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István Ozsváth and Engelbert Schucking

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Gödel's trip

István Ozsváth[a)]
Department of Mathematics, The University of Texas at Dallas, Richardson, Texas 75083-0688

Engelbert Schucking
Department of Physics, New York University, 4 Washingon Place, New York, New York 10003

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We discuss the geometrical light cone structure in Gödel's rigidly rotating cosmos and correct a picture in the standard literature.
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I. INTRODUCTION

On May 10, 1949 Kurt Gödel at the Institute for Advanced Study (IAS) at Princeton University wrote to his mother in Vienna, Austria. He had intended to write long before, but for several weeks he had been beset by a "problem" that had driven everything else out of his mind. He would have preferred not to work on it uninterruptedly, but that had proved impossible. Even when he listened to the radio or went to the movies he did so "with only half an ear." At last though, just a few days before, he had "settled the matter enough to be able to sleep well again." [1]

Gödel did not say what the problem was that had so obsessed him. But the diary of his friend Oscar Morgenstern, two days later provided the answer. Morgenstern had learned from Gödel, "Now in his universe one can travel into the past." [1]

Three days before Gödel wrote to his mother he had demonstrated his conclusion in a lecture at the IAS with Einstein, Oppenheimer, and Chandrasekhar in the audience. The manuscript of this lecture was published in Gödel's collected work,[2] and it is a better introduction to his ideas than his paper on "An example of a new type of cosmological solutions of Einstein's field equations of gravitation" [3] published in the same year.

Time, the deep and mysterious concept in the trinity of space–time–matter, had been twisted with the two others in Gödel's universe into a mind-boggling enigma. Gödel had shown the existence of closed time-like world lines in a space in which matter was in a state of rigid rotation. A time machine, enabling one to travel into the past, defying old-fashioned causality, had now become a theoretical possibility sanctioned by Einstein's theory of gravitation.

Gödel's work in relativity did not immediately inspire much new work. In 1956 Wolfgang Kundt in Hamburg, Germany, calculated the geodesic world lines in Gödel's model. He used an elegant method suggested by Felix Pirani to use the first integrals of the equations derived from the Killing vectors for the five-dimensional isometries of Gödel's model.[4] Five years later Subrahmanyan Chandrasekhar and his student James Wright at the University of Chicago also calculated Gödel's geodesics. They did not use Pirani's elegant method in their lengthy calculations, not having read Kundt's paper. Because they did not find closed time-like geodesics, they announced that Gödel's claim of closed time-like geodesics in his universe was incorrect.[5] However, Gödel had never claimed that the closed time-like world lines in his model were *geodesic*. It took nine years until Howard Stein[6] showed that Chandrasekhar and Wright had been mistaken.

Even in 1973, Misner, Thorne, and Wheeler's standard work on gravitation[7] did not mention Gödel's papers on relativity in its more than 500 references. But in the same year, the monograph, *The Large Scale Structure of Space-Time* by Stephen Hawking and George Ellis[8] did picture Gödel's space–time in an intriguing drawing that we reproduce here (see Fig. 1). It also gave for the first time a geometrical insight into the occurrence of closed time-like world lines in the Gödel universe. Although this picture clearly shows how the closed time-like world lines arise, something is wrong with it, for the matter world-line in the upper left-hand corner of the graph is space-like. Because the authors did not publish the calculations that led to this picture and because it is often used in descriptions of Gödel's model (tilting of the light-cones), we believe it is worthwhile to reconstruct it.

Meanwhile, time machines have become popular theoretical gadgets. The second edition of Paul Nahin's book[9] has more than 2×10^3 references on time travel. More references to Gödel's model can be found in Ref. 10.

II. THE METRIC

A closed time-like line is a smooth world line in the space–time manifold that runs smoothly back into itself. Because all events are functions of space–time, all happenings along a closed time-like line are periodic with a period given by the integral over the proper time differential ds taken from one initial event forward, and thus, back to it. The notion that causality implies that the cause is earlier and not later than the effect and the belief in free will, the ability to affect the future but not the past, are challenged by the existence of closed time-like lines.

Because the closed time-like lines appear as circles, they have their simplest description in cylindrical coordinates. For this reason we choose a coordinate system for the Gödel cosmos that exhibits these coordinates. A lucid introduction to Gödel's model can be found in Ref. 11.

In his first paper[3] about his world model, Gödel introduced these coordinates through a complicated transformation. How we can obtain them more directly is shown in Ref. 10.

