



# 12

## Analog Filter Approximations

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### 12.1 Filter Definitions

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#### 12.1.1 Normalized *Ideal Low-pass Filter* (see [Figure 12.1a](#))

$$\begin{aligned} H(j\omega) &= e^{-j\omega} & 0 \leq |\omega| \leq 1 \\ &= 0 & |\omega| > 1 \end{aligned}$$

#### 12.1.2 Filter Transfer Function

$$H(j\omega) = |H(j\omega)| e^{j\theta(\omega)}$$

$$\theta(\omega) = \text{Arg } H(j\omega)$$

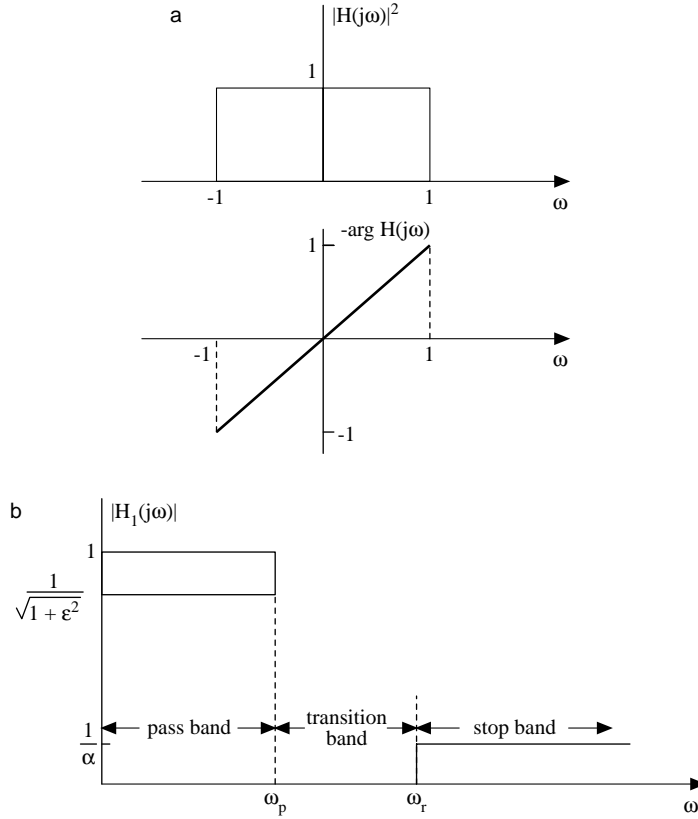
$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = \text{group delay}$$

$$\omega_c = \text{cutoff frequency at which } |H(j\omega_c)|^2 = \frac{1}{2}$$

or

$$20 \log |H(j\omega)|_{\omega=\omega_c} = -3 \text{ dB}$$

$$A(\omega) = -10 \log |H(j\omega)|^2 \quad (= \text{attenuation}) \text{ dB}$$



**FIGURE 12.1**

$$|H(j\omega)| = [H(j\omega)H^*(j\omega)]^{1/2} = [H(j\omega)H(-j\omega)]^{1/2} = [H(s)H(-s)]_{s=j\omega}^{1/2}$$

$$|H(j\omega)|^2 = \frac{K(\omega^2 + z_1^2)(\omega^2 + z_2^2)\cdots}{(\omega^2 + p_1^2)(\omega^2 + p_2^2)\cdots}$$

Complex poles and zeros occur in conjugate pairs. Both the numerator and denominator polynomials of the magnitude squared function of a transfer function are polynomials of  $\omega^2$  with real coefficients, and these polynomials are greater than zero for all  $\omega$ .

## 12.2 Butterworth Approximation

### 12.2.1 Definition of Butterworth Low-Pass Filter

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}; \quad |H(j\omega_c)|^2 = \frac{1}{2}$$

$$10\log|H(j\omega)|^2 \Big|_{\omega=\omega_c} = -3.01 \cong -3.0 \text{ dB}$$

Normalized

$$|H(j\omega)|^2 = \frac{1}{1 + \omega^{2n}}; \quad |H(j1)|^2 = \frac{1}{2}$$

## 12.3 Properties of Butterworth Approximation

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$$12.3.1 \quad |H(j0)|^2 = 1; \quad |H(j1)|^2 = \frac{1}{2}; \quad |H(j\infty)|^2 = 0$$

$$-10 \log |H(j1)|^2 = -10 \log 0.5 = 3.01 \cong 3.0 \text{ dB}$$

12.3.2  $|H(j\omega)|^2$  monotonically decreasing for  $\omega \geq 0$ . Its maximum value is at  $\omega = 0$ .

12.3.3 The first  $(2n - 1)$  derivatives of an  $n^{\text{th}}$ -order low-pass Butterworth filter are zero at  $\omega = 0$  (*maximally flat* magnitude).

12.3.4 The high-frequency roll-off of an  $n^{\text{th}}$ -order filter is  $20n \text{ dB/decade}$

$$-10 \log |H(j\omega)|^2 = -\log \frac{1}{1 + \omega^{2n}} \cong -\log \frac{1}{\omega^{2n}} = 10 \log \omega^{2n} = 20n \log \omega \text{ dB}$$

## 12.4 Transfer Function of Butterworth Approximation

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$$12.4.1 \quad |H(j\omega)|^2 = H(s)H(-s) \Big|_{s=j\omega} = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}} = \frac{1}{1 + (-1)^n \left(\frac{s}{\omega_c}\right)^{2n}}$$

Poles:

$$1 + (-1)^n \left(\frac{s}{\omega_c}\right)^{2n} = 0 \quad \text{or} \quad s_k = \omega_c e^{j\pi(1-n+2K)/2n}, \quad K = 0, 1, \dots, 2n-1$$

### 12.4.2 Stable Function

Left-half-plane poles are used

$$s_K = \omega_c \left[ -\sin \frac{(2K+1)\pi}{2n} + j \cos \frac{(2K+1)\pi}{2n} \right], \quad K = 0, 1, \dots, n-1$$

### 12.4.3 Transfer Function

$$H(s) = (-1)^n \prod_{K=0}^{n-1} \frac{s_K}{s - s_K}$$

### 12.4.4 Butterworth Normalized Low-Pass Filter

[Table 12.1](#) gives the Butterworth polynomials ( $\omega_c = 1$ ) to be used for normalized filters.

