# CT - RESÚMENES

## ANÁLISIS MATEMÁTICO II

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Año de cursado: 2014

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Desde Corriente Tecnológica nos preocupamos por brindarles a los estudiantes la mayor cantidad de herramientas posibles para facilitar y hacer más ameno su camino dentro de la facultad, porque entendemos y compartimos las mismas necesidades, y porque como agrupación estudiantil asumimos el rol y el compromiso de ser quienes pongan estas herramientas a disposición de todos.

Sumando a estos motivos la intención de fomentar las actividades solidarias dentro de nuestra universidad, ponemos al alcance de todos los estudiantes aquellos resúmenes y producciones propias de los alumnos que quieren aportarlos para contribuir a esta causa, brindamos clases de apoyo semanales de materias de ciencias básicas y realizamos ferias de apuntes solidarias a principio de cada cuatrimestre, como algunos de nuestros principales proyectos.

Es nuestro deseo que estos instrumentos puedan ser aprovechados al máximo, razón por la cual, si tenés algún comentario, crítica o sugerencia sobre los mismos, te pedimos que nos lo transmitas para poder mejorarlos con tu ayuda. De igual modo, si estás interesado en compartir tus resúmenes, sumarte a las clases de apoyo o contribuir a cualquiera de estos proyectos, estás invitado a hacerlo.

Recordá siempre que entre todos podemos construir una universidad mejor.

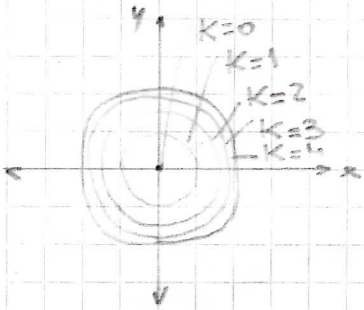
Corriente Tecnológica.

*De los estudiantes, para los estudiantes.*

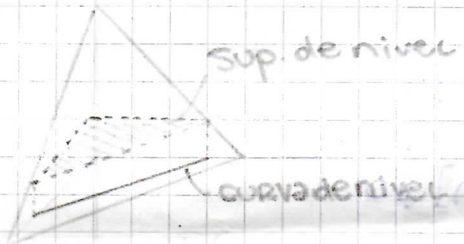
## ② Curvas de nivel:

$\{(x;y)\in\mathbb{R}^2 \wedge \forall z\in\mathbb{R} \wedge \forall k\in\mathbb{R} / z=f(x;y)=k\}$

$z=x^2+y^2$:




## Superficie de nivel:



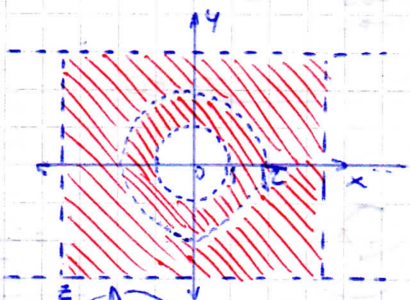

**Existencia** $\forall (x;y)\in\mathbb{R}^2 \; \exists z\in\mathbb{R} / z=f(x;y)$

**Unicidad** $\forall (x;y)\in\mathbb{R}^2 \; \exists z_1\in\mathbb{R} \wedge \exists z_2\in\mathbb{R} / z_1=f_1(x;y) \wedge z_2=f_2(x;y) \Rightarrow z_1=z_2$

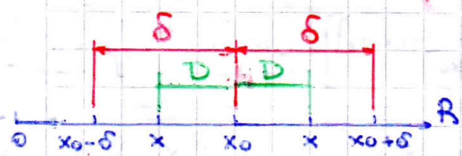
**Dominio** $D_f=\{(x;y)\in\mathbb{R}^2 / \exists z=f(x;y)\}$

$$z=\frac{1}{\ln(-1+x^2+y^2)}$$

$$\left.\begin{array}{l} -1+x^2+y^2\leq 0 \longrightarrow x^2+y^2\leq 1 \\ -1+x^2+y^2=1 \longrightarrow x^2+y^2=2 \end{array}\right\} \text{Restricciones}$$

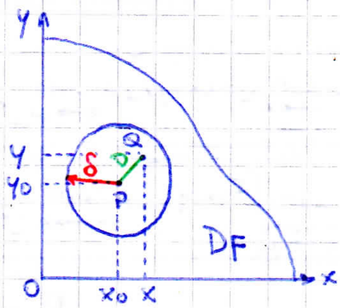



## Entorno:



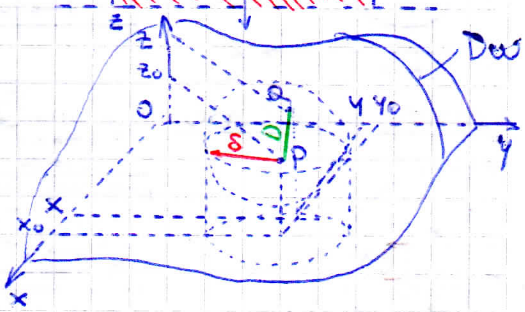

$E(x_0;\delta)$

$D=|x-x_0|<\delta$




$E((x_0;y_0);\delta)$

$D=\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$




$E((x_0;y_0;z_0);\delta)$

$D=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta$

**Entorno reducido:** $\boxed{D>0}$

$0<|x-x_0|<\delta$ $\qquad$ $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ $\qquad$ $0<\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta$

## 3 Límite:

**Definición:** $\lim\limits_{(x;y)\to(x_0;y_0)} [f(x;y)] = L$ si $\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0 : \left[0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\right] \Rightarrow \left| f(x;y) - L \right| < \varepsilon$

**Propiedades:**

1) $\lim\limits_{\substack{x\to a\\ y\to b}} [k] = K$

2) $\lim\limits_{\substack{x\to a\\ y\to b}} [K.f(x;y)] = K.\lim\limits_{\substack{x\to a\\ y\to b}} [f(x;y)]$

3) $\lim\limits_{\substack{x\to a\\ y\to b}} \left[f(x;y) \pm g(x;y)\right] = \lim\limits_{\substack{x\to a\\ y\to b}} [f(x;y)] \pm \lim\limits_{\substack{x\to a\\ y\to b}} [g(x;y)]$

4) $\lim\limits_{\substack{x\to a\\ y\to b}} [f(x;y)\, g(x;y)] = \lim\limits_{\substack{x\to a\\ y\to b}} [f(x;y)] . \lim\limits_{\substack{x\to a\\ y\to b}} [g(x;y)]$

5) $\lim\limits_{\substack{x\to a\\ y\to b}} \left[\dfrac{f(x;y)}{g(x;y)}\right] = \dfrac{\lim\limits_{\substack{x\to a\\ y\to b}} [f(x;y)]}{\lim\limits_{\substack{x\to a\\ y\to b}} [g(x;y)]}$ si $\lim\limits_{\substack{x\to a\\ y\to b}} [g(x;y)] \neq 0$

6) $\lim\limits_{\substack{x\to a\\ y\to b}} \left[f(x;y)^{g(x;y)}\right] = \lim\limits_{\substack{x\to a\\ y\to b}} [f(x;y)]^{\lim\limits_{\substack{x\to a\\ y\to b}} [g(x;y)]}$ si $f(x;y) > 0$

7) $\lim\limits_{\substack{x\to a\\ y\to b}} [\mathrm{Log}(f(x;y))] = \mathrm{Log}\left(\lim\limits_{\substack{x\to a\\ y\to b}} [f(x;y)]\right)$

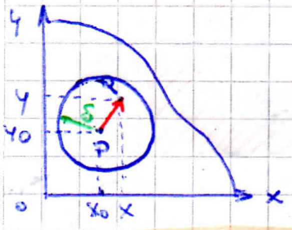
**Límite simultáneo:**

$$\boxed{L = \lim\limits_{\substack{x\to x_0\\ y\to y_0}} [z] = \lim\limits_{\substack{x\to x_0\\ y\to y_0}} [f(x;y)] = f(x;y)_P = f(x_0;y_0) = z(P)}$$

si $z$ es continua en P.

$$\boxed{L = \lim\limits_{\substack{x\to x_0\\ y\to y_0}} [z] = \lim\limits_{\substack{x\to x_0\\ y\to y_0}} [f(x;y)] = \text{Indeterminado}}$$

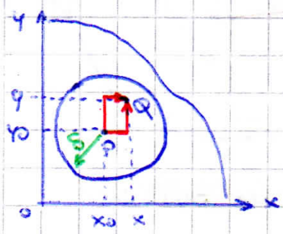
si $z$ no es continua en P.




**Límite Reiterado:**

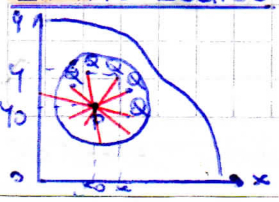
$$\boxed{L_1 = \lim\limits_{\substack{x\to x_0\\ y\to y_0}} [z] = \lim\limits_{\substack{x\to x_0\\ y\to y_0}} [f(x;y)] = \lim\limits_{y\to y_0} \left[\lim\limits_{x\to x_0} [f(x;y)]\right] = \lim\limits_{y\to y_0} [f(x_0;y)] = \lim\limits_{y\to y_0} [f(y)] = R}$$

$$\boxed{L_2 = \lim\limits_{\substack{x\to x_0\\ y\to y_0}} [z] = \lim\limits_{\substack{x\to x_0\\ y\to y_0}} [f(x;y)] = \lim\limits_{x\to x_0} \left[\lim\limits_{y\to y_0} [f(x;y)]\right] = \lim\limits_{x\to x_0} [f(x;y_0)] = \lim\limits_{x\to x_0} [f(x)] = R}$$




$$y - y_0 = m(x - x_0) \quad \begin{cases} R \neq h(m) \longrightarrow R = L \\ z = h(m) \longrightarrow \nexists L \end{cases}$$

**Límite Radial:**



## Continuidad:

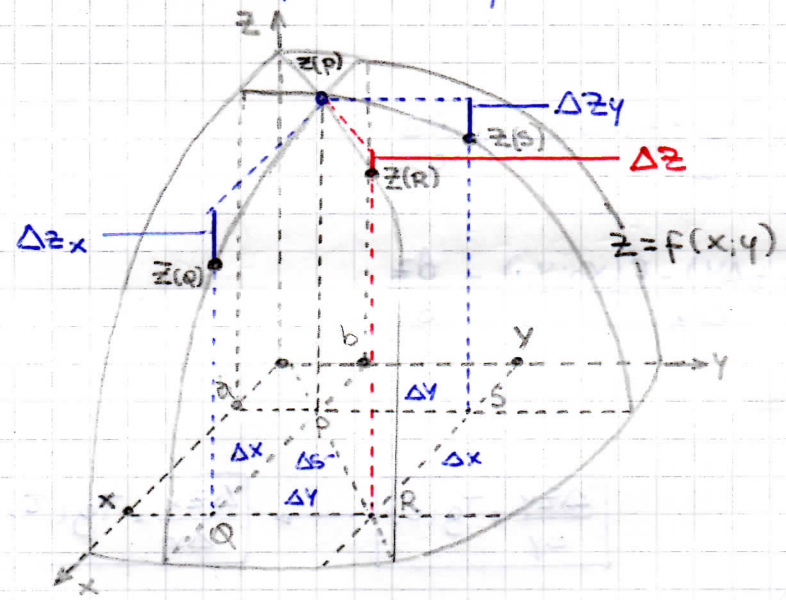
$$(z)_P = f(a;b) = \lim_{\substack{x \to a \\ y \to b}} [f(x;y)]$$

1) $\lim_{\substack{x \to a \\ y \to b}} [f(x;y)] = L$

2) $(z)_P = f(a;b) = R$

3) $R = (z)_P = L$

## 4 Incrementos parciales y total



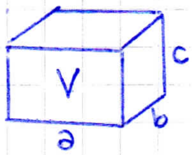

$\Delta z_x = Z_{(Q)} - Z_{(P)} = f_i - f_o = f(a+\Delta x; b) - f(a;b)$

$\Delta z_y = Z_{(S)} - Z_{(P)} = f_i - f_o = f(a; b+\Delta y) - f(a;b)$

$\Delta z = Z_{(R)} - Z_{(P)} = f_i - f_o = f(a+\Delta x; b+\Delta y) - f(a;b) = f(x;y) - f(a;b)$

Generalmente $\Delta z \neq \Delta z_x + \Delta z_y$

**Ejemplo:** $\Delta V = f(a+\Delta a; b+\Delta b; c+\Delta c) - f(a;b;c) = (a+\Delta a)(b+\Delta b)(c+\Delta c) - abc$




$= ab\Delta c + a.c.\Delta b + a.\Delta b.\Delta c + b.c.\Delta a + b.\Delta a.\Delta c + \Delta a \Delta b \Delta c$

$= (1,5+0,1)(2,5+0,18)(3,5+0,2) - 1,5 . 2,5 . 3,5 = 2,7406\ cm^3$

$a = 1,5\ cm$
$b = 2,5\ cm$
$c = 3,5\ cm$
$\Delta a = 0,1\ cm$
$\Delta b = 0,18\ cm$
$\Delta c = 0,2\ cm$

$\Delta V_a = f(a+\Delta a; b; c) - f(a;b;c) = (a+\Delta a)\,b.c - abc = (1,5+0,1).2,5.3,5 - 1,5.2,5.3,5$

$\Delta V_a = 0,875\ cm^3$

$\Delta V_b = f(a; b+\Delta b; c) - f(a;b;c) = a.(b+\Delta b).c - abc = 1,5.(2,5.0,18).3,5 - 1,5.2,5.3,5$

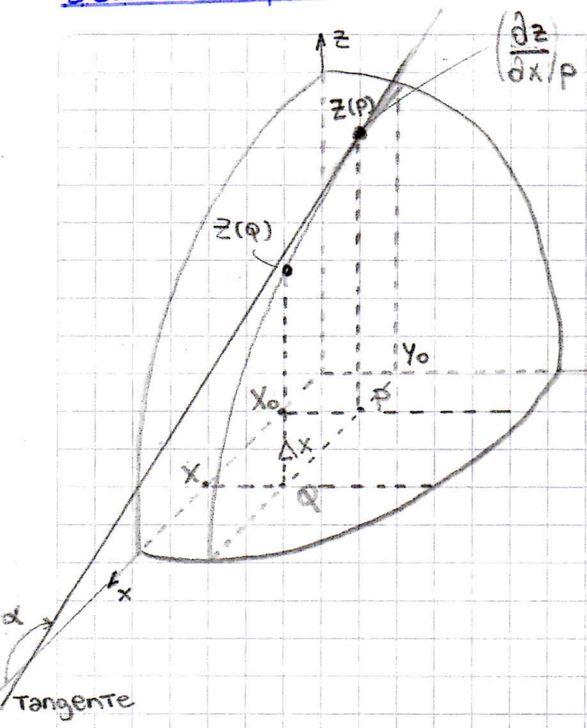
$\Delta V_b = 0,945\ cm^3$

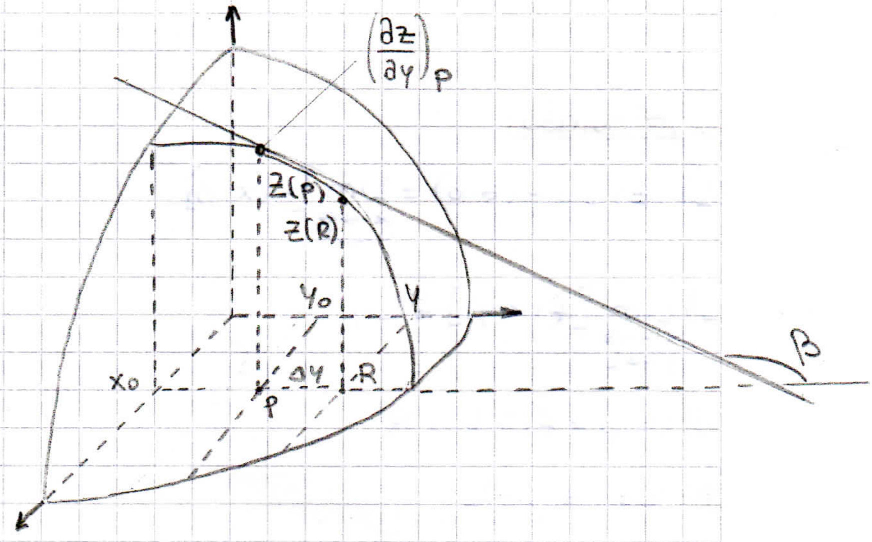
$\Delta V_c = f(a;b;c+\Delta c) - f(a;b;c) = a.b(c+\Delta c) - abc = 1,5.2,5.(3,5+0,2) - 1,5.2,5.3,5$

$\Delta V_c = 0,75\ cm^3$

$S = \Delta V_a + \Delta V_b + \Delta V_c = 0,875 + 0,945 + 0,75 = 2,57\ cm^3 \neq 2,7406\ cm^3$

## Derivadas parciales:

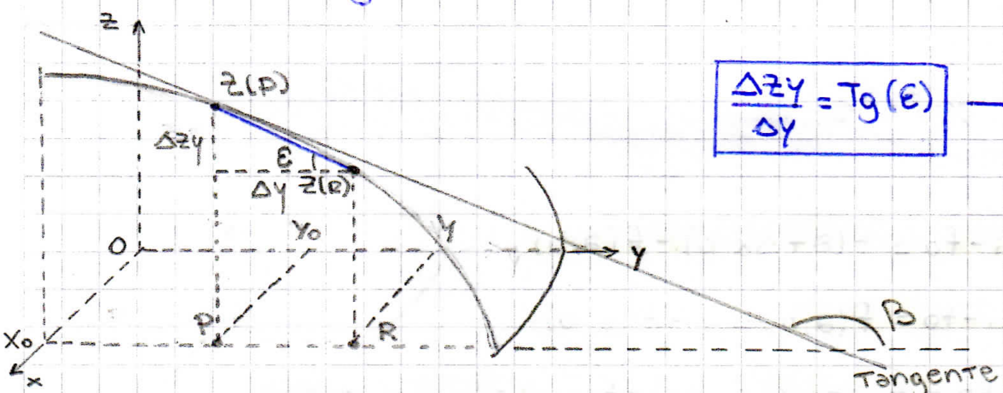



$$\lim_{\Delta x \to 0}\left[\frac{\Delta z_x}{\Delta x}\right] = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x; y_0) - f(x_0; y_0)}{\Delta x} = \frac{\partial z}{\partial x}$$

$$\lim_{\Delta y \to 0}\left[\frac{\Delta z_y}{\Delta y}\right] = \lim_{\Delta y \to 0} \frac{f(x_0; y_0+\Delta y) - f(x_0; y_0)}{\Delta y} = \frac{\partial z}{\partial y}$$

## Interpretación geométrica:




$$\boxed{\frac{\Delta z_y}{\Delta y} = Tg(\varepsilon)} \longrightarrow \boxed{\frac{\Delta z_x}{\Delta x} = Tg(\tau)}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{\Delta z_y}{\Delta y}\right] = \lim_{\Delta y \to 0}\left[Tg(\varepsilon)\right] = Tg(180° - \beta) = -Tg(\beta)$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{\Delta z_x}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[Tg(\tau)\right] = Tg(180 - \alpha) = -Tg(\alpha)$$

## Cálculo de la derivada parcial:

1) **Por definición:** $z = 2x^3 - 3xy^2 + y^{-2}$ con respecto a $y$.

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{\Delta z_y}{\Delta y}\right] = \lim_{\Delta y \to 0}\left[\frac{2x^3 - 3x(y+\Delta y)^2 + (y+\Delta y)^{-2} - (2x^3 - 3xy^2 + y^{-2})}{\Delta y}\right] =$$

$$= \lim_{\Delta y \to 0}\left[\frac{2x^3 - 3x(y^2 + 2y\Delta y + \Delta y^2) + (y+\Delta y)^{-2} - (2x^3 - 3xy^2 + y^{-2})}{\Delta y}\right] =$$

$$= \lim_{\Delta y \to 0}\left[\frac{-3x(y^2 + 2y\Delta y + \Delta y^2) + (y+\Delta y)^{-2} + 3xy^2 - y^{-2}}{\Delta y}\right] =$$

$$= \lim_{\Delta y \to 0}\left[\frac{-3xy^2 - 6xy\Delta y - 3x\Delta y^2 + (y+\Delta y)^{-2} + 3xy^2 - y^{-2}}{\Delta y}\right] =$$

$$= \lim_{\Delta y \to 0}\left[\frac{-6xy\Delta y - 3x\Delta y^2 + (y+\Delta y)^{-2} - y^{-2}}{\Delta y}\right] =$$

$$= \lim_{\Delta y \to 0}\left[-6xy - 3x\Delta y + \frac{1}{\Delta y(y+\Delta y)^2} - \frac{1}{\Delta y \cdot y^2}\right] =$$

$$= \lim_{\Delta y \to 0}\left[-6xy - 3x\Delta y + \frac{\Delta y \cdot y^2 - \Delta y(y+\Delta y)^2}{\Delta y^2 (y+\Delta y)^2 y^2}\right] =$$

$$= \lim_{\Delta y \to 0}\left[-6xy - 3x\Delta y + \frac{\cancel{\Delta y y^2} - \cancel{\Delta y y^2} - 2y\Delta y^2 - \Delta y^3}{\Delta y^2 (y+\Delta y)^2 y^2}\right] =$$

$$= \lim_{\Delta y \to 0}\left[-6xy - 3x\Delta y - \frac{2}{(y+\Delta y)^2 y} - \frac{\Delta y}{(y+\Delta y)^2 y^2}\right] =$$

$-3(2xy$

$$= -6xy - \frac{2}{y^3} = -3\left(xy + \frac{2}{y^3}\right)$$

2) **Por derivación directa:** $z = 2x^3 - 3xy^2 + y^{-2}$ con respecto a $y$

$$\frac{\partial z}{\partial y} = \frac{\partial\, 2x3}{\partial y} - \frac{\partial\, 3xy^2}{\partial y} + \frac{\partial\, y^{-2}}{\partial y} = 0 - 6xy - 2y^{-3} = -3\left(xy - \frac{2}{y^3}\right)$$

3) **Por tabla:** $z = 2x^3 - 3xy^2 + y^{-2}$ con respecto a $y$:

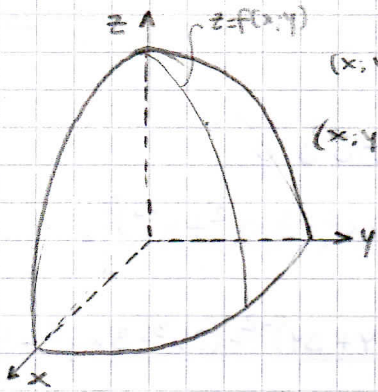
$$2x^3 \rightarrow C \rightarrow \frac{dC}{du} = 0$$

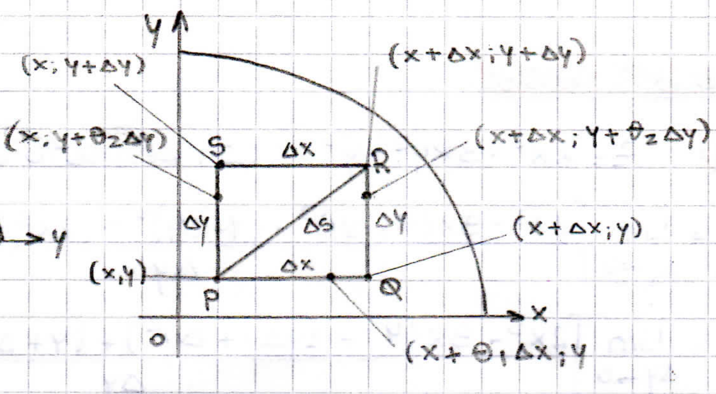
$$-3xy^2 \rightarrow Cy^2 = Cu^2 \rightarrow \frac{dC.f(u)}{du} = C.\frac{d(u^a)}{du} = \frac{C.a.u^{a-1}.du}{du} = C.2.y.1 = -6xy$$

$$y^{-2} \rightarrow u^a \rightarrow \frac{d(u^a)}{du} = \frac{a.u^{a-1}.d(u)}{du} = -2y^{-3}.1 = \frac{-2}{y3}$$

$$\frac{\partial z}{\partial y} = 0 - 6xy - \frac{2}{y3} = -3\left(xy - \frac{2}{y3}\right)$$

## (5) Teorema del valor medio:




$$\Delta z = z(R) - z(P) = f_i - f_o = f(x+\Delta x; y+\Delta y) - f(x;y)$$

$$\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y) + f(x+\Delta x; y) - f(x+\Delta x; y)$$

$$\Delta z = [f(x+\Delta x; y+\Delta y) - f(x+\Delta x; y)] + [f(x+\Delta x; y) - f(x;y)]$$

$$\Delta z = \frac{\partial f(x+\Delta x; y+\theta_2 \Delta y)}{\partial y}\Delta y + \frac{\partial f(x+\theta_1\Delta x; y)}{\partial x}\Delta x$$

$$\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y) = \frac{\partial f(x+\Delta x; \bar{y})}{\partial y}\Delta y + \frac{\partial f(\bar{x}; y)}{\partial x}\Delta x$$