Four-dimensional space–time is the sum of a three-dimensional space–time M and a one-dimensional space-like line x^3. The metric on M does not depend on x^3 and the x^3 lines $M=$constant are all orthogonal to M. Because closed time-like lines occur already for constant x^3, entirely in M, we shall drop the discussion of the four-dimensional manifold and restrict ourselves to consider M.

Gödel's model as a solution of the Einstein field equations

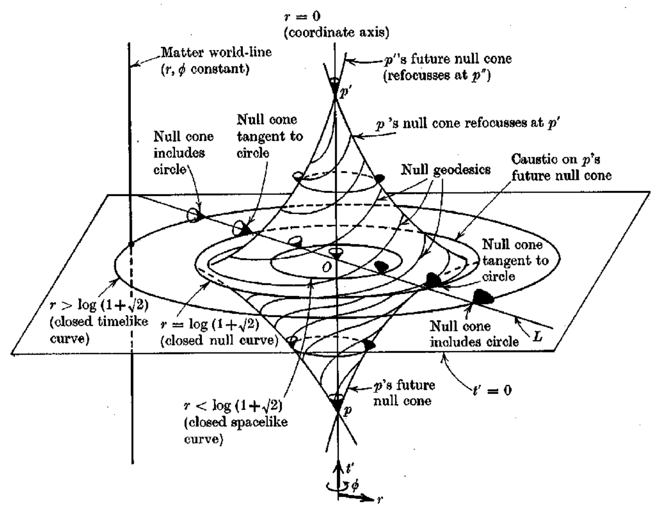

Fig. 1. Gödel's universe with the irrelevant coordinate z suppressed. The space is rotationally symmetric about any point; the diagram represents correctly the rotational symmetry about the axis $r=0$, and the time invariance. The light cone opens out and tips over as r increases (see line L) resulting in closed time-like curves. The diagram does not correctly represent the fact that *all* points are in fact equivalent. (Reproduced by permission of Cambridge University Press.)

depends on the parameter a, which characterizes the mass density ρ and the angular velocity ω of matter,

$$\frac{1}{a^2}=8\pi G\rho, \quad \omega=\frac{\sqrt{2}}{a}, \quad \lambda=-\frac{1}{2a^2}. \tag{1}$$

A Gödel metric for the three-dimensional space–time M is given by

$$\frac{1}{4a^2}ds^2=(dt+\sqrt{2}\sinh^2 r\,d\phi)^2-dr^2-\sinh^2 r\cosh^2 r\,d\phi^2. \tag{2}$$

Here G is Newton's gravitational constant and λ is Einstein's cosmological term. The parameter a appears also as a constant scale factor of the metric. We shall set a equal to 1 because this choice will simplify all formulas.

If we consider the circle C given by

$$x^0=t=\text{constant}, \quad x^1=r=\text{constant}, \tag{3}$$

we see that the tangent vector to this circle,

$$\frac{\partial}{\partial x^2}=\frac{\partial}{\partial\phi}, \tag{4}$$

has the length squared

$$\left(\frac{\partial}{\partial x^2}\right)\left(\frac{\partial}{\partial x^2}\right)=g_{22}=4\sinh^2 r(\sinh^2 r-1). \tag{5}$$

The circle C is, therefore, space-like, null, or time-like depending on whether $\sinh^2 r$ is smaller, equal, or larger than 1.

The acceleration vector γ^j is given by

$$\gamma^j=u^j{}_{;k}u^k, \quad u^j u_j=1 \tag{6}$$

for the time-like unit vector

$$u^j=\delta_2^j|g_{22}|^{-1/2}. \tag{7}$$

The semicolon in (6) indicates covariant differentiation.

We then obtain that the acceleration γ defined by

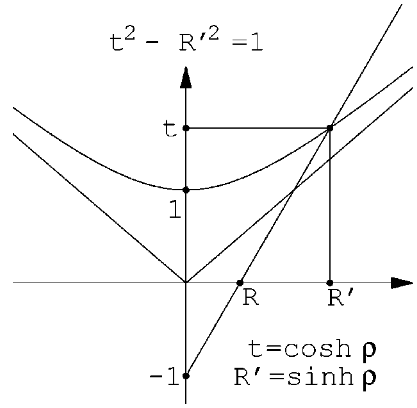

Fig. 2. Stereographic projection of the plane $t^2-R'^2=1$ in three-dimensional space–time on the plane $t=0$ from the point $R'=0$, $t=-1$.

$$(\gamma)^2=-g_{jk}\gamma^j\gamma^k \tag{8}$$

becomes

$$\gamma=\frac{1}{4}\frac{d\ln|g_{22}|}{dr}, \tag{9}$$

or

$$\gamma=\frac{1}{2}\frac{\cosh r}{\sinh r}\frac{2\sinh^2 r-1}{\sinh^2 r-1}. \tag{10}$$

We see that γ does not vanish for $\sinh^2 r>1$, and it asymptotically becomes 1 for large r.

