TURING MACHINE

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FSM → Finite State Machine → DFA, NFA

$0 1^* 0$

array

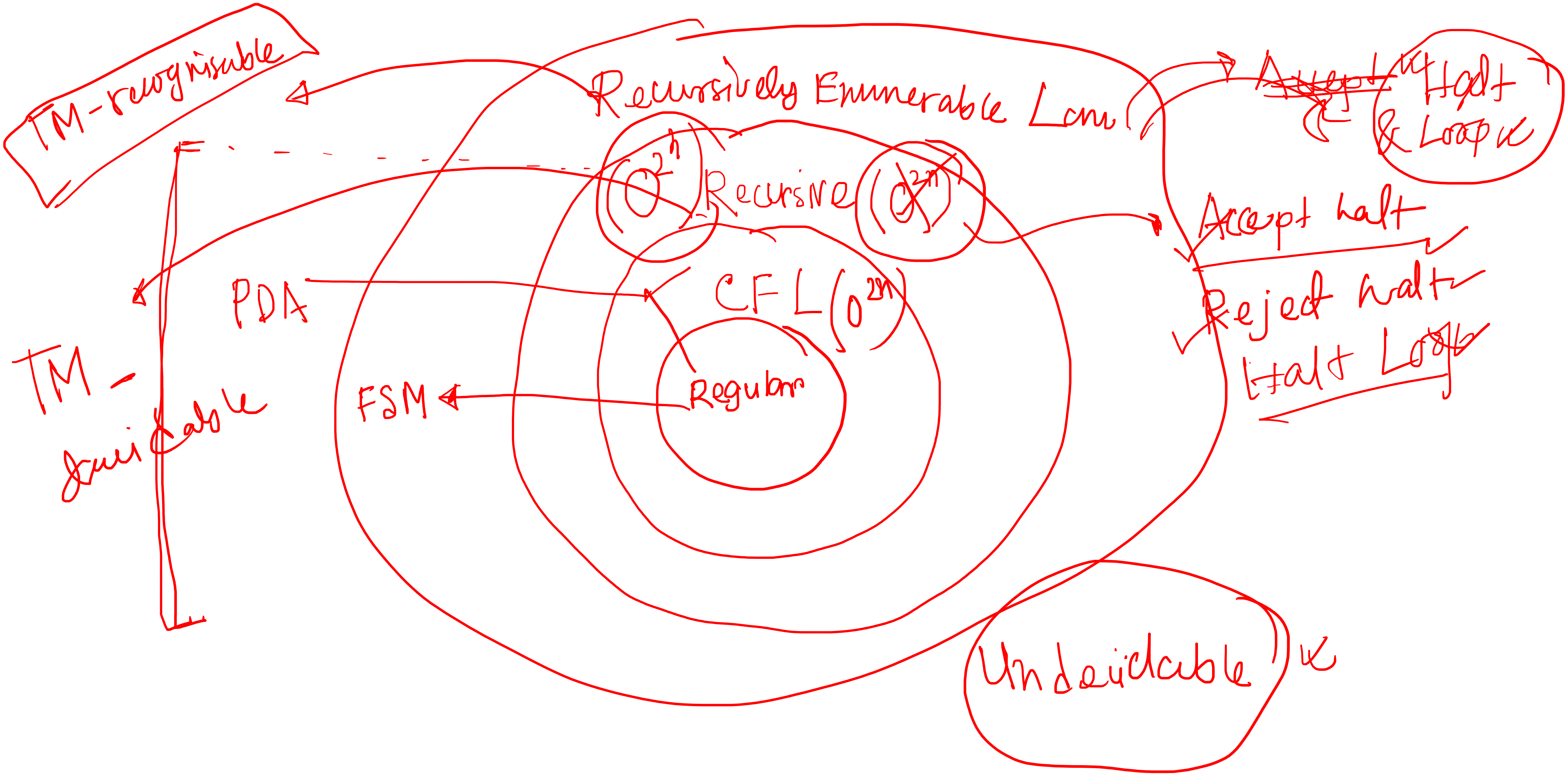
memory

PDA → Pushdown Automata (FSM + stack)

CFG → CFL → counting

$0^n 1^n$

Recursively enumerable Language } → TM


TM-recognisable
Recursively Enumerable Lan
~~Accept~~ Halt
& Loop
Recursive
Accept halt
Reject halt
Halt Loop
PDA
TM-decidable
CFL
FSM
Regular
Undecidable

