A Primer in Quantum Computer
Lesson 4: Matrices and Quantum Gates

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1 Quick Review

In the last few weeks, we introduced the idea of qubits, two level quantum systems:

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \tag{1}$$

This equation makes the following physical statement: when measured, qubit $|\psi\rangle$ can collapse into state $|0\rangle$ with probability $P_0 = |\alpha_0|^2$, and $|1\rangle$ with probability $P_1 = |\alpha_1|^2$.

Last class, we learned that $|\psi\rangle$, $|0\rangle$, and $|1\rangle$ are actually column vectors. By convention

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

These column-type vectors are called *kets*. We can also take their transpose and get row-type vectors called *bras*:

$$\langle 0| = |0\rangle^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{4}$$

$$\langle 1| = |1\rangle^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{5}$$

We'll take one last mathematical detour and talk about how we can apply operations to vectors. These will be the basis for quantum circuits and algorithms!

2 Linear Algebra Detour: Matrices

A matrix *acts on* a vector to produce another vector. We think of matrices as transformations to vectors. Let's start with a simple matrix and see what we can say about it:

$$\hat{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{6}$$

First, we use a little $\hat{}$ symbol to denote that $\hat{A}$ is a matrix (unlike vectors for which we have the $\vec{}$ symbol). Second, this is a 2x2 matrix, meaning that there 2 rows, and 2 columns. Because it has the same number of rows and columns, we call it a *square matrix*.

2.1 Matrix-vector Multiplication

I claimed that if you want to transform a vector, you act with it using a matrix. More concretely, this means we multiply the vector with a matrix. For this running example, we'll use $\vec{v} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ as our vector of interest. How does $\hat{A}$ transform $\vec{v}$? We need to calculate

$$\hat{A}\vec{v} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} \tag{7}$$

This is where the dot product from last time comes in! We'll consider each row in the matrix, and dot it with $\vec{v}$; this is how we'll end up with a vector at the end of the calculation. So here we go:

$$\hat{A}\vec{v} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix} \tag{8}$$

So the $\hat{A}$ matrix is a vector transformation that takes $\begin{pmatrix} 5 \\ 6 \end{pmatrix} \to \begin{pmatrix} 16 \\ 39 \end{pmatrix}$.

2.1.1 Quick Practice 1

1. Compute the following matrix-vector multiplications:

   (a) $\hat{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

   (b) $\hat{A} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

   (c) $\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2. Are the following matrix-vector multiplications even *allowed*? If so, compute the resulting vector. If not, explain what part of the multiplication process goes wrong.

(a) $\hat{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

(b) $\hat{A} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

2.2 Matrix-matrix Multiplication

Not only can we multiply a matrix with a vector, we can extend our process from above to multiply two matrices together. This will be an invaluable skill as we work through our first quantum algorithm. Consider two matrices:

$$\hat{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \ \hat{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \tag{9}$$

So this matrix-matrix multiplication involves computing

$$\hat{A}\hat{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \tag{10}$$

How does one do this? First we need to recognize that matrix times a matrix is going to be another matrix. In fact, the rules go as follows:

- If I multiply a $m \times n$ matrix with an $n \times k$ matrix, the resulting matrix will have dimensions $m \times k$.
- The inner dimensions n must be the same!

2.2.1 Quick Practice 2

1. We would like to perform the matrix-matrix multiplication $\hat{A}\hat{B}$. $\hat{A}$ has dimensions $m \times a$ and $\hat{B}$ has dimensions $b \times n$. Are the following multiplications allowed? If so what are the dimensions of the resulting matrix?

   (a) $m = 1$, $a = 2$, $b = 2$, $n = 3$

   (b) $m = 1$, $a = 1$, $b = 4$, $n = 1$

   (c) $m = 4$, $a = 2$, $b = 2$, $n = 1$

   (d) $m = 1$, $a = 3$, $b = 1$, $n = 4$

So under these rules, we're totally allowed to perform $\hat{A}\hat{B}$... let's do it! The key to matrix-matrix multiplication is repeatedly applying the dot product to get the entries in the new matrix.

$$\hat{A}\hat{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} \tag{11}$$

3 Quantum Operators

We just said matrices act on vectors and transform them in some way. Last class, we saw quantum states, like qubits, can be represented using vectors. So it follows that, to manipulate a qubit, I do so with a matrix! In the language of quantum computing, these matrices are called *gates*. Let's start with one of the simplest (and most important gates):

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

Before I give you its name (to avoid spoilers), let's see how it transforms our basis states $|0\rangle$, $|1\rangle$.

$$\hat{X}|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \tag{13}$$

Interesting! It looks like this gate takes $|0\rangle$ and flips it to the $|1\rangle$.

3.0.1 Quick Practice 3

1. Verify that a similar thing happens to the $|1\rangle$ state (i.e. $\hat{X}|1\rangle = |0\rangle$)

Nice! So to summarize, we found that $\hat{X}$ has the following effect: $|0\rangle \to |1\rangle$ and $|1\rangle \to |0\rangle$. For this reason, we call this the ***quantum NOT gate***. More colloquially, it's also called the ***quantum bit flip*** operator.

3.1 A Survey of Quantum Operators

I'll now introduce some operators that show up all the time, and give a quick sense for what they do: how do they act on our familiar basis vectors: $|0\rangle$, $|1\rangle$?

3.1.1 Pauli Matrices

These are perhaps the 3 (or 4?) most famous quantum operators, and actually you've already met one of them:

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

- $\hat{X}|0\rangle = |1\rangle$ and $\hat{X}|1\rangle = |0\rangle$
- "Bit flip gate" (AKA quantum NOT)

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

- $\hat{Z}|0\rangle = |0\rangle$ and $\hat{Z}|1\rangle = -|1\rangle$

- "Phase flip gate"

$$\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{16}$$

- $\hat{Y}|0\rangle = i|1\rangle$ and $\hat{Y}|1\rangle = -i|0\rangle$
- Both a "bit flip" and a "phase flip"

3.1.2 Other Important Gates

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{17}$$

- $\hat{H}|0\rangle = |+\rangle$ and $\hat{H}|1\rangle = |-\rangle$ where

  $$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{18}$$

  $$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{19}$$

  If you remember, these are just superposition states with equal probability of being in $|0\rangle$ and $|1\rangle$.

- "Hadamard" gate (a special case of the *quantum Fourier transform*)

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

- $\hat{I}|0\rangle = |0\rangle$ and $\hat{I}|1\rangle = |1\rangle$
- "Identity" or the "do-nothing" gate
- Some sources consider $\hat{I}$ to be the fourth Pauli matrix

3.1.3 Quick Practice 4

1. Verify the following operator identities:
   - $\hat{H}\hat{H} = \hat{I}$
   - $\hat{H}\hat{Z}\hat{H} = \hat{X}$

In today's lab, we'll explore how we can "string-together" these gates into so-called *quantum circuits*.