## Diferencial total:

$$\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y) = \frac{\partial f(\bar{x}; y)}{\partial x}\Delta x + \frac{\partial f(x+\Delta x; \bar{y})}{\partial y}\Delta y$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\partial f(\bar{x}; y)}{\partial x} = \frac{\partial f(x;y)}{\partial x} = \frac{\partial f(P)}{\partial x} \quad ; \quad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\partial f(x+\Delta x; \bar{y})}{\partial y} = \frac{\partial f(x;y)}{\partial y} = \frac{\partial f(P)}{\partial y}$$

$$\frac{\partial f(\bar{x}; y)}{\partial x} = \frac{\partial f(x;y)}{\partial x} \pm \varepsilon_1 \quad ; \quad \frac{\partial f(x+\Delta x; \bar{y})}{\partial y} = \frac{\partial f(x;y)}{\partial y} \pm \varepsilon_2$$

$$\Delta z = \frac{\partial f(\bar{x}; y)}{\partial x}\Delta x + \frac{\partial f(x+\Delta x; \bar{y})}{\partial y}\Delta y = \left(\frac{\partial f(x;y)}{\partial x} \pm \varepsilon_1\right)\Delta x + \left(\frac{\partial f(x;y)}{\partial y} \pm \varepsilon_2\right)\Delta y$$

$$\Delta z = \frac{\partial f(x;y)}{\partial x}\Delta x + \frac{\partial f(x;y)}{\partial y}\Delta y + (\pm \varepsilon_1 \Delta x \pm \varepsilon_2 \Delta y)$$

$$\Delta z = \frac{\partial f(x;y)}{\partial x}\Delta x + \frac{\partial f(x;y)}{\partial y}\Delta y \pm IOS$$

$$\Delta z = \frac{\partial f(x;y)}{\partial x}dx + \frac{\partial f(x;y)}{\partial y}dy \pm IOS$$

$$dz = \frac{\partial f(x;y)}{\partial x}dx + \frac{\partial f(x;y)}{\partial y}dy$$

$$\boxed{\Delta z = dz \pm IOS} \longrightarrow \boxed{\Delta z - dz = \pm IOS}$$

## Diferencial aplicado a aproximaciones y a errores:

**Error aproximado:** $$dz = \frac{\partial f(x;y)}{\partial x}dx + \frac{\partial f(x;y)}{\partial y}dy$$

**Ejemplo:** Hallar el error aproximado de la aceleración $a = g \cdot sen(\alpha)$ de un cuerpo si g aumenta 3 cm por seg² y $\alpha = 0,7$ con un error de 1°.

$$da = \frac{\partial a}{\partial g}dg + \frac{\partial a}{\partial \alpha}d\alpha = sen(\alpha).3\frac{cm}{s^2} + g.cos(\alpha).0,01745\ Rad =$$

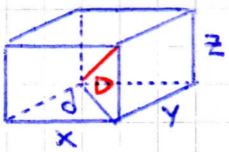
$$= sen(0,7).\frac{3cm}{s^2} + 981\frac{cm}{s^2}.cos(0,7).0,01745\ Rad =$$

$$da = 15,03\frac{cm}{s^2}$$

**Error máximo:** $$\hat{dz} = \left|\frac{\partial f(x;y)}{\partial x}\right||dx| + \left|\frac{\partial f(x;y)}{\partial y}\right||dy|$$

**Ejemplo:**

$$D = \sqrt{d^2+z^2} = \sqrt{\sqrt{x^2+y^2}^{\,2}+z^2} = \sqrt{x^2+y^2+z^2} = f(x;y;z)$$




$$|dD| = \hat{dD} = \left|\frac{\partial D}{\partial x}\right||dx| + \left|\frac{\partial D}{\partial y}\right||dy| + \left|\frac{\partial D}{\partial z}\right||dz| =$$

$x = 9m$
$y = 7m$
$z = 4m$
$0,02m \rightarrow$ ERROR

$$\hat{dD} = \frac{0,02\,(x+y+z)}{\sqrt{x^2+y^2+z^2}} = \frac{0,02(9+7+4)}{\sqrt{9^2+7^2+4^2}} =$$

$$\hat{dD} = 0,0331\,m$$

**Error porcentual:** $$\varepsilon_{p\,dz} = \frac{dz}{z}.100 \qquad \varepsilon_{p\,\hat{dz}} = \frac{\hat{dz}}{z}.100$$

**Ejemplo:**

$$\varepsilon_{p\,dz} = \varepsilon_{p\,da} = \frac{da}{a}.100 = \frac{15,03}{631,97}.100 = 2,38\%$$

$$\varepsilon_{p\,\hat{dz}} = \varepsilon_{p\,\hat{dD}} = \frac{\hat{dD}}{D}.100 = \frac{0,0331\,m}{12,08\,m}.100 = 0,27\%$$

## 6) Derivación de funciones compuestas:

$$z = f(x;y)\ ;\ x = g(r)\ ;\ y = h(r) \longrightarrow z = f(g(r);h(r)) = F(r)$$

$$\Delta r \longrightarrow \Delta x = g(r+\Delta r) - g(r) \ \wedge\ \Delta y = h(r+\Delta r) - h(r)$$

$$\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y) = \left(\frac{\partial z}{\partial x}\right)_P \Delta x + \left(\frac{\partial z}{\partial y}\right)_P \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\lim_{\Delta r \to 0}\left(\frac{\Delta z}{\Delta r}\right) = \lim_{\Delta r \to 0}\left[\frac{\left(\frac{\partial z}{\partial x}\right)_P \Delta x + \left(\frac{\partial z}{\partial y}\right)_P \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\Delta r}\right] = \lim_{\Delta r \to 0}\left[\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta r} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta r} + \varepsilon_1\frac{\Delta x}{\Delta r} + \varepsilon_2\frac{\Delta y}{\Delta r}\right] =$$

$$\lim_{\Delta r\to 0}\left(\frac{\Delta z}{\Delta r}\right)=\frac{dz}{dr}=\lim_{\Delta r\to 0}\left(\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta r}\right)+\lim_{\Delta r\to 0}\left(\frac{\partial z}{\partial y}\frac{\Delta y}{\Delta r}\right)+\lim_{\Delta r\to 0}\left(\varepsilon_1\frac{\Delta x}{\Delta r}\right)+\lim_{\Delta r\to 0}\left(\varepsilon_2\frac{\Delta y}{\Delta r}\right)=$$

$$=\lim_{\Delta r\to 0}\left(\frac{\partial z}{\partial x}\right)\cdot\lim_{\Delta r\to 0}\left(\frac{\Delta x}{\Delta r}\right)+\lim_{\Delta r\to 0}\left(\frac{\partial z}{\partial y}\right)\cdot\lim_{\Delta r\to 0}\left(\frac{\Delta y}{\Delta r}\right)+\lim_{\Delta r\to 0}(\varepsilon_1)\cdot\lim_{\Delta r\to 0}\left(\frac{\Delta x}{\Delta r}\right)+\lim_{\Delta r\to 0}(\varepsilon_2)\cdot\lim_{\Delta r\to 0}\left(\frac{\Delta y}{\Delta r}\right)$$

$$=\frac{\partial z}{\partial x}\frac{dx}{dr}+\frac{\partial z}{\partial y}\frac{dy}{dr}+0\cdot\frac{dx}{dr}+0\cdot\frac{dy}{dr}\longrightarrow\boxed{\frac{dz}{dr}=\frac{\partial z}{\partial x}\frac{dx}{dr}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dr}}$$

$z=f(x;y)\ ;\ x=g(r;s)\ ;\ y=h(r;s)\longrightarrow z=f(g(r;s);h(r;s))=F(r;s)$

$$\boxed{\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}}\qquad\boxed{\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}}$$

$z=f(x;y)\ ;\ x=g(x;s)\ ;\ y=h(x;s)\longrightarrow z=f(g(x;s);h(x;s))=F(x;s)$

$$\boxed{\frac{\partial z}{\partial x}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}}\qquad\boxed{\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}}$$

## 7) Derivación de funciones implícitas:

$y=f(x)\quad\therefore\quad y-f(x)=0\quad\therefore\quad \Gamma(x;y)=y-f(x)=0$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=y'\qquad \text{si } \exists\,\Delta x\Rightarrow\exists\,\Delta y\Rightarrow\exists\,\Delta\Gamma$$

$\Gamma(x;y)=\Gamma(x;f(x))=0$

$\Delta\Gamma=\Gamma(x+\Delta x;y+\Delta y)-\Gamma(x;y)$ ✗

$$\frac{d\Gamma(x;y)}{dx}=\frac{d\Gamma(x;f(x))}{dx}=\frac{d(0)}{dx}=0\longrightarrow\frac{d\Gamma}{dx}=\lim_{\Delta x\to 0}\left(\frac{\Delta\Gamma}{\Delta x}\right)=0\longrightarrow\Delta\Gamma=0$$

$\Delta\Gamma=\Gamma(x+\Delta x;y+\Delta y)-\Gamma(x;y)=0$

$$\Delta\Gamma=\frac{\partial\Gamma}{\partial x}\Delta x+\frac{\partial\Gamma}{\partial y}\Delta y\pm\varepsilon_1\Delta x\pm\varepsilon_2\Delta y=0$$

$$\lim_{\Delta x\to 0}\left(\frac{\Delta\Gamma}{\Delta x}\right)=\lim_{\Delta x\to 0}\left[\frac{\partial\Gamma}{\partial x}+\frac{\partial\Gamma}{\partial y}\frac{\Delta y}{\Delta x}\pm\varepsilon_1\pm\varepsilon_2\frac{\Delta y}{\Delta x}\right]=\lim_{\Delta x\to 0}(0)$$

$$\frac{d\Gamma}{dx}=\frac{\partial\Gamma}{\partial x}+\frac{\partial\Gamma}{\partial y}\frac{dy}{dx}=0$$

$$\boxed{\frac{dy}{dx}=\frac{-\frac{\partial\Gamma}{\partial x}}{\frac{\partial\Gamma}{\partial y}}}$$

## Método General para la derivación de funciones implícitas:

① $r_1(u;v;x;y)=0$ $\qquad u=g(x;y)$

② $r_2(u;v;x;y)=0$ $\qquad v=h(x;y)$

$$\frac{\partial u}{\partial x}\,;\ \frac{\partial u}{\partial y}\,;\ \frac{\partial v}{\partial x}\,;\ \frac{\partial v}{\partial y}$$

① $$\frac{\partial r_1}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial r_1}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial r_1}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial r_1}{\partial y}\frac{dy}{dx}=0$$

$$\frac{\partial r_1}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial r_1}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial r_1}{\partial x}=0$$

② $$\frac{\partial r_2}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial r_2}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial r_2}{\partial x}\frac{dx}{dx}+\frac{\partial r_2}{\partial y}\frac{dy}{dx}=0$$

$$\frac{\partial r_2}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial r_2}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial r_2}{\partial x}=0$$

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix}-\frac{\partial r_1}{\partial x} & \frac{\partial r_1}{\partial v}\\ -\frac{\partial r_2}{\partial x} & \frac{\partial r_2}{\partial v}\end{vmatrix}}{\begin{vmatrix}\frac{\partial r_1}{\partial u} & \frac{\partial r_1}{\partial v}\\ \frac{\partial r_2}{\partial u} & \frac{\partial r_2}{\partial v}\end{vmatrix}}=\frac{-\frac{\partial r_1}{\partial x}\cdot\frac{\partial r_2}{\partial v}+\frac{\partial r_2}{\partial x}\cdot\frac{\partial r_1}{\partial v}}{\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial v}-\frac{\partial r_2}{\partial u}\cdot\frac{\partial r_1}{\partial v}}=\frac{\frac{\partial r_2}{\partial x}\cdot\frac{\partial r_1}{\partial v}-\frac{\partial r_1}{\partial x}\cdot\frac{\partial r_2}{\partial v}}{\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial v}-\frac{\partial r_2}{\partial u}\cdot\frac{\partial r_1}{\partial v}}=\frac{\Delta ux}{\Delta}$$

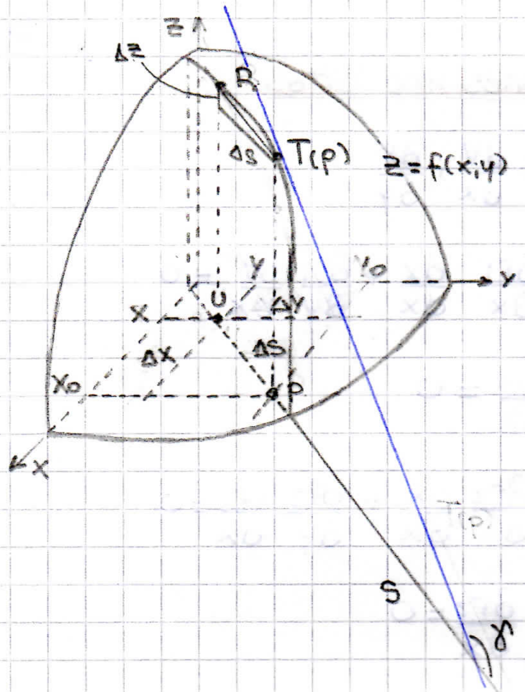
$$\frac{\partial v}{\partial x}=\frac{\begin{vmatrix}\frac{\partial r_1}{\partial u} & -\frac{\partial r_1}{\partial x}\\ \frac{\partial r_2}{\partial u} & -\frac{\partial r_2}{\partial x}\end{vmatrix}}{\begin{vmatrix}\frac{\partial r_1}{\partial u} & \frac{\partial r_1}{\partial v}\\ \frac{\partial r_2}{\partial u} & \frac{\partial r_2}{\partial v}\end{vmatrix}}=\frac{-\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial x}+\frac{\partial r_2}{\partial u}\cdot\frac{\partial r_1}{\partial x}}{\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial v}-\frac{\partial r_2}{\partial u}\cdot\frac{\partial r_1}{\partial v}}=\frac{\frac{\partial r_2}{\partial u}\cdot\frac{\partial r_1}{\partial x}-\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial x}}{\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial v}-\frac{\partial r_2}{\partial u}\cdot\frac{\partial r_1}{\partial v}}=\frac{\Delta vx}{\Delta}$$

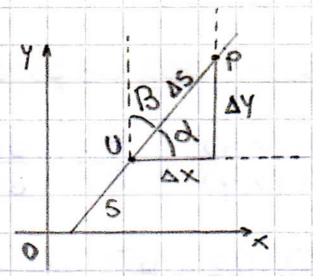
El proceso se repite para obtener $\frac{\partial u}{\partial y}$ y $\frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y}=\frac{\frac{\partial r_1}{\partial v}\cdot\frac{\partial r_2}{\partial y}-\frac{\partial r_1}{\partial y}\cdot\frac{\partial r_2}{\partial v}}{\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial v}-\frac{\partial r_2}{\partial u}\cdot\frac{\partial r_1}{\partial v}}=\frac{\Delta uy}{\Delta}$$

$$\frac{\partial v}{\partial y}=\frac{\frac{\partial r_1}{\partial y}\cdot\frac{\partial r_2}{\partial u}-\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial y}}{\frac{\partial r_1}{\partial u}\cdot\frac{\partial r_2}{\partial v}-\frac{\partial r_2}{\partial u}\cdot\frac{\partial r_1}{\partial v}}=\frac{\Delta vy}{\Delta}$$

## 8) Derivada direccional:




$$\Delta z = f(x+\Delta x;\, y+\Delta y) - f(x;y) = \left(\frac{\partial z}{\partial x}\right)_p \Delta x + \left(\frac{\partial z}{\partial y}\right)_p \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\lim_{\Delta s \to 0}\left(\frac{\Delta z}{\Delta s}\right) = \lim_{\Delta s \to 0}\left[\frac{\left(\frac{\partial z}{\partial x}\right)_p \Delta x + \left(\frac{\partial z}{\partial y}\right)_p \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\Delta s}\right] = \frac{dz}{ds}$$

$$\frac{dz}{ds} = \lim_{\Delta s \to 0}\left[\left(\frac{\partial z}{\partial x}\right)_p \frac{\Delta x}{\Delta s} + \left(\frac{\partial z}{\partial y}\right)_p \frac{\Delta y}{\Delta s} + \varepsilon_1 \frac{\Delta x}{\Delta s} + \varepsilon_2 \frac{\Delta y}{\Delta s}\right]$$

$$\frac{dz}{ds} = \lim_{\Delta s \to 0}\left[\left(\frac{\partial z}{\partial x}\right)_p \cos(\alpha) + \left(\frac{\partial z}{\partial y}\right)_p \operatorname{sen}(\alpha) + \varepsilon_1 \cos(\alpha) + \varepsilon_2 \cdot \operatorname{sen}(\alpha)\right]$$

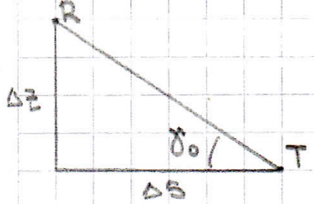
$$\frac{dz}{ds} = \left(\frac{\partial z}{\partial x}\right)_p \cos(\alpha) + \left(\frac{\partial z}{\partial y}\right)_p \operatorname{sen}(\alpha) + 0$$

$$\frac{dz}{ds} = \left(\frac{\partial z}{\partial x}\right)_p \cos(\alpha) + \left(\frac{\partial z}{\partial y}\right)_p \cos(\beta)$$

si $\alpha = 0° = 0$ Rad $\wedge$ $\beta = 90° = \pi/2$ Rad $\longrightarrow \dfrac{dz}{ds} = \left(\dfrac{\partial z}{\partial x}\right)_p$

si $\alpha = 90° = \pi/2$ Rad $\wedge$ $\beta = 0° = 0$ Rad $\longrightarrow \dfrac{dz}{ds} = \left(\dfrac{\partial z}{\partial y}\right)_p$

## Interpretación geométrica de la derivada direccional:




$Tg(\gamma_0) = \dfrac{\Delta z}{\Delta s}$ cuando $\Delta s \to 0 \Rightarrow \gamma_0 \to \gamma_1 = 180° - \gamma$

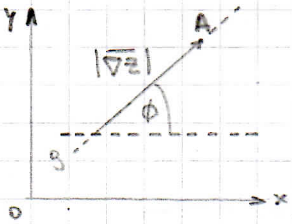
$$\frac{dz}{ds} = \lim_{\Delta s \to 0}\left(\frac{\Delta z}{\Delta s}\right) = \lim_{\Delta s \to 0}\left(Tg(\gamma_0)\right) = Tg(180° - \gamma) = -Tg(\gamma)$$

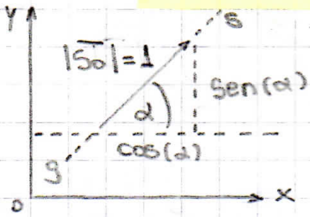
$$\boxed{\frac{dz}{ds} = Tg(\gamma)}$$

## 9 y 10 Gradiente:

$\overline{V_1} = \left(\frac{\partial z}{\partial x}\right)_P i \quad \wedge \quad \overline{V_2} = \left(\frac{\partial z}{\partial y}\right)_P j \quad \therefore \quad \overline{\nabla z} = \overline{V_1} + \overline{V_2} = \left(\frac{\partial z}{\partial x}\right)_P i + \left(\frac{\partial z}{\partial y}\right)_P j$

$|\nabla z| = \sqrt{\left(\frac{\partial z}{\partial x}\right)_P^2 + \left(\frac{\partial z}{\partial y}\right)_P^2} \qquad \phi = \operatorname{Arctg}\left(\frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}}\right)$



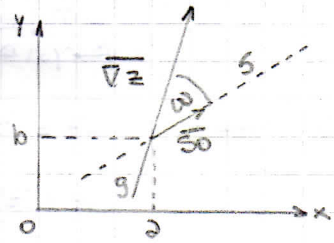



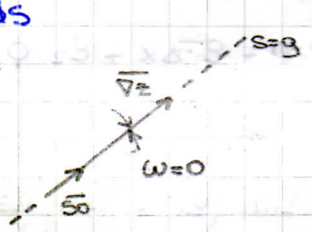

$sen^2(\alpha) + cos^2(\alpha) = 1 = |\overline{S_0}|^2$

$cos(\alpha)\, i + sen(\alpha)\, j = \overline{S_0}$

$$\overline{\nabla z} \cdot \overline{S_0} = \left[\left(\frac{\partial z}{\partial x}\right)_P i + \left(\frac{\partial z}{\partial y}\right)_P j\right] \cdot \left[Cos(\alpha)\, i + Sen(\alpha)\, j\right] = \left(\frac{\partial z}{\partial x}\right)_P cos(\alpha) + \left(\frac{\partial z}{\partial y}\right)_P sen(\alpha) = \frac{dz}{ds}$$

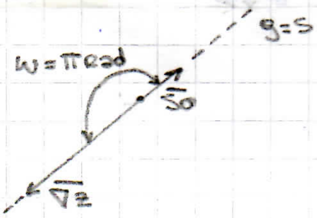
$$\frac{dz}{ds} = \overline{\nabla z} \cdot \overline{S_0} = |\overline{\nabla z}| \cdot |\overline{S_0}| \cdot cos(\omega) = \sqrt{\left(\frac{\partial z}{\partial x}\right)_P^2 + \left(\frac{\partial z}{\partial y}\right)_P^2} \cdot 1 \cdot cos(\omega)$$



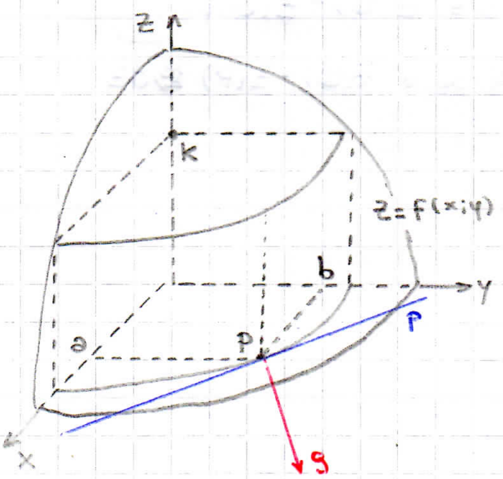

Si $\omega = 0 \rightarrow Cos(\omega) = 1$

$\frac{dz}{ds} = |\overline{\nabla z}| = \frac{\hat{dz}}{ds}$ Derivada Direccional Máxima




Si $\omega = \pi Rad \rightarrow cos(\omega) = -1$

$\frac{dz}{ds} = -|\overline{\nabla z}| = \frac{\check{dz}}{ds}$ Derivada Direccional Mínima




$F(x;y) = F(x; r(x)) = K$

$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = \frac{d(K)}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx}$

$\left(\frac{dy}{dx}\right)_P = -\left(\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)_P = P_t$

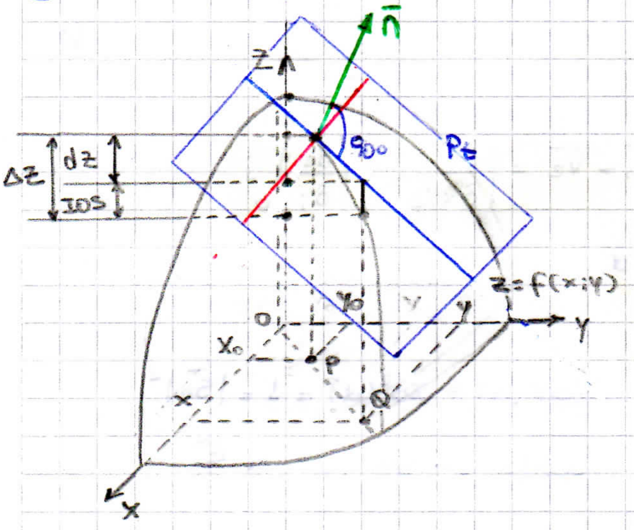
$\left(Tg(\phi)\right)_P = \left(\frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}}\right)_P = P_g$

$\left(\frac{dy}{dx}\right)_P \cdot \left(Tg(\phi)\right)_P = \left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)_P \cdot \left(\frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}}\right)_P = -1$

$\omega = \pi/2\ Rad = 90°$

$\left(\frac{dz}{ds}\right)_P = |\overline{\nabla z}| \cdot cos(\omega) = |\overline{\nabla z}| \cdot Cos(90°) = 0$

## 11 Plano tangente:




$P_t: a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$z-z_0 = \frac{-a}{c}(x-x_0) - \frac{b}{c}(y-y_0) = A(x-x_0)+B(y-y_0)$

$z-z_0 = \Delta z = dz + IOS = \left(\frac{\partial z}{\partial x}\right)_p \Delta x + \left(\frac{\partial z}{\partial y}\right)_p \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\Delta x = x - x_0$ ; $\Delta y = y - y_0$ siendo $Q(x;y) \in E(P;\delta)$

Si $x \to x_0$: $\Delta z = z - z_0 = A.0 + B(y-y_0) = \left(\frac{\partial z}{\partial x}\right)_p . 0 + \left(\frac{\partial z}{\partial y}\right)_p . \Delta y + \varepsilon_1 . 0 + \varepsilon_2$

$\Delta z = z - z_0 = B(y-y_0) = \left(\frac{\partial z}{\partial y}\right)_p \Delta y + \varepsilon_2 \Delta y = B\,\Delta y$

$\lim_{y\to y_0}\left(\frac{\Delta z}{\Delta y}\right) = \lim_{y\to y_0}\left(\frac{z-z_0}{\Delta y}\right) = \lim_{y\to y_0}(B) = \lim_{y\to y_0}\left(\frac{\partial z}{\partial y}\right)_p + \lim_{y\to y_0}(\varepsilon_2) = B = \left(\frac{\partial z}{\partial y}\right)_p$

Si $y \to y_0$: $\Delta z = z - z_0 = A.(x-x_0) + B.0 = \left(\frac{\partial z}{\partial x}\right)_p . \Delta x + \left(\frac{\partial z}{\partial y}\right)_p . 0 + \varepsilon_1 \Delta x + \varepsilon_2 . 0$

$\Delta z = z - z_0 = A(x-x_0) = \left(\frac{\partial z}{\partial x}\right)_p \Delta x + \varepsilon_1 \Delta x = A.\Delta x$

$\lim_{x\to x_0}\left(\frac{\Delta z}{\Delta x}\right) = \lim_{x\to x_0}\left(\frac{z-z_0}{\Delta x}\right) = \lim_{x\to x_0}(A) = \lim_{x\to x_0}\left(\frac{\partial z}{\partial x}\right)_p + \lim_{x\to x_0}(\varepsilon_1) = A = \left(\frac{\partial z}{\partial x}\right)_p$

$P_t: z - z_0 = \left(\frac{\partial z}{\partial x}\right)_p (x-x_0) + \left(\frac{\partial z}{\partial y}\right)_p (y-y_0)$

### Interpretación geométrica del diferencial:

$\Delta z = z(Q) - z(P)$ ; $dz = z(Q)_{pt} - z(P)$ ; $IOS = z(Q) - z(Q)_{pt}$

$\Delta z = dz + IOS = z(Q)_{pt} - z(P) + z(Q) - z(Q)_{pt} = z(Q) - z(P) \approx dz$

### Recta normal:

$\bar{n}: \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

$-c.\frac{x-x_0}{a} = -c.\frac{y-y_0}{b} = -c.\frac{z-z_0}{c}$

$\frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{-1}$

$\frac{x-x_0}{\left(\frac{\partial z}{\partial x}\right)_p} = \frac{y-y_0}{\left(\frac{\partial z}{\partial y}\right)_p} = \frac{z-z_0}{-1}$

$$\frac{x-x_0}{\left(\frac{\partial z}{\partial x}\right)_P} = \frac{z-z_0}{-1} \longrightarrow z = -\frac{x-x_0}{\left(\frac{\partial z}{\partial x}\right)_P} + z_0$$

$$\frac{y-y_0}{\left(\frac{\partial z}{\partial y}\right)_P} = \frac{z-z_0}{-1} \longrightarrow z = -\frac{y-y_0}{\left(\frac{\partial z}{\partial y}\right)_P} + z_0$$

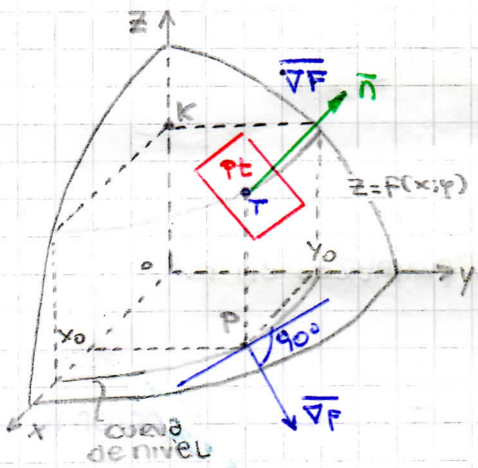
$$\bar{n} = \left[z = -\frac{x-x_0}{\left(\frac{\partial z}{\partial x}\right)_P} + z_0\right] \cap \left[z = -\frac{y-y_0}{\left(\frac{\partial z}{\partial y}\right)_P} + z_0\right]$$

$\bar{n}_1 = a_1 i + a_2 j + a_3 k \longrightarrow$ normal al plano en dirección de las x.

