How does a boundary work?

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Introduction

I found this exercise somewhere on the internet, showing how the product rule applies also to the world of manifolds. If you aren't sure of what those words mean, don't worry as I'll go the other way around. Building some simple intuition that even a 6-year-old can follow, and try to see how much generality we can extract.

Before doing anything else, I'll start explaining two key concepts. First, we need to understand what the Cartesian product is. Look at the following illustration:

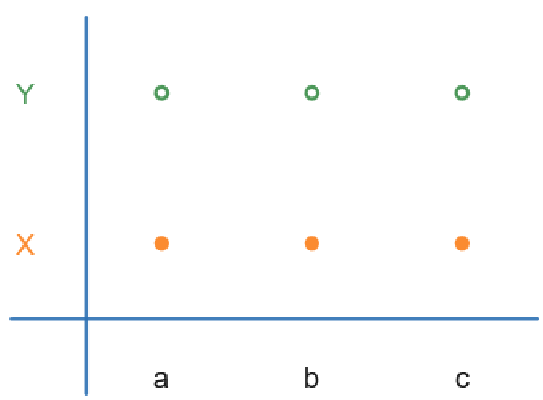

Figure 1. A Cartesian product.

We have 2 types of sets. We can call A the set $\{a, b, c\}$ and B the set $\{X, Y\}$, thus, their Cartesian product is all the different *ordered pairs* we can make. In this case six different pairs are made, which may sound familiar to the multiplication we are used to.

To state things clearly, we can visualize the first set A in a line, and the second set B in a new **perpendicular**, thus the Cartesian product is just checking how they "interact" with each other.

If we used instead of a discrete set (with a finite set of terms) a continuous interval such $[0,1]$, we can construct an area. Let's call this interval I.

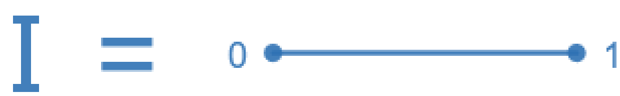

Figure 2. The interval I.

Then the Cartesian product of I with itself would be a square and a cube respectively.

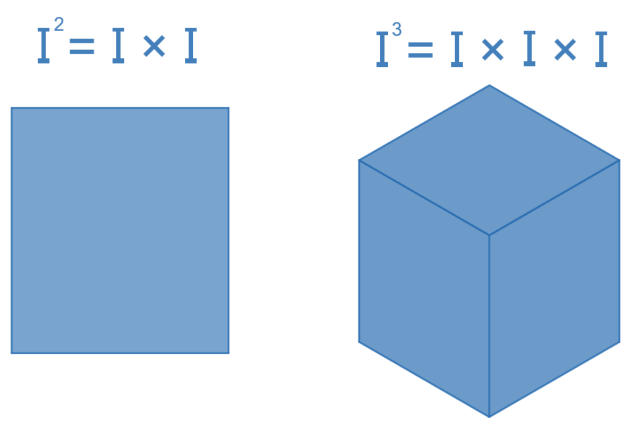

Figure 3. Square and cube made from the Cartesian product.

Now that we have an intuitive sense of the Cartesian product, let's explore the second key concept call the boundary.

The boundary is a very intuitive concept, as it is the intersection of the exterior and interior of any object. We can use the symbol ∂ as an operator, to give us the boundary. See the following example with a disk called D.

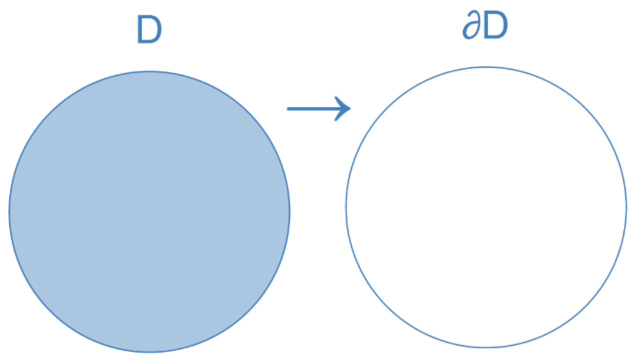

Figure 4. The boundary of a disk is a circle.

We had the solid interior of the disk D, but once we applied the boundary operator ∂, we get only get the circumference. Now imagine the case of a solid ball B, ∂B would denote a hollow ball, just the boundary.

Notice how when we have an object in a certain dimension (n, for example), the boundary of that object is one dimension below (n-1, following the example). This applies to almost any type of object: any shape, solid, or (in general dimensions called a) manifold.