**TABLE 12.1** Butterworth Normalized and Factored Polynomials

$n$	Butterworth Polynomials
1	$s + 1$
2	$s^2 + 1.41421s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.84776s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$
6	$(s^2 + 0.51764s + 1)(s^2 + 1.41421s + 1)(s^2 + 1.93185s + 1)$
7	$(s + 1)(s^2 + 0.44504s + 1)(s^2 + 1.24798s + 1)(s^2 + 1.80194s + 1)$
8	$(s^2 + 0.39018s + 1)(s^2 + 1.11114s + 1)(s^2 + 1.66294s + 1)(s^2 + 1.96157s + 1)$
9	$(s + 1)(s^2 + 0.34730s + 1)(s^2 + s + 1)(s^2 + 1.53209s + 1)(s^2 + 1.87939s + 1)$
10	$(s^2 + 0.31287s + 1)(s^2 + 0.90798s + 1)(s^2 + 1.41421s + 1)(s^2 + 1.78201s + 1)(s^2 + 1.97538s + 1)$

### 12.4.5 Butterworth Filter Specifications (see also [Figure 12.1](#))

$A_p$  = maximum passband attenuation

$f_p$  = passband edge frequency

Maximum allowable attenuation in the stopband

$f_r$  = stopband edge frequency

$$A_p = 10 \log \left[ 1 + \left( \frac{\omega_p}{\omega_c} \right)^{2n} \right] \quad (\text{see also 12.3.4})$$

$$A_r = 10 \log \left[ 1 + \left( \frac{\omega_r}{\omega_c} \right)^{2n} \right]$$

$$\omega_p = 2\pi f_p$$

$$\omega_r = 2\pi f_r$$

Solve  $A_p$  and  $A_r$  to find

$$n = \frac{\left| \log[(10^{0.1A_p} - 1)/(10^{0.1A_r} - 1)] \right|}{\left| \log(\omega_p / \omega_r) \right|}$$

$$k = \text{selectivity parameter} = \frac{\omega_p}{\omega_r} = \frac{f_p}{f_r} < 1$$

$$d = \text{discrimination factor} = \left( \frac{10^{0.1A_p} - 1}{10^{0.1A_r} - 1} \right)$$

**Note:** a) larger values of  $k$  imply steeper roll off, b) smaller  $d$  values imply greater difference between  $A_p$  and  $A_r$

$$n \geq \frac{\left| \log d \right|}{\left| \log k \right|} \quad (\text{accept next higher integer to noninteger } n)$$

$$\omega_c = \frac{\omega_p}{(10^{0.1A_p} - 1)^{1/2n}}$$

$$\omega_c = \frac{\omega_r}{(10^{0.1A_r} - 1)^{1/2n}} \equiv \text{meets stopband attenuation exactly and exceeds the requirement of passband specification}$$

Figure 12.2 shows magnitude-squared characteristics of the Butterworth low-pass filter.

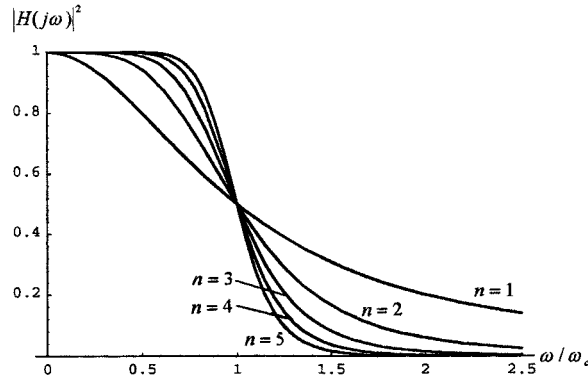


FIGURE 12.2

### Example 12.1 Butterworth Filter Design

Filter requirements: a) no more than 1.5 dB deviation from ideal filter at 1300 Hz; b) at least 35 dB for frequencies above 6000 Hz.

**Solution:**

$$A_p = 1.5 \text{ dB} \quad \omega_p = 2\pi \times 1300 \text{ rad s}^{-1}$$

$$A_r = 35 \text{ dB} \quad \omega_r = 2\pi \times 6000 \text{ rad s}^{-1}$$

$$d = \frac{\sqrt{10^{0.1A_p}} - 1}{\sqrt{10^{0.1A_r}} - 1} = \frac{\sqrt{10^{0.15}} - 1}{\sqrt{10^{3.5}} - 1} = \frac{0.6423}{56.2252} = 1.1424 \times 10^{-2}$$

$$n \geq \frac{|\log d|}{|\log k|} = \frac{1.9422}{0.6576} = 2.953 \quad \Rightarrow \quad n = 3$$

$$s_k = -\sin \frac{(2K+1)\pi}{2n} + j \cos \frac{(2K+1)\pi}{2n} \quad K = 0, 1, \dots, n-1$$

$$s_o = -\sin \frac{\pi}{6} + j \cos \frac{\pi}{6} = -\frac{1}{2} + j \frac{\sqrt{3}}{2}$$

$$s_1 = -\sin \frac{3\pi}{6} + j \cos \frac{3\pi}{6} = -1$$

$$s_2 = -\sin \frac{5\pi}{6} + j \cos \frac{5\pi}{6} = -\frac{1}{2} - j \frac{\sqrt{3}}{2}$$

$$\begin{aligned}
 H(s) &= (-1)^n \prod_{K=0}^{n-1} \frac{s_k}{s - s_k} = -\frac{\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}{s - \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)} \frac{-1}{s - (-1)} \frac{-\frac{1}{2} - j\frac{\sqrt{3}}{2}}{s - \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)} \\
 &= \frac{1}{(s+1)(s^2+s+1)} = \text{normalized}
 \end{aligned}$$

$$\omega_c = \omega_p (10^{0.1A_p} - 1)^{-1/2n} = 2\pi \times 1300 (10^{0.15} - 1)^{-1/6} = 9416 \text{ rad s}^{-1}$$

$$H\left(\frac{s}{\omega_c}\right) = \frac{1}{\left(\frac{s}{9461} + 1\right) \left[\left(\frac{s}{9461}\right)^2 + \frac{s}{9461} + 1\right]}$$