By introducing new coordinates

$$\rho=2r, \tag{11}$$

we put the metric into the form

$$ds^2=(2dt+\sqrt{2}(\cosh\rho-1)d\phi)^2-d\rho^2-\sinh^2\rho\,d\phi^2. \tag{12}$$

The last two terms in Eq. (12) describe the metric on the analogue of a unit sphere, a surface whose constant curvature equals minus one. Such a surface appears, for instance, in the three-dimensional Minkowski space–time with metric

$$ds^2=dt^2-dR'^2-R'^2d\phi^2 \tag{13}$$

as the surface

$$t^2-R'^2=1, \quad t\geqslant 1. \tag{14}$$

By using the parameter representation

$$t=\cosh\rho, \quad R'=\sinh\rho, \tag{15}$$

we obtain the metric on the pseudo-sphere

$$-ds^2=d\rho^2+\sinh^2\rho\,d\phi^2. \tag{16}$$

If we project this surface from the point $(t=-1, R'=0)$ onto the plane $t=0$, we obtain

$$\frac{R}{1}=\frac{R'}{1+t}=\frac{\sinh\rho}{\cosh\rho+1}, \tag{17}$$

where R is the project point of the R' coordinate (see Fig. 2). This stereographic transformation maps the pseudo-sphere into the interior of the unit disk $R<1$. It is known as the Poincaré model, which is famous through Escher's tilings.

For the inverse transformation

$$\rho = \ln(1+R) - \ln(1-R), \tag{18}$$

we have

$$\sinh \rho = \frac{e^{\rho} - e^{-\rho}}{2} = \frac{1}{2}\left(\frac{1+R}{1-R} - \frac{1-R}{1+R}\right) = \frac{2R}{1-R^2}, \tag{19}$$

$$\cosh \rho = \frac{e^{\rho} + e^{-\rho}}{2} = \frac{1}{2}\left(\frac{1+R}{1-R} + \frac{1-R}{1+R}\right) = \frac{1+R^2}{1-R^2}, \tag{20}$$

$$d\rho = \left(\frac{1}{1+R} + \frac{1}{1-R}\right) dR = \frac{2dR}{1-R^2}, \tag{21}$$

and the line element (2) becomes

$$ds^2 = \left(2\,dt + \frac{2\sqrt{2}R^2}{1-R^2}\,d\phi\right)^2 - \frac{4}{(1-R^2)^2}(dR^2 + R^2 d\phi^2), \tag{22}$$

or written conformally

$$ds^2 = \frac{4}{(1-R^2)^2}\{((1-R^2)dt + \sqrt{2}R^2 d\phi)^2 - dR^2 - R^2\,d\phi^2\}. \tag{23}$$

We have extracted a common factor $(2/(1-R)^2)^2$ from the metric, and we shall deal hence with the metric in curly braces. The new metric has the advantage that the entire manifold extends only to $R=1$. Null geodesics are not affected by such a conformal transformation or the character of vectors being time-like, null, space-like, or zero. All angles and velocities are unchanged.

III. NULL GEODESICS

To study the null geodesics, we consider the variational problem for the conformal metric in Eq. (23),

$$\delta \int L\,d\lambda = 0, \quad L = \tfrac{1}{2}\{((1-R^2)\dot{t} + \sqrt{2}R^2\dot{\phi})^2 - \dot{R}^2 - R^2\dot{\phi}^2\}. \tag{24}$$

Here,

$$\dot{t} = \frac{dt}{d\lambda}, \quad \dot{R} = \frac{dR}{d\lambda}, \quad \dot{\phi} = \frac{d\phi}{d\lambda}, \tag{25}$$

with the distinguished parameter λ. (The parameter λ is called distinguished because it is determined up to a linear transformation by the geodesic equation. Unlike proper time it can also serve for null geodesics.)