Turing Machine

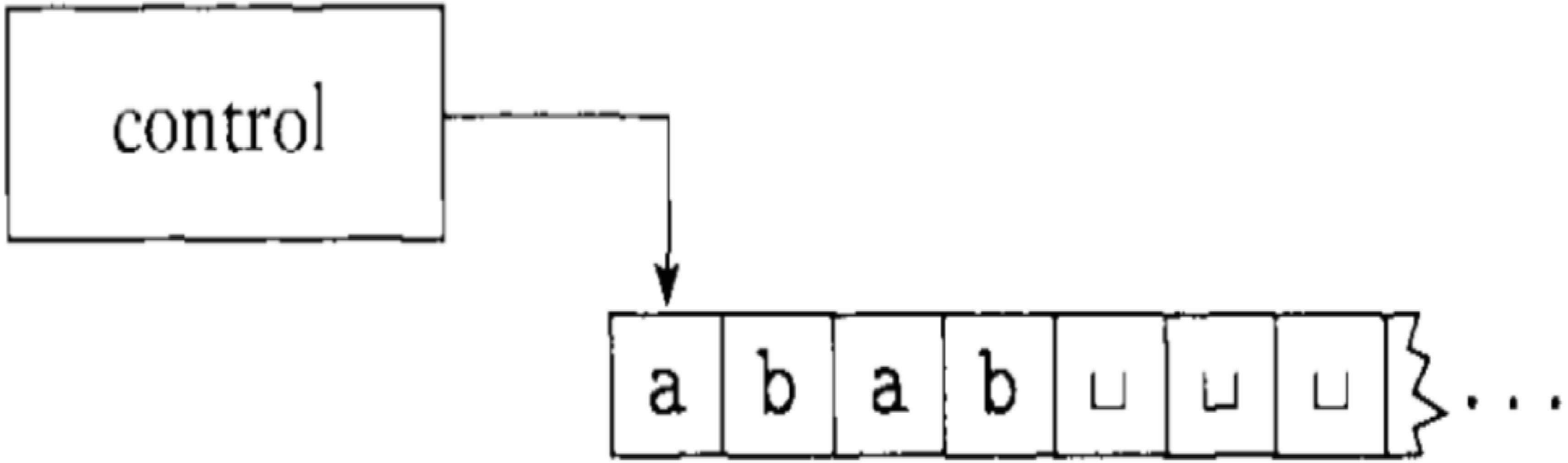


FIGURE 3.1
Schematic of a Turing machine

Differences between Pushdown Automaton and Turing Machine

- A Turing machine can both write on the tape and read from it.
- The read-write head can move both to the left and too the right.
- The tape is infinite.
- The special states for rejecting and accepting takes effect immediately.

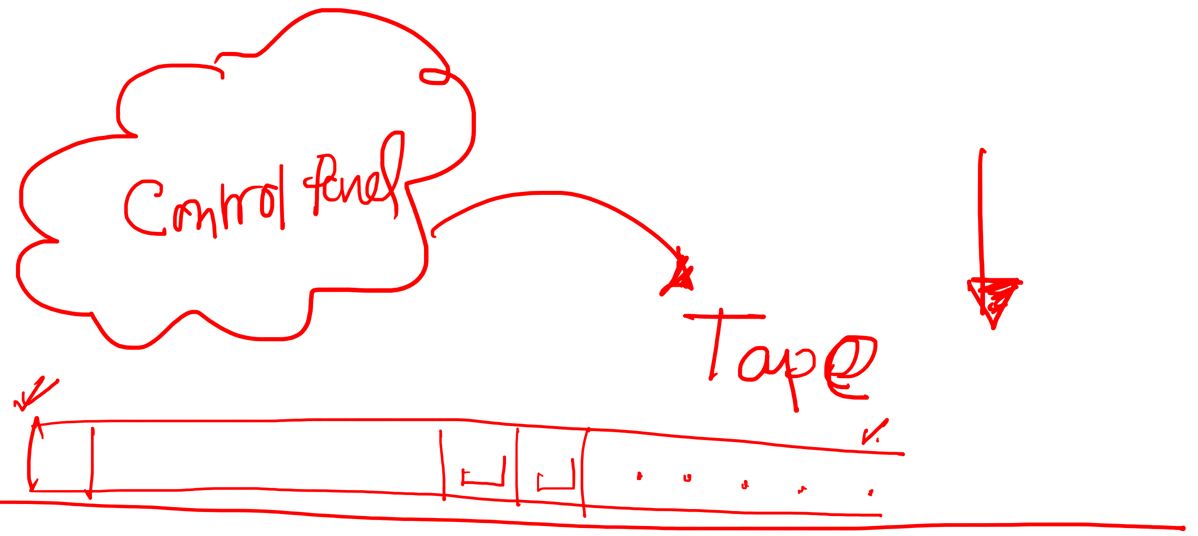
TM

1. ↓ (head) → Read/scan
2. ↓ (head) → write
3. ↓ (head) move
   a. Left
   b. Right