$\bar{n}_2 = b_1 i + b_2 j + b_3 k \longrightarrow$ normal al plano en dirección de las y.

$$\bar{n} = \bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = A_1 i + A_2 j + A_3 k$$

## Relación del plano tangente y la recta normal con el gradiente:




$f(x;y) - z = F(x;y;z) = K$

$$P_t: \left(\frac{\partial F}{\partial x}\right)_T (x-x_0) + \left(\frac{\partial F}{\partial y}\right)_T (y-y_0) + \left(\frac{\partial F}{\partial z}\right)_T (z-z_0) = 0$$

Si $R(x;y;z) \in [P_t]$:

$$\overline{TR} = (x-x_0)i + (y-y_0)j + (z-z_0)k$$

$$(\overline{\nabla F})_T = \left(\frac{\partial F}{\partial x}\right)_T i + \left(\frac{\partial F}{\partial y}\right)_T j + \left(\frac{\partial F}{\partial z}\right)_T k$$

$$(\overline{\nabla F})_T \cdot \overline{TR} = \left(\frac{\partial F}{\partial x}\right)_T (x-x_0) + \left(\frac{\partial F}{\partial y}\right)_T (y-y_0) + \left(\frac{\partial F}{\partial z}\right)_T (z-z_0) = P_t = 0$$

$$(\overline{\nabla F})_T \cdot \overline{TR} = |\overline{\nabla F}|_T \cdot |\overline{TR}| \cdot \cos(\omega) = |\overline{\nabla F}|_T |\overline{TR}| \cdot \cos(90°) = 0 \longrightarrow \boxed{(\overline{\nabla F})_T \perp \overline{TR}}$$

$$(\overline{\nabla F})_T = \bar{n}_T = \left(\frac{\partial F}{\partial x}\right)_T i + \left(\frac{\partial F}{\partial y}\right)_T j + \left(\frac{\partial F}{\partial z}\right)_T \cdot k$$

## 12 Función diferenciable:

Cuando el incremento de una función puede expresarse como la sumatoria de términos lineales con respecto a $\Delta x; \Delta y, \dots$, etc., más infinitésimos de orden superior respecto a tales incrementos de las variables, se dice que la función $z = f(x;y)$ es diferenciable.

Func. monovariables: Derivabilidad en P implica continuidad en P.

Func. multivariables: Continuidad en P implica diferenciabilidad en P y que existan las derivadas primeras en P.

Una función multivariable es diferenciable si:

a) $(z)_P = (f(x;y))_P = f(x_0;y_0)$

b) $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} (z) = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (z) = L$

c) $f(x_0;y_0) = L$

} Lo mismo debe suceder para cada derivada de la función.

Si $Q(x;y) = Q(x_0+\Delta x ; y_0+\Delta y) \wedge Q \in E(P(x_0;y_0);\delta)$

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} [f(x;y)] = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(Q)] = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(x_0+\Delta x ; y_0+\Delta y)] = f(x_0;y_0) = L$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(x_0+\Delta x ; y_0+\Delta y)] - f(x_0;y_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(x_0+\Delta x ; y_0+\Delta y)] - \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(x_0;y_0)] =$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(x_0+\Delta x ; y_0+\Delta y) - f(x_0;y_0)] = 0$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [\Delta z] = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\left(\frac{\partial z}{\partial x}\right)_P \Delta x + \left(\frac{\partial z}{\partial y}\right)_P \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y\right] = 0$$

Si en P la función es diferenciable:

- $\Delta z = dz + IOS$
- $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [\Delta z] = 0$

- Existen las derivadas primeras en P.

Si existen las derivadas primeras en el punto pero la función no es diferenciable:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [\Delta z] = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\left(\frac{\partial z}{\partial x}\right)_P \Delta x + \left(\frac{\partial z}{\partial y}\right)_P \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y\right] \neq 0 \to \boxed{f(x_0;y_0) \neq L}$$

La derivabilidad no implica continuidad

(13) Derivadas sucesivas o de orden superior: $\boxed{D_v^n = v^n}$

**Ejemplo:** Hallar $\dfrac{\partial^5 f}{\partial y^2 \partial x^2 \partial y}$ de $z = f(x;y) = 3x^3y^2$

$$\frac{\partial f}{\partial y} = 6x^3y \; ; \; \frac{\partial^2 f}{\partial y^2} = 6x^3 \; ; \; \frac{\partial^3 f}{\partial y^2 \partial x} = 18x^2 \; ; \; \frac{\partial^4 f}{\partial y^2 \partial x^2} = 36x \; ; \; \frac{\partial^5 f}{\partial y^2 \partial x^2 \partial y} = 0$$

## Teorema de Schwarz:

$\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y)$ ; $\Delta z_x = f(x+\Delta x; y) - f(x;y)$ ; $\Delta z_y = f(x; y+\Delta y) - f(x;y)$

$S = \Delta z - \Delta z_x - \Delta z_y = [f(x+\Delta x; y+\Delta y) - f(x;y)] - [f(x+\Delta x; y) - f(x;y)] - [f(x; y+\Delta y) - f(x;y)]$

$S = f(x+\Delta x; y+\Delta y) - f(x;y) - f(x+\Delta x; y) + f(x;y) - f(x; y+\Delta y) + f(x;y)$

1) $S = [f(x+\Delta x; y+\Delta y) - f(x+\Delta x; y)] - [f(x; y+\Delta y) - f(x;y)]$

$f(x; y+\Delta y) - f(x;y) = G(x) \longrightarrow G(x+\Delta x) = f(x+\Delta x; y+\Delta y) - f(x+\Delta x; y)$

$S(x;y) = G(x+\Delta x) - G(x)$

$S(x;y) = \Delta x . G'_x(\bar{x}) = \Delta x . [f(\bar{x}; y+\Delta y) - f(\bar{x}; y)]'_x$

$S(x;y) = [f'_x(\bar{x}; y+\Delta y) - f'_x(\bar{x}; y)]\Delta x$ para $x \leqslant \bar{x} \leqslant x+\Delta x$

$S(x;y) = \Delta x [\Delta y . f''_{xy}(\bar{x};\bar{y})] = \Delta x \Delta y . f''_{xy}(\bar{x};\bar{y}) = \Delta x \Delta y \dfrac{\partial^2 f(\bar{x};\bar{y})}{\partial x \partial y}$ para $y \leqslant \bar{y} \leqslant y+\Delta y$

2) $S = [f(x+\Delta x; y+\Delta y) - f(x; y+\Delta y)] - [f(x+\Delta x; y) - f(x;y)]$

$f(x+\Delta x; y) - f(x;y) = H(y) \longrightarrow H(y+\Delta y) = f(x+\Delta x; y+\Delta y) - f(x; y+\Delta y)$

$S(x;y) = H(y+\Delta y) - H(y)$

$S(x;y) = \Delta y . H'_y(\bar{y}) = \Delta y [f(x+\Delta x; \bar{y}) - f(x;\bar{y})]'_y$

$S(x;y) = [f'_y(x+\Delta x; \bar{y}) - f'_y(x;\bar{y})]\Delta y$ para $y \leqslant \bar{y} \leqslant y+\Delta y$

$S(x;y) = \Delta y [\Delta x . f''_{yx}(\bar{x}; y)] = \Delta y \Delta x . f''_{yx}(\bar{x};\bar{y}) = \Delta y \Delta x \dfrac{\partial^2 f(\bar{x};\bar{y})}{\partial y \partial x}$ para $x \leqslant \bar{x} \leqslant x+\Delta x$

$$S(x;y) = \Delta x \Delta y \frac{\partial^2 f(\bar{x};\bar{y})}{\partial x \partial y} = \Delta y \Delta x \frac{\partial^2 f(\bar{x};\bar{y})}{\partial y \partial x} \longrightarrow \frac{\partial^2 f(\bar{x};\bar{y})}{\partial x \partial y} = \frac{\partial^2 f(\bar{x};\bar{y})}{\partial y \partial x}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{\partial^2 f(\bar{x};\bar{y})}{\partial x \partial y}\right] = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{\partial^2 f(\bar{x};\bar{y})}{\partial y \partial x}\right] \longrightarrow \frac{\partial^2 f(x;y)}{\partial x \partial y} = \frac{\partial^2 f(x;y)}{\partial y \partial x}$$

## 14) Diferencial de orden superior:

$$d(dz) = d\left[\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right] = d\left[\frac{\partial z}{\partial x}dx\right] + d\left[\frac{\partial z}{\partial y}dy\right] =$$

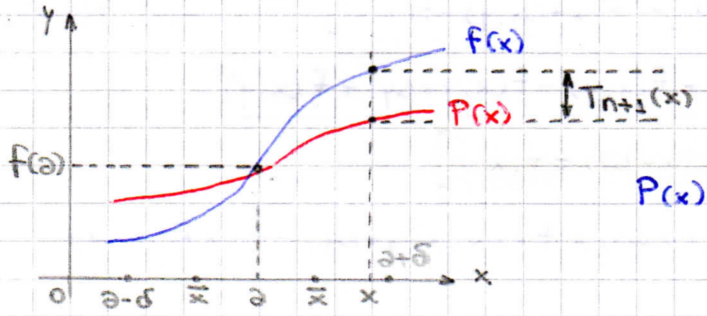
$$d(dz) = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}dx\right]dx + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}dx\right]dy + \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}dy\right]dx + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}dy\right]dy$$

$$d(dz) = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2 = d^2 z$$

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2 \equiv \left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)^{n=2}$$

$$\boxed{d^n z = \left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)^n}$$

## 15 Serie de potencias de Taylor:




$$y = f(x) = P(x)^n + T_{n+1}(x)$$

$$P(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \dots + A_n(x-a)^n$$

Siendo: $x-a = dx = \Delta x$

$$P^{'1}(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \dots + nA_n(x-a)^{n-1}$$
$$P^{'2}(x) = 2A_2 + 6A_3(x-a) + \dots + (n-1)nA_n(x-a)^{n-2}$$
$$P^{'3}(x) = 6A_3 + \dots + (n-2)(n-1)nA_n(x-a)^{n-3}$$
$$\dots$$
$$P^{'n}(x) = \dots(n-3)(n-2)(n-1)nA_n(x-a)^{n-n}$$

$$P(a) = A_0 \;;\; P^{'1}(a) = A_1 \;;\; P^{'2}(a) = 2A_2 \;;\; P^{'3}(a) = 6A_3 \;;\; P^{'n}(a) = \dots(n-3)(n-2)(n-1)nA_n$$

$$A_0 = \frac{P(a)}{1} = \frac{P(a)}{0!} \;;\; A_1 = \frac{P^{'1}(a)}{1} = \frac{P^{'1}(a)}{1!} \;;\; A_2 = \frac{P^{'2}(a)}{1.2} = \frac{P^{'2}(a)}{2!}$$

$$A_3 = \frac{P^{'3}(a)}{1.2.3} = \frac{P^{'3}(a)}{3!} \;;\; A_n = \frac{P^{'n}(a)}{\dots(n-3)(n-2)(n-1)n} = \frac{P^{'n}(a)}{n!}$$

$$P(x) = \frac{P(a)}{0!} + \frac{P^{'1}(a)}{1!}(x-a) + \frac{P^{'2}(a)}{2!}(x-a)^2 + \frac{P^{'3}(a)}{3!}(x-a)^3 + \dots + \frac{P^{'n}(a)}{n!}(x-a)^n$$

$$f(x) = \frac{P(a)}{0!} + \frac{P^{'1}(a)}{1!}(x-a) + \frac{P^{'2}(a)}{2!}(x-a)^2 + \frac{P^{'3}(a)}{3!}(x-a)^3 + \dots + \frac{P^{'n}(a)}{n!}(x-a)^n + T_{n+1}(x)$$

$$f(x) = \frac{f(a)}{0!} + \frac{f^{'1}(a)}{1!}(x-a) + \frac{f^{'2}(a)}{2!}(x-a)^2 + \frac{f^{'3}(a)}{3!}(x-a)^3 + \dots + \frac{f^{'n}(a)}{n!}(x-a)^n + T_{n+1}(x)$$

$$T_{n+1}(x) = f(x) - \left[\frac{f(a)}{0!} + \frac{f^{'1}(a)}{1!}(x-a) + \frac{f^{'2}(a)}{2!}(x-a)^2 + \frac{f^{'3}(a)}{3!}(x-a)^3 + \dots + \frac{f^{'n}(a)}{n!}(x-a)^n\right]$$

$$T_{n+1}(a) = f(a) - f(a) = 0 \;;\; T_{n+1}^{'1}(a) = f^{'1}(a) - f^{'1}(a) = 0 \;;\; T_{n+1}^{'2}(a) = f^{'2}(a) - f^{'2}(a) = 0$$
$$T_{n+1}^{'3}(a) = f^{'3}(a) - f^{'3}(a) = 0 \;;\; T_{n+1}^{'n}(a) = f^{'n}(a) - f^{'n}(a) = 0$$

$$T_{n+1}(x) = \frac{(x-a)^{n+1}}{(n+1)!} \cdot f^{'(n+1)}(\bar{x}) = T_{n+1}(\bar{x})$$

$$\lim_{n\to\infty}\left[T_{n+1}(x)\right] = \lim_{n\to\infty}\left[\frac{(x-a)^{n+1}}{(n+1)!} \cdot f^{'(n+1)}(\bar{x})\right] = 0$$

$$f(x) = \frac{f(a)}{0!} + \frac{df(a)}{dx}\frac{(x-a)}{1!} + \frac{d^2f(a)}{dx^2}\frac{(x-a)^2}{2!} + \frac{d^3f(a)}{dx^3}\frac{(x-a)^3}{3!} + \dots + \frac{d^nf(a)}{dx^n}\frac{(x-a)^n}{n!} + \frac{d^{n+1}f(\bar{x})}{dx^{(n+1)}}\frac{(x-a)^{(n+1)}}{(n+1)!}$$

$$f(x) = \left[\sum_{n=0}^{n} \frac{1}{n!} \cdot d^n f(x)\right]_{x=a}^{n} + T_{n+1}(\bar{x}) \qquad \text{para } a \leq \bar{x} \leq x$$

$$f(x;y) = \left[\sum_{n=0}^{n} \frac{1}{n!} \, d^n f(x;y)\right]_{(a,b)}^{n} + T_{n+1}(\bar{x};\bar{y}) \qquad \begin{array}{l}\text{para } a \leq \bar{x} \leq x \\ \text{para } b \leq \bar{y} \leq y\end{array}$$

$$f(x;y) = \frac{f(a;b)}{0!} + \frac{\partial f(a;b)}{\partial x}(x-a) + \frac{\partial f(a;b)}{\partial y}(y-b) +$$

$$+ \frac{1}{2!}\left[\frac{\partial^2 f(a;b)}{\partial x^2}(x-a)^2 + 2\frac{\partial^2 f(a;b)}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f(a;b)}{\partial y^2}(y-b)^2\right] +$$

$$+ \frac{1}{3!}\left[\frac{\partial^3 f(a;b)}{\partial x^3}(x-a)^3 + 3\frac{\partial^3 f(a;b)}{\partial x^2 \partial y}(x-a)^2(y-b) + 3\frac{\partial^3 f(a;b)}{\partial x \partial y^2}(x-a)(y-b)^2 + \frac{\partial^3 f(a;b)}{\partial y^3}(y-b)^3\right] +$$

$$+ \dots + \frac{1}{n!}\left(\frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)\right)_P^n + \frac{1}{(n+1)!}\left(\frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)\right)_{(\bar{x};\bar{y})}^{n+1}$$

**Demostración:**

$$f(x;y) = \left(\sum_{n=0}^{n} \frac{1}{n!} d^n f(x;y)\right)_{(a;b)}^{2} + T_{n+1}(\bar{y})$$

$$f(x;y) = \overbrace{f(x;b)}^{1} + \overbrace{\frac{\partial f(x;b)}{\partial y}(y-b)}^{2} + \overbrace{\frac{\partial^2 f(x;b)}{\partial y^2}}^{3}\frac{(y-b)^2}{2!} + \frac{\partial^3 f(x;\bar{y})}{\partial y^3}\frac{(y-b)^3}{3!}$$

1) $f(x;b) = f(a;b) + \frac{\partial f(a;b)}{\partial x}(x-a) + \frac{\partial^2 f(a;b)}{\partial x^2}\frac{(x-a)^2}{2!} + \frac{\partial^3 f(\bar{x};b)}{\partial x^3}\frac{(x-a)^3}{3!}$

2) $\frac{\partial f(x;b)}{\partial y} = \frac{\partial f(a;b)}{\partial y} + \frac{\partial^2 f(a;b)}{\partial y \partial x}(x-a) + \frac{\partial^3 f(\bar{x};b)}{\partial y \partial x^2}\frac{(x-a)^2}{2!}$

3) $\frac{\partial^2 f(x;b)}{\partial y^2} = \frac{\partial^2 f(a;b)}{\partial y^2} + \frac{\partial^3 f(\bar{x};b)}{\partial y^2 \partial x}(x-a)$

$$f(x;y) = f(a;b) + \frac{\partial f(a;b)}{\partial x}(x-a) + \frac{\partial^2 f(a;b)}{\partial x^2}\frac{(x-a)^2}{2!} + \frac{\partial^3 f(\bar{x};b)}{\partial x^3}\frac{(x-a)^3}{3!} +$$

$$+ \frac{\partial f(a;b)}{\partial y}(y-b) + \frac{\partial^2 f(a;b)}{\partial y \partial x}(y-b)(x-a) + \frac{\partial^3 f(\bar{x};b)}{\partial y \partial x^2}\frac{(y-b)(x-a)^2}{2!} +$$

$$+ \frac{\partial^2 f(a;b)}{\partial y^2}\frac{(y-b)^2}{2!} + \frac{\partial^3 f(\bar{x};b)}{\partial y^2 \partial x}\frac{(y-b)^2(x-a)}{2!} + \frac{\partial^3 f(x;\bar{y})}{\partial y^3}\frac{(y-b)^3}{3!}$$

$$f(x;y) = f(a;b) + \frac{\partial f(a;b)}{\partial x}(x-a) + \frac{1}{2}\frac{\partial^2 f(a;b)}{\partial x^2}(x-a)^2 + \frac{1}{6}\frac{\partial^3 f(\bar{x};b)}{\partial x^3}(x-a)^3 +$$

$$+ \frac{\partial f(a;b)}{\partial y}(y-b) + \frac{1}{2} \cdot 2 \cdot \frac{\partial^2 f(a;b)}{\partial y \partial x}(y-b)(x-a) + \frac{3}{3} \cdot \frac{1}{2}\frac{\partial^3 f(\bar{x};b)}{\partial y \partial x^2}(y-b)(x-a)^2 +$$

$$+ \frac{1}{2}\frac{\partial^2 f(a;b)}{\partial y^2}(y-b)^2 + \frac{3}{3}\frac{1}{2}\frac{\partial^2 f(a;b)}{\partial y^2 \partial x}(y-b)^2(x-a) + \frac{1}{6}\frac{\partial^3 f(\bar{x};b)}{\partial y^3}(y-b)^3$$

$$f(x;y)=f(a;b)+\frac{\partial f(a;b)}{\partial x}(x-a)+\frac{\partial f(a;b)}{\partial y}(y-b)+$$

$$+\frac{1}{2}\left[\frac{\partial^2 f(a;b)}{\partial x^2}(x-a)^2+2\frac{\partial^2 f(a;b)}{\partial x\partial y}(x-a)(y-b)+\frac{\partial^2 f(a;b)}{\partial y^2}(y-b)^2\right]+$$

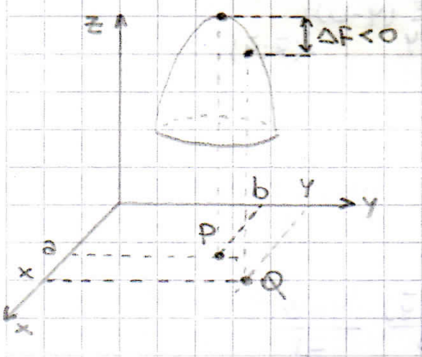
$$+\frac{1}{6}\left[\frac{\partial^3 f(\bar{x};b)}{\partial x^3}(x-a)^3+3\frac{\partial^3 f(\bar{x};b)}{\partial x^2\partial y}(x-a)^2(y-b)+3\frac{\partial^3 f(\bar{x};b)}{\partial y^2\partial x}(x-a)(y-b)+\frac{\partial^3 f(\bar{x};b)}{\partial y^3}(y-b)^3\right]$$

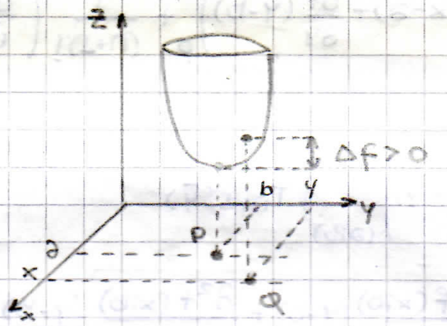
$$f(x;y)=\frac{f(a;b)}{0!}+\frac{df(a;b)}{1!}+\frac{d^2f(a;b)}{2!}+\frac{df(\bar{x};\bar{y})}{3!}=\left(\sum_{n=0}^{n}\frac{1}{n!}d^nf(x;y)\right)^2_{(a;b)}+T_3(\bar{x};\bar{y})$$

## Serie de Potencias de MacLaurin:

$$f(x;y)=\left(\sum_{n=0}^{n}\frac{1}{n!}d^nf(x;y)\right)^n_{(a;b)=(0;0)}+T_{n+1}(\bar{x};\bar{y}) \quad \text{para } \begin{matrix}0\leq\bar{x}\leq x\\ 0\leq\bar{y}\leq y\end{matrix}$$

## 16 Extremos Relativos:




Si $\Delta f<0 \Longrightarrow f(x;y)-f(a;b)<0 \Longrightarrow f(x;y)<f(a;b) \quad \forall\, P(x;y)\in E(P(a;b);\delta)\longrightarrow Mx(a;b;f(a;b))$

