RC Circuit Solution Derivation

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1 Introduction

In class, we introduced a circuit with a resistor (R) and a capacitor (C) in series with each other: the RC circuit. We derived that the differential equation (DE) governing the voltage in the capacitor is:

$$V_{in} = V_c + RC\frac{dV_c}{dt} \tag{1}$$

where V_{in} is the voltage provided by the ideal voltage source and V_c is the voltage in the capacitor at time t.

We've solved DEs before, but they've all been *homogenous*, in that all terms in the equation relate to the variable we want to solve for. As such, when we move all terms relating to said variable to one side, the other side should be 0. For example, for the spring-mass system, our DE

$$0 = \omega^2 x + \frac{d^2x}{dt^2} \tag{2}$$

only has terms containing x and its derivatives. Equation 1 is an *inhomogenous* DE. Unfortunately, with the RC circuit, we have to contend with V_{in}, which is a constant unrelated to V_c directly.

2 Solving Inhomogenous DEs

To solve inhomogenous DEs, we employ a property of linear DEs: superposition. If you're wondering about this, feel free to check out my last write-up on deriving the solution to the spring-mass system. The idea is that the solution to this system V_c is the sum of the homogenous and inhomogenous solutions to this DE.

We get the homogenous solution by 'pretending' that the V_{in} in equation 1 is 0:

$$0 = V_c + RC\frac{dV_c}{dt} \tag{3}$$

If we take our Ansatz (guess) to be $V_c = C_1 e^{\lambda t}$, where C_1 is a constant related to the initial condition of our system, and plug it into equation 3, we get:

$$\begin{aligned} 0 &= C_1 e^{\lambda t} + RC\lambda C_1 e^{\lambda t} && (4) \\ &= 1 + RC\lambda && (5) \end{aligned}$$

$$\lambda = -\frac{1}{RC} \tag{6}$$

Plugging λ back into our guess, we get our homogenous solution:

$$V_{c,h} = C_1 e^{-\frac{1}{RC}t} \tag{7}$$

To get the inhomogenous solution, a good first step is to see what happens if, in equation 1, $V_{c,i}$ is a constant. Supposing that this is true, $\frac{dV_{c,i}}{dt} = 0$ and so equation 1 reduces to:

$$V_{c,i} = V_{in} \tag{8}$$

Employing superposition, the total solution to the system is the sum of the homogenous and inhomogenous solutions:

$$\begin{aligned} V_c &= V_{c,h} + V_{c,i} && (9) \\ &= C_1 e^{-\frac{1}{RC}t} + V_{in} && (10) \end{aligned}$$

Lastly, we can solve for C_1 using the initial condition; let V_0 be the initial voltage in the capacitor.

$$V_c(t=0) = V_0 = C_1 + V_{in} \tag{11}$$

$$C_1 = V_0 - V_{in} \tag{12}$$

Plugging in for C_1, we get the final solution I presented at the end of class:

$$V_c(t) = (V_0 - V_{in}) e^{-\frac{1}{RC}t} + V_{in} \tag{13}$$