# A Primer in Quantum Computing
# Lesson 2: Linear Algebra, Dirac Notation

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## 1 Quick Review

Last time, we introduced qubits: 2-level quantum systems, which we learned could be represented mathematically like

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \tag{1}$$

Physically, we interpret this qubit as a quantum system that, after measurement, can be in two states: $|0\rangle$, $|1\rangle$. **When we measure the qubit, it "collapses" into $|0\rangle$ *or* $|1\rangle$.** Recall that this collapse is probabilistic, which means sometimes it collapses into $|0\rangle$, sometimes $|1\rangle$. The probabilities of these events happening are directly related to the probability amplitudes $\alpha_0$, $\alpha_1$:

$$P_0 = |\alpha_0|^2 \tag{2}$$

$$P_1 = |\alpha_1|^2 \tag{3}$$

Last time, I introduced Equation 1, we talked about what it means physically, but I asked that you please ignore the weird $|...\rangle$ notation. We'll dedicate today to understanding this and the underlying math.

## 2 A Primer in Linear Algebra

### 2.1 Vectors and Basis Vectors

$|\psi\rangle$, $|0\rangle$, and $|1\rangle$ in Equation 1 are all mathematical objects called *vectors*. You can think of these vectors as arrows in the 2D plane. Vectors have a length (AKA magnitude, norm) and a direction.

For example, the vector $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ looks like this:

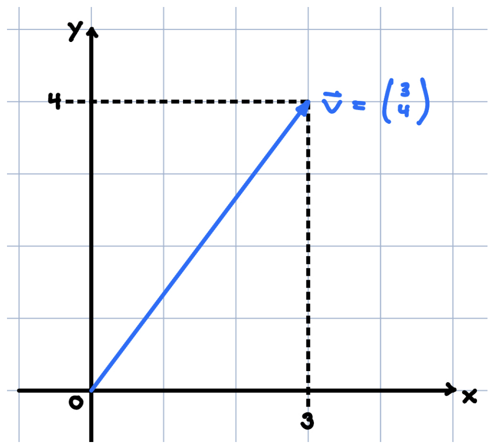


Figure 1: Vector in x-y plane.

Intuitively, we see that the 3 entry tells us how far along x the vector is, and 4 tells u how far along y. So somehow $\begin{pmatrix}1\\0\end{pmatrix}$ corresponds to the x-axis, and $\begin{pmatrix}0\\1\end{pmatrix}$ corresponds to the y-axis. Can you see why? Let's take this idea further. We can write out this vector as:

$$\vec{v} = 3\begin{pmatrix}1\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix} \tag{4}$$

We have decomposed $\vec{v}$ into a sum of *basis vectors*: $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. This weighted sum of basis vectors is called a *linear combination*.

#### 2.1.1 Quick Practice 1

1. Decompose the following vectors as a linear combination of the basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$:
   - $\begin{pmatrix}2\\3\end{pmatrix}$
   - $\begin{pmatrix}-1\\5\end{pmatrix}$
2. Do these decompositions remind you of anything we learned last time? Hint: it had a fancy name starting with an 's.'

Here's the punchline: if we have enough basis vectors, we can write any vector $\vec{v}$ as a linear combination (or a "sup-" in quantum talk) of the basis vectors.

## 2.2 Row Vectors and the "Dot" Product

So what can *do* with vectors? We saw earlier that we can add them together by adding the entries. Seems reasonable. What about multiplication: how do we multiply two vector together? It turns out that there are a few ways to do vector-vector multiplication. We'll introduce the "dot" product here.

Say I have two vectors: $\vec{v}_a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{v}_b = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, and I want to take their dot product, $\vec{v}_a \cdot \vec{v}_b$. The computation goes as follows:

$$\begin{aligned}
\vec{v}_a \cdot \vec{v}_b &= (\vec{v}_a)^T \vec{v}_b && (5)\\
&= \begin{pmatrix} 2 \\ 3 \end{pmatrix}^T \begin{pmatrix} 4 \\ 1 \end{pmatrix} && (6)\\
&= \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} && (7)\\
&= (2)(4) + (3)(1) && (8)\\
&= 11 && (9)
\end{aligned}$$

Line 5 is just the definition of the dot product and in Line 6 we sub-in our vectors, but what about the weird "T"? Notice how it turns a column vector into a row vector. When we apply this "T" to a vector, we're taking its *transpose*: column vectors become row vectors, and vice versa. Another very important thing to note: we started off with two vectors, and our answer was just a number (AKA scalar). For this reason, sometimes the dot product is called the 'scalar product.'

### 2.2.1 Quick Practice 2

1. Take the transpose of the following vectors:

   - $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
   - $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$
   - $\begin{pmatrix} 2 & 3 \end{pmatrix}$
   - $\begin{pmatrix} -1 & 4 \end{pmatrix}$

The dot product has an extremely geometric interpretation; its the answer to the question: how much of $\vec{v}_a$ is along $\vec{v}_b$. And by being a little careful with our vectors (normalizing them), it's telling us: how "similar" they are to each other. Let's build some intuition:

### 2.2.2 Quick Practice 3

1. Compute the dot product, $(\vec{v}_a)^T \vec{v}_b$, for the following pairs of vectors:
   - $\vec{v}_a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$
   - $\vec{v}_a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_b = -\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$
   - $\vec{v}_a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
2. Plot each pair above in the x-y plane. Do you notice any patterns between their geometry and your answers for the dot product.

