SHO Sum of Sinusoids Solution Derivation

Agustin Valdes Martinez

July 2023

1 Introduction

A differential equation (DE) is an equation that contains both a variable and its derivative. In class, we came across the following DE when analyzing the spring-mass system:

$$-\frac{k}{m}x(t) = \frac{d^2}{dt^2}x(t) \tag{1}$$

The variable here of interest, the one to solve, is the displacement x(t). I claimed a solution to this DE is:

$$A\cos(\omega t) + B\cos(\omega t) \tag{2}$$

where A and B are constants. We also found that $\omega = \sqrt{\frac{k}{m}}$ in class by verifying that this solution does indeed work. But where does equation 2 come from?

2 Properties of Linear DE's

We know equation 1 to be a DE. Specifically, it is an ordinary, linear, second-order DE. It is ordinary because are taking full derivatives, not partial derivatives, of x(t). It is linear because the derivatives are all raised to the first power. and it is second order because the highest derivative in this equation is the second derivative of x(t). We can arrive at equation 2 by leveraging the following properties of DE's:

- Superposition: if $x_1(t)$ and $x_2(t)$ are separately solutions to a DE, then $x_1(t) + x_2(t)$ is also a solution to the DE.
- Scaling Invariance: if $x_1(t)$ is a solution to a DE, then $\alpha x_1(t)$ is also a solution, where α is a scalar.

These will be important later.

3 Ansatz Substitution

Ansatz is the German word for approach. And generally a good approach for solving these types of DE's is to set $x(t) = e^{i\omega t}$. Complex exponentials, and exponentials in general, are great for solving linear DE's because recall from calculus that $\frac{d}{dx}e^x = e^x$. Let's make this substitution:

$$-\frac{k}{m}e^{i\omega t} = \frac{d^2}{dt^2}e^{i\omega t} \tag{3}$$

$$-\frac{k}{m}e^{i\omega t} = \frac{d}{dt}i\omega e^{i\omega t} \tag{4}$$

$$-\frac{k}{m}e^{i\omega t} = -\omega^2 e^{i\omega t} \tag{5}$$

$$\frac{k}{m} = \omega^2 \tag{6}$$

So not only was the derivative taking process pretty straightforward, we now see that the $e^{i\omega t}$ terms on the left and right-hand sides cancel. Great so our Ansatz is a good place to start and indeed satisfies the DE. But we can't just leave it as it is to describe our physical system since it's complex, meaning that it has both real and imaginary components. By Euler's equation:

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \tag{7}$$

Euler's equation is itself a whole can of worms, but if you're interested in where it comes from, there's a great video by 3Brown1Blue on YouTube about it. We can extract two solutions from this equation, one by taking the imaginary part, and one by taking the real part of equation 7:

$$\mathrm{Im}[\cos(\omega t) + i\sin(\omega t)] = \sin(\omega t) \tag{8}$$

$$\mathrm{Re}[\cos(\omega t) + i\sin(\omega t)] = \cos(\omega t) \tag{9}$$

And leveraging the properties of superposition and scalar invariance from section 2, we get:

$$A\cos(\omega t) + B\sin(\omega t) \tag{10}$$

Where A and B are scalars that come from the two initial conditions of our system.