↓ read → a

↓ write(b) → b

↓ Left ~~Right~~ ↓ Move ✗

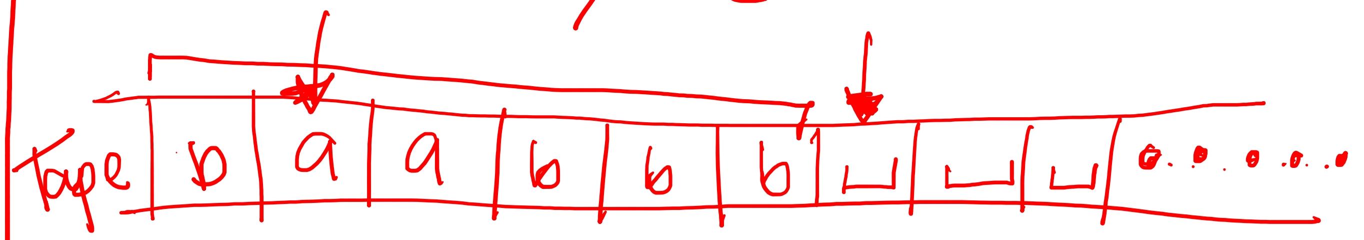


$\Sigma = \{a, b, \ldots\}$ ⊔

$\$ \rightarrow \notin \Sigma$

$\sqcup \rightarrow \notin \Sigma$

$\sqcup \rightarrow R$



Formal Definition

$(Q, \Sigma, \Gamma, \delta, q_0, F, b)$

$Q \rightarrow$ States

$\Sigma \rightarrow$ Alphabet

$\Gamma \rightarrow$ Tape

$b \rightarrow \sqcup$

$(q_0, 0) \rightarrow (q_{00}, 0, R)$

$\cancel{0}\,0\,0\,1\,1\,1$

x

$\boxed{\delta}$

$(q_0, 0) \longrightarrow (q_1, x, R)$

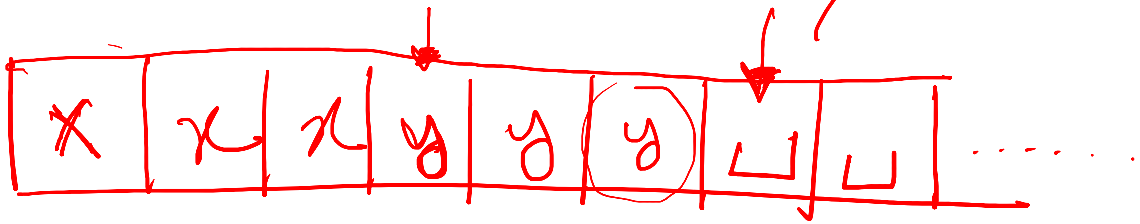
$(q_x, 1) \longrightarrow (q_y, y, L)$

000111

0^n1^n

1 → REJECT

␣ → ACCEPT

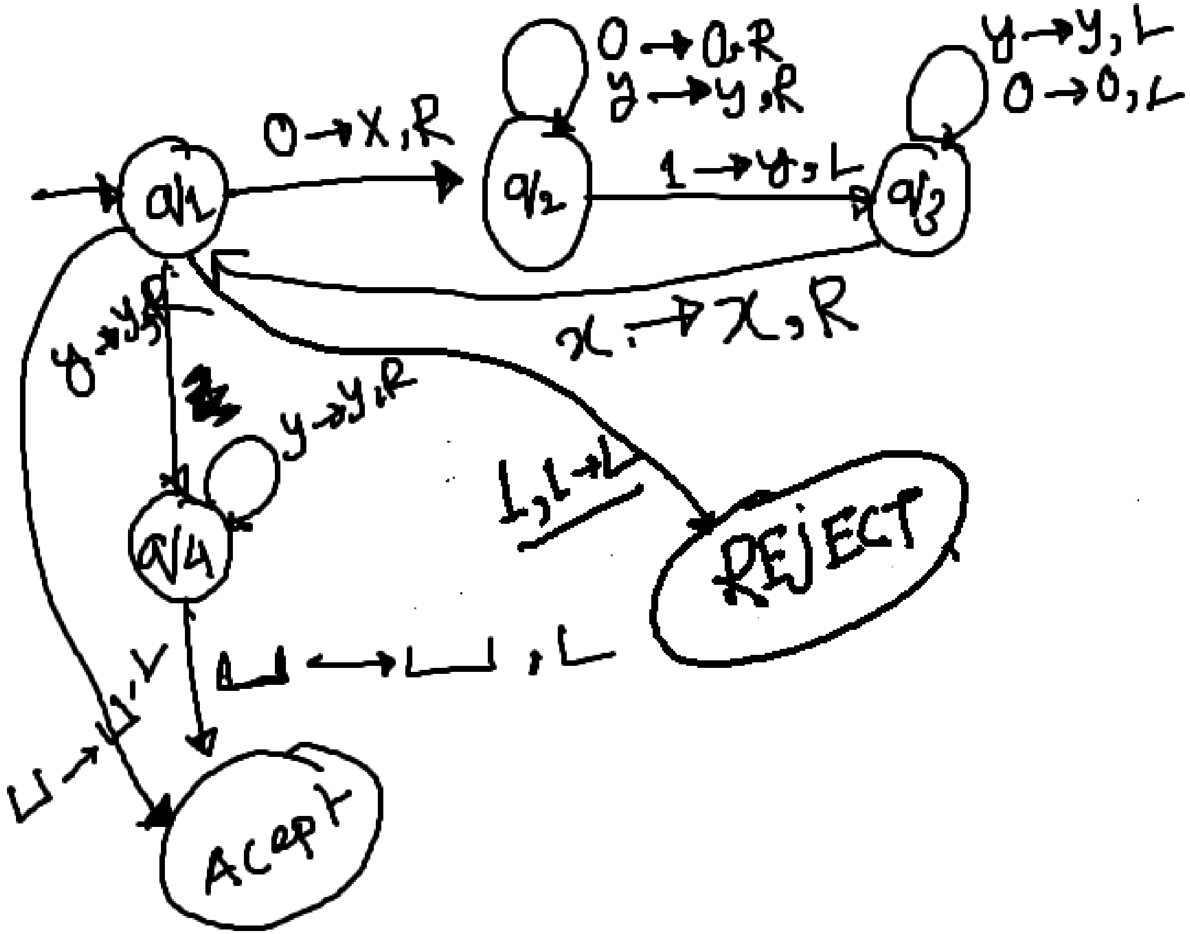


→ REJECT

