
IX.—The Gravitational Field of a Distribution of Particles Rotating about an Axis of Symmetry

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IX.—The Gravitational Field of a Distribution of Particles Rotating about an Axis of Symmetry. By W. J. van Stockum, Mathematical Institute, University of Edinburgh. *Communicated by* Professor E. T. WHITTAKER, F.R.S.

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§ 1. INTRODUCTION.

THE generalisation to the stationary case of the solution given by Weyl (1918) of Einstein's gravitational equations in a statical axially symmetric universe, has been attempted by various writers. It has not, however, so far been possible to reduce the equations to linear form and thus to find the most general solution. A set of special solutions has been obtained by Lewis (1932), valid in a region free of matter, which contain Weyl's solution as a particular case. They depend upon an arbitrary solution of the equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{\partial^2 V}{\partial z^2}+\frac{1}{r}\frac{\partial V}{\partial r}=0.$$

In the first part of the present paper the gravitational equations are considered in the interior of an axially symmetric distribution of particles rotating with constant angular velocity about its axis of symmetry. Solutions of the equations are found depending upon an arbitrary solution of the equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{\partial^2 V}{\partial z^2}-\frac{1}{r}\frac{\partial V}{\partial r}=0.$$

In the second part the field of an infinite rotating cylinder is considered, and the internal solution obtained by the method of the present paper is associated with the external solution given by Lewis. The boundary conditions, namely the continuity of the coefficients of the fundamental form and their first derivatives across the surface of the cylinder, determine uniquely the constants occurring in Lewis's solution in terms of the physical dimensions of the system. It appears that there are two essentially different types of external field of a rotating cylinder, according as the radius of the cylinder is less or greater than a certain critical value. In the first case, the geodesic planes normal to the axis of symmetry are infinite and tend to euclidean planes at infinity; in the second case, these planes are finite and closed. The external field of a rotating cylinder given by Lewis corresponds to the case where the radius is less than the critical value.

§ 2. Definition of Axial Symmetry.

It is customary to define stationary axially symmetric space-time to be such that by a suitable choice of co-ordinates the fundamental form assumes a certain simple expression. As however, in what follows, we shall allow ourselves to be guided by geometrical intuition in defining the energy tensor of a rotating system of particles, it may be more consistent to give a geometrical definition of axial symmetry, and to deduce the particularisation of the fundamental form from our definition.

Stationary axially symmetric space-time we define as follows. The universe contains a privileged observer O, whose world-line is a time-like geodesic g. The observer O separates space-time into space and time by referring to the 3-spaces S formed by all geodesics at O normal to g as space and to some suitably chosen parameter t defining his position on g as time. The universe is said to be stationary if with the passing of time the observer O detects no change in the intrinsic geometry of the space S. If the parameter t is used as one of the co-ordinates, it follows that the coefficients of the fundamental form must be independent of t. We now say that, in addition to being stationary, the universe is axially symmetric if at any instant there exists in S a privileged geodesic a, passing through O, which is such that at any point of it all directions in S normal to a are intrinsically indistinguishable. We have then at every point of g a privileged geodesic a normal to it. We shall now show that it follows from the definition of axial symmetry that the unit tangent vectors at g to the geodesics a are parallel in Levi-Civita's sense. For suppose this is not the case; then selecting the S, passing through the point O on g, we obtain a cone of directions at O by propagating the unit tangent vectors to the geodesics a parallelly along g to O. Considering the geodesics in S defined by this cone of directions, we obtain a surface in S, containing the privileged geodesic a of S. But this surface would define, at all points of a, privileged directions in S normal to a, namely those tangent to the surface, which contradicts the assumption of axial symmetry.

We may now choose a system of co-ordinates as follows. At an arbitrary point of g we select two unit vectors in S which, with the unit tangent vector to a, form an orthogonal triad. We can use the triad to set up in this S a system of geodesic polar co-ordinates, r being the length of the geodesic joining an arbitrary point to O, θ the angle between the tangent to this geodesic and the tangent to a at O, and ϕ the azimuthal angle. Propagating the triad parallelly along g, we have defined a triad in every S and r, θ, ϕ, t, can now be used as co-ordinates of space-time. From the definition of axial symmetry it follows at once that the coefficients of the fundamental form must be independent of ϕ. We can show that in the present system of co-ordinates the t-lines will be normal to the r-lines. This is seen by noting that the t-line of a point, the r, θ, ϕ co-ordinates of which are kept fixed, is described by propagating the geodesic radius vector, which is normal to g, parallelly along it, and it is well known that its extremity then describes a curve which is at all points normal to the geodesic radius vector. It can further be shown to follow from the definition of axial symmetry that the t-lines are normal to the θ-lines. Consider the surface Σ in S formed by all the points geodesically equidistant from O, and let this surface intersect a in a point P. The directions of the t-lines at points of S can be projected on to S. Suppose this is done at all points of S lying on Σ. Then since the t-lines are normal to the r-lines the projections will lie in Σ, forming a congruence on Σ. Now, since all directions at P in Σ must be equivalent, this congruence must lie symmetrically about P, and hence must cut the θ-lines orthogonally. It follows that the t-lines are normal to the θ-lines. From these last two results it follows that the product terms $drdt$ and $d\theta dt$ are absent in the fundamental form, which will then be of the form

$$ds^2 = dr^2 + Ad\theta^2 + Bd\phi^2 + Cd\phi dt + Ddt^2$$

where the coefficients are functions or r and θ only.

§ 3. Calculation of Ricci Tensor.

For analytical purposes the co-ordinate system established in the preceding section is not the most convenient. We therefore apply the transformation

$$x^1 = x^1(r, \theta), \qquad x^2 = x^2(r, \theta), \qquad x^3 = \phi, \qquad x^4 = t, \tag{3.1}$$

thus obtaining the slightly more general form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{mn}dx^m dx^n, \qquad \begin{pmatrix} \alpha, \beta = 1, 2 \\ m, n = 3, 4 \end{pmatrix} \tag{3.2}$$

where the $g_{\alpha\beta}$ and the g_{mn} are functions of the co-ordinates x^1 and x^2 only. In what follows, unless the contrary be explicitly stated, it will be understood that Greek indices are to have the range 1, 2 and Roman indices the range 3, 4. The summation convention is adhered to, with the understanding that repeated Greek and Roman indices are to be summed through the ranges 1, 2 and 3, 4 respectively.