## 12.5 Chebyshev Filter Approximation

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### 12.5.1 Definition of Chebyshev Filters (equi-ripple passband)

$$C_o(\omega) = 1 \text{ and } C_1(\omega) = \omega. \quad |H(j\omega)|^2 = \frac{1}{1 + \varepsilon C_n^2(\omega)} = \text{normalized}$$

$$\begin{aligned}
 C_n(\omega) &= \text{Chebyshev polynomials} = \cos(n \cos^{-1} \omega) & 0 \leq \omega \leq 1 \\
 &= \cosh(n \cosh^{-1} \omega) & \omega > 1
 \end{aligned}$$

$\varepsilon = \text{ripple factor}$

If we set  $u = \cos^{-1} \omega$ , then  $C_n(\omega) = \cos nu$  and thus

$$\begin{aligned}
 C_o(\omega) &= \cos 0 = 1, C_1(\omega) = \cos u = \cos(\cos^{-1} \omega) = \omega, \quad C_2(\omega) = \cos 2u = 2 \cos^2 u - 1 = 2\omega^2 - 1, \\
 C_3(\omega) &= \cos 3u = 4 \cos^3 u - 3 \cos u = 4\omega^3 - 3\omega, \text{ etc.}
 \end{aligned}$$

### 12.5.2 Recursive Formula for Chebyshev Polynomials

From  $\cos[(n+1)u] = 2 \cos nu \cos u - \cos[(n-1)u]$ , we get

$$C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega) \quad n = 0, 1, 2, \dots$$

with  $C_o(\omega) = 1$  and  $C_1(\omega) = \omega$ . [Figure 12.3](#) shows the first five Chebyshev polynomials.

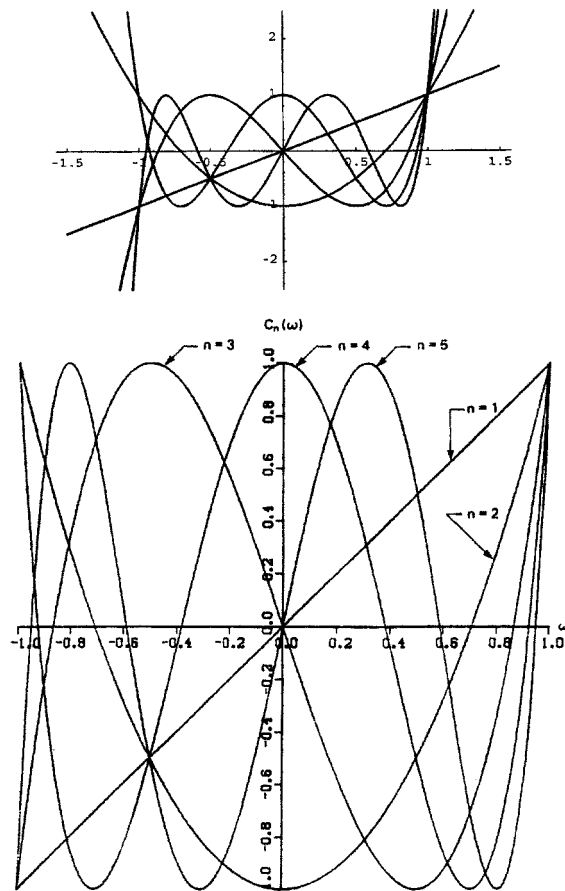


FIGURE 12.3 Chebyshev polynomials.

12.5.3 Table 12.2 gives the first ten Chebyshev polynomials

TABLE 12.2 Chebyshev Polynomials $C_n(\omega)$	
$n$	Chebyshev Polynomials $C_n(\omega)$
0	1
1	$\omega$
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$
9	$256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$
10	$512\omega^{10} - 1280\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$



## 12.5.4 Properties of the Chebyshev Polynomials

1. For any  $n$

$$\begin{aligned} 0 \leq |C_n(\omega)| \leq 1 & \quad \text{for } 0 \leq |\omega| \leq 1 \\ |C_n(\omega)| > 1 & \quad \text{for } |\omega| > 1 \end{aligned}$$

2.  $C_n(1) = 1$  for any  $n$
3.  $|C_n(\omega)|$  increases monotonically for  $|\omega| > 1$
4.  $C_n(\omega)$  is an even (odd) polynomial if  $n$  is even (odd)
5.  $|C_n(0)| = 0$  for odd  $n$
6.  $|C_n(0)| = 1$  for even  $n$

## 12.5.5 Chebyshev Magnitude Response Properties

1.  $|H(j\omega)|_{\omega=0} = 1$  when  $n$  is odd

$$= \frac{1}{\sqrt{1+\epsilon^2}} \quad \text{when } n \text{ is even}$$

2. Since  $C_n(1) = 1$  for any  $n$

$$|H(j1)| = \frac{1}{\sqrt{1+\epsilon^2}} \quad \text{for any } n$$

3.  $|H(j\omega)|$  decreases monotonically

## 12.5.6 Pole Location of Chebyshev Filters

$$|H(j\omega)|^2 = \frac{1}{1 + \epsilon C_n^2(\omega)} = \frac{1}{1 + \epsilon C_n^2(-js)} \bigg|_{s=j\omega}$$

$$s = \sigma + j\omega$$

$$\sigma_K = \pm \sin \left[ (2K+1) \frac{\pi}{2n} \right] \sinh \left[ \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon} \right]$$

$$\omega_K = \cos \left[ (2K+1) \frac{\pi}{2n} \right] \cosh \left[ \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon} \right] \quad K = 0, 1, \dots, 2n-1$$

$$\frac{\sigma_K^2}{\sinh^2 y} + \frac{\omega_K^2}{\cosh^2 y} = 1 \quad \text{an ellipse on the } \sigma - \omega \text{ plane}$$

$$y = \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon}$$

## 12.5.7 Design Relations of Chebyshev Filters

$$|H(j\omega)|^2 = \frac{1}{1 + \epsilon^2 C_n^2\left(\frac{\omega}{\omega_p}\right)}$$

$$\left|H(j\omega_p)\right|^2 = \frac{1}{1 + \epsilon^2 C_n^2\left(\frac{\omega}{\omega_p}\right)} \bigg|_{\omega=\omega_p} = \frac{1}{1 + \epsilon^2}$$

$$A_p = 10 \log(1 + \epsilon^2)$$

$$\epsilon = \sqrt{10^{0.1A_p} - 1}$$

$$A_r = 10 \log \left[ 1 + \epsilon^2 C_n^2\left(\frac{\omega_r}{\omega_p}\right) \right]$$

$$= 10 \log \left[ 1 + \epsilon^2 \cosh^2 \left[ n \cosh^{-1} \left( \frac{\omega_r}{\omega_p} \right) \right] \right]$$

$$n \geq \frac{\cosh^{-1} \left( \frac{10^{0.1A_r} - 1}{\epsilon^2} \right)^{1/2}}{\cosh^{-1} \left( \frac{\omega_r}{\omega_p} \right)}$$

$$k = \frac{\omega_p}{\omega_r} = \frac{f_p}{f_r}, \quad d = \left( \frac{10^{0.1A_p} - 1}{10^{0.1A_r} - 1} \right)^{1/2}$$

or

$$n \geq \frac{\cosh^{-1} \left( \frac{1}{d} \right)}{\cosh^{-1} \left( \frac{1}{k} \right)}$$