Si $\Delta f>0 \Longrightarrow f(x;y)-f(a;b)>0 \Longrightarrow f(x;y)>f(a;b) \quad \forall\, P(x;y)\in E(P(a;b);\delta)\longrightarrow Mn(a;b;f(a;b))$

$$z=f(x;y)=\left(\sum_{n=0}^{n}\frac{1}{n!}d^nf(x;y)\right)^2_P+T_3(\bar{x};\bar{y}) \qquad \text{como } \lim_{n\to\infty}\left[T_3(\bar{x};\bar{y})\right]=0:$$

$$z=f(x;y)=\left(\sum_{n=0}^{n}\frac{1}{n!}d^nf(x;y)\right)^2_P=f(a;b)+\frac{df(a;b)}{1!}+\frac{d^2f(a;b)}{2!}$$

$$z=f(x;y)=f(a;b)+\left(\frac{\partial f}{\partial x}(x-a)+\frac{\partial f}{\partial y}(y-b)+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x-a)^2+2\frac{\partial^2 f}{\partial x\partial y}(x-a)(y-b)+\frac{\partial^2 f}{\partial y^2}(y-b)^2\right]\right)_P$$

$$z=f(x;y)=f(a;b)+\left(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}dx^2+2\frac{\partial^2 f}{\partial x\partial y}dx\,dy+\frac{\partial^2 f}{\partial y^2}dy^2\right]\right)_P$$

$$f(x;y)-f(a;b)=\left(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}dx^2+2\frac{\partial^2 f}{\partial x\partial y}dx\,dy+\frac{\partial^2 f}{\partial y^2}dy^2\right]\right)_P$$

## Condición necesaria:

$$P_t:\ z-z(a;b)=\left[\frac{\partial f}{\partial x}(x-a)+\frac{\partial f}{\partial y}(y-b)\right]_P=\left[\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\right]_P$$

$$\left[\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\right]_P=0 \Longleftrightarrow \frac{\partial f}{\partial x}=0 \ \wedge\ \frac{\partial f}{\partial y}=0 \Longleftrightarrow P_t\,/\!/\,[xy]$$

**Condición suficiente:**

$$\operatorname{Sig} \Delta f = \operatorname{Sig}\left[\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2\right)\right]_P$$

como $\frac{\partial f}{\partial x} = 0 \wedge \frac{\partial f}{\partial y} = 0$ (condición necesaria):

$$\operatorname{Sig} \Delta f = \operatorname{Sig}\left[\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2\right)\right]_P$$

$$\operatorname{Sig} \Delta f = \operatorname{Sig}\left[\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2\right]_P$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_P = A \; ; \; \left(\frac{\partial^2 f}{\partial x \partial y}\right)_P = B \; ; \; \left(\frac{\partial^2 f}{\partial y^2}\right)_P = C \; :$$

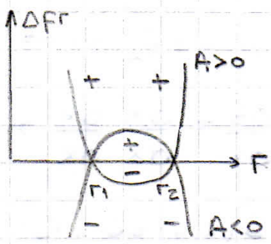
$$\operatorname{Sig} \Delta f = \operatorname{Sig}\left[A dx^2 + 2B\, dxdy + C dy^2\right]$$

$$\operatorname{Sig} \Delta f = dy^2 \operatorname{Sig}\left[A\left(\frac{dx}{dy}\right)^2 + 2B\left(\frac{dx}{dy}\right) + C\right] \qquad \text{Si } \frac{dx}{dy} = r$$

$$\operatorname{Sig} \Delta f = \operatorname{Sig}\left[A.r^2 + 2Br + C\right]$$

$$(r_1; r_2) = \frac{-2B \pm \sqrt{(2B)^2 - 4AC}}{2A} = \frac{-2B \pm \sqrt{4B^2 - 4AC}}{2A} = \frac{-2B \pm \sqrt{D}}{2A} \longrightarrow D = 4B^2 - 4AC = 4(B^2 - AC)$$

1) Si $D > 0 \longrightarrow$ 2RRD

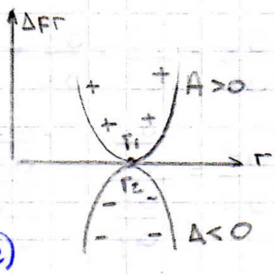



$$D = 4B^2 - 4AC = -4(AC - B^2) = -4H \longrightarrow H_{(x;y)} = (AC - B^2)_{(x;y)}$$

$$H_{(x;y)} = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

como $D = -4H > 0 \longrightarrow H < 0 \longrightarrow$ No hay extremos.

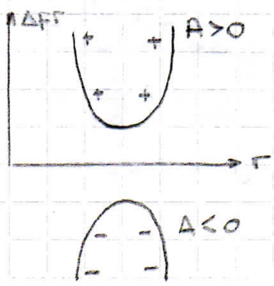
2) Si $D = 0 \longrightarrow$ 2RRI




como $D = -4H = 0 \longrightarrow H = 0 \longrightarrow$ Incertidumbre

3) Si $D < 0 \longrightarrow$ 2RCC

como $D = -4H < 0 \longrightarrow$ $H > 0 \longrightarrow$ Hay extremo (condición suficiente)



## Categorización de extremos:

$$\operatorname{sig}\Delta f = \operatorname{sig}\left[Ar^2 + 2Br + C\right]$$

$$\operatorname{sig}\Delta f = \operatorname{sig}\left[\frac{(Ar)^2 + 2BAr + AC}{A}\right]$$

$$\operatorname{sig}\Delta f = \operatorname{sig}\left[\frac{(Ar)^2 + 2.A.B.r + AC + B^2 - B^2}{A}\right]$$

$$\begin{cases} \operatorname{sig}\Delta f = \operatorname{sig}\left[\dfrac{(Ar+B)^2 + AC - B^2}{A}\right] & (Ar+B)^2 > 0 \wedge (AC - B^2) = H > 0 \\ \operatorname{sig}\Delta f = \operatorname{sig}\left[\dfrac{(Cr+B)^2 + AC - B^2}{C}\right] & (Cr+B)^2 > 0 \wedge (AC - B^2) = H > 0 \end{cases}$$

Como $AC - B^2 > 0 \longrightarrow \operatorname{sig} A = \operatorname{sig} C \longrightarrow \operatorname{sig}\left(\frac{\partial^2 f}{\partial x^2}\right) = \operatorname{sig}\left(\frac{\partial^2 f}{\partial y^2}\right)$

- $H > 0$, si $A > 0$ (o $C > 0$) $\rightarrow \Delta f > 0 \rightarrow$ Mn
- $H > 0$, si $A < 0$ (o $C < 0$) $\rightarrow \Delta f < 0 \rightarrow$ Mx

## 17) Extremos condicionados (o Ligados)

### 1) Por extremos relativos:

$$C = g(x;y) \longrightarrow \begin{array}{l} y = f_1(x) \longrightarrow z = f(x;y) = f(x; f_1(x)) = F_1(x) \\ x = f_2(y) \longrightarrow z = f(x;y) = f(f_2(y); y) = F_2(y) \end{array}$$

### 2) Método de Lagrange:

Si $y = f_1(x) \rightarrow z = f(x; f_1(x)) = F(x)$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 0 \quad \text{(por condición necesaria)}$$

$C(x;y) = 0$

$$\frac{dC}{dx} = \frac{\partial C}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial C}{\partial y}\frac{dy}{dx} = \frac{\partial C}{\partial x} + \frac{\partial C}{\partial y}\frac{dy}{dx} = 0 \quad \text{(por condición necesaria)}$$

$$\frac{dy}{dx} = \frac{-\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{-\frac{\partial C}{\partial x}}{\frac{\partial C}{\partial y}} \longrightarrow \frac{\frac{\partial z}{\partial x}}{\frac{\partial C}{\partial x}} = \frac{\frac{\partial z}{\partial y}}{\frac{\partial C}{\partial y}} = -\lambda \longrightarrow \frac{\partial z}{\partial x} + \frac{\partial C}{\partial x}\lambda = 0 \wedge \frac{\partial z}{\partial y} + \frac{\partial C}{\partial y}\lambda = 0$$

$$\begin{cases} \dfrac{\partial z}{\partial x} + \lambda\dfrac{\partial C}{\partial x} = 0 \\ \dfrac{\partial z}{\partial y} + \lambda\dfrac{\partial C}{\partial y} = 0 \\ C(x;y) = 0 \end{cases} \longrightarrow L(x;y;\lambda) = f(x;y) + \lambda C(x;y) \longrightarrow \begin{cases} \dfrac{\partial L}{\partial x} = \dfrac{\partial z}{\partial x} + \lambda\dfrac{\partial C}{\partial x} = 0 \\ \dfrac{\partial L}{\partial y} = \dfrac{\partial z}{\partial y} + \lambda\dfrac{\partial C}{\partial y} = 0 \\ \dfrac{\partial L}{\partial \lambda} = C(x;y) = 0 \end{cases}$$

a) **Diferencial 2° de la función auxiliar:**

$F(x;y) = L(x;y;\lambda^*) = f(x;y) + \lambda^* C(x;y)$

Si $d^2F(x;y) = d^2[F(x;y) + \lambda^* C(x;y)] > 0 \longrightarrow$ Mínimo condicionado local.
Si $d^2F(x;y) = d^2[F(x;y) + \lambda^* C(x;y)] < 0 \longrightarrow$ Máximo condicionado local.

b) **Hessiano orlado:**

$$H_0(x;y;\lambda^*) = \begin{vmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial C}{\partial x} \\ \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial y^2} & \frac{\partial C}{\partial y} \\ \frac{\partial C}{\partial x} & \frac{\partial C}{\partial y} & 0 \end{vmatrix} \longrightarrow \begin{cases} H_0 > 0 \longrightarrow \text{Máximo condicionado local } (M_x) \\ H_0 < 0 \longrightarrow \text{Mínimo condicionado local } (M_n) \\ H_0 = 0 \longrightarrow \text{Incertidumbre } (I) \end{cases}$$

**Gradiente (otra forma de fundamentar el método de lagrange).**

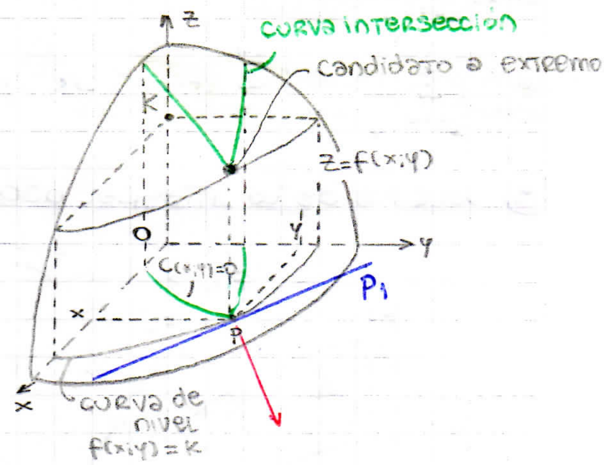
$z = f(x;y) = K \longrightarrow f(x;y) - K = 0$

Si $z$ está condicionada por $C(x;y) = 0$ y sabemos que $y = f_1(x)$:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 0 \longrightarrow \frac{dy}{dx} = \frac{-\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = P_1$$

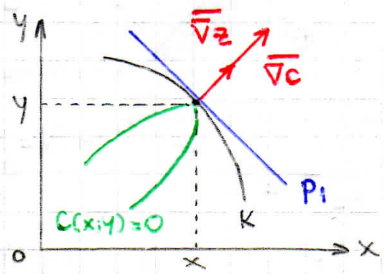
$$\overline{\nabla z} = \frac{\partial z}{\partial x} i + \frac{\partial z}{\partial y} j \longrightarrow Tg(\phi) = \frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}} = P_2$$

$$P_1 \cdot P_2 = \frac{-\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} \cdot \frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}} = -1$$




$$\frac{dC}{dx} = \frac{\partial C}{\partial x} + \frac{\partial C}{\partial y}\frac{dy}{dx} = 0 \longrightarrow \frac{dy}{dx} = \frac{-\frac{\partial C}{\partial x}}{\frac{\partial C}{\partial y}} = P_3$$

$$\overline{\nabla C} = \frac{\partial C}{\partial x} i + \frac{\partial C}{\partial y} j \longrightarrow Tg(\phi_c) = \frac{\frac{\partial C}{\partial y}}{\frac{\partial C}{\partial x}} = P_4$$



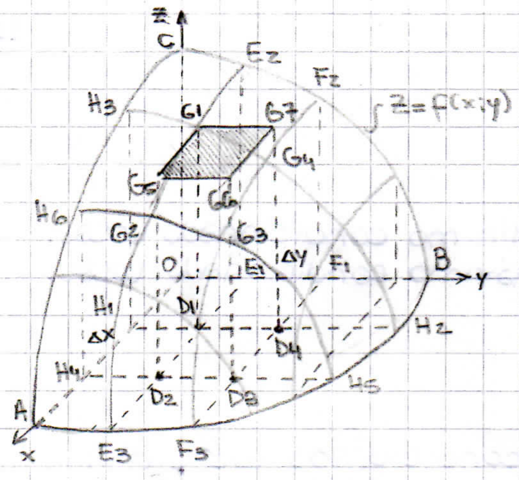

$$P_3 \cdot P_4 = \frac{-\frac{\partial C}{\partial x}}{\frac{\partial C}{\partial y}} \cdot \frac{\frac{\partial C}{\partial y}}{\frac{\partial C}{\partial x}} = -1$$

Como $P_1 \parallel P_3$ $(P_1 = P_3) \longrightarrow P_2 \parallel P_4 \longrightarrow \overline{\nabla z} \parallel \overline{\nabla C}$

$$\overline{\nabla z} = -\lambda \overline{\nabla C} \longrightarrow \frac{\partial z}{\partial x} i + \frac{\partial z}{\partial y} j = -\lambda\left(\frac{\partial C}{\partial x} i + \frac{\partial C}{\partial y} j\right) \longrightarrow \frac{\partial z}{\partial x} = -\lambda \frac{\partial C}{\partial x} \wedge \frac{\partial z}{\partial y} = -\lambda \frac{\partial C}{\partial y}$$

$$\frac{\partial z}{\partial x} + \lambda \frac{\partial C}{\partial x} = 0 \quad \wedge \quad \frac{\partial z}{\partial y} + \lambda \frac{\partial C}{\partial y} = 0$$

## (18) Integral doble: Definición.




$$\Delta V_{c_{ij}} = (\Delta A \cdot a)_{ij} = (\Delta x\, \Delta y \cdot a)_{ij} = (\Delta x\, \Delta y\, f(x;y))_{ij}$$

$$\Delta V_r = \dots + \dots + \Delta V_{ca} + \Delta V_c + \Delta V_{cp} + \dots = \sum_{j=1}^{j=m} (\Delta V_c)_{ij}$$

$$\Delta V_r = \sum_{j=1}^{j=m} [f(x;y) \cdot \Delta x \Delta y]$$

$$\Delta V_R = \lim_{m\to\infty} \sum_{j=1}^{j=m} (\Delta V_c)_{ij} = \lim_{\Delta y\to 0} \sum_{j=1}^{j=m} [(f(x;y)\,\Delta y)_{ij} (\Delta x)_{ij}] =$$

$$= \left[\lim_{\Delta y\to 0}\sum_{j=1}^{j=m}[(f(x;y)\,\Delta y)_{ij}]\right]\left[\lim_{\Delta y\to 0}\sum_{j=1}^{j=m}(\Delta x)_{ij}\right] = \left[\int f(x;y)\,dy\,(\Delta x)\right]_{ij}$$

$$V_T = \dots+\dots+\Delta V_{Ra}+\Delta V_R+\Delta V_{Rp}+\dots = \sum_{i=1}^{i=n}(\Delta V_R)_i$$

$$V_T = \sum_{i=1}^{i=n} \int f(x;y)\,dy \cdot \Delta x_i$$

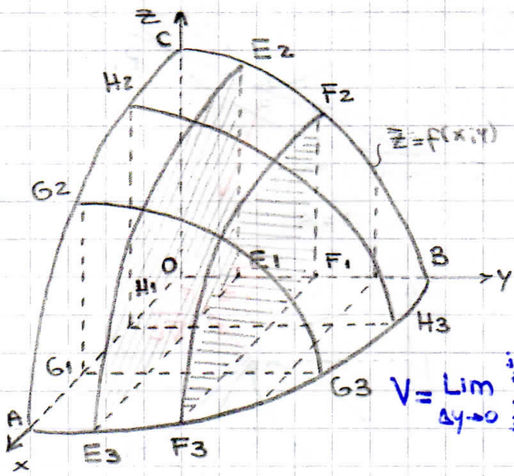
$$V = \lim_{n\to\infty}\sum_{i=1}^{i=n}\int f(x;y)\,dy\,\Delta x_i = \lim_{\Delta x\to 0}\sum_{i=1}^{i=n}\int f(x;y)\,dy \cdot \Delta x_i = \int\left[\int f(x;y)\,dy\right]dx$$

$$V = \iint_D f(x;y)\,dy\,dx$$

$$\lim_{\Delta x\to 0}\left\{\sum_{i=1}^{i=n}\left[\lim_{\Delta y\to 0}\left(\sum_{j=1}^{j=m}(f(x;y)\cdot\Delta y\cdot\Delta x)_{ij}\right)\right]\right\} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\sum_{i=1}^{i=n}\sum_{j=1}^{j=m}[f(x;y)\,\Delta y\,\Delta x]_{ij} = \iint_D f(x;y)\,dy\,dx = I_D$$

$$\lim_{\Delta y\to 0}\left\{\sum_{j=1}^{j=m}\left[\lim_{\Delta x\to 0}\left(\sum_{i=1}^{i=n}(f(x;y)\cdot\Delta x\cdot\Delta y)_{ij}\right)\right]\right\} = \lim_{\substack{\Delta y\to 0\\ \Delta x\to 0}}\sum_{j=1}^{j=m}\sum_{i=1}^{i=n}[f(x;y)\,\Delta x\,\Delta y]_{ij} = \iint_D f(x;y)\,dx\,dy = I_D$$

## (19) Cálculo de la integral doble




$$A_{E_j} = \left[\int_{x_{E_1}}^{x_{E_3}} f(x;y)\,dx\right]_j = \left[\int_{x_{E_1}}^{x_{E_3}} f(x;k)\,dx\right]_j$$

$$\Delta V_{r_j} = [A_E\, \Delta y]_j = \left[\int_{x_{E_1}=x_i}^{x_{E_3}=x_s} f(x;y)\,dx\,\Delta y\right]_j$$

$$VT = \Delta V_{r_1}+\Delta V_{r_2}+\dots+\Delta V_{r_m} = \sum_{j=1}^{j=m}\Delta V_{r_j}$$

$$VT = \sum_{j=1}^{j=m}\left[\int_{x_i}^{x_s} f(x;y)\,dx\,\Delta y\right]_j$$

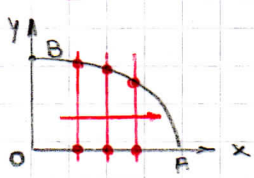
$$V = \lim_{\Delta y\to 0}\sum_{j=1}^{j=m}\left[\int_{x_i}^{x_s} f(x;y)\,dx\,\Delta y\right]_j = \int_{y_i}^{y_s}\left[\int_{x_i}^{x_s} f(x;y)\,dx\right]dy$$

$$V = \int_{x_O}^{x_A}\left[\int_{y_{H_1}}^{y_{H_3}} f(x;y)\,dy\right]dx = \int_{x_i}^{x_s}dx\int_{y_i}^{y_s} f(x;y)\,dy = \iint_D f(x;y)\,dy\,dx$$

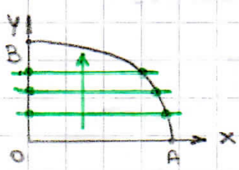
$$V = \int_{y_O}^{y_B}\left[\int_{x_{E_1}}^{x_{E_3}} f(x;y)\,dx\right]dy = \int_{y_i}^{y_s}dy\int_{x_i}^{x_s} f(x;y)\,dx = \iint_D f(x;y)\,dx\,dy$$

## Dominio:

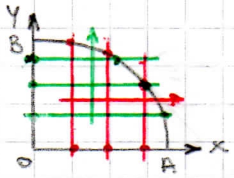
### • Dominio regular:




Regular en y



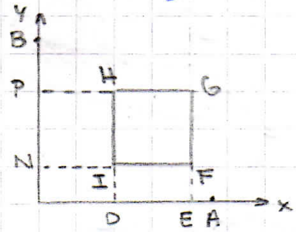

Regular en x




Regular

En dominios regulares:

$$I_D = \iint_D f(x;y)\,dydx = \iint_D f(x;y)\,dxdy$$




$$I_D = \iint_D f(x;y)\,dxdy = \int_{y_i = y_N}^{y_s = y_P} dy \int_{x_i = x_D}^{x_s = x_E} f(x;y)\,dx = \text{N° Real}$$

$$I_D = \iint_D f(x;y)\,dydx = \int_{x_i = x_D}^{x_s = x_E} dx \int_{y_i = y_N}^{y_s = y_P} f(x;y)\,dy = \text{N° Real}$$

### • Dominio no regular:

$D_{NR} = D_1 + D_2 + \ldots + D_n$ donde $D_1; D_2, \ldots, D_n$ son dominios regulares

$$\iint_{D_{NR}} f(x;y)\,dxdy = \iint_{D_1} f(x;y)\,dxdy + \iint_{D_2} f(x;y)\,dxdy + \ldots + \iint_{D_n} f(x;y)\,dxdy$$

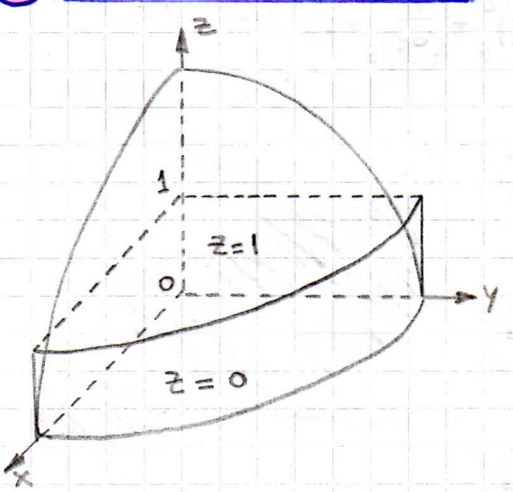
(Teorema de la partición de dominios)

## 20 Cálculo de volumen:

$$V = \iint_D f(x;y)\,dxdy = \iint_D [f_s(x;y) - f_i(x;y)]\,dydx = \iint_D [f_T(x;y) - f_P(x;y)]\,dydx = R$$

## 21 Cálculo de área plana:

$z = f(x;y)$ ; $f_1(x;y) = 1$



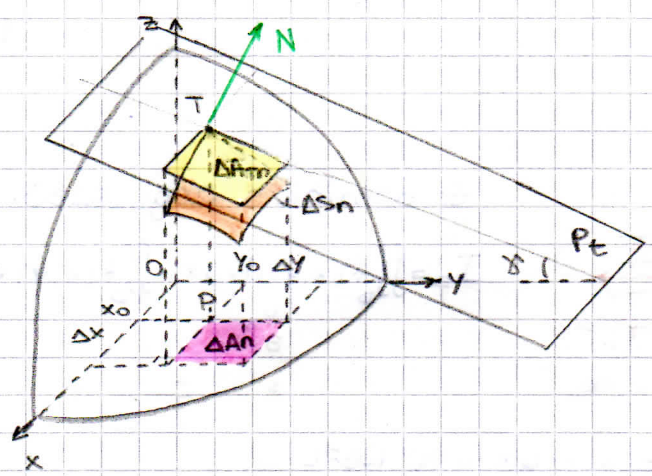

$$V = \iint_D f(x;y)\,dxdy = \iint_D [f_T(x;y) - f_P(x;y)]\,dydx =$$

$$= \iint_D [f_1(x;y) - 0]\,dydx = \iint_D f_1(x;y)\,dA = \iint_D 1 \,.\, dA =$$

$$= \iint_D dA = \iint_D dydx = \int_{y_i}^{y_s} dy \int_{x_i}^{x_s} dx = \int_{x_i}^{x_s} dx \int_{y_i}^{y_s} dy = R\,[V]$$

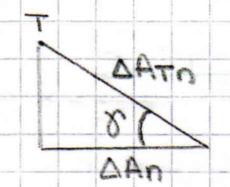
$$A = \iint_D dxdy = R\,[A]$$

## (22) Cálculo de superficie:




$$S = \lim_{n \to \infty} \sum_{n=1}^{n=n} \Delta S_n$$

$$S = \lim_{n \to \infty} \sum_{n=1}^{n=n} \Delta S_n \approx \lim_{n \to \infty} \sum_{n=1}^{n=n} \Delta A_{Tn}$$




$$\cos(\gamma)_n = \frac{\Delta A_n}{\Delta A_{Tn}} \longrightarrow \Delta A_{Tn} = \frac{\Delta A_n}{\cos(\gamma)_n} = \frac{(\Delta x \Delta y)_n}{\cos(\gamma)_n}$$

$$S = \lim_{n \to \infty}\left[\sum_{n=1}^{n=n} \Delta S_n\right] \approx \lim_{n \to \infty}\left[\sum_{n=1}^{n=n} \Delta A_{Tn}\right] = \lim_{n \to \infty}\left[\sum_{n=1}^{n=n} \frac{(\Delta x \Delta y)_n}{\cos(\gamma)_n}\right] = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}\left[\sum_{i=1}^{i=\infty}\sum_{j=1}^{j=\infty} \frac{(\Delta x \Delta y)_{ij}}{\cos(\gamma)_{ij}}\right]$$

$$P_t:\ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$P_t:\ z - z_0 = -\frac{a}{c}(x-x_0) - \frac{b}{c}(y-y_0) \longrightarrow -\frac{a}{c} = \frac{\partial z}{\partial x} \ \wedge\ -\frac{b}{c} = \frac{\partial z}{\partial y}$$

$$P_t:\ z - z_0 = \left(\frac{\partial z}{\partial x}\right)_p (x-x_0) + \left(\frac{\partial z}{\partial y}\right)_p (y-y_0)$$

$$\cos(\gamma) = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{\dfrac{a^2+b^2+c^2}{c^2}}} = \frac{1}{\sqrt{\left(\dfrac{a}{c}\right)^2 + \left(\dfrac{b}{c}\right)^2 + 1}} = \frac{1}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)_p^2 + \left(\dfrac{\partial z}{\partial y}\right)_p^2 + 1}}$$