We note that the transformation (3.1) can always be chosen so that $g_{11} = g_{22}$ and $g_{12} = 0$. If we suppose this to have been done, we may write (3.2)

$$ds^2 = e^{2\psi}(dx^1dx^1 + dx^2dx^2) + l dx^3dx^3 + 2m dx^3dx^4 - f dx^4dx^4. \tag{3.3}$$

It will be convenient first of all to obtain the expressions for the components of the Ricci tensor of the form (3.2), in tensor form with regard to transformations of the type (3.1). We notice that with respect to this group of transformations the functions g_{mn}, g^{mn}, D, where D is defined by

$$D^2 = -\mid g_{mn} \mid,$$

transform like invariants, and that their partial derivatives transform like vectors in the (x^1, x^2)-surfaces. We denote partial differentiation by small Greek suffixes. These suffixes can then be raised and lowered in the usual way with respect to the form

$$d\sigma^2 = g_{\alpha\beta}dx^\alpha dx^\beta. \tag{3.4}$$

Covariant derivation with respect to (3.4) we denote by a comma preceding the suffix, or alternatively by the symbol $\dfrac{\delta}{\delta x^\alpha}$. For the Christoffel symbols of the form (3.2), containing the indices 3 or 4, we have the formulæ

$$\left\{\begin{array}{lll} \left\{\begin{matrix} l \\ mn \end{matrix}\right\} = 0, & \left\{\begin{matrix} m \\ \alpha\beta \end{matrix}\right\} = 0, & \left\{\begin{matrix} \alpha \\ \beta m \end{matrix}\right\} = 0, \\ \left\{\begin{matrix} m \\ \alpha n \end{matrix}\right\} = \frac{1}{2}g^{ms}g_{ns\alpha}, & \left\{\begin{matrix} \alpha \\ mn \end{matrix}\right\} = -\frac{1}{2}g^{\alpha}_{mn}. & \end{array}\right. \tag{3.5}$$

Substituting these values in the usual expressions for the components of the Ricci tensor in terms of the Christoffel symbols, we obtain, after some simplification, the following formulæ for the non-vanishing components

$$\left\{\begin{array}{l} R^\alpha_\beta = K^\alpha_\beta + D^{-1}D^\alpha_{.,\beta} - D^{-1}D^\alpha D_\beta - \frac{1}{4}g^\alpha_{mn}g^{mn}_\beta, \\ R^m_n = \frac{1}{2}D^{-1}\dfrac{\delta}{\delta x^\alpha}(Dg^{ms}g^\alpha_{ns}), \end{array}\right. \tag{3.6}$$

where K^α_β is the Ricci tensor of (3.4). Writing these formulæ explicitly for the form (3.3), we obtain

$$\left\{\begin{array}{l} \sqrt{(-g)}R^\alpha_\beta = D\Delta\psi\delta^\alpha_\beta + D_{\alpha,\beta} - \frac{1}{4}D^{-1}(l_\alpha f_\beta + l_\beta f_\alpha + 2m_\alpha m_\beta), \\ \sqrt{(-g)}R^3_3 = \dfrac{1}{2}\dfrac{\partial}{\partial x^\alpha}\left[\dfrac{fl_\alpha + mm_\alpha}{D}\right], \\ \sqrt{(-g)}R^3_4 = \dfrac{1}{2}\dfrac{\partial}{\partial x^\alpha}\left[\dfrac{fm_\alpha - mf_\alpha}{D}\right], \\ \sqrt{(-g)}R^4_3 = \dfrac{1}{2}\dfrac{\partial}{\partial x^\alpha}\left[\dfrac{ml_\alpha - lm_\alpha}{D}\right], \\ \sqrt{(-g)}R^4_4 = \dfrac{1}{2}\dfrac{\partial}{\partial x^\alpha}\left[\dfrac{lf_\alpha + mm_\alpha}{D}\right], \\ \sqrt{(-g)}(R^3_3 + R^4_4) = \Delta D, \end{array}\right. \tag{3.7}$$

where Δ is the ordinary Laplacian operator in the variables x^1 and x^2.

§ 4. The Energy Tensor.

We consider now the energy tensor of a distribution of particles rotating round the axis of symmetry. We suppose that from the point of view of the observer O of § 2, the particles are describing the ϕ-lines in the space S. It follows then that the first two components of the unit tangent vector to their world-lines vanish. We furthermore suppose that the particles are describing their paths without mutual interaction, the world-line of each particle being a geodesic in space-time. The energy tensor of such a system of particles is of the form

$$T^r_s = \mu\lambda^r\lambda_s, \qquad (r, s = 1, 2, 3, 4)$$

where μ is the density of the particles and λ^r is the unit tangent vector to their world-lines. In order to satisfy the condition of axial symmetry, it is of course necessary that the density μ be a function of x^1 and x^2 only. In the present case we have $\lambda^1 = \lambda^2 = 0$. If we write $\Omega = \lambda^3/\lambda^4$, we find for the components of the unit tangent vector

$$\lambda^3 = \frac{\Omega}{\sqrt{(f - 2\Omega m - \Omega^2 l)}},$$
$$\lambda^4 = \frac{1}{\sqrt{(f - 2\Omega m - \Omega^2 l)}}.$$

We obtain for the non-vanishing components of the energy tensor the expressions

$$(4.1) \qquad \begin{cases} T^3_3 = \mu\dfrac{\Omega^2 l + \Omega m}{f - 2\Omega m - \Omega^2 l}, & T^3_4 = \mu\dfrac{\Omega^2 m - \Omega f}{f - 2\Omega m - \Omega^2 l}, \\ T^4_3 = \mu\dfrac{\Omega l + m}{f - 2\Omega m - \Omega^2 l}, & T^4_4 = \mu\dfrac{\Omega m - f}{f - 2\Omega m - \Omega^2 l}. \end{cases}$$