### Left-Hand Poles for the Transfer Function

$$s_K = \sin \left[ (2K+1) \frac{\pi}{2n} \right] \sinh \left[ \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon} \right] + j \cos \left[ (2K+1) \frac{\pi}{2n} \right] \cosh \left[ \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon} \right]$$

$$H(s) = - \prod_{K=0}^{n-1} \frac{s_K}{s - s_K}, \quad n \text{ odd}$$

$$H(s) = \frac{1}{\sqrt{1+\epsilon^2}} \prod_{K=0}^{n-1} \frac{s_K}{s-s_K}, \quad n \text{ even}$$

For non-normalized transfer function set  $s/\omega_p$  in place of  $s$

$$|H(j\omega_c)|^2 = \frac{1}{2} = \frac{1}{1+\epsilon^2 C_n^2(\omega_c)}, \quad 3-dB \text{ cutoff}$$

$$\omega_c = \cosh\left(\frac{1}{n} \cosh^{-1} \frac{1}{\epsilon}\right)$$

### Example 12.2 (Chebyshev Filter Design):

Filter requirements: a) ripple not to exceed 2 dB up to  $\omega_p$ ; b) 50 dB rejection above  $5 \omega_p$ .

#### Solution

$$A_p \leq 2 \text{ dB} \quad \text{at} \quad \omega = \omega_p$$

$$A_r \geq 50 \text{ dB} \quad \text{at} \quad \omega = \omega_r = 5\omega_p$$

$$\epsilon = (10^{0.1A_p} - 1)^{1/2} = (10^{0.2} - 1)^{1/2} = 0.765$$

$$k = \frac{\omega_p}{\omega_r} = \frac{\omega_p}{5\omega_p} = 0.2$$

$$d = \frac{0.765}{(10^{0.1A_r} - 1)^{1/2}} = \frac{0.765}{(10^5 - 1)^{1/2}} = 2.42 \times 10^{-3}$$

$$n \geq \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} = \frac{\ln(1/d + \sqrt{1/d^2 - 1})}{\ln(1/k + \sqrt{1/k^2 - 1})} = \frac{2.718}{2.312} = 2.91$$

accept  $n = 3$

From 12.5.6

$$y = \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon} = \frac{1}{n} \ln\left(\frac{1}{\epsilon} + \sqrt{\frac{1}{\epsilon^2} + 1}\right) = 0.361$$

$$\sinh y = \frac{e^y - e^{-y}}{2} = 0.3689 \quad \cosh y = \frac{e^y + e^{-y}}{2} = 1.0659$$

$$s_0 = \sin\left(\frac{\pi}{6}\right)(0.3689) + j \cos\left(\frac{\pi}{6}\right)(1.0659) = -0.1844 + j0.9231$$

$$s_1 = \sin\left(\frac{\pi}{2}\right)(0.3689) + j \cos\left(\frac{\pi}{2}\right)(1.0659) = -0.3689$$

$$s_2 = -\sin\left(\frac{\pi}{6}\right)(0.3689) + j \cos\left(\frac{\pi}{6}\right)(1.0659) = -0.1844 - j0.9231 = s_0^*$$

$$H(s) = \frac{0.3289}{(s + 0.3689)(s^2 + 0.3689s + 0.8861)}$$

To denormalize  $H(s)$  we set  $\omega_p = 2\pi f_p$  we set  $s/\omega_p$  in place of  $s$ . Table 12.3 gives the denominator or the normalized Chebyshev low-pass filters. Figure 12.4 shows the third-order filter with  $\omega_p = 2\pi \times 2$ .