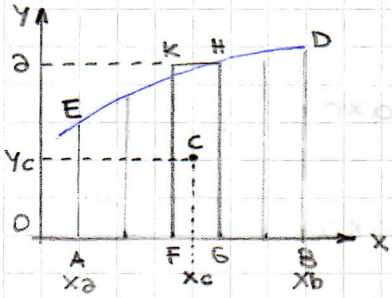
$$S = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}\left[\sum_{i=1}^{i=\infty}\sum_{j=1}^{j=\infty} \frac{(\Delta x\, \Delta y)_{ij}}{\dfrac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)_p^2 + \left(\frac{\partial z}{\partial y}\right)_p^2 + 1}}}\right] = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i=1}^{i=\infty}\sum_{j=1}^{j=\infty} \left[\sqrt{\left(\frac{\partial z}{\partial x}\right)_p^2 + \left(\frac{\partial z}{\partial y}\right)_p^2 + 1}\right]_{ij} (\Delta x \Delta y)_{ij}$$

$$S = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)_p^2 + \left(\frac{\partial z}{\partial y}\right)_p^2 + 1}\ dx\,dy = R[A] \longrightarrow [xy]$$

$$S = \iint_D \sqrt{\left(\frac{\partial x}{\partial y}\right)_p^2 + \left(\frac{\partial x}{\partial z}\right)_p^2 + 1}\ dy\,dz = R[A] \longrightarrow [yz]$$

$$S = \iint_D \sqrt{\left(\frac{\partial y}{\partial x}\right)_p^2 + \left(\frac{\partial y}{\partial z}\right)_p^2 + 1}\ dx\,dz = R[A] \longrightarrow [xz]$$

## 23 Cálculo de momento estático (primer momento)




$$\boxed{x_c = \frac{\Delta M y_i}{\Delta A_i}} \qquad \boxed{y_c = \frac{\Delta M x_i}{\Delta A_i}}$$

$$\Delta A_i = (a \cdot \Delta x)_i = f_i(x)\,\Delta x_i = [f(x)\,\Delta x]_i = y_i\,\Delta x_i$$

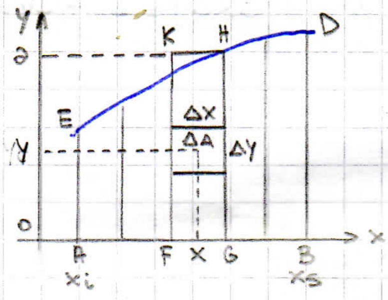
$$y_c = \frac{a_i}{2} = \frac{f_i(x)}{2} = \frac{y_i}{2}$$

$$\Delta M x_i = y_c \cdot \Delta A_i = \frac{y_i}{2} \cdot y_i\,\Delta x_i = \frac{1}{2} y_i^2\,\Delta x_i$$

$$M_x = \lim_{n \to \infty} \sum_{i=1}^{i=n} \frac{1}{2} y_i^2\,\Delta x_i = \lim_{\Delta x \to 0} \sum_{i=1}^{i=n} \frac{1}{2} y_i^2\,\Delta x_i = \frac{1}{2}\int_{x_a}^{x_b} y^2\,dx$$

$$\Delta M y_i = x_c\,\Delta A_i = x_i \cdot y_i\,\Delta x_i$$

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{i=n} x_i y_i\,\Delta x_i = \lim_{\Delta x \to 0} \sum_{i=1}^{i=n} x_i y_i\,\Delta x_i = \int_{x_a}^{x_b} x y\,dx$$




$$\Delta A_{ij} = (\Delta x\,\Delta y)_{ij}$$

$$\Delta M x_{ij} = (y \cdot \Delta A)_{ij} = (y\,\Delta x\,\Delta y)_{ij}$$

$$\Delta M y_{ij} = (x \cdot \Delta A)_{ij} = (x\,\Delta x\,\Delta y)_{ij}$$

$$M_x = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j (y\,\Delta x\,\Delta y)_{ij} = \iint_D y\,dx\,dy$$

$$M_y = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j (x\,\Delta x\,\Delta y)_{ij} = \iint_D x\,dx\,dy$$

$$M_x = \iint_D y\,dx\,dy = \int_{x_i}^{x_s} dx \int_{y_i=0}^{y_s=f(x)} y\,dy = \int_{x_i}^{x_s} dx \cdot \left[\frac{1}{2} y^2\right]_0^{f(x)} = \frac{1}{2}\int_{x_i}^{x_s} f^2(x)\,dx$$

$$M_y = \iint_D x\,dx\,dy = \int_{x_i}^{x_s} x\,dx \int_{y_i=0}^{y_s=f(x)} dy = \int_{x_i}^{x_s} x y\,dx = \int_{x_i}^{x_s} x \cdot f(x) \cdot dx$$

## 26 Cálculo de masa:

$$ds(x;y)_{ij} = \frac{\Delta m_{ij}}{\Delta S_{ij}} = \frac{\Delta m_{ij}}{(\Delta x\,\Delta y)_{ij}} \longrightarrow \Delta m_{ij} = ds(x;y)_{ij}\,(\Delta x\,\Delta y)_{ij}$$

$$M_a = \sum_i \sum_j ds(x;y)_{ij}\,(\Delta x\,\Delta y)_{ij} \longrightarrow M = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij}\,(\Delta x\,\Delta y)_{ij} = \iint_D ds(x;y)\,dx\,dy$$

$$\Delta M x_{ij} = \Delta m_{ij}\,y_{ij} = ds(x;y)_{ij}\,(\Delta x\,\Delta y)_{ij} \cdot y_{ij}$$

$$\Delta M y_{ij} = \Delta m_{ij}\,x_{ij} = ds(x;y)_{ij}\,(\Delta x\,\Delta y)_{ij}\,x_{ij}$$

$$M_x = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij}(\Delta x\,\Delta y)_{ij}\,y_{ij} = \iint_D ds(x;y)\,y\,dx\,dy$$

$$M_y = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij}(\Delta x\,\Delta y)_{ij}\,x_{ij} = \iint_D ds(x;y)\,x\,dx\,dy$$

## (24) Cálculo de momento de inercia (segundo momento):

$I = mr^2$

$\Delta Ix_{ij} = \Delta m_{ij} \cdot y^2_{ij} = ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, y^2_{ij}$
$\Delta Iy_{ij} = \Delta m_{ij} \cdot x^2_{ij} = ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, x^2_{ij}$

$$Ix = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, y^2_{ij} = \iint_D ds(x;y)\, y^2\, dx\, dy$$

$$Iy = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, x^2_{ij} = \iint_D ds(x;y)\, x^2\, dx\, dy$$

$r^2_{ij} = (x^2 + y^2)_{ij}$

$\Delta Io_{ij} = \Delta m_{ij} \cdot r^2_{ij} = ds(x;y)_{ij} (\Delta x \Delta y)_{ij} (x^2+y^2)_{ij}$

$$Io = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij} (\Delta x \Delta y)_{ij} (x^2+y^2)_{ij} = \iint ds(x;y)(x^2+y^2)\, dx\, dy = Ix + Iy = Iz$$

## (25) Cálculo de radios de giro:

$I = mr^2 \rightarrow r = \sqrt{I/m}$

$$r_x = \sqrt{\frac{Ix}{M}} = \sqrt{\frac{\iint_D ds(x;y)\, y^2\, dx\, dy}{\iint_D ds(x;y)\, dx\, dy}} \qquad r_y = \sqrt{\frac{Iy}{M}} = \sqrt{\frac{\iint_D ds(x;y)\, x^2\, dx\, dy}{\iint_D ds(x;y)\, dx\, dy}}$$

## (*) Cálculo del centro de gravedad:

$\Delta Mx = \Delta m_{ij} \cdot y_{ij} = ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, y_{ij}$
$\Delta My = \Delta m_{ij} \cdot x_{ij} = ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, x_{ij}$

$$yc_{ij} = \frac{\Delta Mx}{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}} = \frac{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, y_{ij}}{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}}$$

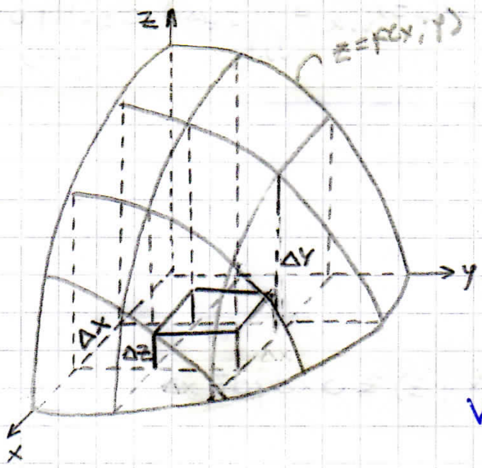
$$xc_{ij} = \frac{\Delta My}{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}} = \frac{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, x_{ij}}{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}}$$

$$xc = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j \frac{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, x_{ij}}{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}} = \frac{\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, x_{ij}}{\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij} (\Delta x \Delta y)_{ij}} = \frac{\iint_D ds(x;y)\, x\, dx\, dy}{\iint_D ds(x;y)\, dx\, dy}$$

$$yc = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j \frac{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, y_{ij}}{ds(x;y)_{ij} (\Delta x \Delta y)_{ij}} = \frac{\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij} (\Delta x \Delta y)_{ij}\, y_{ij}}{\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i \sum_j ds(x;y)_{ij} (\Delta x \Delta y)_{ij}} = \frac{\iint_D ds(x;y)\, y\, dx\, dy}{\iint_D ds(x;y)\, dx\, dy}$$

$$xc = \frac{My}{M} \quad \wedge \quad yc = \frac{Mx}{M}$$

## 27 Integrales Triples




$$\Delta V_{ijk} = (\Delta A_{ij}\,\Delta z_k) = (\Delta x\,\Delta y\,\Delta z)_{ijk}$$

$$\Delta V_{ca} = \sum_{k=1}^{k=p} (\Delta x \Delta y \Delta z)_{ijk}$$

$$\Delta V_{Ra} = \sum_{j=1}^{j=m} (\Delta V_{ca}) = \sum_{j=1}^{j=m}\sum_{k=1}^{k=p} (\Delta x \Delta y \Delta z)_{ijk}$$

$$V_R = \lim_{\substack{\Delta y \to 0\\ \Delta z \to 0}} \sum_{j=1}^{j=m}\sum_{k=1}^{k=p} (\Delta x \Delta y \Delta z)_{ijk} = \left[\iint dy\,dz\right]\Delta x_i$$

$$V_a = \sum_{i=1}^{i=n} V_R = \sum_{i=1}^{i=n} \left[\iint dy\,dz\right]\Delta x_i = \sum_{i=1}^{i=n} \lim_{\substack{\Delta y \to 0\\ \Delta z \to 0}} \sum_{j=1}^{j=m}\sum_{k=1}^{k=p} (\Delta x \Delta y \Delta z)_{ijk}$$

$$V = \lim_{\substack{\Delta x \to 0\\ \Delta y \to 0\\ \Delta z \to 0}} \sum_{i=1}^{i=n}\sum_{j=1}^{j=m}\sum_{k=1}^{k=p} (\Delta x \Delta y \Delta z)_{ijk} = \iiint_D dx\,dy\,dz$$

### Definición General:

$$\omega = f(x;y;z) \longrightarrow \Delta P_{ijk} = \left[f(x;y;z) \cdot \Delta x \cdot \Delta y \cdot \Delta z\right]_{ijk}$$

$$P_a = \sum_{i=1}^{i=n}\sum_{j=1}^{j=m}\sum_{k=1}^{k=p} f(x;y;z)_{ijk} \cdot (\Delta x \Delta y \Delta z)_{ijk}$$

$$P = \lim_{\substack{\Delta x \to 0\\ \Delta y \to 0\\ \Delta z \to 0}} \sum_{i=1}^{i=n}\sum_{j=1}^{j=m}\sum_{k=1}^{k=p} f(x;y;z)_{ijk} (\Delta x \Delta y \Delta z)_{ijk} = \iiint_D f(x;y;z)\,dx\,dy\,dz = I_T$$

## 28 Cálculo de la integral Triple - Dominio

$$I_T = \iiint_D f(x;y;z)\,dx\,dy\,dz = \int_{z_i}^{z_s} dz \int_{y_i}^{y_s} dy \int_{x_i}^{x_s} f(x;y;z)\,dx = \int_{x_i}^{x_s} dx \int_{y_i}^{y_s} dy \int_{z_i}^{z_s} f(x;y;z)\,dz = \ldots = \text{etc.}$$

(para dominio regular)

$D_{NR} = D_1 + D_2 + \ldots + D_n$ donde $D_1, D_2, \ldots, D_n$ son dominios regulares

$$\iiint_D f(x;y;z)\,dx\,dy\,dz = \iiint_{D_1} f(x;y;z)\,dx\,dy\,dz + \iiint_{D_2} f(x;y;z)\,dx\,dy\,dz + \ldots + \iiint_{D_n} f(x;y;z)\,dx\,dy\,dz$$

(Teorema de la partición de dominios)

## 29 Aplicaciones:

- **Volumen:** $\Delta P_{ijk} = f(x;y;z)_{ijk}\,(\Delta x \Delta y \Delta z)_{ijk} = 1\,(\Delta x \Delta y \Delta z)_{ijk} = (\Delta x \Delta y \Delta z)_{ijk}$

$\boxed{f(x;y;z)=1}$

$$P = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum_{i=1}^{i=n}\sum_{j=1}^{j=m}\sum_{k=1}^{k=p} f(x;y;z)_{ijk}\,(\Delta x \Delta y \Delta z)_{ijk} = \iiint_D 1\,dx\,dy\,dz = \iiint_D dx\,dy\,dz = V$$

- $f(x;y;z)_{ijk} = dV(x;y;z)_{ijk} = \dfrac{\Delta m_{ijk}}{\Delta V_{ijk}}$

- **Masa:** $M = \iiint_D dV(x;y;z)\,dx\,dy\,dz$

- **Momentos estáticos:**

$$M_{xy} = \iiint_D dV(x;y;z)\,z\,dx\,dy\,dz$$

$$M_{xz} = \iiint_D dV(x;y;z)\,y\,dx\,dy\,dz$$

$$M_{yz} = \iiint_D dV(x;y;z)\,x\,dx\,dy\,dz$$

- **Centro de gravedad:**

$$x_c = \frac{M_{yz}}{M} = \frac{\iiint_D dV(x;y;z)\,x\,dx\,dy\,dz}{\iiint_D dV(x;y;z)\,dx\,dy\,dz}$$

$$y_c = \frac{M_{xz}}{M} = \frac{\iiint_D dV(x;y;z)\,y\,dx\,dy\,dz}{\iiint_D dV(x;y;z)\,dx\,dy\,dz}$$

$$z_c = \frac{M_{xy}}{M} = \frac{\iiint_D dV(x;y;z)\,z\,dx\,dy\,dz}{\iiint_D dV(x;y;z)\,dx\,dy\,dz}$$

- **Momentos de inercia:**

$$I_x = \iiint_D dV(x;y;z)\,(y^2+z^2)\,dx\,dy\,dz$$

$$I_y = \iiint_D dV(x;y;z)\,(x^2+z^2)\,dx\,dy\,dz$$

$$I_z = \iiint_D dV(x;y;z)\,(x^2+y^2)\,dx\,dy\,dz$$

$$I_R = \iiint_D dV(x;y;z)\,(x^2+y^2+z^2)\,dx\,dy\,dz$$

- **Radios de giro:**

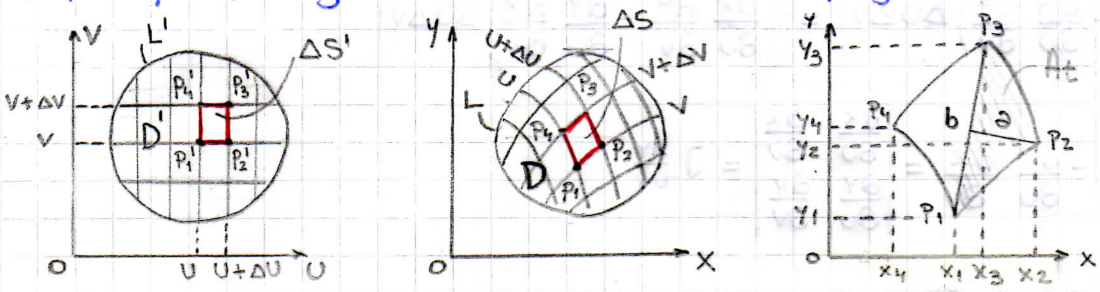
$$r_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{\iiint_D dV(x;y;z)\,(y^2+z^2)\,dx\,dy\,dz}{\iiint_D dV(x;y;z)\,dx\,dy\,dz}} \qquad r_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{\iiint_D dV(x;y;z)\,(x^2+z^2)\,dx\,dy\,dz}{\iiint_D dV(x;y;z)\,dx\,dy\,dz}}$$

$$r_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{\iiint_D dV(x;y;z)\,(x^2+y^2)\,dx\,dy\,dz}{\iiint_D dV(x;y;z)\,dx\,dy\,dz}} \qquad r_R = \sqrt{\frac{I_R}{M}} = \sqrt{\frac{\iiint_D dV(x;y;z)\,(x^2+y^2+z^2)\,dx\,dy\,dz}{\iiint_D dV(x;y;z)\,dx\,dy\,dz}}$$

## 30 Método de Ostrogradsky de transformación de coordenadas:

$I_D = \iint_D f(x;y)\,dx\,dy$

$z = f(x;y) \quad ; \quad x = g(u;v) \wedge y = h(u;v) \longrightarrow z = f(g(u;v);h(u;v)) = F(u;v)$




$dx\,dy \approx \Delta S \quad \therefore \quad \Delta S \approx 2A_t \; ;$

$$A_t = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_1y_3 - x_3y_2}{2}$$

$$= \frac{x_1(y_2-y_3) - y_1(x_2-x_3) + x_2y_3 - x_3y_2 + x_3y_3 - x_3y_3}{2} =$$

$$= \frac{x_1(y_2-y_3) - y_1(x_2-x_3) + y_3(x_2-x_3) + x_3(y_3-y_2)}{2} =$$

$$= \frac{(x_2-x_3)(y_3-y_1) + x_3(y_3-y_2) + x_1(y_2-y_3)}{2} =$$

$$= \frac{-(x_3-x_2)(y_3-y_1) + (y_3-y_2)(x_3-x_1)}{2} =$$

$$= \frac{(x_3-x_1)(y_3-y_2) - (y_3-y_1)(x_3-x_2)}{2} =$$

$$A_t = \left| \frac{(x_3-x_1)(y_3-y_2) - (y_3-y_1)(x_3-x_2)}{2} \right|$$

$$\Delta S \approx 2A_t = \left|(x_3-x_1)(y_3-y_2) - (y_3-y_1)(x_3-x_2)\right|$$

$$x_3 - x_1 = g(u+\Delta u; v+\Delta v) - g(u;v) = dg + IOS \approx \boxed{\frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v}$$

$$y_3 - y_2 = h(u+\Delta u; v+\Delta v) - h(u+\Delta u; v) = dh + IOS \approx \frac{\partial h}{\partial u}\Delta u + \frac{\partial h}{\partial v}\Delta v = \boxed{\frac{\partial h}{\partial v}\Delta v}$$

$$y_3 - y_1 = h(u+\Delta u, v+\Delta v) - h(u;v) = dh + IOS \approx \boxed{\frac{\partial h}{\partial u}\Delta u + \frac{\partial h}{\partial v}\Delta v}$$

$$x_3 - x_2 = g(u+\Delta u; v+\Delta v) - g(u+\Delta u; v) = dg + IOS \approx \frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v = \boxed{\frac{\partial g}{\partial v}\Delta v}$$

$$\Delta S \approx \left|(x_3-x_1)(y_3-y_2)-(y_3-y_1)(x_3-x_2)\right| =$$

$$\Delta S = \left|\left(\frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v\right)\left(\frac{\partial h}{\partial v}\Delta v\right) - \left(\frac{\partial h}{\partial u}\Delta u + \frac{\partial h}{\partial v}\Delta v\right)\left(\frac{\partial g}{\partial v}\Delta v\right)\right| =$$

$$= \left|\frac{\partial g}{\partial u}\frac{\partial h}{\partial v}\Delta u\Delta v + \cancel{\frac{\partial g}{\partial v}\frac{\partial h}{\partial v}\Delta v^2} - \left(\frac{\partial h}{\partial u}\frac{\partial g}{\partial v}\Delta u\Delta v + \cancel{\frac{\partial h}{\partial v}\frac{\partial g}{\partial v}\Delta v^2}\right)\right|$$

$$= \left|\left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial h}{\partial u}\frac{\partial g}{\partial v}\right)\Delta u\Delta v\right| = \left|\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\Delta u\Delta v\right|$$

$$\Delta = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = J\,\frac{xy}{uv}$$

$$\Delta S \approx \left|J\,\frac{xy}{uv}\right|\Delta u\Delta v = \left|J\,\frac{xy}{uv}\right|\Delta S'$$

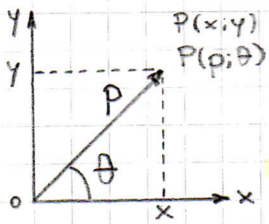
$$f(x;y)\,\Delta x\,\Delta y = f(x;y)\,\Delta S \approx F(u;v)\left|J\,\frac{xy}{uv}\right|\Delta u\Delta v = F(u;v)\left|J\,\frac{xy}{uv}\right|\Delta S'$$

$$I_D = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sum_i\sum_j f(x;y)\,\Delta x\,\Delta y = \iint_D f(x;y)\,dx\,dy =$$

$$= \lim_{\substack{\Delta u\to 0\\ \Delta v\to 0}} \sum_i\sum_j F(u;v)\left|J\,\frac{xy}{uv}\right|\Delta u\Delta v = \iint_{D'} F(u;v)\left|J\,\frac{xy}{uv}\right| du\,dv$$

$$\iint_D f(x;y)\,dx\,dy = \iint_{D'} F(u;v)\left|J\,\frac{xy}{uv}\right| du\,dv$$

## 3.1 Coordenadas polares $(p;\theta)$




$$x = \cos(\theta).p = g(p;\theta) \quad ; \quad y = \mathrm{sen}(\theta)p = h(p;\theta)$$

$$f(x;y) = f(g(p;\theta);h(p;\theta)) = F(p;\theta)$$

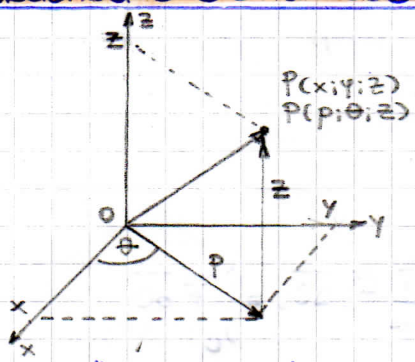
$$I_D = \iint_D f(x;y)\,dx\,dy = \iint_D F(p;\theta)\left|J\,\frac{xy}{p\theta}\right| dp\,d\theta$$

$$J\,\frac{xy}{p\theta} = \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial p}\cdot\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial p}\frac{\partial x}{\partial \theta} = \cos(\theta).p.\cos(\theta) - \mathrm{sen}(\theta)(-\mathrm{sen}(\theta)p) =$$

$$= p.\cos^2(\theta) + p\,\mathrm{sen}^2(\theta) = p(\cos^2(\theta)+\mathrm{sen}^2(\theta)) = p$$

$$I_D = \iint_D f(x;y)\,dx\,dy = \iint_D F(p;\theta)\,p\,dp\,d\theta$$

## Coordenadas cilíndricas $(\rho;\theta;z)$:




$$x=\rho\cos(\theta)=g(\rho;\theta)\ ;\ y=\rho\,\text{sen}(\theta)=h(\rho;\theta)$$

$$f(x;y;z)=f(g(\rho;\theta);h(\rho;\theta);z)=F(\rho;\theta;z)$$

$$I_T=\iiint_D f(x;y;z)\,dx\,dy\,dz=\iiint_D F(\rho;\theta;z)\left|J\frac{xyz}{\rho\theta z}\right|d\rho\,d\theta\,dz$$