The coordinates t and ϕ do not occur in the Lagrange function L. We have, therefore, from the Euler–Lagrange equations the two integrals

$$\frac{\partial L}{\partial \dot{t}} = [(1-R^2)\dot{t} + \sqrt{2}R^2\dot{\phi}](1-R^2) = A = \text{constant}, \tag{26}$$

$$\frac{\partial L}{\partial \dot{\phi}} = [(1-R^2)\dot{t} + \sqrt{2}R^2\dot{\phi}]\sqrt{2}R^2 - R^2\dot{\phi} = B = \text{constant}. \tag{27}$$

Because we are dealing with null geodesics, we also have

$$[(1-R^2)\dot{t} + \sqrt{2}R^2\dot{\phi}]^2 - \dot{R}^2 - R^2\dot{\phi}^2 = 0. \tag{28}$$

We are interested in the null geodesics that come from $R=0$. They are characterized by having

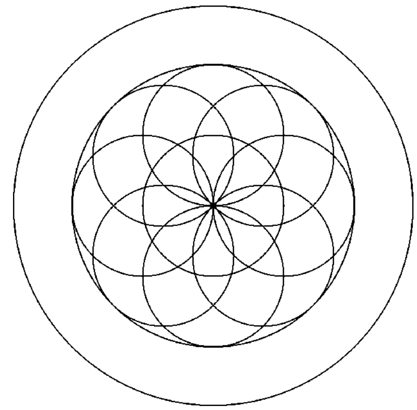

Fig. 3. The projection of the null-lines onto the $x-y$ plane are circles of radius $1/2\sqrt{2}$ through the origin. Their envelope is a circle of radius $1/\sqrt{2}$ about the origin.

$$B = 0. \tag{29}$$

We obtain from Eq. (27)

$$[(1-R^2)\dot{t} + \sqrt{2}R^2\dot{\phi}] = \frac{\dot{\phi}}{\sqrt{2}}. \tag{30}$$

If we substitute this expression into Eqs. (26) and (28), we obtain

$$\dot{\phi} = \frac{\sqrt{2}A}{1-R^2}, \tag{31}$$

$$(\dot{\phi})^2(\tfrac{1}{2} - R^2) - \dot{R}^2 = 0. \tag{32}$$

Equation (32) gives

$$\left(\frac{dR}{d\phi}\right)^2 + R^2 = \frac{1}{2}, \tag{33}$$

or, with the integration constant ϕ_0,

$$R = \frac{1}{\sqrt{2}}\sin(\phi - \phi_0). \tag{34}$$

We put $\phi_0 = 0$ and obtain in Cartesian coordinates

$$R = \sqrt{x^2+y^2} = \frac{1}{\sqrt{2}}\,\frac{y}{\sqrt{x^2+y^2}}, \tag{35}$$

or

$$x^2 + \left(y - \frac{1}{2\sqrt{2}}\right)^2 = \left(\frac{1}{2\sqrt{2}}\right)^2. \tag{36}$$

Equation (36) is the equation of a circle with radius $1/2\sqrt{2}$ through the origin whose center is displaced on the positive y axis. Light emitted at the origin in the direction of the positive x axis (say East) will run in a circle reaching its maximum distance from the origin on the y axis at $y = 1/\sqrt{2}$ (in the North) and will return, coming in along the negative x axis (from the West). Thus, Gödel's model provides an extreme example of light bending: A galaxy located in the North at $y = 1/\sqrt{2}$ would at the origin be seen in the West (see Fig. 3).

If we now drop the restriction that $\phi_0 = 0$, we obtain all circles with radius $1/2\sqrt{2}$ going through the origin. Light from here in any direction of the $x-y$ plane will return from the opposite direction—as expected from cylindrical symmetry.

We have from Eq. (30) that

$$\frac{dt}{d\phi}=\frac{1}{1-R^2}\left(\frac{1}{\sqrt{2}}-\sqrt{2}R^2\right)=\frac{1-2R^2}{\sqrt{2}(1-R^2)}, \tag{37}$$

or, using Eq. (34),

$$\frac{dt}{d\phi}=\frac{\cos^2(\phi-\phi_0)}{\sqrt{2}(1-\frac{1}{2}\sin^2(\phi-\phi_0))}=\sqrt{2}\left[1-\frac{1}{1+\cos^2(\phi-\phi_0)}\right]. \tag{38}$$

Equation (38) can be integrated to give

$$t-t_0=\sqrt{2}\left[\phi-\phi_0-\frac{1}{\sqrt{2}}\tan^{-1}\frac{\tan(\phi-\phi_0)}{\sqrt{2}}\right]. \tag{39}$$