**TABLE 12.3** Factors of the Denominator Polynomials Normalized Chebyshev Low-Pass Filters

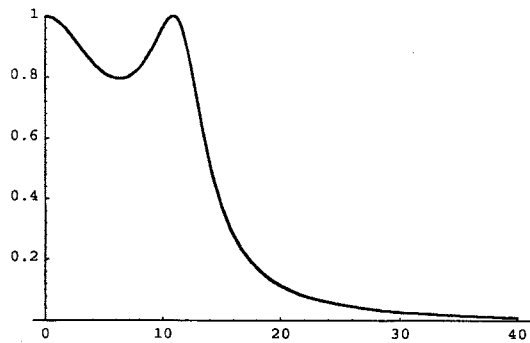
0.1-dB Ripple ( $\epsilon = 0.15262$ )	
n	
1	$s + 6.55220$
2	$s^2 + 2.37236s + 3.31403$
3	$(s + 0.96941)(s^2 + 0.96941s + 1.68975)$
4	$(s^2 + 0.52831s + 1.33003)(s^2 + 1.27546s + 0.62292)$
5	$(s + 0.53891)(s^2 + 0.33307s + 1.19494)(s^2 + 0.87198s + 0.63592)$
6	$(s^2 + 0.22939s + 1.12939)(s^2 + 0.62670s + 0.69637)(s^2 + 0.85608s + 0.26336)$
7	$(s + 0.37678)(s^2 + 0.16768s + 1.09245)(s^2 + 0.46983s + 0.75322)(s^2 + 0.67893s + 0.33022)$
8	$(s^2 + 0.12796s + 1.06949)(s^2 + 0.36440s + 0.79889)(s^2 + 0.54536s + 0.41621)(s^2 + 0.64330s + 0.14561)$
9	$(s + 0.29046)(s^2 + 0.10088s + 1.05421)(s^2 + 0.29046s + 0.83437)$ $\cdot (s^2 + 0.44501s + 0.49754)(s^2 + 0.54589s + 0.20134)$
10	$(s^2 + 0.08158s + 1.04351)(s^2 + 0.23675s + 0.86188)(s^2 + 0.36874s + 0.56799)$ $\cdot (s^2 + 0.46464s + 0.27409)(s^2 + 0.51506s + 0.09246)$
0.2-dB Ripple ( $\epsilon = 0.21709$ )	
n	
1	$s + 4.60636$
2	$s^2 + 1.92709s + 2.35683$
3	$(s + 0.81463)(s^2 + 0.81463s + 1.41363)$
4	$(s^2 + 0.44962s + 1.19866)(s^2 + 1.08548s + 0.49155)$
5	$(s + 0.46141)(s^2 + 0.28517s + 1.11741)(s^2 + 0.74658s + 0.55839)$
6	$(s^2 + 0.19705s + 1.07792)(s^2 + 0.53835s + 0.64491)(s^2 + 0.73540s + 0.21190)$
7	$(s + 0.32431)(s^2 + 0.14433s + 1.05566)(s^2 + 0.40441s + 0.71644)(s^2 + 0.58439s + 0.29343)$
8	$(s^2 + 0.11028s + 1.04183)(s^2 + 0.31407s + 0.77124)(s^2 + 0.47004s + 0.38855)(s^2 + 0.55445s + 0.11795)$
9	$(s + 0.25057)(s^2 + 0.08702s + 1.03263)(s^2 + 0.25057s + 0.81278)$ $\cdot (s^2 + 0.38389s + 0.47596)(s^2 + 0.47092s + 0.17976)$
10	$(s^2 + 0.44461s + 0.07513)(s^2 + 0.40109s + 0.25677)(s^2 + 0.31830s + 0.55066)$ $\cdot (s^2 + 0.20436s + 0.84455)(s^2 + 0.07042s + 1.02619)$
0.5-dB Ripple ( $\epsilon = 0.34931$ )	
n	
1	$s + 2.86278$
2	$s^2 + 1.42562s + 1.51620$
3	$(s + 0.62646)(s^2 + 0.62646s + 1.14245)$
4	$(s^2 + 0.35071s + 1.06352)(s^2 + 0.84668s + 0.35641)$
5	$(s + 0.36232)(s^2 + 0.22393s + 1.03578)(s^2 + 0.58625s + 0.47677)$
6	$(s^2 + 0.15530s + 1.02302)(s^2 + 0.42429s + 0.59001)(s^2 + 0.57959s + 0.15610)$
7	$(s + 0.25617)(s^2 + 0.11401s + 1.01611)(s^2 + 0.31944s + 0.67688)(s^2 + 0.46160s + 0.25388)$
8	$(s^2 + 0.08724s + 1.01193)(s^2 + 0.24844s + 0.74133)(s^2 + 0.37182s + 0.35865)(s^2 + 0.43859s + 0.08805)$
9	$(s + 0.19841)(s^2 + 0.06891s + 1.00921)(s^2 + 0.19841s + 0.78937)$ $\cdot (s^2 + 0.30398s + 0.45254)(s^2 + 0.37288s + 0.15634)$
10	$(s^2 + 0.05580s + 1.00734)(s^2 + 0.161934s + 0.82570)(s^2 + 0.25222s + 0.53181)$ $\cdot (s^2 + 0.31781s + 0.23791)(s^2 + 0.35230s + 0.05628)$
1-dB Ripple ( $\epsilon = 0.50885$ )	
n	
1	$s + 1.96523$
2	$s^2 + 1.09773s + 1.10251$
3	$(s + 0.49417)(s^2 + 0.49417s + 0.99421)$
4	$(s^2 + 0.27907s + 0.98651)(s^2 + 0.67374s + 0.27940)$
5	$(s + 0.28949)(s^2 + 0.17892s + 0.98832)(s^2 + 0.46841s + 0.42930)$

