), we find after some further reductions

$$\frac{dx_0}{d\sigma} = - \left(\frac{2D^2}{D^2 - B^2} \right)^{1/2} + 2\sqrt{2\alpha} \frac{\sec^2 \sigma}{1 + \alpha \tan^2 \sigma}. \quad (29)$$

The integral of this last equation is readily found to be

$$x_0 = - \left(\frac{2D^2}{D^2 - B^2} \right)^{1/2} \sigma + 2\sqrt{2} \tan^{-1} (\sqrt{\alpha} \tan \sigma) + c_0, \quad (30)$$

where c_0 is a constant.

7. It remains to integrate equation (14) for x_2 . Rewriting this equation in the form

$$\frac{dx_2}{d\sigma} = 2 \left(\frac{2B^2}{D^2 - B^2} \right)^{1/2} e^{-x_1} \cos \theta, \quad (31)$$

and making use of equations (23) and (26), we have

$$\frac{dx_2}{d\sigma} = 2 \left(\frac{2B^2}{D^2 - B^2} \right)^{1/2} e^{-c_1} \frac{(1 - \alpha \tan^2 \sigma) \sec^2 \sigma}{(1 + \alpha \tan^2 \sigma)^2}. \quad (32)$$

This equation is readily integrated and we find

$$x_2 = 2e^{-c_1} \left(\frac{2B^2}{D^2 - B^2} \right)^{1/2} \frac{\tan \sigma}{1 + \alpha \tan^2 \sigma} + c_2, \quad (33)$$

where c_2 is a further constant.

8. Equations (8), (26), (30), and (33) represent the complete solution of the geodesic equations. In Figure 1, we illustrate the behavior of these solutions for the case $\alpha = 1/4$ and $c_0 = c_1 = c_2 = c_3 = 0$ (the slope for x_3 is arbitrary within limits).

We have already drawn attention to certain limitations on the admissible values of the constants arising from the definition (cf. equation (12))

$$D^2 = 2(B^2 + C^2 + 1). \quad (34)$$

This definition implies further restrictions on the possible range of the constant α . Thus,

$$\alpha = \frac{D - B}{D + B} = \frac{B^2 + 2C^2 + 2}{[\sqrt{2}(B^2 + C^2 + 1)^{1/2} + B]^2}. \quad (35)$$

The admissible range of α , so long as B is positive, is, therefore,

$$1 \geq \alpha \geq (\sqrt{2} - 1)^2 = 0.1716. \quad (36)$$

The two sides of this inequality arise from the limits $B = 0$, and $C = 0$ and $B \rightarrow \infty$, respectively.

When negative values of B are also allowed, the full admissible range of α is clearly

$$(\sqrt{2} + 1)^2 \geq \alpha \geq (\sqrt{2} - 1)^2. \quad (37)$$

9. Gödel has exhibited the rotational symmetry of his metric by transforming to a system of cylindrical coordinates, r , φ , and t in the subspace $x_3 = \text{constants}$. The requisite transformations are:

$$e^{x_1} = \cosh 2r + \sinh 2r \cos \varphi, \quad (38)$$

$$\frac{x_2 e^{x_1}}{\sqrt{2}} = \sinh 2r \sin \varphi, \quad (39)$$

$$\frac{1}{2} \varphi + \frac{1}{2\sqrt{2}} (x_0 - 2t) = \tan^{-1} \left(e^{-2r} \tan \frac{1}{2} \varphi \right), \quad (40)$$

$$\text{and} \quad x_3 = 2y. \quad (41)$$

In these new coordinates, the metric (2) becomes

$$ds^2 = 4 \{ dt^2 - dr^2 - dy^2 + (\sinh^4 r - \sinh^2 r) d\varphi^2 + 2\sqrt{2} \sinh^2 r d\varphi dt \}. \quad (42)$$

By considering the special geodesics of the metric (42) which correspond to "circular orbits," $r = \text{constant}$, described uniformly, Gödel has drawn the conclusion that these orbits allow one "to travel into the past, or otherwise influence the past." It is, therefore, of interest to relate Gödel's special geodesics with the general ones we have found.