The vanishing of the divergence of the energy tensor gives the equations

$$(4.2) \qquad T^{rs}_{\cdot\cdot, s} = \frac{\partial\mu}{\partial x^s}\lambda^r\lambda^s + \mu\lambda^r_{\cdot, s}\lambda^s + \mu\lambda^r\lambda^s_{\cdot, s} = 0. \quad (r, s = 1, 2, 3, 4).$$

Since $\lambda^\alpha = 0$, and $\dfrac{\partial\mu}{\partial x^m} = 0$, the first term in (4.2) disappears. Using the fact that $\lambda^\alpha = \lambda_\alpha = 0$ and that λ^m is a function of x^1 and x^2 only, we find that

$$(4.3) \qquad \begin{cases} \lambda^m{}_{,n} = \lambda^\alpha{}_{,\beta} = 0, \\ \lambda^m{}_{,\alpha} = \dfrac{\partial\lambda^m}{\partial x^\alpha} + \begin{Bmatrix} m \\ \alpha n \end{Bmatrix}\lambda^n, \\ \lambda^\alpha{}_{,m} = \begin{Bmatrix} \alpha \\ mn \end{Bmatrix}\lambda^n. \end{cases}$$

Substituting from (4.3) in (4.2), the latter reduce to the two equations

$$\mu\begin{Bmatrix} a \\ mn \end{Bmatrix}\lambda^m\lambda^n = 0 \tag{4.4}$$

which, using (3.5), may be written

$$\mu(f_a - 2\Omega m_a - \Omega^2 l_a) = 0. \tag{4.5}$$

We now suppose that Ω is a constant, and that $\mu \neq 0$, then the equations (4.5) can be integrated at once, giving

$$f - 2\Omega m - \Omega^2 l = \text{constant}. \tag{4.6}$$

It is important to notice that the equation (4.6) holds only when $\mu \neq 0$. We now adopt a new system of co-ordinates with the transformation

$$\bar{x}^1 = x^1, \qquad \bar{x}^2 = x^2, \qquad \bar{x}^3 = x^3 - \Omega x^4, \qquad \bar{x}^4 = x^4. \tag{4.7}$$

This transformation constitutes a change to a rotating system of reference, relative to which the matter is at rest. If the bars be omitted, which may be done without danger of ambiguity, the fundamental form becomes

$$ds^2 = e^{2\psi}(dx^1dx^1 + dx^2dx^2) + Ldx^3dx^3 + 2Mdx^3dx^4 - Fdx^4dx^4, \tag{4.8}$$

where

$$\begin{cases} L = l, \\ M = m + \Omega l, \\ F = f - 2\Omega m - \Omega^2 l. \end{cases} \tag{4.9}$$

If we denote the components of the energy tensor in the new system of co-ordinates by accented letters, we find

$$\begin{cases} 'T^3_3 = T^3_3 - \Omega T^4_3, \\ 'T^3_4 = \Omega T^3_3 + T^3_4 - \Omega^2 T^4_3 - \Omega T^4_4, \\ 'T^4_3 = T^4_3, \\ 'T^4_4 = \Omega T^4_3 + T^4_4, \end{cases} \tag{4.10}$$

the remaining components being zero as before. If we now substitute from (4.1), using (4.9), we find that the quantity Ω disappears from the expressions, and we obtain, finally,

$$\begin{cases} 'T^3_3 = 0, & 'T^3_4 = 0, \\ 'T^4_3 = \mu\dfrac{M}{F}, & 'T^4_4 = -\mu. \end{cases} \tag{4.11}$$

The equation (4.6), yielded by the vanishing of the divergence of the energy tensor, becomes in the present system of co-ordinates

$$F = \text{constant}. \tag{4.12}$$

§ 5. THE GRAVITATIONAL EQUATIONS.

Since the fundamental form (4.8) is of exactly the same type as (3.3), we obtain the expressions for the components of the Ricci tensor in the system of co-ordinates of § 4, by writing F, L, M, for f, l, m, in (3.7). We write the gravitational equations in the form

$$R^i_j = -\kappa(T^i_j - \tfrac{1}{2}T\delta^i_j), \qquad (i, j = 1, 2, 3, 4) \tag{5.1}$$

where we may, without ambiguity, omit the accents relating to the present system of co-ordinates. We have, from (4.11),

$$T = T^4_4 = -\mu,$$

and hence we deduce from (5.1),

$$R^3_3 + R^4_4 = 0,$$

and this gives, by (3.7),

$$\Delta D = 0. \tag{5.2}$$

On the basis of this equation we may proceed with the introduction of Weyl's canonical co-ordinates, the application of which to the stationary case was first noted by Lewis (*loc. cit.*). We define a transformation

$$r = D, \qquad z = D', \tag{5.3}$$

where D' is a function of x^1 and x^2 such that

$$D + iD' = f(x^1 + ix^2),$$

which, if (5.2) is satisfied, is always possible. This transformation leaves the fundamental form of the (x^1, x^2)-surfaces in the isothermal form, and hence occasions no change in the expressions for the Ricci tensor. We suppose the transformation to the co-ordinates r, z to have been effected, but we will retain the indicial notation whenever convenient, suffixes 1 and 2 referring to differentiation with respect to r and z respectively. We may then put $D = r$ in all formulæ thus far calculated. For $D_{\alpha, \beta}$ we have

$$D_{1,1} = -\psi_1, \qquad D_{1,2} = -\psi_2, \qquad D_{2,2} = \psi_1. \tag{5.4}$$

Considering now the remaining gravitational equations, if we substitute from (3.7), (4.11), and (5.4) in (5.1), and write for convenience $\rho = \kappa\mu\sqrt{(-g)}$, we obtain after some simplification:

$$\Delta\psi = \frac{1}{4r^2}(L_1F_1 + L_2F_2 + M_1^2 + M_2^2) - \frac{\rho}{2r}, \tag{5.5}$$

$$(5.6) \qquad \psi_1 = -\frac{1}{4r}(L_1F_1 + M_1^2 - L_2F_2 - M_2^2),$$

$$(5.7) \qquad \psi_2 = -\frac{1}{4r}(L_1F_2 + L_2F_1 + 2M_1M_2),$$

$$(5.8) \qquad \frac{\partial}{\partial x^a}\left[\frac{FM_a - MF_a}{r}\right] = 0,$$

$$(5.9) \qquad \frac{\partial}{\partial x^a}\left[\frac{FL_a - LF_a}{r}\right] = -2\rho,$$

$$(5.10) \qquad \frac{\partial}{\partial x^a}\left[\frac{ML_a - LM_a}{r}\right] = -2\rho\frac{M}{F}.$$

§ 6. Solution of the Gravitational Equations.