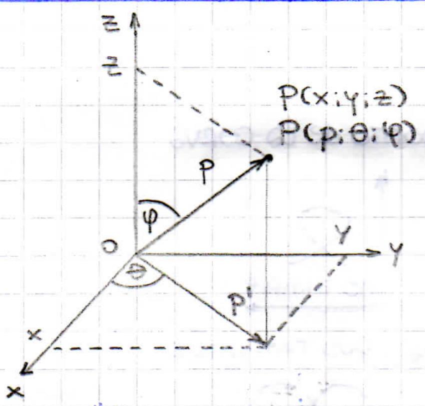
$$\left|J\frac{xyz}{\rho\theta z}\right|=\left\|\begin{matrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}\end{matrix}\right\|=\left\|\begin{matrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z}\\ 0 & 0 & 1\end{matrix}\right\|=\left|\frac{\partial x}{\partial \rho}\frac{\partial y}{\partial \theta}-\frac{\partial y}{\partial \rho}\frac{\partial x}{\partial \theta}\right|=\left|\cos(\theta)\,\rho\cos(\theta)-\text{sen}(\theta)\,\rho(-\text{sen}(\theta))\right|$$

$$=\left|\rho\cos^2(\theta)+\rho\,\text{sen}^2(\theta)\right|$$

$$=\left|\rho(\cos^2(\theta)+\text{sen}^2(\theta))\right|=|\rho|=\rho$$

$$I_T=\iiint_D f(x;y;z)\,dx\,dy\,dz=\iiint_D F(\rho;\theta;z)\,\rho\,d\rho\,d\theta\,dz$$

## Coordenadas esféricas $(\rho;\theta;\varphi)$:




$$x=\rho\,\text{sen}(\varphi)\cos(\theta)=g(\rho;\theta;\varphi)$$
$$y=\rho\,\text{sen}(\varphi)\,\text{sen}(\theta)=h(\rho;\theta;\varphi)$$
$$z=\rho\cos(\varphi)=K(\rho;\varphi)$$

$$f(x;y;z)=f(g(\rho;\theta;\varphi);h(\rho;\theta;\varphi);K(\rho;\varphi))=F(\rho;\theta;\varphi)$$

$$I_T=\iiint_D f(x;y;z)\,dx\,dy\,dz=\iiint_D F(\rho;\theta;\varphi)\left|J\frac{xyz}{\rho\theta\varphi}\right|d\rho\,d\theta\,d\varphi$$

$$\left|J\frac{xyz}{\rho\theta\varphi}\right|=\left\|\begin{matrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi}\\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi}\end{matrix}\right\|=\left\|\begin{matrix}\text{sen}(\varphi)\cos(\theta) & -\rho\,\text{sen}(\varphi)\,\text{sen}(\theta) & \rho\cos(\varphi)\cos(\theta)\\ \text{sen}(\varphi)\,\text{sen}(\theta) & \rho\,\text{sen}(\varphi)\cos(\theta) & \rho\cos(\varphi)\,\text{sen}(\theta)\\ \cos(\varphi) & 0 & -\rho\,\text{sen}(\varphi)\end{matrix}\right\|=$$

$$=\left|\begin{matrix}-\rho^2\,\text{sen}^3(\varphi)\cos^2(\theta)-\rho^2\cos^2(\varphi)\,\text{sen}(\varphi)\,\text{sen}^2(\theta)\\ -\rho^2\,\text{sen}^3(\varphi)\,\text{sen}^2(\theta)-\rho^2\cos^2(\varphi)\,\text{sen}(\varphi)\cos^2(\theta)\end{matrix}\right|=$$

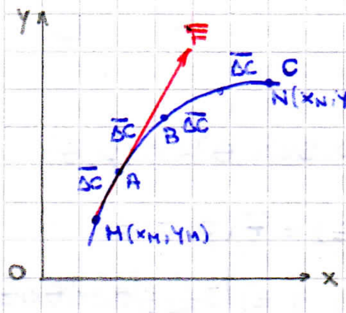
$$=\left|-\rho^2\,\text{sen}^3(\varphi)\left(\cos^2(\theta)+\text{sen}^2(\theta)\right)-\rho^2\cos^2(\varphi)\,\text{sen}(\varphi)\left(\text{sen}^2(\theta)+\cos^2(\theta)\right)\right|=$$

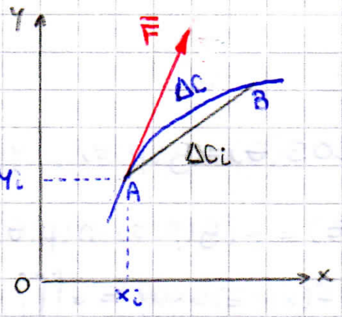
$$=\left|-\rho^2\,\text{sen}^3(\varphi)-\rho^2\cos^2(\varphi)\,\text{sen}(\varphi)\right|$$

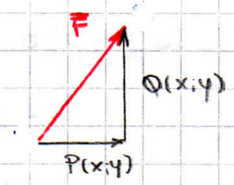
$$=\left|-\rho^2\,\text{sen}(\varphi)\left(\text{sen}^2(\varphi)+\cos^2(\varphi)\right)\right|=\left|-\rho^2\,\text{sen}(\varphi)\right|=\rho^2\,\text{sen}(\varphi)$$

$$I_T=\iiint_D f(x;y;z)\,dx\,dy\,dz=\iiint_D F(\rho;\theta;\varphi)\,\rho^2\,\text{sen}(\varphi)\,d\rho\,d\theta\,d\varphi$$

## 32) Integrales curvilíneas:




$$T_i \approx \overline{F_i}\,\overline{\Delta c} \approx \overline{F_i}\,\Delta C_i \quad \therefore \quad T_a \approx \sum_{i=1}^{i=n} \overline{F_i}\,\overline{\Delta C_i} \quad \therefore \quad T = \lim_{\substack{\Delta C \to 0 \\ (n \to \infty)}} \sum_{i=1}^{i=n} \overline{F_i}\,\overline{\Delta C_i} = \int_C \overline{F}\,\overline{dc}$$

$$\vec{F} = P(x;y)\,i + Q(x;y)\,j \qquad \wedge \qquad \overline{\Delta C_i} = \Delta x_i + \Delta y_j$$

$$T_i \approx \overline{F_i}\,\overline{\Delta C_i} = (P(x;y)\,i + Q(x;y)\,j)(\Delta x_i + \Delta y_j) = P(x_i;y_i)\,\Delta x_i + Q(x_i;y_i)\,\Delta y_i$$

$$T_a \approx \sum_{i=1}^{i=n} P(x_i;y_i)\,\Delta x_i + Q(x_i;y_i)\,\Delta y_i$$

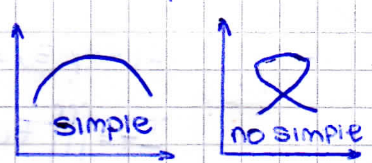
$$T = \lim_{\Delta C \to 0} \sum_{i=1}^{i=n} P(x_i;y_i)\,\Delta x_i + Q(x_i;y_i)\,\Delta y_i = \int_C \left[P(x;y)\,dx + Q(x;y)\,dy\right] = R\,[T]$$

$$I_C = \int_C \overline{F}.\overline{dc} = \int_C \left[P(x;y)\,dx + Q(x;y)\,dy\right] = \int_{M(x_M;y_M)}^{N(x_N;y_N)} \left[P(x;y)\,dx + Q(x;y)\,dy\right]$$

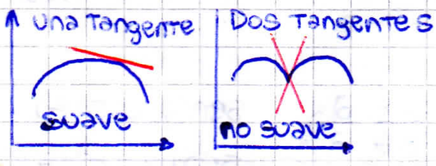
### Condiciones de existencia:

1) $P(x;y)$ y $Q(x;y)$ deben ser continuas en cada punto de la curva

2) La curva C tiene que ser simple




3) La curva C tiene que ser suave




## 33) Cálculo de la integral curvilínea:

a) $y = f(x) \quad ; \quad dy = f'(x)\,dx$

$$P(x;y)\,dx + Q(x;y)\,dy = P(x;f(x))\,dx + Q(x;f(x))\,f'(x)\,dx = P_1(x)\,dx + Q_1(x)\,f'(x)\,dx =$$

$$= \left[P_1(x) + Q_1(x)\,.\,f'(x)\right]dx = U(x)\,dx$$

$$I_C = \int_{(x;y)_M}^{(x;y)_N} \left[P(x;y)\,dx + Q(x;y)\,dy\right] = \int_{(x)_M}^{(x)_N} U(x)\,dx$$

b) $x = g(y)$ ; $dx = g'(y)\,dy$

$P(x;y)\,dx + Q(x;y)\,dy = P(g(y);y)\,g'(y)\,dy + Q(g(y);y)\,dy = P_1(y)\,g'(y)\,dy + Q_1(y)\,dy =$

$= [P_1(y)\,g'(y) + Q_1(y)]\,dy = U(y)\,dy$

$$I_C = \int_{(x;y)_M}^{(x;y)_N} [P(x;y)\,dx + Q(x;y)\,dy] = \int_{(y)_M}^{(y)_N} U(y)\,dy$$

c) $y = f(t) \wedge x = g(t)$ ; $dx = g'(t)\,dt$ ; $dy = f'(t)\,dt$

$P(x;y)\,dx + Q(x;y)\,dy = P(g(t);f(t))\,g'(t)\,dt + Q((g(t);f(t))\,f'(t)\,dt = P_1(t)\,g'(t)\,dt + Q_1(t)\,f'(t)\,dt$

$= [P_1(t)\,g'(t) + Q_1(t)\,f'(t)]\,dt = U(t)\,dt$

$$I_C = \int_{(x;y)_M}^{(x;y)_N} [P(x;y)\,dx + Q(x;y)\,dy] = \int_{(t)_M}^{(t)_N} U(t)\,dt$$

**Propiedades:**

- Si $P(x;y) = x$ ; $Q(x;y) = -3y^2$ ; $C: y = x$ ; $M(x;y) = M(0;0)$ ; $N(x;y) = N(2;2)$

$$\left.\begin{aligned} \int_{(0;0)}^{(2;2)} x\,dx + (-3y^2)\,dy &= \int_0^2 x\,dx + (-3x^2)\,dx = \left[\frac{x^2}{2} - x^3\right]_0^2 = -6 \\ \int_{(2;2)}^{(0;0)} x\,dx + (-3y^2)\,dy &= \int_2^0 x\,dx + (-3x^2)\,dx = \left[\frac{x^2}{2} - x^3\right]_2^0 = 6 \end{aligned}\right\} \int_M^N = -\int_N^M$$

- Si $P(x;y) = x$ ; $Q(x;y) = -3y^2$ ; $C: y = 0{,}5x^2$ ; $M(x;y) = M(0;0)$ ; $N(x;y) = N(2;2)$

$$\int_{(0;0)}^{(2;2)} x\,dx + (-3y^2)\,dy = \int_0^2 x\,dx + \left(\frac{-3}{4}x^4\right)x\,dx = \left[\frac{x^2}{2} - \frac{1}{8}x^6\right]_0^2 = -6$$

Independencia de trayectoria

- Si $P(x;y) = x^2y$ ; $Q(x;y) = (x^2 - y^2)$ ; $C: y = 3x^2$ ; $M(x;y) = M(0;0)$ ; $N(x;y) = N(1;3)$

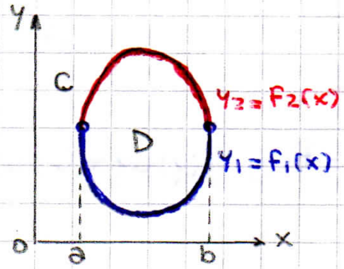
$$\int_{(0;0)}^{(1;3)} x^2y\,dx + (x^2 - y^2)\,dy = \int_0^1 x^2 \cdot 3x^2\,dx + (x^2 - 9x^4)\,6x\,dx = \left[\frac{3}{5}x^5 + \frac{6}{4}x^4 - \frac{54}{6}x^6\right]_0^1 = -6{,}9$$

- Si $P(x;y) = x^2y$ ; $Q(x;y) = (x^2 - y^2)$ ; $C: y = 3x$ ; $M(x;y) = M(0;0)$ ; $N(x;y) = N(1;3)$

$$\int_{(0;0)}^{(1;3)} x^2y\,dx + (x^2 - y^2)\,dy = \int_0^1 x^2\,3x\,dx + (x^2 - 9x^2)\,3\,dx = \left[\frac{3}{4}x^4 - 8x^3\right]_0^1 = -7{,}25$$

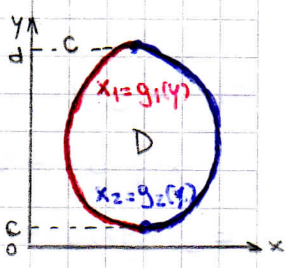
Dependencia de trayectoria

## Teorema de Green:




$$I_C = \oint_C^{\uparrow} \overline{F}\,\overline{dC} = \oint_C^{\uparrow} [P(x;y)\,dx + Q(x;y)\,dy]$$

$$I_C = \oint_C^{\uparrow} P(x;y)\,dx + \oint_C^{\uparrow} Q(x;y)\,dy$$




$$\oint_C^{\uparrow} P(x;y)\,dx = \int_a^b P(x;y_1)\,dx + \int_b^a P(x;y_2)\,dx = \int_b^a P(x;y_2)\,dx - \int_b^a P(x;y_1)\,dx = \int_b^a [P(x;y_2) - P(x;y_1)]\,dx$$

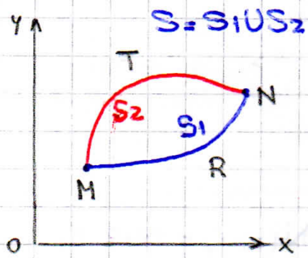
$$P(x;y_2) - P(x;y_1) = [P(x;y)]_{y_1}^{y_2} = \int_{y_1}^{y_2} \frac{\partial P(x;y)}{\partial y}\,dy \longrightarrow \oint_C^{\uparrow} P(x;y)\,dx = -\int_a^b \int_{y_1}^{y_2} \frac{\partial P(x;y)}{\partial y}\,dy\,dx$$

$$\oint_C^{\uparrow} Q(x;y)\,dy = \int_c^d Q(x_2;y)\,dy + \int_d^c Q(x_1;y)\,dy = \int_c^d Q(x_2;y)\,dy - \int_c^d Q(x_1;y)\,dy = \int_c^d [Q(x_2;y) - Q(x_1;y)]\,dy$$

$$Q(x_2,y) - Q(x_1;y) = [Q(x;y)]_{x_1}^{x_2} = \int_{x_1}^{x_2} \frac{\partial Q(x;y)}{\partial x}\,dx \longrightarrow \oint_C^{\uparrow} Q(x;y)\,dy = \int_c^d \int_{x_1}^{x_2} \frac{\partial Q(x;y)}{\partial x}\,dx\,dy$$

$$I_C = \oint_C^{\uparrow} \overline{F}.\overline{dC} = \oint_C^{\uparrow} [P(x;y)\,dx + Q(x;y)\,dy] = \iint_D \frac{\partial Q(x;y)}{\partial y}\,dxdy - \iint_D \frac{\partial P(x;y)}{\partial x}\,dxdy = \iint_D \left(\frac{\partial Q(x;y)}{\partial x} - \frac{\partial P(x;y)}{\partial y}\right) dxdy$$

## 34 Independencia del camino:




$$I_{C_1} = \int_{S_1} (\overline{F}\,\overline{dC}) = \int_{MRN} [P(x;y)\,dx + Q(x;y)\,dy]$$

$$I_{C_2} = \int_{S_2} (\overline{F}\,\overline{dC}) = \int_{MTN} [P(x;y)\,dx + Q(x;y)\,dy]$$

$$I_{C_1} = I_{C_2} = \int_{MRN} [P(x;y)\,dx + Q(x;y)\,dy] = \int_{MTN} [P(x;y)\,dx + Q(x;y)\,dy]$$

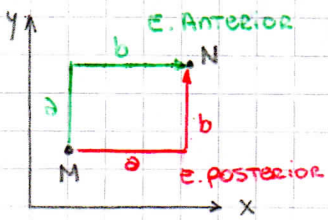
$$\int_{MRN} [P(x;y)\,dx + Q(x;y)\,dy] - \int_{MTN} [P(x;y)\,dx + Q(x;y)\,dy] = 0$$

$$\int_{MRN} [P(x;y)\,dx + Q(x;y)\,dy] + \int_{NTM} [P(x;y)\,dx + Q(x;y)\,dy] = 0$$

$$\oint_S^{\uparrow} [P(x;y)\,dx + Q(x;y)\,dy] = 0 = \iint_D \left[\frac{\partial Q(x;y)}{\partial y} - \frac{\partial P(x;y)}{\partial x}\right] dxdy \Longrightarrow \boxed{\frac{\partial Q(x;y)}{\partial y} = \frac{\partial P(x;y)}{\partial x}}$$

condición de simetría

## Trayectorias escalonadas:




**Escalón anterior:**

$a \longrightarrow x = x_M \,;\, dx = 0$
$\longrightarrow y \longrightarrow y_M \wedge y_N$

$b \longrightarrow x \longrightarrow x_M \wedge x_N$
$\longrightarrow y = y_N \,;\, dy = 0$

**Escalón Posterior:**

$a \longrightarrow x \longrightarrow x_M \wedge x_N$
$\longrightarrow y = y_M \,;\, dy = 0$

$b \longrightarrow x = x_N \,;\, dx = 0$
$\longrightarrow y \longrightarrow y_M \wedge y_N$

Escalón anterior: $I_C=\int_C(\overline{F}\,\overline{dC})=\int_{M(x;y)}^{N(x;y)}[P(x;y)\,dx+Q(x;y)\,dy]=$

$$=\int_{y_M}^{y_N}[P(x_M;y)\,dx+Q(x_M;y)\,dy]+\int_{x_M}^{x_N}[P(x;y_N)\,dx+Q(x;y_N)\,dy]=$$

$$=\int_{y_M}^{y_N}Q(x_M;y)\,dy+\int_{x_M}^{x_N}P(x;y_N)\,dx$$

Escalón posterior: $I_C=\int_C(\overline{F}\,\overline{dC})=\int_{M(x;y)}^{N(x;y)}[P(x;y)\,dx+Q(x;y)\,dy]=$

$$=\int_{x_M}^{x_N}[P(x;y_M)\,dx+Q(x;y_M)\,dy]+\int_{y_M}^{y_N}[P(x_N;y)\,dx+Q(x_N;y)\,dy]=$$

$$=\int_{x_M}^{x_N}P(x;y_M)\,dx+\int_{y_M}^{y_N}Q(x_N;y)\,dy$$

35 Función potencial:

$$\overline{F}.\overline{dC}=P(x;y)\,dx+Q(x;y)\,dy=dU(x;y)=\frac{\partial U(x;y)}{\partial x}dx+\frac{\partial U(x;y)}{\partial y}dy$$

$$\overline{F}=P(x;y)\,i+Q(x;y)\,j=\frac{\partial U(x;y)}{\partial x}i+\frac{\partial U(x;y)}{\partial y}j=\overline{\nabla U(x;y)}$$

Si $\overline{F}=\overline{\nabla U(x;y)}\longrightarrow U(x;y)=f_p$

$$I_C=\int_C(\overline{F}\,\overline{dC})=\int_C\overline{\nabla U}.\overline{dC}=\int_C\left(\frac{\partial U}{\partial x}i+\frac{\partial U}{\partial y}j\right)(dx\,i+dy\,j)=\int_C\overline{\nabla f_p}.\overline{dC}=\int_C P(x;y)\,dx+Q(x;y)\,dy$$

---

Si $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}\Longrightarrow\int_{MRN}\overline{F}\,\overline{dC}=\int_{MTN}\overline{F}\,\overline{dC}$

$$\int_{MRN}\overline{F}\,\overline{dC}-\int_{NTN}\overline{F}\,\overline{dC}=0=\int_{MRN}\overline{F}\,\overline{dC}+\int_{NTM}\overline{F}\,\overline{dC}=\oint_C\overline{F}\,\overline{dC}=\oint\overline{\nabla U}\,\overline{dC}=\oint\overline{\nabla f_p}\,\overline{dC}=0$$

$\boxed{\text{Campo conservativo}}$

Cálculo de la función potencial:

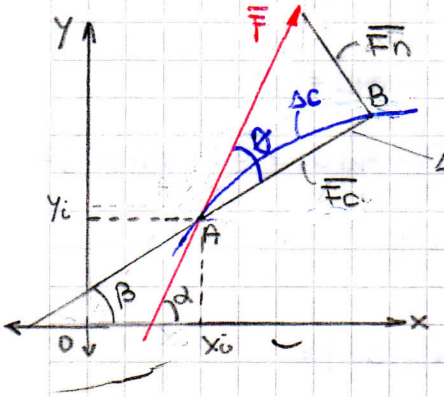
$$Q(x;y)=\frac{\partial U(x;y)}{\partial y}\longrightarrow\partial U(x;y)=Q(x;y)\,\partial y\longrightarrow U(x;y)=\int Q(x;y)\,\partial y=R_1+C(x)$$

$$\frac{\partial U(x;y)}{\partial x}=\frac{\partial}{\partial x}\int Q(x;y)\,\partial y=\frac{\partial}{\partial x}(R_1+C(x))=\int\frac{\partial Q(x;y)}{\partial x}\partial y+C'(x)=P(x;y)$$

$$C(x)=\int C'(x)\,dx=\int\left(P(x;y)-\int\frac{\partial Q(x;y)}{\partial x}\partial y\right)dx+K$$

$$U(x;y)=\int Q(x;y)\,\partial y+\int\left(P(x;y)-\int\frac{\partial Q(x;y)}{\partial x}\partial y\right)dx+K$$

## Cálculo de trabajo:




$$\bar{F}\,\bar{dc} = |\bar{F}||\bar{dc}|.\cos(\theta) = F.dC.\cos(\theta)$$

$$\Delta T_{AB_i} = [F.\cos(\theta).\Delta C]_i$$

$$T = \lim_{\substack{\Delta C \to 0 \\ (n\to\infty)}} \sum_{i=1}^{i=n} [F.\cos(\theta)\Delta C]_i = \int_C F.\cos(\theta)\,dC$$

$$\cos(\theta) = \cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \operatorname{sen}(\alpha)\operatorname{sen}(\beta)$$

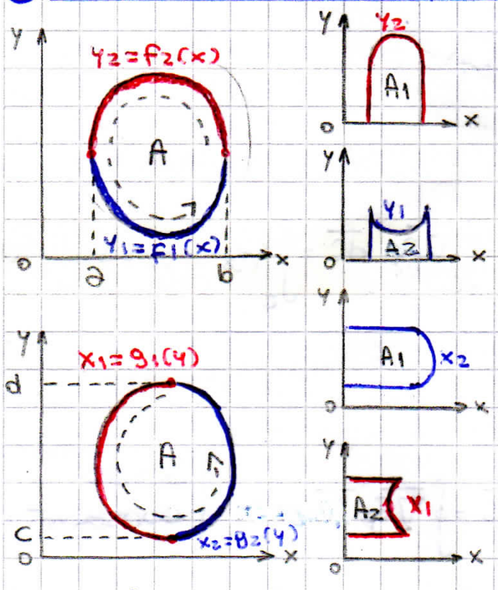
$$F.\cos(\theta).\Delta C = F\left[\cos(\alpha)\cos(\beta) + \operatorname{sen}(\alpha)\operatorname{sen}(\beta)\right]\Delta C$$

$$F.\cos(\theta)\Delta C = F.\cos(\alpha)\Delta C\cos(\beta) + F\operatorname{sen}(\alpha)\Delta C\operatorname{sen}(\beta)$$

$$F.\cos(\theta)\Delta C = P(x;y)\,\Delta x + Q(x;y)\,\Delta y$$

$$T = \lim_{\Delta C\to 0}\sum_{i=1}^{i=n}[\bar{F}\,\bar{\Delta C}]_i = \lim_{\Delta C\to 0}\sum_{i=1}^{i=n}[F.\cos(\theta).\Delta C]_i = \lim_{\Delta C\to 0}\sum_{i=1}^{i=n}[P(x;y)\Delta x + Q(x;y)\Delta y]_i = \int_C P(x;y)dx + Q(x;y)dy$$

## Cálculo de áreas planas:




$$A = A_1 - A_2 = \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx = \int_a^b y_2\,dx - \int_a^b y_1\,dx$$

$$A = -\int_b^a y_2\,dx - \int_a^b y_1\,dx = -\oint_C (y_2+y_1)\,dx = -\oint_C^{\uparrow} y\,dx$$

$$A = A_1 - A_2 = \int_c^d g_2(y)\,dy - \int_c^d g_1(y)\,dy = \int_c^d x_2\,dy - \int_c^d x_1\,dy$$

$$A = \int_c^d x_2\,dy + \int_d^c x_1\,dy = \oint_C^{\uparrow}(x_2+x_1)\,dy = \oint_C^{\uparrow} x\,dy$$

$$A + A = \oint_C^{\uparrow} x\,dy - \oint_C^{\uparrow} y\,dx = \oint_C^{\uparrow}(x\,dy - y\,dx) = 2A \longrightarrow A = \frac{1}{2}\oint_C^{\uparrow}(x\,dy - y\,dx)$$