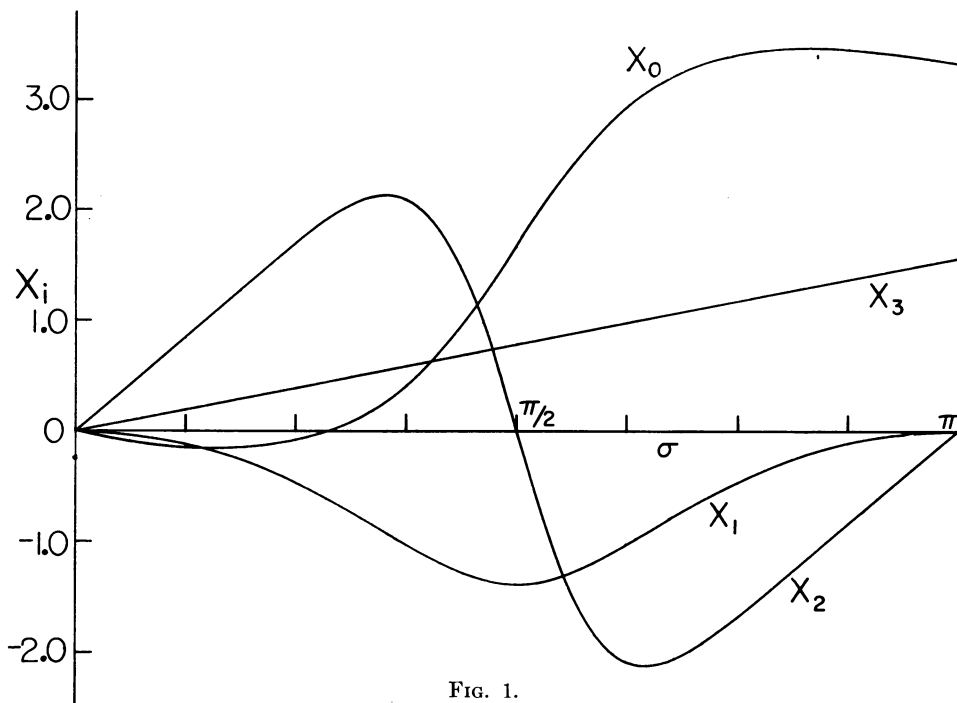


FIG. 1.

10. First, we may observe that according to equations (38) and (39),

$$2e^{x_1} \cosh 2r = 1 + e^{2x_1} + \frac{1}{2} x_2 e^{2x_1}, \quad (43)$$

whereas according to the solutions for x_1 and x_2 given by equations (27) and (33) (with c_2 set equal to zero),

$$\frac{x_2 e^{x_1}}{\sqrt{2}} = \frac{B}{(D^2 - B^2)^{1/2}} \sin 2\sigma = \frac{1 - \alpha}{2\sqrt{\alpha}} \sin 2\sigma. \quad (44)$$

(In equation (44) and in the sequel, the signs when square roots are taken are those appropriate for $B > 0$; the final results will be independent of this choice but the signs in the intermediate equations are not.)

From equations (35) and (40), it now follows that along the geodesics.

$$\sinh 2r \sin \varphi = \frac{1 - \alpha}{2\sqrt{\alpha}} \sin 2\sigma. \quad (45)$$

Similarly, from equations (27), (43), and (44) we find

$$\cosh 2r = \frac{1 + e^{2c_1}(\cos^2 \sigma + \alpha \sin^2 \sigma)^2 + [(\alpha - 1)^2/4\alpha] \sin^2 2\sigma}{2e^{c_1}(\cos^2 \sigma + \alpha \sin^2 \sigma)}. \quad (46)$$

11. According to equation (46), r is in general a function of σ (i.e., of s). However, for a special choice of the constant c_1 , we shall show that r becomes independent of σ .

Now if r should be independent of σ , then from a comparison of the values of $\cosh 2r$ for $\sigma = 0$ and $\sigma = \pi/2$, we get

$$1 + e^{2c_1} = \frac{1}{\alpha} (1 + \alpha^2 e^{2c_1}), \quad (47)$$

or

$$e^{2c_1} = 1/\alpha; \quad (48)$$

and when e^{2c_1} has this value, equation (46) is found to reduce to

$$\cosh 2r = \frac{1}{2} \left(\frac{1}{\sqrt{\alpha}} + \sqrt{\alpha} \right); \quad (49)$$

and r is indeed a constant.