We first of all remark that the equations (5.8) to (5.10) are not independent; (5.10) may be deduced from (5.8) and (5.9). To obtain the general solution of the equations we make use of the relation (4.12). Putting

$$(6.1) \qquad F = 1,$$

the equations (5.5) to (5.9) become

$$(6.2) \qquad \Delta\psi = \frac{1}{4r^2}(M_1^2 + M_2^2) - \frac{\rho}{2r},$$

$$(6.3) \qquad \psi_1 = -\frac{1}{4r}(M_1^2 - M_2^2),$$

$$(6.4) \qquad \psi_2 = -\frac{1}{2r}M_1M_2,$$

$$(6.5) \qquad M_{11} + M_{22} - \frac{1}{r}M_1 = 0,$$

$$(6.6) \qquad L_{11} + L_{22} - \frac{1}{r}L_1 = -2r\rho.$$

We notice that the equation (6.5) expresses the condition of integrability of the equations (6.3) and (6.4). We may therefore attempt to obtain solutions of the equations by choosing a function M satisfying (6.5). The equations (6.3) and (6.4) being integrable then determine ψ. We are then left with the equations (6.2) and (6.6) to determine L and ρ.

The functions L and M, however, are not independent. Owing to the particular choice of the co-ordinate system we have from (5.3)

$$r^2 = FL + M^2, \tag{6.7}$$

and hence by (6.1)

$$L = r^2 - M^2. \tag{6.8}$$

Substituting from (6.8) in (6.6), and using the fact that M satisfies (6.5), the equation (6.6) becomes

$$M_1^2 + M_2^2 = r\rho. \tag{6.9}$$

We may consider this equation to define the density distribution, and then the equations (6.3) to (6.6) are all satisfied. If we now calculate $\Delta\psi$ from (6.3) and (6.4), and substitute the result in (6.2), this equation becomes identical with (6.9), so that it also is satisfied. We have therefore shown that the general solution of the equations depends upon an arbitrary solution of the equation (6.5).

If we write $\phi = x^3$, $ct = x^4$, we have for the fundamental form

$$ds^2 = e^{2\psi}(dr^2 + dz^2) + (r^2 - M^2)d\phi^2 + 2Mcd\phi dt - c^2dt^2, \tag{6.10}$$

where M is any solution of (6.5), ψ is determined from (6.3) and (6.4), and where the density μ is given by the equation

$$\kappa\mu = \frac{1}{r^2}e^{-2\psi}(M_1^2 + M_2^2). \tag{6.11}$$

We cannot deduce solutions for the external field from the present solution by putting $\mu = 0$, for we then see from (6.11) that this implies $M =$ constant, and the resulting solution is trivial, space-time being then galilean. The reason is that the equation (6.1) was deduced from the vanishing of the divergence of the energy tensor on the supposition $\mu \neq 0$, and hence does not necessarily hold when the energy tensor vanishes.

§ 7 The Field of an Infinite Rotating Cylinder.

We now consider the particular solution of the equations which is obtained by supposing M to be a function of r only. The equation (6.5) then reads

$$M_{11} - \frac{1}{r}M_1 = 0,$$

and this yields on integration

$$M = ar^2, \tag{7.1}$$

where a is a constant of integration. Determining the function ψ from (6.3) and (6.4), we find

$$e^{2\psi}=e^{-a^2r^2}.$$

The equation (6.11) for the density now reads

$$\kappa\mu=4a^2e^{-a^2r^2}. \tag{7.2}$$

We obtain therefore the fundamental form

$$ds^2=H(dr^2+dz^2)+Ld\phi^2+2Md\phi dt-Fdt^2, \tag{7.3}$$

where

$$\begin{cases} H=e^{-a^2r^2}, & L=r^2(1-a^2r^2), \\ M=acr^2, & F=c^2. \end{cases} \tag{7.4}$$

The present system of co-ordinates constitutes, as we have seen, a system of reference relative to which the matter composing the cylinder is at rest. We define the angular velocity of the cylinder to be the angular velocity relative to a non-rotating system of reference associated with an observer on the axis of symmetry, using Walker's definition of non-rotating (1935). Walker defines a non-rotating system of reference for an observer to be such that in it the acceleration of a free isolated particle in the neighbourhood of the observer is independent of its velocity. This clearly corresponds to what is meant by a non-rotating system of reference in Newtonian dynamics. We may call such a system a dynamical rest frame for the observer. Walker has shown that for such a system the *unit tangent vectors in the direction of the space axes must be defined* by Fermi-transport along the world-line of the observer, which, since the world-line in the present instance is a geodesic, reduces to ordinary parallel transport.