**TABLE 12.3** Factors of the Denominator Polynomials Normalized Chebyshev Low-Pass Filters (continued)

1-dB Ripple ( $\epsilon = 0.50885$ )	
n	
6	$(s^2 + 0.12436s + 0.99073) (s^2 + 0.33976s + 0.55772) (s^2 + 0.46413s + 0.12471)$
7	$(s + 0.20541) (s^2 + 0.09142s + 0.99268) (s^2 + 0.25615s + 0.65346) (s^2 + 0.37014s + 0.23045)$
8	$(s^2 + 0.07002s + 0.99414) (s^2 + 0.19939s + 0.72354) (s^2 + 0.29841s + 0.34086) (s^2 + 0.35110s + 0.07026)$
9	$(s + 0.15933) (s^2 + 0.05533s + 0.99523) (s^2 + 0.15933s + 0.77539)$ $\cdot (s^2 + 0.24411s + 0.43856) (s^2 + 0.29944s + 0.14236)$
10	$(s^2 + 0.04483s + 0.99606) (s^2 + 0.13010s + 0.81442) (s^2 + 0.20263s + 0.52053)$ $\cdot (s^2 + 0.25533s + 0.22664) (s^2 + 0.28304s + 0.04500)$
1.5-dB Ripple ( $\epsilon = 0.64229$ )	
n	
1	$s + 1.55693$
2	$s^2 + 0.92218s + 0.92521$
3	$(s + 0.42011) (s^2 + 0.42011s + 0.92649)$
4	$(s^2 + 0.23826s + 0.95046) (s^2 + 0.57521s + 0.24336)$
5	$(s + 0.24765) (s^2 + 0.15306s + 0.96584) (s^2 + 0.40071s + 0.40682)$
6	$(s^2 + 0.10650s + 0.97534) (s^2 + 0.29097s + 0.54233) (s^2 + 0.39747s + 0.10932)$
7	$(s + 0.17603) (s^2 + 0.07834s + 0.98147) (s^2 + 0.21951s + 0.64225) (s^2 + 0.31720s + 0.21924)$
8	$(s^2 + 0.06003s + 0.98561) (s^2 + 0.17094s + 0.71501) (s^2 + 0.25583s + 0.33233) (s^2 + 0.30177s + 0.06173)$
9	$(s + 0.13667) (s^2 + 0.04745s + 0.98852) (s^2 + 0.13664s + 0.76867)$ $\cdot (s^2 + 0.20934s + 0.43185) (s^2 + 0.25679s + 0.13565)$
10	$(s^2 + 0.03845s + 0.99063) (s^2 + 0.11159s + 0.80900) (s^2 + 0.17381s + 0.51510)$ $\cdot (s^2 + 0.21901s + 0.22121) (s^2 + 0.24277s + 0.03958)$
2-dB Ripple ( $\epsilon = 0.76478$ )	
n	
1	$s + 1.30756$
2	$s^2 + 0.80382s + 0.82306$
3	$(s + 0.36891) (s^2 + 0.36891s + 0.88610)$
4	$(s^2 + 0.20978s + 0.92868) (s^2 + 0.50644s + 0.22157)$
5	$(s + 0.21831) (s^2 + 0.13492s + 0.95217) (s^2 + 0.35323s + 0.39315)$
6	$(s^2 + 0.09395s + 0.96595) (s^2 + 0.25667s + 0.53294) (s^2 + 0.35061s + 0.09993)$
7	$(s + 0.15533) (s^2 + 0.06913s + 0.97462) (s^2 + 0.19371s + 0.63539) (s^2 + 0.27991s + 0.21239)$
8	$(s^2 + 0.05298s + 0.98038) (s^2 + 0.15089s + 0.70978) (s^2 + 0.22582s + 0.32710) (s^2 + 0.26637s + 0.05650)$
9	$(s + 0.12063) (s^2 + 0.04189s + 0.98440) (s^2 + 0.12063s + 0.76455)$ $\cdot (s^2 + 0.18482s + 0.42773) (s^2 + 0.22671s + 0.13153)$
10	$(s^2 + 0.03395s + 0.98730) (s^2 + 0.09853s + 0.80567) (s^2 + 0.15347s + 0.51178)$ $\cdot (s^2 + 0.19338s + 0.21788) (s^2 + 0.21436s + 0.03625)$
2.5-dB Ripple ( $\epsilon = 0.88220$ )	
n	
1	$(s + 1.13353)$
2	$(s^2 + 0.71525s + 0.75579)$
3	$(s + 0.32995) (s^2 + 0.32995s + 0.85887)$
4	$(s^2 + 0.18796s + 0.91386) (s^2 + 0.45378s + 0.20676)$
5	$(s + 0.19577) (s^2 + 0.12099s + 0.94284) (s^2 + 0.31677s + 0.38382)$
6	$(s^2 + 0.08429s + 0.95953) (s^2 + 0.23028s + 0.52651) (s^2 + 0.31456s + 0.09350)$
7	$(s + 0.13941) (s^2 + 0.06204s + 0.96992) (s^2 + 0.17384 + 0.63070) (s^2 + 0.25120s + 0.20769)$
8	$(s^2 + 0.04756s + 0.97680) (s^2 + 0.13054s + 0.70620) (s^2 + 0.20269s + 0.32352) (s^2 + 0.23909s + 0.05292)$
9	$(s + 0.10829) (s^2 + 0.03761s + 0.98157) (s^2 + 0.10829s + 0.76173)$ $\cdot (s^2 + 0.16591s + 0.42490) (s^2 + 0.20352s + 0.12870)$
10	$(s^2 + 0.19245s + 0.03396) (s^2 + 0.17361s + 0.21560) (s^2 + 0.13778s + 0.50949)$ $\cdot (s^2 + 0.08846s + 0.80338) (s^2 + 0.03048s + 0.98502)$

**TABLE 12.3** Factors of the Denominator Polynomials Normalized Chebyshev Low-Pass Filters (continued)

n	3-dB Ripple ( $\epsilon = 0.99763$ )
1	$(s + 1.00238)$
2	$(s^2 + 0.64490s + 0.70795)$
3	$(s + 0.29862)(s^2 + 0.29862s + 0.83917)$
4	$(s^2 + 0.17034s + 0.90309)(s^2 + 0.41124s + 0.19598)$
5	$(s + 0.17753)(s^2 + 0.10970s + 0.93603)(s^2 + 0.28725s + 0.37701)$
6	$(s^2 + 0.07646s + 0.95483)(s^2 + 0.20889s + 0.52182)(s^2 + 0.28535s + 0.08880)$
7	$(s + 0.12649)(s^2 + 0.05629s + 0.96648)(s^2 + 0.15773 + 0.62726)(s^2 + 0.22792s + 0.20425)$
8	$(s^2 + 0.04316s + 0.97417)(s^2 + 0.12290s + 0.70358)(s^2 + 0.18393s + 0.32089)(s^2 + 0.21696s + 0.05029)$
9	$(s + 0.09827)(s^2 + 0.03413s + 0.97950)(s^2 + 0.09827s + 0.75966)$ $\cdot (s^2 + 0.15057s + 0.42283)(s^2 + 0.18470s + 0.12664)$
10	$(s^2 + 0.02766s + 0.98335)(s^2 + 0.08028s + 0.80171)(s^2 + 0.12504s + 0.50782)$ $\cdot (s^2 + 0.15757s + 0.21393)(s^2 + 0.17466s + 0.03229)$



**FIGURE 12.4**  $f_p = 2H_z$ ;  $\omega_p = 12.5664$

## 12.6 Inverse-Chebyshev Approximation

### 12.6.1 Definition

The inverse-Chebyshev filter is flat in the passband and equi-ripple in the stopband.

### 12.6.2 The Magnitude-Squared Transfer Function

$$|H(j\omega)|^2 = \frac{\epsilon^2 C_n^2(\omega_r / \omega)}{1 + \epsilon^2 C_n^2(\omega_r / \omega)}$$

$C_n(\omega)$  = Chebyshev polynomial;  $\omega_r$  = stopband edge frequency

### 12.6.3 Attenuation

$$A(\omega) = 10 \log \left( 1 + \frac{1}{\epsilon^2 C_n^2(\omega_r / \omega)} \right) \text{ dB}$$

$\epsilon$  = ripple factor calculated at  $\omega = \omega_r$

$$A_r(\omega) = 10 \log \left( 1 + \frac{1}{\epsilon^2 C_n^2(1)} \right), \quad C_n^2(1) = 1$$

$$\varepsilon = \frac{1}{\sqrt{10^{0.1A_r} - 1}}$$

### 12.6.4 Filter Order

$$n \geq \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)}$$

$$k = \frac{f_p}{f_r}, \quad d = \left( \frac{10^{0.1A_p} - 1}{10^{0.1A_r} - 1} \right)^{1/2}$$

### 12.6.5 Poles and Zeros

$$H(s)H(-s) = \frac{\varepsilon^2 C_n^2(j\omega_r/s)}{1 + \varepsilon^2 C_n^2(j\omega_r/s)}$$

#### Zeros

$$C_n(j\omega_r/s) = 0 = \cos(n \cos^{-1}(j\omega_r/s))$$

$$\cos^{-1}(j\omega_r/s) = m\pi/2n, \quad m \text{ odd}$$

$$s_m = \text{zeros} = j\omega_r \sec(m\pi/2n), \quad m = 1, 3, \dots, 2n-1$$

#### Poles

$$1 + \varepsilon^2 C_n^2(j\omega_r/s) = 0$$

same poles as in 12.5.6 except that  $-s$  is replaced by  $1/s$ .