When $B > 0$, $\alpha < 1$ and equation (49) implies that

$$e^{-2r} = \sqrt{\alpha} \quad \text{and} \quad \sinh 2r = \frac{1 - \alpha}{2\sqrt{\alpha}}. \quad (50)$$

When r has this constant value, equation (45) gives

$$\sin \varphi = \sin 2\sigma. \quad (51)$$

If we restrict ourselves to $0 \leq 2\pi$, then we may write

$$\varphi = 2\sigma. \quad (52)$$

Turning finally to equation (40), we have along these geodesics

$$2\sqrt{2}\sigma + x_0 - 2t = 2\sqrt{2} \tan^{-1} (\sqrt{\alpha} \tan \sigma). \quad (53)$$

Combining this result with the general integral (30), we obtain

$$t = \left\{ \sqrt{2} - \frac{1}{2} \left(\frac{2D^2}{D^2 - B^2} \right)^{1/2} \right\} \sigma = \sqrt{2} \left(1 - \frac{\alpha + 1}{4\sqrt{\alpha}} \right) \sigma. \quad (54)$$

Alternatively, we can write

$$t = \sqrt{2}(1 - 1/2 \cosh 2r)\sigma = \beta\sigma \quad (\text{say}). \quad (55)$$

12. The limits (37) on α implies that there is a limitation on the radius of the circular orbits we have considered. For, according to equation (49), we must have

$$1 \leq \cosh 2r \leq \sqrt{2} \quad (56)$$

The origin of this upper bound on r becomes apparent when we evaluate (cf. equation (42))

$$d\Sigma^2 = dt^2 + (\sinh^4 r - \sinh^2 r)d\varphi^2 + 2\sqrt{2} \sinh^2 r d\varphi dt \quad (57)$$

for the orbit described by equations (49), (52), and (55). We find

$$d\Sigma^2 = (1 - \frac{1}{2} \cosh^2 2r)d\sigma^2. \quad (58)$$

Accordingly, when $\cosh 2r$ has its maximum value specified by (56), $d\Sigma^2 = 0$. In other words the *circular orbit of the maximum radius is the null geodesic*.

13. Finally, it is important to remark that for the range of r allowed by (56), the constant of proportionality β between t and σ ($= \frac{1}{2}\varphi$) is *always positive*. This last fact seems to be contrary to some statements of Gödel from which he has drawn the conclusion we have quoted earlier.

¹ Gödel, Kurt, "An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation," *Rev. Mod. Phys.*, **21**, 447-450 (1949).

² Robertson, H. P., "Relativistic Cosmology," *Rev. Mod. Phys.*, **5**, 62-90 (1933), see p. 65.

³ Gödel, Kurt, "A Remark about the Relationship between Relativity Theory and Idealistic Philosophy," *Albert Einstein: Philosopher-Scientist* (Evanston, Illinois: The Library of Living Philosophers, Inc., 1949), pp. 557-562.

ABSENCE OF DISPERSIVE PROPERTIES OF SPACE FOR ELECTROMAGNETIC RADIATION TESTED TO $\pm 14 \times 10^{-5}$; COMMENTS ON A PROPOSAL OF SOFTKY AND SQUIRE*

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In session I 1 of the Berkeley meeting of December 30, 1960, S. D. Softky and R. K. Squire proposed a test for dispersive properties of space for electromagnetic radiation by detonating a nuclear explosive at a distance of 10^6 miles from the earth and noting the arrival times of different types of radiation at detectors above the atmosphere. In justification of such an experiment (which will no doubt cost the taxpayer several tens of millions of dollars), they assert the following, which I quote directly from their published abstract: "Measurements of c for different frequencies of radiation (radio waves, light, and ratio of esu to emu) have not demonstrated that c is independent of frequency . . . Astronomical tests of the invariability of c with frequency have been done only for optical frequencies . . . In view of the importance of c in many physical theories, it is thought that an accurate comparison for radio, optical, X-ray, and γ -ray frequencies would be worth while."