Let us denote the unit tangent vector to the r-lines by ξ^i ($i=1, 2, 3, 4$). Then if ξ^i is transported parallelly along the world-line of the observer the equations

$$\frac{\partial\xi^i}{\partial x^a}\frac{dx^a}{ds}+\left\{ {i \atop ab} \right\}\xi^a\frac{dx^b}{ds}=0 \quad (i, a, b=1, 2, 3, 4) \tag{7.5}$$

must be satisfied at the origin. We have

$$\begin{cases} \xi^i=(H^{-\frac{1}{2}}, 0, 0, 0), \\ \dfrac{dx^a}{ds}=(0, 0, 0, F^{-\frac{1}{2}}). \end{cases} \tag{7.6}$$

Substituting from (7.6) in (7.5), and using (3.5), we obtain the two equations

$$D^{-2}(FM_1-MF_1)H^{-\frac{1}{2}}F^{-\frac{1}{2}}=0, \tag{7.7}$$

$$D^{-2}(MM_1+LF_1)H^{-\frac{1}{2}}F^{-\frac{1}{2}}=0. \tag{7.8}$$

We now apply the transformation (4.7), where we write Ω' instead of Ω, to the form (7.3), and calculate the expressions on the left-hand side of (7.7) and (7.8) for this new form, obtaining the equations

$$r(c - a\Omega' r^2) = 0, \tag{7.9}$$

$$\frac{\Omega' + ac}{r} - 2a^2 r\Omega' + a^3 r^3 c^{-1}\Omega'^2 = 0. \tag{7.10}$$

These equations are satisfied at the origin only if

$$\Omega' = -ac. \tag{7.11}$$

It may be shown similarly that the unit tangent vector to the z-lines is propagated parallelly along the world-line of the observer, independent of the value of Ω'. We therefore obtain the expression for the fundamental form in a non-rotating system of co-ordinates, by applying the transformation (4.7), where we substitute for Ω the value given by (7.11). When this is done we obtain the form (7.3), where

$$\begin{cases} H = e^{-a^2 r^2}, & L = r^2(1 - a^2 r^2), \\ M = a^3 c r^4, & F = c^2(1 + a^2 r^2 + a^4 r^4). \end{cases} \tag{7.12}$$

§ 8. Interpretation of the Solution.

We see from (7.11) that the angular velocity ω of the cylinder is given by

$$\omega = ac. \tag{8.1}$$

If we denote the density of the cylinder on the axis of symmetry by μ_0, we find from (7.2), if we substitute for κ its value in terms of Newton's gravitational constant γ, that

$$a^2 c^2 = 2\pi\gamma\mu_0, \tag{8.2}$$

and hence we have for the angular velocity

$$\omega = \sqrt{2\pi\gamma\mu_0}. \tag{8.3}$$

If we suppose the cylinder to be of the density of water on the axis of symmetry, so that $\mu_0 = 1$, the period of rotation of the cylinder is approximately 2 hours 42 minutes. If we denote by R the value of r on the boundary of the cylinder, then we see from (7.12) that we must have

$$aR < 1,$$

otherwise the coefficient of $d\phi^2$ in the fundamental form is negative in the interior of the cylinder. For a cylinder of given density therefore there is an upper limit for the radius. Since $\omega = ac$, the inequality may also be written

$$\omega R < c,$$

so that the upper limit of R is the same as the upper limit of the radius of a rotating cylinder in the special theory of relativity. The quantity R, however, is not the radius of the cylinder but connected with it by the equation

$$R' = \int_0^R e^{-\frac{1}{2}a^2r^2}dr,$$

where R' denotes the radius. If $\mu_0 = 1$ we find that the maximum radius is approximately $3{\cdot}5 \times 10^8$ K.M.

We now consider with what angular velocity a particle is to describe a ϕ-line if its world-line is to be a geodesic in space-time. We write the equations of the geodesics in the Lagrangian form

$$\frac{d}{ds}\frac{\partial T}{\partial x'^i} - \frac{\partial T}{\partial x^i} = 0, \qquad (i = 1, 2, 3, 4)$$

where

$$T = H(r'^2 + z'^2) + L\phi'^2 + 2M\phi' t' - Ft'^2,$$

dashes denoting differentiation with respect to the arc s. We attempt to satisfy these equations by putting r = constant, z = constant. The only pertinent equation is then seen to be

$$\frac{d}{ds}\frac{\partial T}{\partial r'} - \frac{\partial T}{\partial r} = 0,$$

which gives a quadratic for $d\phi/dt$, namely

$$(8.4) \qquad L_1 d\phi^2 + 2M_1 d\phi dt - F_1 dt^2 = 0.$$

Substituting from (7.12) in (8.4), and solving the quadratic, we obtain the roots

$$(8.5) \qquad \omega_1 = ac,$$

$$(8.6) \qquad \omega_2 = -\frac{1 + 2a^2r^2}{1 - 2a^2r^2}ac.$$

The first root gives the angular velocity of the cylinder, verifying that the world-lines of the particles composing the cylinder are geodesics. If we suppose a thin tube hollowed out in the cylinder along a ϕ-line, the second root gives the angular velocity with which a particle must be endowed in order to traverse the tube in a sense contrary to the sense of rotation of the cylinder.

The unit tangent vectors to the world-lines of the particles composing the cylinder must always be time-like. To investigate whether this is the case, we consider the null-directions in the (ϕ, t)-surfaces. These are given by

$$(8.7) \qquad L d\phi^2 + 2M d\phi dt - F dt^2 = 0.$$

Solving this quadratic we obtain the values of $d\phi/dt$ corresponding to the null-directions. They are found to be

$$\Omega_1 = \frac{1 - a^3r^3}{1 - a^2r^2}\frac{ac}{ar}, \tag{8.8}$$

$$\Omega_2 = -\frac{1 + a^3r^3}{1 - a^2r^2}\frac{ac}{ar}. \tag{8.9}$$

We see from (8.8) that, for all values of r, we have $\Omega_1 > ac$, and hence the tangent vectors to the world-lines of the particles are always time-like. Comparing (8.6) and (8.9), we see that when $ar = \frac{1}{2}$, we have $\Omega_2 = \omega_2$. Hence when $ar \longrightarrow \frac{1}{2}$, the velocity with which a particle must be projected in order to describe a ϕ-line, in a sense opposite to the sense of rotation of the cylinder, tends to the velocity of light. At points where r exceeds this value it will be impossible for a particle to describe a ϕ-line in this sense. If we consider the case of a cylinder whose density and radius are such that $aR = \frac{1}{2}$, then on the boundary a light signal sent out in the direction of a ϕ-line, and in a sense opposite to the sense of rotation of the cylinder, will travel along the ϕ-line. An observer therefore on the surface of the cylinder will be able to look right round the cylinder. Assuming $\mu_0 = 1$, we find that if the observer is at rest on the surface of the cylinder, a ray of light sent out by him returns in approximately 40 minutes.