Denormalization is accomplished with respect to stopband edge frequency  $\omega_r$ .

## 12.7 Elliptic Filters

---

### 12.7.1 Square Magnitude Response Function for Elliptic Filters

$$|H(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 R_n^2(\omega)}$$

$$R_n(\omega) = \text{rational function}; \quad \varepsilon = \text{ripple factor}$$

### 12.7.2 Properties of the Rational Function $R_n(\omega)$

1.  $R_n(\omega)$  = even for  $n$  even.  $R_n(\omega)$  = odd for  $n$  odd.
2. The zeros of  $R_n(\omega)$  are in the range  $|\omega| < 1$ .  
The poles of  $R_n(\omega)$  are in the range  $|\omega| > 1$ .
3. The function  $R_n(\omega)$  oscillates between  $\pm 1$  in the passband.
4.  $R_n(\omega) = 1$  at  $\omega = 1$ .
5.  $R_n(\omega)$  oscillates between  $\pm 1/d$  and infinity in the stopband, where  $d$  is given in 12.5.7.

### 12.7.3 The Rational Normalized Function $R_n(\omega)$ with Respect to Center Frequency $\omega_0 = 1$

$$R_n(\omega) = \omega \prod_{i=1}^{(n-1)/2} \frac{\omega_i^2 - \omega^2}{1 - \omega_i^2 \omega^2} \quad \text{for } n \text{ odd}$$

$$R_n(\omega) = \prod_{i=1}^{n/2} \frac{\omega_i^2 - \omega^2}{1 - \omega_i^2 \omega^2} \quad \text{for } n \text{ even}$$

### 12.7.4 Steps to Calculate the Elliptic Filter

1. Find the selectivity factor  $k$

$$k = \frac{\omega_p}{\omega_r}, \quad \omega_p = \text{passband frequency}, \quad \omega_r = \text{stopband frequency}$$

2. Define

$$q_o = \frac{1}{2} \frac{1 - (1 - k^2)^{1/4}}{1 + (1 - k^2)^{1/4}}$$

3. Find the expression

$$q = q_o + 2q_o^5 + 15q_o^9 + 150q_o^{13}$$

4. Find  $d$

$$d = \left( \frac{10^{0.1A_p} - 1}{10^{0.1A_r} - 1} \right)^{1/2}$$

5. Find the filter order  $n$

$$n \geq \frac{\log(16/d^2)}{\log(1/q)}$$

6. Calculate  $\varepsilon$

$$\varepsilon = \left( 10^{0.1A_p} - 1 \right)^{1/2}$$

$$A_p = 10 \log(1 + \varepsilon^2)$$

7. Define

$$\beta = \frac{1}{2n} \ell n \frac{(1 + \varepsilon^2)^{1/2} + 1}{(1 + \varepsilon^2)^{1/2} - 1}$$



8. Calculate

$$a = \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sinh[(2m+1)\beta]}{1 + \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cosh(2m\beta)}$$

9. Define

$$U = \left[ (1 + ka^2) \left( 1 + \frac{a^2}{k} \right) \right]^{1/2}$$

10. Calculate

$$\omega_i = \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sin[(2m+1)\pi\ell/n]}{1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cos(2m\pi\ell/n)}$$

$$\ell = i - \frac{1}{2}, i = 1, 2, \dots, \frac{n}{2}, n = \text{even}, \ell = i, i = 1, 2, \dots, (n-1)/2, n = \text{odd}$$

11. Define

$$V_i = \left[ (1 - k\omega_i^2) \left( 1 - \frac{\omega_i^2}{k} \right) \right]^{1/2}$$

12. Set

$$a_i = \frac{1}{\omega_i^2}$$

13. Set

$$b_i = \frac{2aV_i}{1 + a^2\omega_i^2}$$

14. Set

$$c_i = \frac{(aV_i)^2 + (\omega_i U)^2}{(1 + a^2\omega_i^2)^2}$$

15. Find

$$H_o = a \prod_{i=1}^{(n-1)/2} \frac{c_i}{a_i} \quad \text{for } n = \text{odd}$$

$$H_o = \frac{1}{\sqrt{1+\epsilon^2}} \prod_{i=1}^{n/2} \frac{c_i}{a_i} \quad \text{for } n = \text{even}$$

$$H(s) = H_o \prod_{i=1}^{n/2} \frac{s^2 + a_i}{s^2 + b_i s + c_i} \quad \text{for } n = \text{even}$$

$$H(s) = \frac{H_o}{s + a} \prod_{i=1}^{(n-1)/2} \frac{s^2 + a_i}{s^2 + b_i s + c_i} \quad \text{for } n \text{ odd}$$

## 12.7.5 Unnormalized Transfer Function

Replace  $s$  in the 15th step above (in Section 12.7.4) with  $s/\omega_0$  where  $\omega_o = \sqrt{\omega_p \omega_r}$ .

**Note:** Summations in steps 8 and 10 above converge fast, and up to four or five terms of the series will provide good accuracy.