Side tangent:

Manifolds have some particular, but general enough, features that make them well-behaved when trying to find the boundary. This may sound perhaps a little odd, but since mathematicians love generality there are ways in which we can change some particular assumptions made here. Like, for example: how we define lengths. We could come up with new ways to define the metric of these objects, like the "famous" -norms. In other words, there are strange objects out there!

Now we have everything to tackle the problem!

The problem

I may seem reasonable (if not obvious) that we can express: $I^3 = I^2 \times I$.

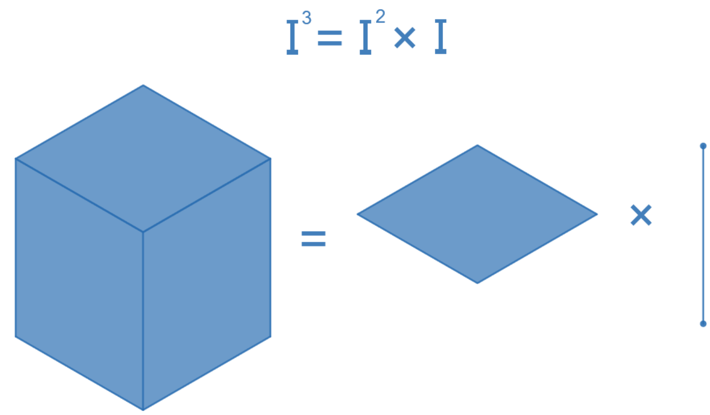

Figure 5. A solid cube is the Cartesian product between a square and a line.

It's also simple finding the boundary of the solid cube, $\partial(I^3)$ would be just the exterior part of the cube, like an empty box. Now, the problem is checking how does the boundary operator applies to $I^2 \times I$. In other words, the issue is how can we manipulate (make some algebra) in the expression, $\partial(I^2 \times I)$?

We are going to make some trial and error, to see how we can get there.

Our first assumption could be checking if the boundary operator just distributes over the Cartesian product: $\partial(I^2 \times I) = \partial(I^2) \times \partial(I)$. Let's visualize the result:

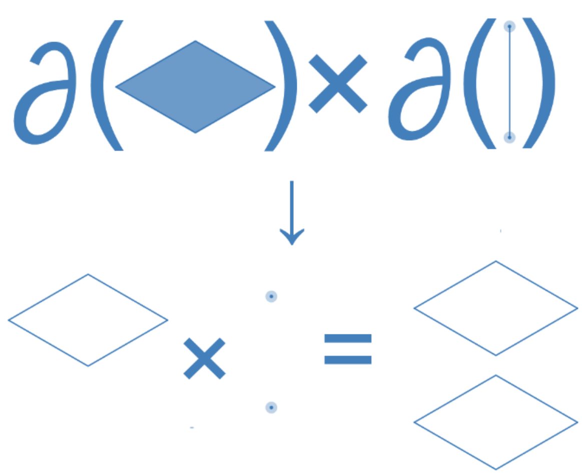

Figure 6. Since the boundary of a one-dimensional line are the endpoints, the result gives us two copies of the boundary of the plane.

Notice how this result is different from the hollow box (the boundary of the solid cube). We can infer that this is not the correct way of manipulating the boundary operator. Said symbolically: $\partial(I^3) = \partial(I^2 \times I) \neq \partial(I^2) \times \partial(I)$.

Let's try to play a little. What would happen if we make the Cartesian product of $\partial(I^2)$ with I?

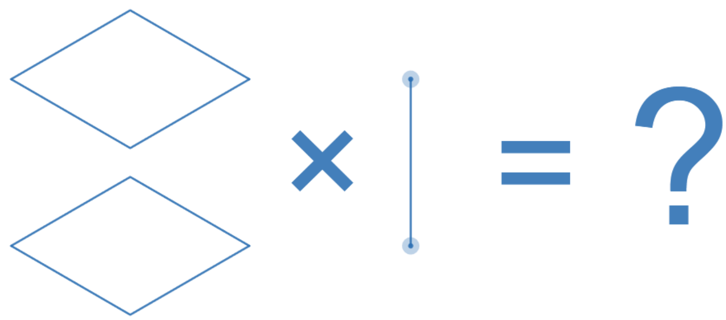

Figure 7. What would the result of Cartesian product of the silhouette of a square and a line?

This would get us closer to the answer we are looking for, as we would now have 4 of the 6 sides of the empty box.