Returning to equation (8.6), we see that when $ar = \frac{1}{\sqrt{2}}$, the angular velocity ω_2 becomes infinite. The ϕ-line is then a space-like geodesic. The length l of a ϕ-line is given by

$$l = \int_0^{2\pi} r(1 - a^2r^2)^{\frac{1}{2}} d\phi = 2\pi r(1 - a^2r^2)^{\frac{1}{2}},$$

and we see that l is a maximum when $ar = \frac{1}{\sqrt{2}}$. As r increases beyond this value the length of successive ϕ-lines diminishes. When $ar \longrightarrow 1$ the length of the corresponding ϕ-line $\longrightarrow 0$. If the cylinder is such that $aR = 1$, then all geodesics issuing from the origin in the planes $z =$ constant, meet again on the boundary, which then reduces to a line, the antipodal line of the axis of symmetry. We will return to this point when we consider the external solution.

Before proceeding to the question of the external field and boundary conditions, we may consider the Newtonian analogue of the present solution. If in Newtonian potential theory an infinite liquid cylinder of uniform density is endowed with a constant angular velocity Ω, there exists a definite value for Ω, which will reduce the pressure everywhere to zero. If μ is the density of the cylinder and γ the gravitational constant, we find for Ω the value

$$\Omega = \sqrt{2\pi\mu\gamma}.$$

The present paper is concerned with the gravitational field of such a cylinder according to the general theory of relativity. We see that we obtain the same value for the angular velocity, but in the present case the density is not constant, it decreases with increasing distance from the axis. If the radius of the cylinder is small, however, the density does not vary very much from its value on the axis of symmetry.

§ 9. Lewis's Solution for the External Field.

In order to complete the solution we must now consider the external field and the boundary conditions. We make use of Lewis's solution for the external field of a rotating cylinder (Lewis, 1932). The form in which Lewis gives this solution is not a convenient one from the point of view of determining the constants occurring in it by means of the boundary conditions. We will therefore here obtain this solution by a different method, and in a form more convenient for our purposes.

We return to the equations (5.5) to (5.10), where we put $\rho = 0$. With Lewis we remark that the conditions of integrability of (5.6) and (5.7), and the condition of compatibility of these equations with (5.5), are contained in the system (5.8) to (5.10). These last three equations are not independent, any one being a consequence of the remaining two. We consider (5.9) and (5.10), and make the substitution

$$u = \frac{F}{L}, \qquad v = \frac{M}{L}. \tag{9.1}$$

Using the relation (6.7), the equations (5.9) and (5.10) become

$$\frac{\partial}{\partial x^a}\left[\frac{ru_a}{u+v^2}\right] = 0, \qquad \frac{\partial}{\partial x^a}\left[\frac{rv_a}{u+v^2}\right] = 0. \tag{9.2}$$

We attempt to obtain solutions of these equations by putting

$$u_a = \Theta_a(u+v^2), \qquad v_a = \Phi_a(u+v^2). \tag{9.3}$$

Substituting from (9.3) in (9.2), we see that the functions Θ and Φ must satisfy the equation

$$\Theta_{11} + \Theta_{22} + \frac{1}{r}\Theta_1 = 0. \tag{9.4}$$

Choosing two arbitrary solutions of this equation, we now attempt to obtain u and v from the system of equations (9.3). This system of first order partial differential equations is complete, that is to say, the conditions of integrability are identically satisfied if

$$\Theta_1\Phi_2 - \Theta_2\Phi_1 = 0,$$

as may easily be verified. This relation involves

$$\Phi = f(\Theta),$$

and since Θ and Φ are both solutions of (9.4), it is easily seen that we have

$$\Phi = A\Theta + B,$$

where A and B are constants. It is then evident from (9.3) that this implies

(9.5) . . . . . . $v = Au + B.$

The two sets of equations in (9.3) are then equivalent, and they reduce to an ordinary differential equation which may be written

(9.6) . . . . $$\frac{du}{A^2u^2 + (2AB + 1)u + B^2} = d\Theta.$$

In the integration of (9.6) three cases arise, according as $4AB + 1$ is $>$, $=$ or < 0. In the case $4AB + 1 < 0$, we obtain, introducing different constants,

$$-u = \alpha^2 + \beta^2 + 2\alpha\beta \coth 2\Theta,$$
$$-v = \alpha \coth 2\Theta + \beta,$$

and these give

(9.7) . . . . $$\begin{cases} r^{-1}L = \dfrac{1}{\alpha} \sinh 2\Theta, \\ r^{-1}M = -\cosh 2\Theta - \dfrac{\beta}{\alpha} \sinh 2\Theta, \\ r^{-1}F = -2\beta \cosh 2\Theta - \dfrac{\alpha^2 + \beta^2}{\alpha} \sinh 2\Theta. \end{cases}$$

Substituting from (9.7) in (5.6) and (5.7) we obtain for ψ the equations

(9.8) . . . . . $$\begin{cases} \psi_1 = -\dfrac{1}{4r} + r(\Theta_1^2 - \Theta_2^2), \\ \psi_2 = 2r\Theta_1\Theta_2. \end{cases}$$

In the case $4AB + 1 < 0$, we have similarly

(9.9) . . . . $$\begin{cases} r^{-1}L = \dfrac{1}{\alpha} \cos 2\Theta, \\ r^{-1}M = \sin 2\Theta - \dfrac{\beta}{\alpha} \cos 2\Theta, \\ r^{-1}F = 2\beta \sin 2\Theta + \dfrac{\alpha^2 - \beta^2}{\alpha} \cos 2\Theta. \end{cases}$$

The equations for ψ are now

$$(9.10) \quad \begin{cases} \psi_1 = -\dfrac{1}{4r} - r(\Theta_1^2 - \Theta_2^2), \\ \psi_2 = -2r\Theta_1\Theta_2. \end{cases}$$

The case $4AB+1=0$ does not, in the present instance, require separate treatment.