## 12.8 Elliptic Filters (Second Approach\*)

### 12.8.1 Transfer Function

$$H(s) = \frac{H_o}{D_o(s)} \prod_{i=1}^r \frac{s^2 + a_{oi}}{s^2 + b_{ir}s + b_{oi}}$$

$$r = \begin{cases} \frac{n-1}{2} & \text{for odd } n \\ \frac{n}{2} & \text{for even } n \end{cases}$$

$$D_o(s) = \begin{cases} s + \sigma_o & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$

### 12.8.2 Steps of Implementation

#### Given

$\omega_p$  = passband frequency;  $\omega_r$  = stopband frequency;

$A_p$  = maximum passband loss (dB);  $A_r$  = minimum stopband loss (dB)

$k$  = selectivity factor =  $\omega_p/\omega_r$

#### Steps

$$1. \quad k^1 = \sqrt{1 - k^2}$$

$$2. \quad q_o = \frac{1}{2} \left( \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \right)$$

$$3. \quad q = q_o + 2q_o^5 + 15q_o^9 + 150q_o^{13}$$

---

\* Antoniou (1993)

$$4. \quad D = \frac{10^{0.1A_r} - 1}{10^{0.1A_p} - 1}$$

$$5. \quad n \geq \frac{\log 16D}{\log(1/q)}$$

$$6. \quad \Lambda = \frac{1}{2n} \ell n \frac{10^{0.05A_p} + 1}{10^{0.05A_p} - 1}$$

$$7. \quad \sigma_o = \left| \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sinh[(2m+1)\Lambda]}{1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cosh 2m\Lambda} \right|$$

$$8. \quad W = \sqrt{(1 + k\sigma_o^2) \left( 1 + \frac{\sigma_o^2}{k} \right)}$$

$$9. \quad \Omega_i = \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sin \left[ \frac{(2m+1)\pi\mu}{n} \right]}{1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cos \left[ \frac{2m\pi\mu}{n} \right]}$$

$$\mu = \begin{cases} i & \text{for odd } n \\ i - \frac{1}{2} & \text{for even } n \end{cases} \quad i = 1, 2, \dots, r$$

$$10. \quad V_i = \sqrt{(1 - k\Omega_i^2) \left( 1 - \frac{\Omega_i^2}{k} \right)}$$

$$11. \quad a_{oi} = \frac{1}{\Omega_i^2}$$

$$b_{oi} = \frac{(\sigma_o V_i)^2 + (\Omega_i W)^2}{(1 + \sigma_o^2 \Omega_i^2)^2}$$

$$b_{li} = \frac{2\sigma_o V_i}{1 + \sigma_o^2 \Omega_i^2}$$

$$12. \quad H_o = \begin{cases} \sigma_o \prod_{i=1}^r \frac{b_{oi}}{a_{oi}} & \text{for odd } n \\ 10^{-0.05A_p} \prod_{i=1}^r \frac{b_{oi}}{a_{oi}} & \text{for even } n \end{cases}$$

The series in steps 7 and 9 converge rapidly, and three to four terms are sufficient for most purposes.

### Example 12.3 Requirements for an Elliptic Filter:

$$\omega_p = \sqrt{0.9} \text{ rad/s}, \quad \omega_r = 1/\sqrt{0.9} \text{ rad/s}, \quad A_p = 0.1 \text{ dB}, \quad \text{and} \quad A_r \geq 50.0 \text{ dB}$$

Results from steps 1 through 5 above:

$$k = 0.9, \quad k' = 0.43589, \quad q_o = 0.10233, \quad q = 0.102352, \quad D = 4,293,090, \quad n \geq 7.92 \quad \text{or} \quad n = 8$$

The coefficients for  $H(s)$  are found from the rest of the steps and are:

$i$	$a_{oi}$	$b_{oi}$	$b_{li}$
1	$1.434825 \times 10$	$2.914919 \times 10^{-1}$	$8.711574 \times 10^{-1}$
2	2.231643	$6.123726 \times 10^{-1}$	$4.729136 \times 10^{-1}$
3	1.320447	$8.397386 \times 10^{-1}$	$1.825141 \times 10^{-1}$
4	1.128832	$9.264592 \times 10^{-1}$	$4.471442 \times 10^{-2}$

$H_o = 2.876332 \times 10^{-3}$

Hence from 12.8.1

$$\begin{aligned}
 H(s) = & 2.87633 \times 10^{-3} \left( \frac{s^2 + 14.34825}{s^2 + 0.8711574s + 0.2914919} \right) \left( \frac{s^2 + 2.231643}{s^2 + 0.4729136s + 0.6123726} \right) \\
 & \times \left( \frac{s^2 + 1.320447}{s^2 + 0.1825141s + 0.8397386} \right) \left( \frac{s^2 + 1.128832}{s^2 + 0.04471442s + 0.9264592} \right)
 \end{aligned}$$

and  $|H(j\omega)|$  is plotted in [Figure 12.5](#).

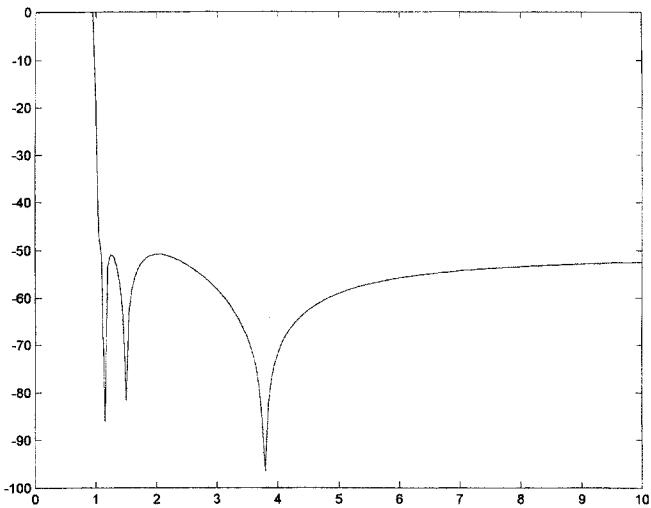


FIGURE 12.5

## 12.9 Transformations

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### 12.9.1 Lowpass to Lowpass

$$\text{Set } s \rightarrow \frac{s}{\omega_p}, \quad \omega_p = \text{new passband frequency}$$

### 12.9.2 Lowpass to Highpass

$$\text{Set } s \rightarrow \frac{\omega_p}{s}, \quad \omega_p = \text{new passband frequency}$$

### 12.9.3 Lowpass to Bandpass

$$\text{Set } s \rightarrow \frac{s^2 + \omega_m^2}{Bs}$$

$\omega_m$  = geometric mean of the upper band edge frequency  $\omega_u$

and the lower band edge frequency  $\omega_\ell = \sqrt{\omega_u \omega_\ell}$

$B = \omega_u - \omega_\ell$  = filter bandwidth

### 12.9.4 Lowpass to Bandstop

$$s \rightarrow \frac{Bs}{s^2 + \omega_m^2}$$

$\omega_m$  and  $B$  are the same as in 12.9.3

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