LC Device Law and Energy Derivations

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1 Introduction

In class, I quoted the following equations without deriving them from first principles

$$E_L = \frac{1}{2} L I_L^2 \tag{1}$$

$$E_C = \frac{1}{2} C V_C^2 \tag{2}$$

$$I_C = C \frac{dV_C}{dt} \tag{3}$$

$$V_L = -L \frac{dI_L}{dt} \tag{4}$$

where C is the capacitance, L is the inductance, I_L is the current through the inductor, and V_C is the voltage across the capacitor. The first two equations are how we analyzed the energy oscillations in the LC circuit, and we used the last two equations (device laws) to set up the circuit's differential equation.

2 Energy Equations

2.1 Inductor

We start with 2 definitions of power and equate the two. First, power is the product of voltage and current: $P = V_L I_L$. It's also the change in work over time: $P = \frac{dW}{dt}$. We are interested in the work because, if we can figure out how much work it takes to fill the inductor, we find how much energy we stored in it.

$$\begin{aligned} \frac{dW}{dt} &= V_L I_L && (5) \\ \int \frac{dW}{dt} dt &= \int V_L I_L dt && (6) \\ W &= \int V_L I_L dt && (7) \end{aligned}$$

Recall the magnitude of V_L is given by $L\frac{dI_L}{dt}$. Making this substitution, and taking L to be a constant in time, we get:

$$\begin{aligned} W &= L\int I_L \frac{dI_L}{dt} dt && (8) \\ &= L\int I_L dI_L && (9) \\ &= \frac{1}{2} L I_L^2 && (10) \end{aligned}$$

Again, now that we know the work it took to fill the inductor, we know the energy stored within it:

$$E_L = \frac{1}{2} L I_L^2 \tag{11}$$

2.2 Capacitor

We start by thinking about the differential (tiny) amount of work it takes to put electric charge in a capacitor. The key idea is that the amount of work it takes to fill the capacitor is equal to the amount of energy stored in it:

$$\begin{aligned} dW &= V dQ && (12) \\ \int dW &= \int V dQ && (13) \\ W &= \int V dQ && (14) \end{aligned}$$

Next, we use the ratio definition of capacitance, $C = \frac{Q}{V}$, rearranging for V, substituting into the integral:

$$\begin{aligned} W &= \int \frac{Q}{C} dQ && (15) \\ &= \frac{Q^2}{2C} && (16) \end{aligned}$$

Now that we know the amount of work it takes to fill the capacitor, we know the energy stored within it. As a final step let's substitute $Q = CV$ to get our energy equation in terms of voltages:

$$E_C = \frac{1}{2} C V_C^2 \tag{17}$$

3 Device Laws

3.1 Inductor

We start with Faraday's Law, one of Maxwell's equations:

$$\epsilon = -\frac{d\Phi_B}{dt} \tag{18}$$

On the left-hand-side, we have the electromotive force, which we can simply think of as the voltage across the inductor V_L. On the right-hand-side, we have the change in time of the magnetic flux. The magnetic flux approximately tells us how many magnetic field-lines go through a cross-section of our inductor/coil.

The magnetic flux is important because we use it to define the inductance:

$$L = \frac{\Phi_B}{I_L} \tag{19}$$

where I_L is the current through the inductor. Rearranging for Φ_B,

$$\Phi_B = LI_L \tag{20}$$

we can make the substitution into Faraday's Law. Note that L is constant in time:

$$V_L = -L\frac{dI_L}{dt} \tag{21}$$

And in the equation above we arrive at the device law for an inductor.

3.2 Capacitor

We start with the definition of capacitance as the ratio between electric charge and voltage:

$$C = \frac{Q}{V} \tag{22}$$

Next, we can rearrange the above equation and take the derivative in time of both sides. Note that we take the capacitance C to be a constant in time:

$$\begin{aligned} C &= QV & (23) \\ \frac{d}{dt}[Q] &= \frac{d}{dt}[CV] & (24) \\ \frac{dQ}{dt} &= C\frac{dV}{dt} & (25) \end{aligned}$$

Recall that $\frac{dQ}{dt}$ is the current going through the capacitor, leaving us with the familiar device law for the capacitor:

$$I_C = C\frac{dV_C}{dt} \tag{26}$$