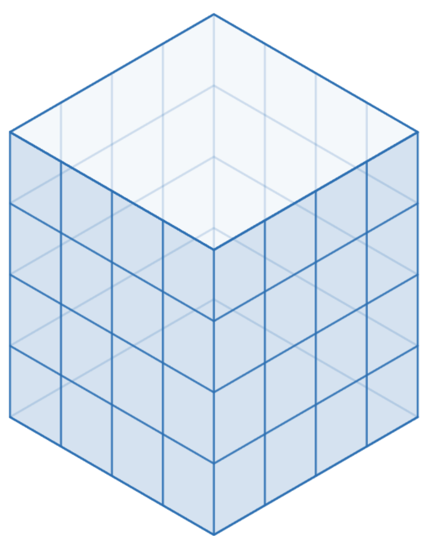

Figure 8. The result of $\partial(I^2)\times I$, gives us a box without the top and the bottom.

Algebraically this would mean that: $\partial(I^2 \times I) = \partial(I^2) \times I + 2 \cdot faces$. Technically we need to use the union (∪) symbol, instead of the +, since we are dealing with sets, but works essentially the same.

Now we can try something interesting. What if the boundary operator, takes turns? First affecting I^2 and then I? Sounds bold, but we can try it out:

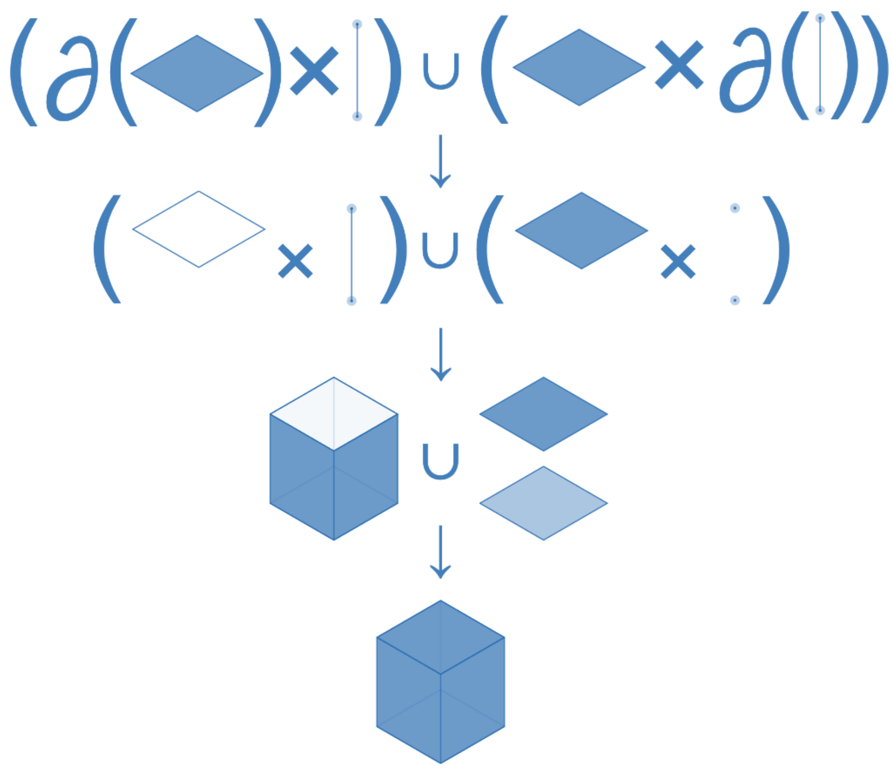

Figure 9. The term on the right, from $I^2 \times \partial(I)$, gives us the two faces we were missing to get to the ∂I^3 hollow box.

We can observe how this gives us the expected answer!

Thus, the correct expansion is: $\partial(I^2 \times I) = (\partial I^2 \times I) \cup (I^2 \times \partial I)$.

Conclusions

All this work of working with planes, and cubes, and boundaries could be (or not) captivating, but what's the point to all of this? The thing is that, as it often happens in mathematics, these observations have some deeper consequences.

If you did learn some differential calculus before, the expression we arrived may sound familiar, as it has the same structure as the product rule: $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

The astounding thing is that the abstract topological sets we have been working (cubes, disk and so on), are related to functions. Both of them are two faces of the same coin called differential forms. Moreover, we have **a more general theory** to look at these objects called de **Rham cohomology**, that which validates how for two manifolds (in which we can get derivatives, aka differential manifold) M and N, the following statement is true: $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$.

Which is the obvious generalization of statement we found above. I must admit that this last part of conclusions may sound like a bit of gibberish, but now you can claim that you find a demonstration of the product rule by adding the faces of a box!