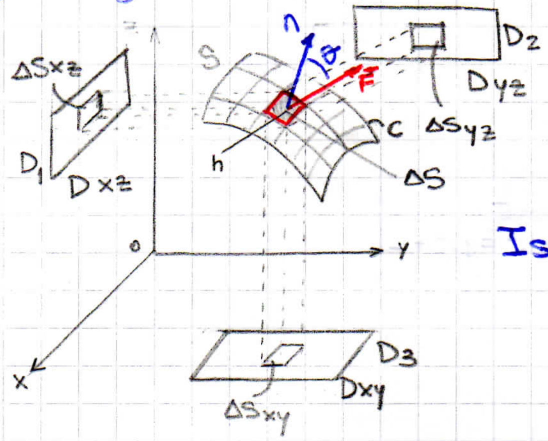
## Aplicación del teorema de Green:

$$I_C = \oint_C^{\uparrow}[P(x;y)\,dx + Q(x;y)\,dy] = \iint_D \left(\frac{\partial Q(x;y)}{\partial x} - \frac{\partial P(x;y)}{\partial y}\right)dx\,dy$$

$$P(x;y) = -y \;\wedge\; Q(x;y) = x \longrightarrow$$

$$A = \frac{1}{2}\oint_C^{\uparrow}(x\,dy - y\,dx) = \frac{1}{2}\iint_D\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)dx\,dy = \frac{1}{2}\iint_D (1+1)\,dx\,dy = \iint_D dx\,dy$$

## 36) Integrales de superficie:




$$\phi = \bar{F}.\bar{S} = [F.\cos(\theta)\,\Delta S]_h$$

$$\Delta\phi_h = (\bar{F}.\bar{n}.\Delta S)_h$$

$$\phi_a = \sum_{h=1}^{h=m} (\bar{F}\bar{n}\Delta S)_h$$

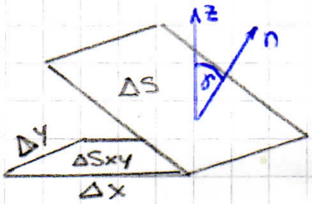
$$I_S = \lim_{m\to\infty} \sum_{h=1}^{h=m} (\bar{F}\bar{n}\Delta S)_h = \lim_{\Delta S\to 0} \sum_{h=1}^{h=\infty} (\bar{F}\bar{n}\Delta S)_h = \iint_S \bar{F}\,\bar{n}\,dS = \phi$$

$$\bar{F}_h = P(x;y;z)\,i + Q(x;y;z)\,j + R(x;y;z)\,K$$

$$\bar{n} = n_x i + n_y j + n_z K = n.\cos(\alpha) i + n\cos(\beta) j + n.\cos(\gamma) K = \cos(\alpha) i + \cos(\beta) j + \cos(\gamma) K$$

$$I_S = \lim_{\Delta S\to 0} \sum_{h=1}^{h=\infty} (\bar{F}\bar{n}\Delta S)_h = \lim_{\Delta S\to 0} \sum_{h=1}^{h=\infty} \Big[\big(P(x;y;z)i + Q(x;y;z)j + R(x;y;z)k\big)\big(\cos(\alpha)i + \cos(\beta)j + \cos(\gamma)k\big)\Delta S\Big]_h =$$

$$= \lim_{\Delta S\to 0} \sum_{h=1}^{h=\infty} \Big[P(x;y;z)\cos(\alpha)\Delta S + Q(x;y;z)\cos(\beta)\Delta S + R(x;y;z)\cos(\gamma)\Delta S\Big]_h =$$




$$\cos(\gamma)\Delta S = \Delta x\Delta y\,;\ \cos(\alpha)\Delta S = \Delta y\Delta z\,;\ \cos(\beta)\Delta S = \Delta x\Delta z$$

$$= \lim_{\Delta S\to 0} \sum_{h=1}^{h=\infty} \Big[P(x;y;z)\,\Delta S_{yz} + Q(x;y;z)\,\Delta S_{xz} + R(x;y;z)\,\Delta S_{xy}\Big]_h$$

$$= \lim_{\substack{\Delta y\to 0\\ \Delta z\to 0}} \sum_j^{\infty}\sum_k^{\infty} \big[P(x;y;z)\,\Delta y\Delta z\big]_{jk} + \lim_{\substack{\Delta x\to 0\\ \Delta z\to 0}} \sum_i^{\infty}\sum_k^{\infty} \big[Q(x;y;z)\,\Delta x\Delta z\big]_{ik} + \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sum_i^{\infty}\sum_j^{\infty} \big[R(x;y;z)\,\Delta x\Delta y\big]_{ij} =$$

$$= \iint_{D_{yz}} P(x;y;z)\,dydz + \iint_{D_{xz}} Q(x;y;z)\,dxdz + \iint_{D_{xy}} R(x;y;z)\,dxdy =$$

$$= \iint_S P(x;y;z)\,dydz + Q(x;y;z)\,dxdz + R(x;y;z)\,dxdy$$

$$= \iint_S P(f_1;y;z)\,dydz + Q(x;f_2;z)\,dxdz + R(x;y;f_3)\,dxdy$$

### Cálculo de la integral de superficie:

1) $$I_S = \iint_S \bar{F}.\bar{n}.dS = \iint_S P(f_1;y;z)\,dydz + Q(x;f_2;z)\,dxdz + R(x;y;f_3)\,dxdy$$

$$\iint_S R(x;y;f_3)\,dxdy = \iint_S R(x;y;z)\,dS.\cos(\gamma) = \pm \lim_{\Delta S\to 0} \sum_{h=1}^{h=\infty} \big[R(x;y;z)\,\Delta S\,|\cos(\gamma)|\big]_h =$$

$$= \pm \lim_{\Delta S\to 0} \sum_{h=1}^{h=\infty} \big[R(x;y;f_3(x;y))\,|\Delta S_{xy}|\big]_h = \pm \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sum_i^{\infty}\sum_j^{\infty} \big[R(x;y;f_3(x;y))\,\Delta x\Delta y\big]_{ij} = \pm \iint_S R(x;y;f_3(x;y))\,dxdy$$

$$\iint_S P(f_1;y;z)\,dydz = \pm \iint_S P(f_1(y;z);y;z)\,dydz \ \wedge\ \iint_S Q(x;f_2;z)\,dxdz = \pm \iint_S Q(x;f_2(x;z);z)\,dxdz$$

$$I_S=\iint_S \bar{F}\,\bar{n}\,dS = \pm\iint_S P(f_1(y;z);y;z)\,dy\,dz \pm\iint_S Q(x;f_2(x;z);z)\,dx\,dz \pm\iint_S R(x;y;f_3(x;y))\,dx\,dy$$

2) $$I_S=\iint_S \bar{F}\,\bar{n}\,dS=\iint_S\left[P(x;y;z).\cos(\alpha)+Q(x;y;z).\cos(\beta)+R(x;y;z).\cos(\gamma)\right]dS$$

$$\cos(\gamma)\,\Delta S=\Delta S_{xy}=\Delta x\Delta y \longrightarrow \Delta S=\frac{\Delta x\Delta y}{\cos(\gamma)}$$

$$\cos(\gamma)=\frac{c}{\sqrt{a^2+b^2+c^2}}=\frac{1}{\sqrt{\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2+1}}=\boxed{\frac{1}{\sqrt{(-z'_x)^2+(-z'_y)^2+1}}}$$

$$\Delta S=\frac{\Delta x\Delta y}{\frac{1}{\sqrt{(-z'_x)^2+(-z'_y)^2+1}}}=\boxed{\Delta x\Delta y\sqrt{(z'_x)^2+(z'_y)^2+1}}$$

$$\boxed{\cos(\alpha)=\frac{-z'_x}{\sqrt{(z'_x)^2+(z'_y)^2+1}}}\ \wedge\ \boxed{\cos(\beta)=\frac{-z'_y}{\sqrt{(z'_x)^2+(z'_y)^2+1}}}$$

$$I_S=\iint_S\left(-P(x;y;z)\frac{-z'_x}{\sqrt{(z'_x)^2+(z'_y)^2+1}}+Q(x;y;z)\frac{-z'_y}{\sqrt{(z'_x)^2+(z'_y)^2+1}}+R(x;y;z)\frac{1}{\sqrt{(z'_x)^2+(z'_y)^2+1}}\right)dx\,dy\sqrt{(z'_x)^2+(z'_y)^2+1}$$

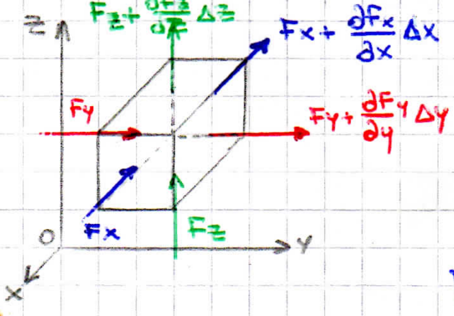
$$I_S=\iint_S\left[-z'_x P(x;y;z)-z'_y\,Q(x;y;z)+R(x;y;z)\right]dx\,dy \qquad [XY]$$

$$I_S=\iint_S\left[P(x;y;z)-x'_y\,Q(x;y;z)-x'_z\,R(x;y;z)\right]dy\,dz \qquad [YZ]$$

$$I_S=\iint_S\left[-y'_x\,P(x;y;z)+Q(x;y;z)-y'_z\,R(x;y;z)\right]dx\,dz \qquad [XZ]$$

## 37) Teorema de Gauss (divergencia):

$$\phi=F.A$$

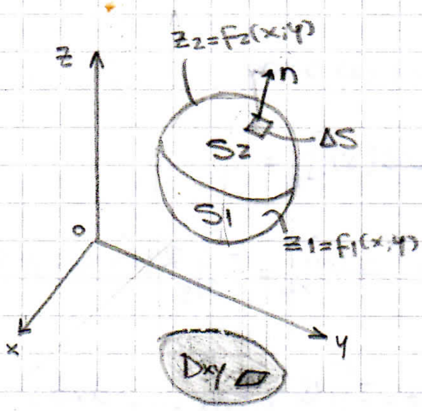



$$\phi_e=-F_x\Delta y\Delta z-F_y\Delta x\Delta z-F_z\Delta x\Delta y$$

$$\phi_s=\left(F_x+\frac{\partial F_x}{\partial x}\Delta x\right)\Delta y\Delta z+\left(F_y+\frac{\partial F_y}{\partial y}\Delta y\right)\Delta x\Delta z+\left(F_z\frac{\partial F_z}{\partial z}\Delta z\right)\Delta x\Delta y$$

$$\Delta\phi=\phi_e+\phi_s=\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)\Delta x\Delta y\Delta z$$

$$D_V=\frac{\Delta\phi}{\Delta V}=\frac{\Delta\phi}{\Delta x\Delta y\Delta z}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}=\overline{\nabla}.\overline{F}$$




$$\phi=\iint_S(\bar{F}\,\bar{n}\,dS)=\iint_S P(x;y;z)\cos(\alpha)\,dS+\iint_S Q(x;y;z)\cos(\beta)\,dS+\iint_S R(x;y;z)\cos(\gamma)\,dS$$

$$S_2:\ z_2=f_2(x;y)$$

$$dS=\frac{dA}{|\cos(\gamma)|}=\frac{dx\,dy}{|\cos(\gamma)|}\longrightarrow \cos(\gamma)\,dS=dx\,dy$$

$$S_1:\ z_1=f_1(x;y)$$

$$dS=\frac{dA}{-\cos(\gamma)}=\frac{dx\,dy}{-\cos(\gamma)}\longrightarrow \cos(\gamma)\,dS=-dx\,dy$$

$$\iint_S R(x;y;z)\cos(\gamma)dS = \iint_{S2} R(x;y;z_2)\cos(\gamma)ds + \iint_{S1} R(x;y;z_1)\cos(\gamma)\,ds =$$

$$= \iint_{Dxy} R(x;y;z_2)\,dxdy - \iint_{Dxy} R(x;y;z_1)\,dxdy = \iint_{Dxy} \left[R(x;y;z)\right]_{z_1}^{z_2} dxdy =$$

$$= \iint_{Dxy}\int_{z_1}^{z_2}\left(\frac{\partial R(x;y;z)}{\partial z}dz\right)dxdy = \boxed{\iiint_D \frac{\partial R(x;y;z)}{\partial z}\,dxdydz}$$

$$\boxed{\iint_S Q(x;y)\cos(\beta)dS = \iiint_D \frac{\partial Q(x;y;z)}{\partial y}dxdydz} ; \boxed{\iint_S P(x;y)\cos(\alpha)dS = \iiint_D \frac{\partial P(x;y;z)}{\partial x}dxdydz}$$

$$\phi = \iiint_D \frac{\partial P(x;y;z)}{\partial x}dxdydz + \iiint_D \frac{\partial Q(x;y;z)}{\partial y}dxdydz + \iiint_D \frac{\partial R(x;y;z)}{\partial z}dxdydz =$$

$$= \iiint_D \left(\frac{\partial P(x;y;z)}{\partial x} + \frac{\partial Q(x;y;z)}{\partial y} + \frac{\partial R(x;y;z)}{\partial z}\right)dxdydz = \iiint_D \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)dxdydz =$$

$$= \iint_S \bar{F}\,\bar{n}\,ds = \iiint_D (\bar{\nabla}.\bar{F})\,dV = \phi$$

$$\bar{F} = P(x;y;z)i + Q(x;y;z)j + R(x;y;z)k = F_x i + F_y j + F_z k$$

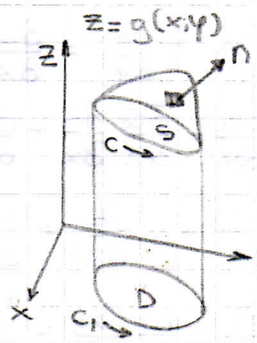
$$\bar{\nabla}.\bar{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(F_x i + F_y j + F_z k) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \longrightarrow \text{divergencia}$$

## 38 Teorema de Stokes:

$$\overline{\nabla \times F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\times\left(P(x;y;z)i + Q(x;y;z)j + R(x;y;z)k\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x;y;z) & Q(x;y;z) & R(x;y;z) \end{vmatrix} =$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k \longrightarrow \text{ROTOR}$$

$$\oint_C \bar{F}\,\overline{dc} = \iint_S \overline{\nabla \times F}\,\overline{ds} = \iint_S \overline{\nabla \times F}\,\bar{n}\,dS$$




**1)** $\iint_S \overline{\nabla \times F}\,\bar{n}\,ds$:

$$\bar{n} = \frac{\overline{\nabla F(x;y;z)}}{|\overline{\nabla F(x;y;z)}|} = \frac{\frac{\partial F}{\partial x}i + \frac{\partial F}{\partial y}j + \frac{\partial F}{\partial z}k}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}} = \frac{-\frac{\partial g}{\partial x}i - \frac{\partial g}{\partial y}j + k}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

$$S = \iint_D \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\right)dxdy \longrightarrow dS = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dxdy$$

$$\overline{\nabla \times F}.\bar{n}.dS = \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k\right].\frac{-\frac{\partial g}{\partial x}i - \frac{\partial g}{\partial y}j + k}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}.\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dxdy =$$

$$= \left[ \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) i + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) j + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k \right] \left[ -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k \right] dxdy =$$

$$= \left[ -\frac{\partial g}{\partial x} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) - \frac{\partial g}{\partial y} \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dxdy$$

$$\iint_S \overline{\nabla \times F}\, \bar{n}\, dS = \iint_D \left[ -\frac{\partial g}{\partial x} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) - \frac{\partial g}{\partial y} \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dxdy$$

2) $\oint_C \bar{F}\, \overline{dC}$: $\quad \bar{F} = F_x i + F_y j + F_z k = P(x;y;z)\, i + Q(x;y;z)\, j + R(x;y;z)\, k$
$\overline{dC} = dx\, i + dy\, j + dz\, k$

$\bar{r} = x(t) i + y(t) j + z(t) k \quad ; \quad \overline{dr} = dx(t) i + dy(t) j + dz(t) k$

$x = x(t) \therefore dx = \frac{dx}{dt} dt \; ; \; y = y(t) \therefore dy = \frac{dy}{dt} dt \; ; \; z = g(x(t);y(t)) \therefore dz = \left[ \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \right] dt$

$$\bar{F} = P_{[x(t);y(t);g(x(t);y(t))]}\, i + Q_{[x(t);y(t);g(x(t);y(t))]}\, j + R_{[x(t);y(t);g(x(t);y(t))]}\, k$$

$$\bar{F} \cdot \overline{dC} = \bar{F} \cdot \overline{dr} = \left\{ P_{[x(t);y(t);z(t)]}\, i + Q_{[x(t);y(t);z(t)]}\, j + R_{[x(t);y(t);z(t)]}\, k \right\} \left[ dx(t) i + dy(t) j + dz(t) k \right] =$$

$$= P_{(x(t);y(t);z(t))}\, dx(t) + Q_{(x(t);y(t);z(t))}\, dy(t) + R_{(x(t);y(t);z(t))}\, dz(t) =$$

$$= \left[ P_{(x(t);y(t);z(t))} \frac{dx}{dt} + Q_{(x(t);y(t);z(t))} \frac{dy}{dt} + R_{(x(t);y(t);z(t))} \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \right] dt =$$

$$= \left[ P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{\partial z}{\partial x} \frac{dx}{dt} + R \frac{\partial z}{\partial y} \frac{dy}{dt} \right] dt =$$

$$= \left[ \left( P + R \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left( Q + R \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] dt = \left( P + R \frac{\partial z}{\partial x} \right) dx + \left( Q + R \frac{\partial z}{\partial y} \right) dy$$

$$\oint_C \bar{F} \cdot \overline{dr} = \oint_C \left[ \left( P + R \frac{\partial z}{\partial x} \right) dx + \left( Q + R \frac{\partial z}{\partial y} \right) dy \right] = \iint_D \left[ \frac{\partial}{\partial x} \left( Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left( P + R \frac{\partial z}{\partial x} \right) \right] dxdy$$

a) $$\frac{\partial}{\partial x} \left( Q + R \frac{\partial z}{\partial y} \right) = \frac{\partial Q}{\partial x} \frac{dx}{dx} + \frac{\partial Q}{\partial y} \frac{dy}{dx} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \left( \frac{\partial R}{\partial x} \frac{dx}{dx} + \frac{\partial R}{\partial y} \frac{dy}{dx} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \right) \cdot \frac{\partial z}{\partial y} + R \cdot \frac{\partial^2 z}{\partial x \partial y} =$$

$$= \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial x \partial y}$$

b) $$\frac{\partial}{\partial y} \left( P + R \frac{\partial z}{\partial x} \right) = \frac{\partial P}{\partial x} \frac{dx}{dy} + \frac{\partial P}{\partial y} \frac{dy}{dy} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \left( \frac{\partial R}{\partial x} \frac{dx}{dy} + \frac{\partial R}{\partial y} \frac{dy}{dy} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \right) \frac{\partial z}{\partial x} + R \frac{\partial^2 z}{\partial y \partial x}$$

$$= \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + R \frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial}{\partial x} \left( Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left( P + R \frac{\partial z}{\partial x} \right) = \left( \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + R \frac{\partial^2 z}{\partial x \partial y} \right) - \left( \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial x \partial y} \right) =$$

$$= \frac{\partial Q}{\partial x} + \left( \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) \frac{\partial z}{\partial x} + \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \frac{\partial z}{\partial y} - \frac{\partial P}{\partial y}$$

$$\oint_C \bar{F}\, \overline{dr} = \iint_D \left[ \frac{\partial Q}{\partial x} + \left( \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) \frac{\partial z}{\partial x} + \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \frac{\partial z}{\partial y} - \frac{\partial P}{\partial y} \right] dxdy$$

$$\oint_C \vec{F}\,\vec{dC} = \iint_S \overline{\nabla\times F}\cdot\vec{dS} = \iint_D \left[-\frac{\partial z}{\partial x}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)-\frac{\partial z}{\partial y}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy$$

## 39 EDOPO a variables separables: (EDVS):

Formato completo: $\boxed{F(x;y;y')=0}$

$y' = \frac{dy}{dx} = F(x;y) = g(x).h(y)$ explícita

$dy = g(x)h(y)\,dx$ operando alg.

$\frac{dy}{h(y)} = g(x)dx$ separamos variables

$g(x)dx - \frac{dy}{h(y)} = 0$

$g(x) = P(x) \wedge -\frac{1}{h(y)} = Q(y)$ renombro

$\boxed{P(x)dx + Q(y)dy = 0}$ EDVS

$\int Q(y)dy = \int -P(x)dx$ integrando

$q(y) + C_1 = p(x) + C_2$

$q(y) = p(x) + C$

**Formatos de las soluciones generales**

1) $y = f(x) + C$ ; $y = F(x;C)$

2) $C = f(x;y)$

3) $x = f(y) + C$ ; $x = f(y;C)$

4) $f(x;y) + C = 0$ ; $f(x;y;C) = 0$

**EJEMPLO:** $\frac{2}{3}y\cdot y' = 2x$ CI $(y_{(1)}=3)$

$\frac{2}{3}y\frac{dy}{dx} = 2x \therefore \frac{dy}{dx} = \frac{3x}{y} \therefore y\,dy = 3x\,dx \therefore \int y\,dy = \int 3x\,dx \therefore \frac{y^2}{2} + C_1 = \frac{3}{2}x^2 + C_2$

$\therefore \frac{y^2}{2} = \frac{3}{2}x^2 + C_0 \therefore y^2 = 3x^2 + 2C_0 \therefore \boxed{y = \sqrt{3x^2+C}}$ SG

$3 = \sqrt{3.1^2 + C} \therefore 9 = 3 + C \therefore \boxed{C=6} \longrightarrow \boxed{y = \sqrt{3x^2+6}}$ SP

## 40 EDOPO Homogéneas de grado cero: (EDH°):

Formato completo: $\boxed{F(x;y;y')=0}$

$y' = \frac{dy}{dx} = F(x;y) = g(x;y).h(x;y)$

$dy = g(x;y)h(x;y)\,dx$ separo diferenciales

$\frac{dy}{h(x;y)} = g(x;y)dx$

$g(x;y)dx - \frac{dy}{h(x;y)} = 0$

$g(x;y) = P(x;y) \wedge -\frac{1}{h(x;y)} = Q(x;y)$

$\boxed{P(x;y)dx + Q(x;y)dy = 0}$

H: si $x = xt \wedge y = yt$

$P(x;y) = P(xt;yt) = t^n P(x;y)$

$Q(x;y) = Q(xt;yt) = t^n Q(x;y)$

$\frac{dy}{dx} = -\frac{P(x;y)}{Q(x;y)} = -\frac{t^n P(x;y)}{t^n Q(x;y)} = -t^0\frac{P(x;y)}{Q(x;y)} = -\frac{P(x;y)}{Q(x;y)}$

$\longrightarrow y = U(x) \cdot V(x) = U(x) \cdot x$

$U(x) = \frac{y}{x}$ ; $\frac{dy}{dx} = y' = U'(x) \cdot x + U(x)$

$$\frac{dy}{dx} = \frac{-P(x;y)}{Q(x;y)} = \frac{-P(x;U \cdot x)}{Q(x;U \cdot x)} = \frac{-x^n P(1;U)}{x^n Q(1;U)} = \frac{-P(1;U)}{Q(1;U)} = G(1;U) = F(U) = U'x + U$$

$U'x = F(U) - U$

$$\frac{dU}{dx} x = F(U) - U = L(U) \longrightarrow \frac{dU}{L(U)} = \frac{dx}{x} \longrightarrow M(U)\,dU = \frac{1}{x} dx$$

$\int M(U)\,dU = \int \frac{1}{x} dx \quad \therefore$

$T(U) + C_1 = Ln(x) + C_2$

$T(U) = Ln(x) + C_0 \longrightarrow \boxed{T(\frac{y}{x}) = Ln(x) + C_0}$

1. $\boxed{y = f(x) + C} \rightarrow \boxed{y = R(x)(Ln(x) + C_0)}$
2. $\boxed{x = e^{T(\frac{y}{x})} C}$
3. $\boxed{C = T(\frac{y}{x}) - Ln(x)}$

$Ln(x) = T(\frac{y}{x}) - C_0$

$e^{Ln(x)} = e^{T(\frac{y}{x}) - C_0}$

$x = e^{T(\frac{y}{x})} \cdot e^{-C_0} \longrightarrow \boxed{x = e^{T(\frac{y}{x})} \cdot C}$

---

**EJEMPLO:** $x \cdot y' + y - 2x = 0 \qquad CI\ (y(-1) = -4)$

$$y' = \frac{2x - y}{x} = \frac{2xt - yt}{xt} = \frac{t'(2x - y)}{t' x} = \frac{t^0 (2x - y)}{x} = \frac{2x - y}{x} \longrightarrow H^0$$

$\longrightarrow y = Ux$ ; $\frac{dy}{dx} = U'x + U$

$$y' = \frac{dy}{dx} = \frac{2x - y}{x} = U'x + U = \frac{2x - Ux}{x} = \frac{x(2 - U)}{x} = 2 - U$$

$U'x = 2 - U - U = 2(1 - U)$

$$\frac{dU}{dx} x = 2(1 - U) \quad \therefore \quad \frac{1}{1 - U} dU = \frac{2}{x} dx \quad \therefore \int \frac{1}{1 - U} dU = \int \frac{2}{x} dx \quad \therefore \quad -Ln(1 - U) + C_1 = 2\,Ln(x) + C_2$$

$Ln(1 - U) = -2\,Ln(x) - C_0$

$Ln(1 - U) = Ln\left(\frac{1}{x^2}\right) + Ln(C_3)$

$Ln(1 - U) = Ln\left(\frac{C_3}{x^2}\right)$

$e^{Ln(1 - U)} = e^{Ln(\frac{C_3}{x^2})}$

$$1 - U = \frac{C_3}{x^2} \quad \therefore \quad U = \frac{y}{x} = 1 - \frac{C_3}{x^2} \quad \therefore \quad y = x - \frac{C_3}{x} \quad \therefore \quad \boxed{y = \frac{x^2 + C}{x}} \text{ SG}$$

$$-4 = \frac{(-1)^2 + C}{-1} \quad \therefore \quad \boxed{C = 3} \longrightarrow \boxed{y = \frac{x^2 + 3}{x}} \text{ SP}$$

