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Problem. *Explain why the sample mean may not be a consistent estimator of the population mean for a Pareto distribution. (Klugman 5th ed, 10.14)*

Solution. *By definition, an estimator is (weakly) consistent if it satisfies the following condition:*

Let $\hat{\theta_n}$ *be an estimator of* θ*, if* $\forall\delta > 0$ *and any* θ*,* $\lim_{n\to\infty} Pr(|\hat{\theta_n} - \theta| > \delta) = 0$*, then* $\hat{\theta_n}$ *is a (weakly) consistent estimator of* θ*.*

Aside from definition, we can try to prove that the sample mean may not be a consistent estimator of the population mean for a Pareto distribution using the following theorem:

If $\lim_{n\to\infty} E[\hat{\theta_n}] = \theta$ *and* $\lim_{n\to\infty} Var[\hat{\theta_n}] = 0$*, then* $\hat{\theta_n}$ *is (weakly) consistent.*

The expectation and variation of a Pareto distribution with parameter (α, θ) *are (with* $\alpha \geq 1 \cap 2$*):*

$$E[X] = \frac{\theta}{\alpha - 1} \tag{1}$$

$$Var[X] = \frac{\theta^2 \times \alpha}{(\alpha - 1)^2 \times (\alpha - 2)} \tag{2}$$

We can immediately answer the question without using the above definition nor theorem (we may do so anyway later) and without involving the more complicated form of the expectation and variation of a Pareto distribution, observe that, if $\alpha = 1$*, the expectation would not exist, thus it is impossible for the sample mean, which will always exist to accurately estimate it.*

If we go to the Pareto Wikipedia page for Pareto distribution, we can also see that if $\alpha <= 1$*, then the mean for the Pareto distribution would not exist, thus if we only use the sample mean, which does not account for the value of* α *and will always exist, we cannot accurately estimate the mean for the Pareto distribution.*