§ 10. Boundary Conditions. The Case $aR < \frac{1}{2}$.

The external field of the cylinder is now obtained by choosing a solution of (9.4) which is a function of r alone. We find

$$(10.1) \quad \Theta = n \log\left(\frac{r}{r_0}\right),$$

We consider first of all the solution given by (9.7). Obtaining ψ from (9.8), we find for H

$$(10.2) \quad H = k\left(\frac{r}{R}\right)^{2n^2-\frac{1}{2}}.$$

We suppose that on the boundary of the cylinder we have $r=R$. We write $2\Theta = \theta_0 + \theta$, where

$$(10.3) \quad \begin{cases} \theta_0 = 2n \log\left(\dfrac{R}{r_0}\right), \\ \theta = 2n \log\left(\dfrac{r}{R}\right), \end{cases}$$

so that $\theta = 0$ on the boundary. Substituting in (9.7) we equate the values of the coefficients on the boundary with the internal boundary values given by (7.4). If we write for convenience $ct = x^4$, we obtain the equations

$$(10.4) \quad \begin{cases} \dfrac{R}{a}\sinh\theta_0 = R^2(1-a^2R^2), \\ -\cosh\theta_0 - \dfrac{a}{\beta}\sinh\theta_0 = aR, \\ -2\beta\cosh\theta_0 - \dfrac{a^2+\beta^2}{a}\sinh\theta_0 = R^{-1}. \end{cases}$$

We now calculate the derivatives of the coefficients of the form and equate their values on the boundary, obtaining the equations

$$(10.5) \quad \begin{cases} \dfrac{1}{a}(\sinh\theta_0 + 2n\cosh\theta_0) = 2R(1-2a^2R^2), \\ -(\cosh\theta_0 + 2n\sinh\theta_0) - \dfrac{a}{\beta}(\sinh\theta_0 + 2n\cosh\theta_0) = 2aR, \\ -2\beta(\cosh\theta_0 + 2n\sinh\theta_0) - \dfrac{a^2+\beta^2}{a}(\sinh\theta_0 + 2n\cosh\theta_0) = 0. \end{cases}$$

The continuity of H and its derivative across the boundary gives two more equations

(10.6) . . . . . . $k = e^{-a^2R^2}$,

(10.7) . . . . . $2n = \sqrt{(1 - 4a^2R^2)}$.

The constants k and n are defined by these equations. To satisfy the six remaining equations we have therefore only the three constants α, β, θ_0 at our disposal. We shall show that the equations are consistent. Eliminating θ_0 from the equations (10.4), and again from the equations (10.5), we obtain the two equations

$$2aR^2\beta + R^2(1 - a^2R^2)(\beta^2 - \alpha^2) = 1,$$
$$2aR^2\beta + R^2(1 - 2a^2R^2)(\beta^2 - \alpha^2) = 0.$$

Solving these two equations for α and β, we find

(10.8) . . . . $$\begin{cases} \beta = -\dfrac{1 - 2a^2R^2}{2a^3R^4}, \\ \alpha = \dfrac{\sqrt{(1 - 4a^2R^2)}}{2a^3R^4}. \end{cases}$$

Substituting these values of α and β in (10.4), we find

(10.9) . . . $$\begin{cases} \sinh\theta_0 = \dfrac{\sqrt{(1 - 4a^2R^2)}}{2a^3R^3}(1 - a^2R^2), \\ \cosh\theta_0 = \dfrac{1 - 3a^2R^2}{2a^3R^3}. \end{cases}$$

All the constants have now been determined in terms of the physical dimensions of the system. If the values of the constants that we have found are substituted in the remaining equations of (10.4) and (10.5), it will be found that these are also satisfied. Substituting from (10.8) and (10.9) in (9.7), we obtain the explicit expression for the coefficients of the fundamental form of the external region. It will be convenient to introduce a new constant ϵ, defined by

(10.10) . . . . $\tanh\epsilon = \sqrt{(1 - 4a^2R^2)}$.

We then obtain for the external field the form (7.3), where

(10.11) . . . . $$H = e^{-a^2R^2}\left(\frac{r}{R}\right)^{-2a^2R^2},$$

(10.12) . . . . $$L = Rr\frac{\sinh(3\epsilon + \theta)}{2\sinh 2\epsilon\cosh\epsilon},$$

(10.13) . . . . $$M = rc\frac{\sinh(\epsilon + \theta)}{\sinh 2\epsilon},$$

$$F=\frac{rc^2}{R}\frac{\sinh(\epsilon-\theta)}{\sinh\epsilon}, \tag{10.14}$$

and where θ is defined by

$$\theta=\sqrt{(1-4a^2R^2)}\log\frac{r}{R}. \tag{10.15}$$

The present system of co-ordinates is, as we have seen, to be interpreted as a rotating system. In order to obtain the field in a system of co-ordinates which is a dynamical rest frame for the observer on the axis of symmetry, we must apply the transformation (4.7) where $\Omega=-ac$. When this is done it will be found that for sufficiently great values of r, the coefficient of dt^2 changes its sign. This does not mean that the fundamental form changes its signature, for by (6.7) the determinant of the quadratic (8.7) is positive for all values of r, and hence, as long as L is positive a transformation of the type (4.7) can always be found which will transform the quadratic into the difference of two squares. We see, however, that the unit tangent vector to the world-line of a particle with fixed space co-ordinates in Walker's dynamical rest frame is space-like at sufficiently great distances from the axis of symmetry. The separation of space-time into space and time by means of Walker's rest frame therefore holds only in the neighbourhood of the observer. We can, however, find a system of co-ordinates in which the coefficient F remains positive throughout space. We apply the transformation (4.7), where we put

$$\Omega=-\frac{c}{R}\frac{\sinh\epsilon+\cosh\epsilon}{\sinh 3\epsilon+\cosh 3\epsilon}2\cosh\epsilon. \tag{10.16}$$