For $\phi-\phi_0=\pi/2$, the null geodesic reaches its largest distance $R=1/\sqrt{2}$—its apogee. The time interval

$$t(\pi/2)-t(0)=\frac{\pi}{2}(\sqrt{2}-1) \tag{40}$$

is half the period for the return of a light signal. Due to cylindrical symmetry, we have complete focusing. The rays sent out simultaneously in different directions of the plane return later at the same moment. At the critical distance $R=1/\sqrt{2}$ from the world line $R=0$, we have circles for $t=\text{constant}$. These circles are null lines as we see from Eq. (23). However, they are not geodesic. They form a cylinder with an axis at $R=0$, which we shall call the "light cylinder."

The null geodesic originating from $R=0$ shows a vanishing $dt/d\phi$ at the light cylinder touching the null circles. The light cones with vertices at $R=0$ are twisted spindles in the light cylinder. To find the shape of the spindle—which is a surface of rotation—we need R as a function of t. If we combine Eq. (34) with Eq. (39), we obtain

$$t-t_0=\sqrt{2}\arcsin\sqrt{2}R-\arctan\frac{R}{\sqrt{1-2R^2}}. \tag{41}$$

From Eqs. (33) and (37), we obtain without differentiation

$$\frac{dR}{dt}=\frac{dR}{d\phi}\frac{d\phi}{dt}=\frac{\sqrt{1/2-R^2}\sqrt{2}(1-R^2)}{1-2R^2}=\frac{1-R^2}{\sqrt{1-2R^2}}. \tag{42}$$

The expression on the right-hand side becomes infinite for $R=1/\sqrt{2}$. The meridian of the light spindle is given by the function $R(t)$ and therefore has a cusp, and the twisted surface of rotation has a sharp edge around its equator. At the origin $R=0$, the slope of $R(t)$ is equal to one, giving the usual light cone with opening angle 45°. A parameter representation for the meridian of the rotation surface is given by Eqs. (34) and (39) (see Fig. 4),

$$t=\sqrt{2}\phi-\arctan\left(\frac{\tan\phi}{\sqrt{2}}\right),\quad R=\frac{1}{\sqrt{2}}\sin\phi,\quad 0\leqslant\phi\leqslant\pi. \tag{43}$$

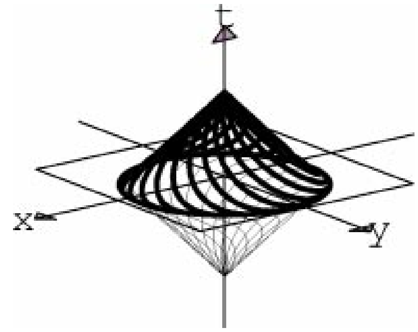

Fig. 4. Spindle with 20 geodesics.

IV. THE ROUND TRIP

We had seen in Eq. (5) that Gödel's model has time-like circles. According to Eq. (5), the circle C becomes time-like if its radius r becomes

$$\sinh r>1$$

or

$$r>\ln(1+\sqrt{2}). \tag{44}$$

In terms of the conformal coordinate R which reaches from zero to one, we have according to Eqs. (11) and (19), time-like circles for $R>1/\sqrt{2}$. Circles with conformal radius $R=1/\sqrt{2}$ are null while those with $R<1/\sqrt{2}$ are space-like.

Because Gödel's manifold M is simply connected, his time-like circles are not the results of event identifications as could easily be achieved, for example, in Minkowski space–time by making the time coordinate periodic. It is clear that something strange must occur in the behavior of the light cones at large distances from the source. The drawing from Ref. 8 reproduced in Fig. 1 suggests that the light cones simply tilt forward in the direction of rotation. This picture is, for instance, essentially repeated in Ref. 9. But it, clearly, does not make sense, because the world lines of the matter, being time-like, lie in this picture outside of the light cone.

However, it is easy to see what actually happens to the light cones by choosing a picture as in Fig. 1 or 4. The manifold M is a cylinder of radius one described by coordinates t, R, and ϕ. The world lines of matter run parallel to the axis of the cylinder at constant coordinates R and ϕ. The light cones are defined by Eq. (23):

$$[(1-R^2)dt+\sqrt{2}R^2d\phi]^2-dR^2-R^2d\phi^2=0. \tag{45}$$

In the radial direction, $d\phi=0$, and null lines going inward and outward have slopes

$$\frac{dt}{dR}=\pm\frac{1}{1-R^2}. \tag{46}$$

From the center at $R=0$ where the cones open under an angle of 45°, they narrow in the radial direction and collapse into their time axis at $R=1$.

