1.2.3. Exact solution

The wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

Considering:

$$u(t,x,y) = e^{\,i(k_x x + k_y y - \omega t)}$$

Time derivatives:

$$\frac{\partial u}{\partial t} = -i\omega u, \qquad \frac{\partial^2 u}{\partial t^2} = (-i\omega)^2 u = -\omega^2 u$$

Spatial derivatives:

$$\nabla^2 u = u_{xx} + u_{yy} = (ik_x)^2 u + (ik_y)^2 u = -(k_x^2 + k_y^2)u$$

Inserting in the wave equation:

$$-\omega^2 u = c^2 \big[-(k_x^2 + k_y^2)u \big]$$

So, if $\omega^2 = c^2\,(k_x^2 + k_y^2)$, then $u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$ satisfies the wave equation.

1.2.4. Dispersion coefficient

According to Eq. (1.7), we have:

$$u_{ij}^n \;=\; e^{\,i\left(kh(i+j) - \tilde{\omega}\, n\Delta t\right)}$$

So:

$$u_{i,j}^n \;=\; \exp\Big(i\,[kh(i+j) - \tilde{\omega}\, n\Delta t] \Big)$$

For a shift in time, we can develop

$$\begin{aligned} u_{i,j}^{n+1} &= \exp\Big(i\,[kh(i+j) - \tilde{\omega}\,(n+1)\Delta t] \Big) \\ &= \exp(-i\tilde{\omega}\Delta t)\, \exp\Big(i\,[kh(i+j) - \tilde{\omega}\, n\Delta t] \Big) \\ &= e^{-i\tilde{\omega}\Delta t}\, u_{i,j}^n \end{aligned}$$

and similarly

$$u_{i,j}^{n-1} \;=\; e^{+i\tilde{\omega}\Delta t}\, u_{i,j}^n$$

By substituting on the left-hand side of the discretized equation and using $e^{i\theta} + e^{-i\theta} = 2\cos\theta$:

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = u_{i,j}^n\, \frac{e^{-i\tilde{\omega}\Delta t} - 2 + e^{+i\tilde{\omega}\Delta t}}{\Delta t^2} = u_{i,j}^n\, \frac{2\big(\cos(\tilde{\omega}\Delta t) - 1\big)}{\Delta t^2}$$

For shifts in x we have:

$$u_{i+1,j}^n = e^{+ikh}\, u_{i,j}^n, \qquad u_{i-1,j}^n = e^{-ikh}\, u_{i,j}^n,$$

and shifts in y are identical

$$u^n_{i,j+1} = e^{+ikh}\, u^n_{i,j}, \qquad u^n_{i,j-1} = e^{-ikh}\, u^n_{i,j}.$$

on the right-hand side of the discretized equation. For the x-part

$$\frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{h^2} = u^n_{i,j}\, \frac{e^{ikh} - 2 + e^{-ikh}}{h^2} = u^n_{i,j}\, \frac{2\big(\cos(kh) - 1\big)}{h^2}$$

For the y-part, similarly

$$\frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{h^2} = u^n_{i,j}\, \frac{2\big(\cos(kh) - 1\big)}{h^2}$$

By setting LHS $= c^2 \times$ RHS and cancel $u^n_{i,j}$:

$$\frac{2\big(\cos(\tilde{\omega}\Delta t) - 1\big)}{\Delta t^2} = \frac{4c^2\big(\cos(kh) - 1\big)}{h^2}$$

So we get:

$$\cos(\tilde{\omega}\Delta t) = 1 - 4C^2 \sin^2\!\left(\frac{kh}{2}\right), \qquad C = \frac{c\,\Delta t}{h}$$

If we choose $C = \frac{1}{\sqrt{2}}$, then:

$$\cos(\tilde{\omega}\Delta t) = 1 - 2\sin^2\!\left(\frac{kh}{2}\right) = \cos(kh)$$

So:

$$\tilde{\omega}\,\Delta t = kh \quad \Rightarrow \quad \tilde{\omega} = \frac{kh}{\Delta t} = \frac{ck}{C} = c\sqrt{2}\,k$$

Relation to the exact dispersion:
With $k_x = k_y = k$ in $\omega = c\sqrt{k_x^2 + k_y^2}$ we get

$$\omega = c\sqrt{2}\,k,$$

so the numerical $\tilde{\omega}$ equals the exact ω when $C = 1/\sqrt{2}$.