### 2.2.3 Vector Length and Dot Products

Let's look at $\vec{v}_a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ more closely: what is its length (denoted $|\vec{v}_a|$)? Well the Pythagorean theorem tell us that $|\vec{v}_a| = \sqrt{(2)^2 + (3)^2} = 5$, which of course is correct. But we could've arrived at the same result in a slightly more elegant way using the dot product:

$$|\vec{v}| = \sqrt{(\vec{v})^T \vec{v}} \tag{10}$$

**So if we dot a vector with itself, we get the length squared!**

# 3 Quantum Interpretation and Dirac Notation

So what do vectors and dot products have to do with quantum computing? **Linear algebra is the language of quantum mechanics**; here are the highlights:

1. Quantum states (e.g. qubits) $\rightarrow$ vectors
2. Quantum measurements $\rightarrow$ dot products

These are (some) of the *axioms of quantum mechanics*. We'll cover some more next time!

## 3.1 Quantum States are Vectors

This whole time, $|\psi\rangle$, $|0\rangle$, and $|1\rangle$ in Equation 1 were secretly vectors! In fact, $|0\rangle$ and $|1\rangle$ are what we've chosen as the basis vectors; they are

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{12}$$

That means that our qubit (still Equation 1), can be written as

$$|\psi\rangle = \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \tag{13}$$

So we can refine the first axiom a little more: **quantum states are vectors, whose entries are the probability amplitudes corresponding to being in a certain state.**

Further, we can even think of qubits as "arrows" in a plane defined by the basis vectors, almost identically as in Figure 1:

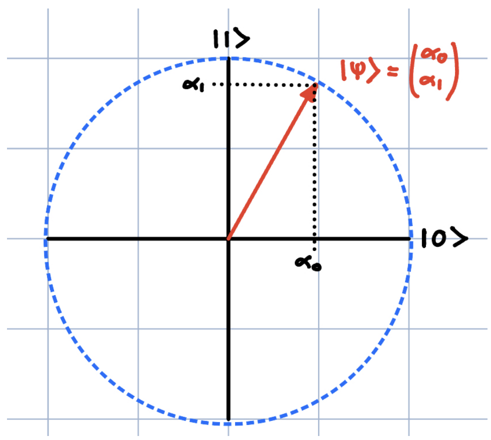


Figure 2: Qubit in the $|0\rangle$, $|1\rangle$ plane.

#### 3.1.1 Worked Example: Condition on Qubit Length (Normalization)

***All*** qubits must lie along the blue circle, can you see why? What is the circle's radius? From what we've learned about dot products so far, the "length" of the qubit is given by

$$\sqrt{\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}^T \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}} = \sqrt{\begin{pmatrix} \alpha_0 & \alpha_1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}} = \sqrt{\alpha_0^2 + \alpha_1^2} = \sqrt{P_0 + P_1} = \sqrt{1} = 1 \tag{14}$$

which we see is nothing else than the (square root of the) sum of probabilities of measuring states $|0\rangle$ and $|1\rangle$... this better be 1, no matter the qubit! **Thus, the radius of the circle is 1** and must be true for *all* qubits.

#### 3.1.2 'Bra's and 'Ket's

Quantum states come in two varieties as far as notation: 'bra's and 'ket's. You've already seen what kets look like; they're column vectors:

$$|\psi\rangle = \begin{pmatrix} \vdots \end{pmatrix} \tag{15}$$

Bra's are row vectors and have a 'reversed' notation:

$$\langle\psi| = \begin{pmatrix}...\end{pmatrix} \tag{16}$$

Bra's and ket's are related to each other by the transpose:

$$|\psi\rangle = \langle\psi|^T \tag{17}$$

$$\langle\psi| = |\psi\rangle^T \tag{18}$$

Note that in this class, we'll only be working with real numbers, not complex numbers. If we were using complex numbers, we'd not only need to do the transpose, but also take the complex conjugate of all the entries in the vector – so-called *hermitian conjugation*.

## 3.2 Quantum Measurement and Dot Products

The distinction between bra's and ket's is important because if I multiply a bra with a ket, I get just a regular number. Take a moment to appreciate why this is true: **a bra times a ket is exactly the dot product we talked about above!**

This 'dotting' process is closely connected with quantum measurement, but for now we can just think of it as a way of getting probability amplitudes. Let's do an example.

### 3.2.1 Worked Example: Extracting $P_0$, $P_1$

Suppose someone hands you many identical copies of a qubit (Equation 1); you would like to figure out what the probability amplitudes are $(\alpha_0, \alpha_1)$, because from these, you can get the probabilities are measuring the qubit in states $|0\rangle$ and $|1\rangle$:

- What is the probability of measuring the qubit in state $|0\rangle$?

  We can do the following: take our qubit $|\psi\rangle$ and multiply it with $\langle 0|$, (which is nothing else but the 'bra' version of $|0\rangle$):

  $$\langle 0||\psi\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\alpha_0 \\ \alpha_1\end{pmatrix} = 1 * \alpha_0 + 0 * \alpha_1 = \alpha_0 \tag{19}$$

  And we know that the corresponding probability is just $P_0 = |\alpha_0|^2$

- What about the probability of measuring the qubit in state $|1\rangle$?

  As you might guess, instead of multiplying $|\psi\rangle$ with $\langle 0|$, we multiply it by the $\langle 1|$:

  $$\langle 1||\psi\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\alpha_0 \\ \alpha_1\end{pmatrix} = 0 * \alpha_0 + 1 * \alpha_1 = \alpha_1 \tag{20}$$

  and $P_1 = |\alpha_1|^2$

These results shouldn't be terribly surprising; they're what I claimed to be true last class. But now we have mathematical way of summarizing everything that's in "Quick Review" that applies not just to qubits, but also *qudits* (quantum states with more than 2 outcomes). We can write down our results from the worked example in a single equation:

$$P_i = |\langle i|\psi\rangle|^2 \tag{21}$$

where, for a qubit, $i$ is either 0 or 1. This equation is actually a special case for something more general, but for today we can stop here.