In the azimuthal direction ($dR=0$), the null rays in the forward and backward directions behave asymmetrically. We have from Eq. (45):

$$\frac{dt}{Rd\phi}=-\frac{\sqrt{2}R\mp1}{1-R^2}. \tag{47}$$

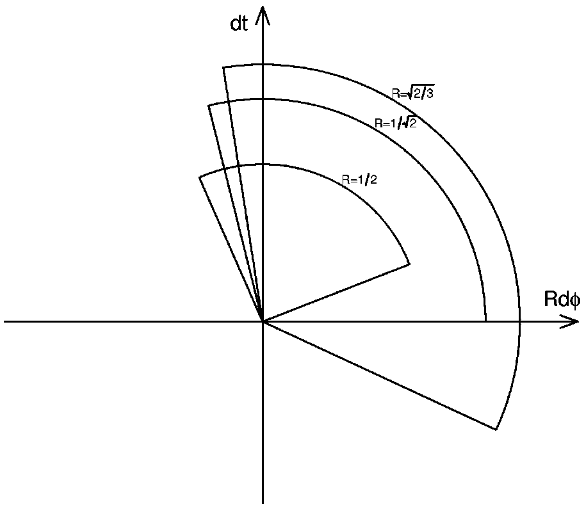

Fig. 5. In the azimuthal direction $R\,d\phi$, the light cone opens with increasing values of R. The drawn-out arcs bounded by azimuthal null directions show the range of future time-like vectors.

In Fig. 5 the azimuthal future time-like directions are indicated by the length of arcs that are bounded by the future null directions for different values of R. The graph shows clearly that the null cones open in the azimuthal direction for increasing R. For $R > 1/\sqrt{2}$, they contain time-like directions for decreasing t pointing into the past. The world lines of matter stay within the light cones. So, it is the *opening* of the light cones in the azimuthal direction that makes travel into the past possible.

The time-like circles C of Eq. (3) with $\sinh r > 1$ are not geodesics. We calculated their acceleration in Eq. (10) and found that it did not vanish. Such a closed time-like line taken as the world line of a person would enable this individual to travel into his/her own past if the acceleration were tolerable and the proper time for the round trip was less than its life-time. For further discussion of this possibility we refer to the paper by David Malament[12] and a book by J. Richard Gott[13] and close by letting Gödel describe his amazing discovery:[14]

> ''Namely, by making a round trip on a rocket ship in a sufficiently wide curve, it is possible in these worlds to travel into any region of the past, present and future, and back again, exactly as it is possible in other worlds to travel to distant parts of space.
>
> This state of affairs seems to imply an absurdity. For it enables one e.g., to travel into the near past of those places where he has himself lived. There he would find a person who would be himself at some earlier period of his life. Now he could do something to this person which, by his memory, he knows has not happened to him. This and similar contradictions, however, in order to prove impossibility of the worlds under consideration, presuppose the actual feasibility of the journey into one's own past. But the velocities which would be necessary in order to complete the voyage in reasonable length of time are far beyond everything that can be expected ever to become practical possibility [Ref. 15]. Therefore it cannot be excluded *a priori*, on the ground of the argument given, that the space-time structure of the real world is of the type described.''

ACKNOWLEDGMENTS

We are indebted to Professor Larry P. Ammann and Professor Peter S. Ozsváth for their generous help. A knowledgeable referee pointed out to us that the mistake in Fig. 1 had already been caught by David Malament (Ref. 12). Malament's very readable paper contains a correct sketch of the light cones in Gödel's model and an enlightening description of the time travel into the past.

[a]Electronic mail: ozsvath@utdallas.edu

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[15]See Ref. 14, p. 561, Gödel's footnote 11. If we base the calculation on a mean density of matter equal to that observed in our world, and assume that we were able to transform matter completely into energy, the weight of the fuel of the rocket ship would have to be the order of $10^{22}/t^2$ times the weight of the ship to complete the voyage in t years (as measured by the traveler) (if stopping, too, is effected by recoil). This estimate applies to $t \ll 10^{11}$. Irrespective of the value of t, the velocity of the ship must be at least $1/\sqrt{2}$ of the velocity of the light.