## 41 EDOPO Lineales: (EDL):

$$\boxed{y' + P(x).y = Q(x)}$$

- Si $Q(x)=0 \longrightarrow y' + P(x).y = 0$ (EDVS)

$\frac{dy}{dx} + P(x)y = 0 \therefore \frac{1}{y}dy = -P(x)\,dx \therefore \int \frac{1}{y}dy = \int -P(x)\,dx \therefore Ln(y) = \int -P(x)\,dx + C_0$

$e^{Ln(y)} = e^{\int -P(x)dx + C_0} = e^{\int -P(x)dx}\, e^{C_0}$ elevando a la e

$$\boxed{y = c.e^{-\int P(x)dx}}$$

- Si $Q(x) \neq 0 \longrightarrow y = U(x)V(x)$ ; $y' = \frac{dy}{dx} = \frac{dU}{dx}.V + U.\frac{dV}{dx}$

$y' + P(x).y = \frac{dU}{dx}V + U\frac{dV}{dx} + P(x)U.V = Q(x)$

búsqueda de una EDVS

$V\left[\frac{dU}{dx} + P(x).U\right] + U\frac{dV}{dx} = Q(x)$

(EDVS) $\left[\frac{dU}{dx} + P(x)U\right] = 0 \therefore \frac{1}{U}dU = -P(x)\,dx \therefore \int \frac{1}{U}dU = \int -P(x)\,dx \therefore Ln(U) = \int -P(x)\,dx + C_0$

$e^{Ln(U)} = e^{\int -P(x)dx + C_0} = e^{\int -P(x)dx}\, e^{C_0}$

$U = c.e^{-\int P(x)dx} \longrightarrow c=1 \longrightarrow \boxed{U = e^{-\int P(x)dx}}$

$V.0 + U\frac{dV}{dx} = Q(x) \therefore dV = \frac{Q(x)}{U}dx \therefore \int dV = \int \frac{Q(x)}{U}dx$

$$\boxed{V = \int Q(x).e^{\int P(x)dx}\,dx + C}$$

$$\boxed{y = U.V = e^{-\int P(x)dx}\left[\int Q(x)\,e^{\int P(x)dx}\,dx + C\right]}$$

**EJEMPLO:** $xy' + y - 2x = 0$

$y' + \frac{y}{x} - 2 = 0 \longrightarrow P(x) = 1/x \wedge Q(x) = 2$

$\longrightarrow y = U(x)V(x)$ ; $y' = U'V + UV'$

$y' + \frac{y}{x} - 2 = U'V + UV' + U.V.1/x - 2$

$V\left[U' + U 1/x\right] + UV' = 2 \longrightarrow \left[U' + U.\frac{1}{x}\right] = 0$ (EDVS)

$\frac{dU}{dx} + \frac{U}{x} = 0 \therefore \frac{1}{U}dU = -\frac{1}{x}dx \therefore \int \frac{1}{U}dU = -\int \frac{1}{x}dx \therefore Ln(U) = -Ln(x) = Ln\left(\frac{1}{x}\right) \rightarrow \boxed{U = \frac{1}{x}}$

$V.0 + UV' = 2 \therefore \frac{dV}{dx} = \frac{2}{U} \therefore \int dV = \int \frac{2}{U}dx \therefore V = \int 2.x\,dx = x^2 + C_0 \rightarrow \boxed{V = x^2 + C_0}$

$y = U.V = \frac{1}{x}(x^2 + C_0) \longrightarrow \boxed{y = \frac{x^2 + C}{x}}$ SG

## 42 EDOPO Exactas o Totales: (EDE):

Formato completo: $\boxed{F(x;y;y')=0}$

$$\boxed{P_{(x;y)}\,dx + Q_{(x;y)}\,dy = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = dU_{(x;y)} = 0} \longrightarrow \boxed{U_{(x;y)}=K}$$

$\dfrac{\partial^2 U}{\partial x\partial y} = \dfrac{\partial P(x;y)}{\partial y}$ ; $\dfrac{\partial^2 U}{\partial y\partial x} = \dfrac{\partial Q(x;y)}{\partial x}$ ; Por Schwarz: $\dfrac{\partial^2 U}{\partial x\partial y} = \dfrac{\partial^2 U}{\partial y\partial x} \longrightarrow \dfrac{\partial P(x;y)}{\partial y} = \dfrac{\partial Q(x;y)}{\partial x}$

$$\frac{\partial U_{(x;y)}}{\partial x} = P(x;y) \therefore U(x;y) = \int P(x;y)\,\partial x \therefore U(x;y) = \int P(x;y)\,\partial x + C(y)$$

$$Q(x;y) = \frac{\partial U(x;y)}{\partial y} = \frac{\partial}{\partial y}\left[\int P(x;y)\,\partial x + C(y)\right] = \int \frac{\partial P(x;y)}{\partial y}\partial x + C'(y)$$

$$C'(y) = Q(x;y) - \int \frac{\partial P(x;y)}{\partial y}\partial x$$

$$C(y) = \int C'(y)\,dy = \int\left[Q(x;y) - \int\frac{\partial P(x;y)}{\partial y}\partial x\right]dx + C_0$$

$$U(x;y) = \int P(x;y)\,\partial y + C(y) = \int P(x;y)\,\partial y + \int\left[Q(x;y) - \int\frac{\partial P(x;y)}{\partial y}\partial x\right]dy + C_0 = K$$

$$\boxed{C = \int P(x;y)\,\partial x + \int\left[Q(x;y) - \int\frac{\partial P(x;y)}{\partial y}\partial x\right]dy}$$

---

**EJEMPLO:** $xy' + y - 2x = 0$

$$x\frac{dy}{dx} + y - 2x = 0 \therefore x\,dy + (y-2x)\,dx = 0 \therefore (y-2x)\,dx + x\,dy = 0$$

$$\frac{\partial P(x;y)}{\partial y} = 1 = \frac{\partial Q(x;y)}{\partial x} \longrightarrow \text{EDE}$$

$$(y-2x)\,dx + x\,dy = dU(x;y) = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = 0 \longrightarrow U(x;y) = K$$

$$\frac{\partial U}{\partial x} = P(x;y) \therefore U(x;y) = \int P(x;y)\,\partial x + C(y) = \int (y-2x)\,\partial x + C(y) = yx - x^2 + C(y)$$

$$Q(x;y) = \frac{\partial U}{\partial y} = \frac{\partial(yx - x^2 + C(y))}{\partial y} = x + C'(y) \longrightarrow C'(y) = Q(x;y) - x$$

$$C(y) = \int C'(y)\,dy = \int[Q(x;y) - x]\,dy = \int (x-x)\,dy = \int 0\,dy = C_1$$

$$U(x;y) = yx - x^2 + C_1 = K \therefore y = \frac{x^2 + K - C_1}{x} \therefore \boxed{y = \frac{x^2 + C}{x}} \text{ SG}$$

## 43) EDOPO de Bernoulli:

$$\boxed{y' + P(x)y = Q(x)y^n}$$

$$\frac{y'}{y^n} + \frac{P(x)y}{y^n} = \frac{Q(x)y^n}{y^n} \therefore \frac{1}{y^n}y' + P(x)y^{1-n} = Q(x) \qquad \text{Si } z = y^{1-n};$$

$$\frac{y'}{y^n} + P(x)z = Q(x); \quad \frac{dz}{dx} = 1-n \cdot y^{-n}\frac{dy}{dx} \therefore \frac{dy}{dx} = \frac{y^n}{1-n}\frac{dz}{dx}$$

$$\frac{dy}{dx}\frac{1}{y^n} + P(x)z = Q(x) \therefore \frac{y^n}{1-n}\frac{dz}{dx}\cdot\frac{1}{y^n} + P(x)z = Q(x) \therefore \frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x)$$

$$\frac{(1-n)}{(1-n)}\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x) \therefore z' + (1-n)P(x)z = (1-n)Q(x)$$

(EDL) $\rightarrow z = u.v \therefore z' = u'v + uv'$

$$u'.v + u.v' + (1-n)P(x).u.v = (1-n)Q(x)$$

$$u[v' + (1-n)P(x)v] + u'v = (1-n)Q(x)$$

(EDVS) $[v' + (1-n)P(x)v] = 0 \therefore \frac{dv}{dx} = -(1-n)P(x)v \therefore \frac{1}{v}dv = -(1-n)P(x)dx$

$$\int\frac{1}{v}dv = \int -(1-n)P(x)dx \therefore \ln(v) = \int -(1-n)P(x)dx + C_0$$

$$e^{\ln(v)} = e^{\int -(1-n)P(x) + C_0} = e^{\int -(1-n)P(x)dx}e^{C_0}$$

$$v = C.e^{\int -(1-n)P(x)dx} \rightarrow C=1 \rightarrow \boxed{v = e^{\int -(1-n)P(x)dx}}$$

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$$u.0 + u'v = (1-n)Q(x) \therefore \frac{du}{dx}v = (1-n)Q(x) \therefore du = \frac{(1-n)Q(x)}{v}dx$$

$$\int du = \int\frac{(1-n)Q(x)}{v}dx = \boxed{u = \int(1-n)e^{\int(1-n)P(x)dx}Q(x) + C}$$

$$z = u.v = \left[(1-n)\int e^{\int(1-n)P(x)dx}.Q(x)dx + C\right]e^{\int -(1-n)P(x)dx} = y^{1-n}$$

$$\boxed{y = \left\{\left[(1-n)\int e^{\int(1-n)P(x)dx}.Q(x)dx + C\right]e^{\int -(1-n)P(x)dx}\right\}^{\frac{1}{1-n}}}$$

---

**EJEMPLO:** $y' + \frac{1}{x}y = 4\ln(x)y^2$

$$\frac{y'}{y^2} + \frac{1}{xy} = 4\ln(x) \qquad \text{Si } z = y^{-1}: \rightarrow \frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \rightarrow \frac{dy}{dx} = -y^2\frac{dz}{dx}$$

$$-y^2\frac{dz}{dx}\cdot\frac{1}{y^2} + \frac{1}{x}.z = 4\ln(x) \therefore -\frac{dz}{dx} + \frac{1}{x}z = 4\ln(x)$$

$\rightarrow z = U.V \quad ; \quad z' = U'V + UV'$

$-(U'.V + U.V') + \frac{1}{x}.U.V = 4 Ln(x) \quad \therefore \quad U'V + UV' - \frac{1}{x} U.V = -4Ln(x)$

$U\left[V' - \frac{1}{x}.V\right] + U'.V = 4 Ln(x)$

$\left[V' - \frac{1}{x}V\right] = 0 \quad \therefore \quad \frac{dv}{dx} = \frac{1}{x}V \quad \therefore \quad \frac{dv}{V} = \frac{1}{x}.dx \quad \therefore \int \frac{dv}{V} = \int \frac{1}{x}.dx$

$Ln(v) = Ln(x) \quad \therefore \quad \boxed{V = x}$

$U.0 + \frac{du}{dx}.V = -4 Ln(x) \quad \therefore \quad du = -\frac{4Ln(x)}{V} dx \quad \therefore \int du = \int -\frac{4Ln(x)}{x}.dx$

$U = -4\frac{Ln^2(x)}{2} + C = -2Ln^2(x) + C \rightarrow \boxed{U = C - Ln^2(x^2)}$

$z = U.V = (C - Ln^2(x^2))\,x = \frac{1}{y} \longrightarrow \boxed{y = \frac{1}{x(C - Ln^2(x^2))}}$

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## 44 EDOSO: Homogeneas a coeficientes constante (HCC):

$ay'' + by' + cy = 0$

$\rightarrow y = e^{rx} \quad ; \quad y' = r e^{rx} \quad ; \quad y'' = r^2 e^{rx}$

$a.r^2e^{rx} + bre^{rx} + ce^{rx} = 0 \quad \therefore \quad e^{rx}(ar^2 + br + c) = 0 \quad \therefore \quad ar^2 + br + C = 0 \rightarrow r_1 \wedge r_2$

$y_1 = e^{r_1 x} \wedge y_2 = e^{r_2 x}$

**Teorema 1:**

$y_1 = e^{r_1 x} \longrightarrow y_a = C_1 y_1$

$ay_1'' + by_1' + cy_1 = 0 \longrightarrow aC_1y_1'' + bC_1y_1' + CC_1y_1 = 0$

$C_1(ay_1'' + by_1' + cy_1) = 0$

$y_a = C_1.y_1 = C_1.e^{r_1 x} \quad ; \quad y_b = C_2.y_2 = C_2.e^{r_2 x}$

**Teorema 2:**

$y = y_a + y_b = C_1y_1 + C_2y_2 \quad ; \quad y' = C_1y_1' + C_2y_2' \quad ; \quad y'' = C_1y_1'' + C_2.y_2''$

$a(C_1y_1'' + C_2y_2'') + b(C_1y_1' + C_2y_2') + c(C_1y_1 + C_2y_2) = 0$

$aC_1y_1'' + aC_2y_2'' + bC_1y_1' + bC_2y_2' + CC_1y_1 + cC_2y_2 = 0$

$C_2(ay_2'' + by_2' + cy_2) + C_1(ay_1'' + by_1' + cy_1) = 0 \longrightarrow C_1(0) + C_2(0) = 0$

---

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & Ky_1 \\ y_1' & Ky_1' \end{vmatrix} = y_1.Ky_1' - y_1'.Ky_1 = 0 \longrightarrow \frac{y_1}{y_2} = K$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = C_1.e^{r_1 x}.C_2.r_2e^{r_2 x} - C_1.r_1.e^{r_1 x}.C_2e^{r_2 x} = C_1.C_2.e^{r_1 x}e^{r_2 x}(r_2 - r_1) \neq 0$

**Primer caso: 2RRD:** $\boxed{SG: y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$

- **Teorema de Cauchy:** $x = x_0$

$y_0 = C_1 e^{r_1 x_0} + C_2 e^{r_2 x_0}$ ; $y_0' = r_1 C_1 e^{r_1 x_0} + r_2 C_2 e^{r_2 x_0}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = C_1 e^{r_1 x_0} . C_2 r_2 e^{r_2 x_0} - C_1 r_1 e^{r_1 x_0} . C_2 e^{r_2 x_0} = C_1 C_2 e^{r_1 x_0} e^{r_2 x_0} (r_2 - r_1) \neq 0$$

$$\begin{cases} y_0 = C_1 e^{r_1 x_0} + C_2 e^{r_2 x_0} \\ y_0' = C_1 r_1 e^{r_1 x_0} + C_2 r_2 e^{r_2 x_0} \end{cases}$$

$$C_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} y_0 & e^{r_2 x_0} \\ y_0' & r_2 e^{r_2 x_0} \end{vmatrix}}{\begin{vmatrix} e^{r_1 x_0} & e^{r_2 x_0} \\ r_1 e^{r_1 x_0} & r_2 e^{r_2 x_0} \end{vmatrix}} \quad ; \quad C_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} e^{r_1 x_0} & y_0 \\ r_2 e^{r_2 x_0} & y_0' \end{vmatrix}}{\begin{vmatrix} e^{r_1 x_0} & e^{r_2 x_0} \\ r_1 e^{r_1 x_0} & r_2 e^{r_2 x_0} \end{vmatrix}}$$

$$\boxed{SP: \quad y = C_1(x_0)\, e^{r_1 x} + C_2(x_0)\, e^{r_2 x}}$$

**Segundo caso: 2RRI:** $\boxed{SG: y = C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{r_1 x} + C_2 e^{r_1 x} = e^{r_1 x}(C_1 + C_2) = C . e^{r_1 x}}$

$y_a = C_1 e^{r_1 x}$ ; $\rightarrow y_b = C_2 x e^{r_1 x}$

$$W = \begin{vmatrix} y_1 & x . y_2 \\ y_1' & y_2 + x y_2' \end{vmatrix} = y_1 y_2 + y_1 . x y_2' - y_1' x y_2 = y_1 y_2 + x(y_1 y_2' - y_1' y_2) =$$

$$= y_1 y_1 + x(y_1 y_1' - y_1' y_1) = y_1^2 + x(0) = y_1^2 = e^{r_1 x^2} \neq 0$$

$y_b' = C_2 e^{r_1 x} + C_2 x r_1 e^{r_1 x} = C_2 e^{r_1 x}(1 + x r_1)$

$y_b'' = C_2 r_1 e^{r_1 x}(1 + x r_1) + C_2 e^{r_1 x} r_1 = C_2 e^{r_1 x}(r_1 + x r_1^2 + r_1) = C_2 e^{r_1 x}(2r_1 + x r_1^2)$

$a\, C_2 e^{r_1 x}(2r_1 + x r_1^2) + b\, C_2 e^{r_1 x}(1 + x r_1) + c\, C_2 x e^{r_1 x} =$

$= C_2 e^{r_1 x}(a(2r_1 + x r_1^2) + b(1 + x r_1) + c(x)) =$

$= C_2 e^{r_1 x}(a 2 r_1 + a x r_1^2 + b + b x r_1 + c x) =$

$= C_2 e^{r_1 x}(x(a r_1^2 + b r_1 + c) + a 2 r_1 + b) = C_2 e^{r_1 x}(x.(0) + 0) = 0$

$\boxed{y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 x e^{r_1 x} = e^{r_1 x}(C_1 + C_2 x)}$ SG

$\boxed{y = e^{r_1 x}(C_1(x_0) + C_2(x_0) . x)}$ SP

## Tercer caso: 2RCC:

$$(r_1;r_2)= \frac{-b \pm i\sqrt{b^2-4ac}}{2a} = \alpha \pm \beta i \longrightarrow r_1=\alpha-\beta i \quad y \quad r_2=\alpha+\beta i$$

$$y= C_1y_1+C_2y_2 = C_1.e^{r_1x} + C_2\,e^{r_2x} = C_1\,e^{(\alpha-\beta i)x} + C_2 e^{(\alpha+\beta i)x} = C_1\,e^{\alpha x}e^{-\beta i x} + C_2\,e^{\alpha x}e^{\beta i x} =$$

$$\boxed{y= e^{\alpha x}(C_1.e^{-\beta i x} + C_2\,e^{\beta i x})} \text{ SG}$$

$$e^{-\beta i x} = \cos(\beta x) - \operatorname{sen}(\beta x)i$$
$$e^{\beta i x} = \cos(\beta x) + \operatorname{sen}(\beta x)i$$

$$y= e^{\alpha x}(C_1(\cos(\beta x)-\operatorname{sen}(\beta x)i) + C_2(\cos(\beta x)+\operatorname{sen}(\beta x)i))$$

$$y= e^{\alpha x}(C_1.\cos(\beta x) - C_1.\operatorname{sen}(\beta x)i + C_2.\cos(\beta x) + C_2.\operatorname{sen}(\beta x)i)$$

$$y= e^{\alpha x}(\cos(\beta x)(C_1+C_2) + \operatorname{sen}(\beta x)(-C_1 i + C_2 i)) = e^{\alpha x}(K_1\cos(\beta x) + K_2.\operatorname{sen}(\beta x))$$

$$\boxed{y= e^{\alpha x}(C_1.\cos(\beta x) + C_2\operatorname{sen}(\beta x))} \text{ SG}$$

$$\boxed{y= e^{\alpha x}(C_{1(x_0)}\cos(\beta x) + C_{2(x_0)}\operatorname{sen}(\beta x))} \text{ SP}$$

## 45 EDOSO: No Homogéneas a coeficientes constantes (NHCC)

### Método de variación de las constantes (Euler * Lagrange).

$ay''+by'+cy=f(x)$ SG: $y = y_h + y_p$

$y_h = C_1y_1 + C_2y_2$
$y_p = C_1(x)\,y_1 + C_2(x)\,y_2$

$y_p' = C_1'(x)\,y_1 + C_2'(x)y_2 + C_1(x)y_1' + C_2(x)y_2'$ si $C_1'(x)\,y_1 + C_2'(x)\,y_2 = 0$

$y_p' = C_1(x)\,y_1' + C_2(x)\,y_2'$

$y_p'' = C_1'(x)y_1' + C_2'(x)\,y_2' + C_1(x)\,y_1'' + C_2(x)\,y_2''$

$a[C_1'(x)y_1' + C_2(x)'y_2' + C_1(x)y_1'' + C_2(x)y_2''] + b[C_1(x)\,y_1' + C_2(x)\,y_2'] + c[C_1(x)y_1 + C_2(x)y_2] = f(x)$

$aC_1'y_1' + aC_2'y_2' + aC_1y_1'' + aC_2y_2'' + bC_1y_1' + bC_2y_2' + cC_1y_1 + cC_2y_2 = f(x)$

$C_1[ay_1'' + by_1' + cy_1] + C_2[ay_2'' + by_2' + cy_2] + aC_1'y_1' + aC_2'y_2' = f(x)$

$$\begin{cases} aC_1'y_1' + aC_2'y_2' = f(x) \\ C_1'y_1 + C_2'y_2 = 0 \end{cases}$$

$$C_1' = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & ay_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ ay_1' & ay_2' \end{vmatrix}} = \frac{-y_2f(x)}{ay_1y_2' - ay_2y_1'} = \frac{h(x)}{a\,\omega(y_1;y_2)} = \frac{h(x)}{m(x)} = F_1(x)$$

$$C_2' = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} y_1 & 0 \\ ay_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ ay_1' & ay_2' \end{vmatrix}} = \frac{y_1f(x)}{ay_1y_2' - ay_2y_1'} = \frac{n(x)}{a\,\omega(y_1;y_2)} = \frac{n(x)}{t(x)} = F_2(x)$$

$C_1(x) = \int C_1'(x)\,dx = \int F_1(x)\,dx$ ; $C_2(x) = \int C_2'(x)\,dx = \int F_2(x)\,dx$

$y_p = C_1(x)y_1 + C_2(x)y_2 = y_1\int F_1(x)\,dx + y_2\int F_2(x)\,dx$

$\boxed{y = y_h + y_p = C_1 \cdot y_1 + C_2 y_2 + y_1\int F_1(x)\,dx + y_2\int F_2(x)\,dx}$ SG

## 46 EDOSO No Homogéneas a coeficientes constantes (NHCC):

**Método de los coeficientes indeterminados (selección y comparación):**

**a) f(x) es polinómica:** $f(x) = P_n(x) \longrightarrow y_p = x^m \cdot R_n(x)$

$m=0$ si $r_1 \neq 0 \wedge r_2 \neq 0$
$m=1$ si $r_1 = 0 \vee r_2 = 0$ $\quad ay'' + by' = 0$
$m=2$ si $r_1 = 0 \wedge r_2 = 0$ $\quad ay'' = 0$

**b) f(x) es exponencial:** $f(x) = d \cdot e^{h \cdot x} \longrightarrow y_p = x^m A e^{hx}$

$m=0$ si $r_1 \neq h \wedge r_2 \neq h$
$m=1$ si $r_1 = h \vee r_2 = h$
$m=2$ si $r_1 = h \wedge r_2 = h$

**c) f(x) es trigonométrica:**

1) Si $f(x) = C \cdot \text{sen}(hx) \vee f(x) = D \cdot \cos(hx) \vee f(x) = C \cdot \text{sen}(hx) + D\cos(hx)$

$\longrightarrow y_p = x^m [A\,\text{sen}(hx) + B\cos(hx)]$

$m=0$ si $r_1 \neq -\beta i \wedge r_2 \neq \beta i \quad (h \neq \beta)$
$m=1$ si $r_1 = -\beta i \wedge r_2 = \beta i \quad (h = \beta)$

2) Si $f(x) = C\,\text{sen}(hx) + D\cos(kx)$

$\longrightarrow y_p = A\,\text{sen}(hx) + B\cos(hx) + G\,\text{sen}(kx) + H\cos(kx)$

- $y_p ; y_p' ; y_p'' \curvearrowright$ ED $\longrightarrow$ No Absurdo $\longrightarrow$ Determinar A, B, G, H
  $\longrightarrow$ Absurdo:
  $y_p = x[A\,\text{sen}(hx) + B\cos(hx) + G\,\text{sen}(kx) + H \cdot \cos(kx)]$

- - $y_p ; y_p' ; y_p'' \curvearrowright$ ED $\longrightarrow$ No Absurdo $\longrightarrow$ Determinar A, B, G, H
  $\longrightarrow$ Absurdo:
  $y_p = x^2[A\,\text{sen}(hx) + B\cos(hx) + G\,\text{sen}(kx) + H\cos(kx)]$
  Determinar: A; B; G; H

## d) f(x) es combinación de polinomio y exponencial:

1) Si $f(x) = P_n(x) + d.e^{hx}$

$\longrightarrow f(x) = g(x) + K(x) \Longrightarrow y_p = y_g + y_K = x^m R_n(x) + x^m A e^{hx}$

2) Si $f(x) = P_n(x).d.e^{hx}$

$\longrightarrow y_p = x^m R_n(x) A e^{hx}$

$m=0$ si $r_1 \neq h \wedge r_2 \neq h$
$m=1$ si $r_1 = h \vee r_2 = h$
$m=2$ si $r_1 = h \wedge r_2 = h$

## e) Caso general de combinación como producto:

$f(x) = e^{hx}[P_p(x)\cos(Kx) + Q_q(x)\,\text{sen}(Kx)]$

$\longrightarrow y_p = x^m \{e^{hx}[R_t(x)\cos(Kx) + S_t(x)\,\text{sen}(Kx)]\}$

$m=0$ si $h \neq \alpha \wedge K \neq \beta$
$m=1$ si $h = \alpha \wedge K = \beta$