We then obtain the form (7.3), where H is given by (10.11) and

$$\left\{\begin{aligned} L&=R^2\left[\frac{\sinh 3\epsilon+\cosh 3\epsilon}{4\sinh 2\epsilon\cosh\epsilon}\left(\frac{r}{R}\right)^{1+2n}+\frac{\sinh 3\epsilon-\cosh 3\epsilon}{4\sinh 2\epsilon\cosh\epsilon}\left(\frac{r}{R}\right)^{1-2n}\right],\\ M&=-\frac{Rc}{\sinh 3\epsilon+\cosh 3\epsilon}\left(\frac{r}{R}\right)^{1-2n},\\ F&=c^2\frac{4\sinh 2\epsilon}{\sinh 3\epsilon+\cosh 3\epsilon}\left(\frac{r}{R}\right)^{1-2n},\end{aligned}\right. \tag{10.17}$$

and where n is defined by (10.7). We easily verify that in the present system of co-ordinates the cosine of the angle between the ϕ-lines and the t-lines tends to zero as r tends to infinity. Furthermore the angular velocities with which a particle must describe a ϕ-line in order that its world-line may be a geodesic in space-time, tend to become equal and opposite and both tend to zero as r tends to infinity. Hence the present system of co-ordinates may be described as one which is not rotating with respect to the fixed stars. We call the present system of co-ordinates an astronomical rest frame for the observer on the axis of symmetry.

The angular velocity ω' with which the cylinder is rotating in the present system of co-ordinates is obtained from (10.16). Expressing ω' in terms of R by using (10.10) we obtain

$$\omega' = \frac{ac}{2a^4R^4}[1 - 2a^2R^2 - \sqrt{(1 - 4a^2R^2)}],$$

$$(10.18) \quad \ldots \quad = ac(1 + 2a^2R^2 + 10a^4R^4 + \ldots).$$

Comparing (10.18) and (8.1), we see that Walker's dynamical rest frame rotates with an angular velocity ω'' relative to the astronomical rest frame, where ω'' is given by

$$(10.19) \quad \ldots \quad \omega'' = 2a^2R^2(1 + 5a^2R^2 + \ldots)ac.$$

If aR is small compared with unity, so that the radius of the cylinder is small compared with its maximum permissible value, then ω'' will be very small compared with ω'. If we suppose the cylinder to be of the density of water on the axis of symmetry, and its radius to be that of the earth, then Walker's system of reference will complete a revolution relative to the fixed stars in $7{\cdot}7 \times 10^5$ years.

§ 11. Boundary Conditions. The Case $aR \geqslant \frac{1}{2}$.

The equations (10.8) and (10.9) show that the solution of the preceding section for the external field is valid only if $aR < \frac{1}{2}$, otherwise the constants occurring in the solution are imaginary. The external field in the case $aR = \frac{1}{2}$ is simply obtained from the equations (10.11) to (10.14) by a limiting process. We find

$$(11.1) \quad \ldots \quad \left\{\begin{aligned} H &= e^{-\frac{1}{4}}\left(\frac{r}{R}\right)^{-\frac{1}{2}}, \\ L &= \tfrac{1}{4}Rr\left(3 + \log\frac{r}{R}\right), \\ M &= \tfrac{1}{2}rc\left(1 + \log\frac{r}{R}\right), \\ F &= \frac{rc^2}{R}\left(1 - \log\frac{r}{R}\right). \end{aligned}\right.$$

To obtain the external field in the case $aR > \frac{1}{2}$, we return to the solution (9.9). Proceeding exactly as before, we find

$$(11.2) \quad \ldots \quad \left\{\begin{aligned} H &= e^{-a^2R^2}\left(\frac{r}{R}\right)^{-2a^2R^2}, \\ L &= Rr\frac{\sin(3\epsilon + \theta)}{2\sin 2\epsilon\cos\epsilon}, \\ M &= rc\frac{\sin(\epsilon + \theta)}{\sin 2\epsilon}, \\ F &= \frac{rc^2}{R}\,\frac{\sin(\epsilon - \theta)}{\sin\epsilon}, \end{aligned}\right.$$

where

(11.3) . . . . . $$\theta=\sqrt{(4a^2R^2-1)}\log\frac{r}{R},$$

(11.4) . . . . . $$\tan\epsilon=\sqrt{(4a^2R^2-1)}.$$

The present solution only holds when $aR > \frac{1}{2}$.

We see from (11.2) that L, the coefficient of $d\phi^2$ in the fundamental form, is zero when

(11.5) . . . . . . $$3\epsilon+\theta=\pi.$$

Denoting the value of r at this point by r', we find

(11.6) . . . . $$r'=R\exp\left[\frac{\pi-3\tan^{-1}\sqrt{(4a^2R^2-1)}}{\sqrt{(4a^2R^2-1)}}\right].$$

When $r > r'$ the coefficient of $d\phi^2$ becomes negative and the fundamental form changes its signature. It is easy to show, however, that all geodesics in the surfaces $z=$constant, issuing from the origin, meet again at the point $r=r'$. It follows that these surfaces are closed, the point $r=r'$ being the antipodal point of the origin. We see from (11.6) that when $aR \longrightarrow \frac{1}{2}$, then $r' \longrightarrow \infty$, and hence the external region becomes infinite as the radius of the cylinder approaches its critical value. When $aR \longrightarrow 1$ it is seen that $r' \longrightarrow R$. Hence as the radius of the cylinder increases, the external region diminishes and finally vanishes when the radius reaches its maximum value. The cylinder then fills space completely. We see from (8.5) and (8.8) that both ω_1 and Ω_1 remain finite when $aR \longrightarrow 1$. It follows that the world-lines of the particles on the antipodal line are null geodesics. Hence as $aR \longrightarrow 1$, the velocity of the particles on the boundary tends to the velocity of light. The cylinder can therefore never fill space completely, there must always remain a small filament of empty space surrounding the cylinder.

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