$\cancel{0}\ \cancel{0}\ \cancel{0}\ \cancel{1}\ \cancel{1}$ ␣

x x x y y

1. ↓ (0) → x, Replace (Read, write)
2. Move Right until (the First '1') / (you find '1') (Move right Read)
3. If (1 Found → y) | else → REJECT (Read)
4. Move Left until (you find 'x') / you find the Leftmost 0 and read 0



$\boxed{\text{Tape Traversal}}$

$q_1\,0\,0\,1\,1$

$x\,q_2\,0\,1\,1$

$x\,0\,q_2\,1\,1$

$x\,q_3\,0\,y\,1$

$q_3\,x\,0\,y\,1$

$x\,q_1\,0\,y\,1$

$x\,x\,q_2\,y\,1$

$x\,x\,y\,q_2\,1$

$x\,x\,q_3\,y\,y$

$x\,q_3\,x\,y\,y$

$x\,x\,q_1\,y\,y$

$x\,x\,y\,q_4\,y$

$x\,x\,y\,y\,q_4\,\sqcup$

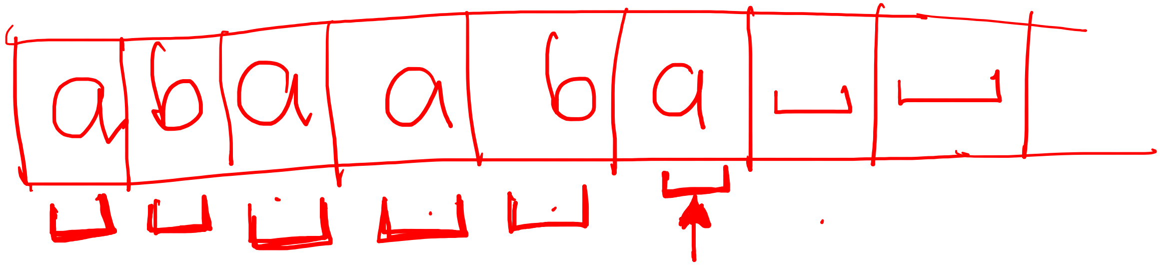
$x\,x\,y\,y\,q_{accept}$

Palindrome [Even Palindrome]

✓② State Diagram

✓③ Tape Traversal

$\Sigma = \{a, b\}$

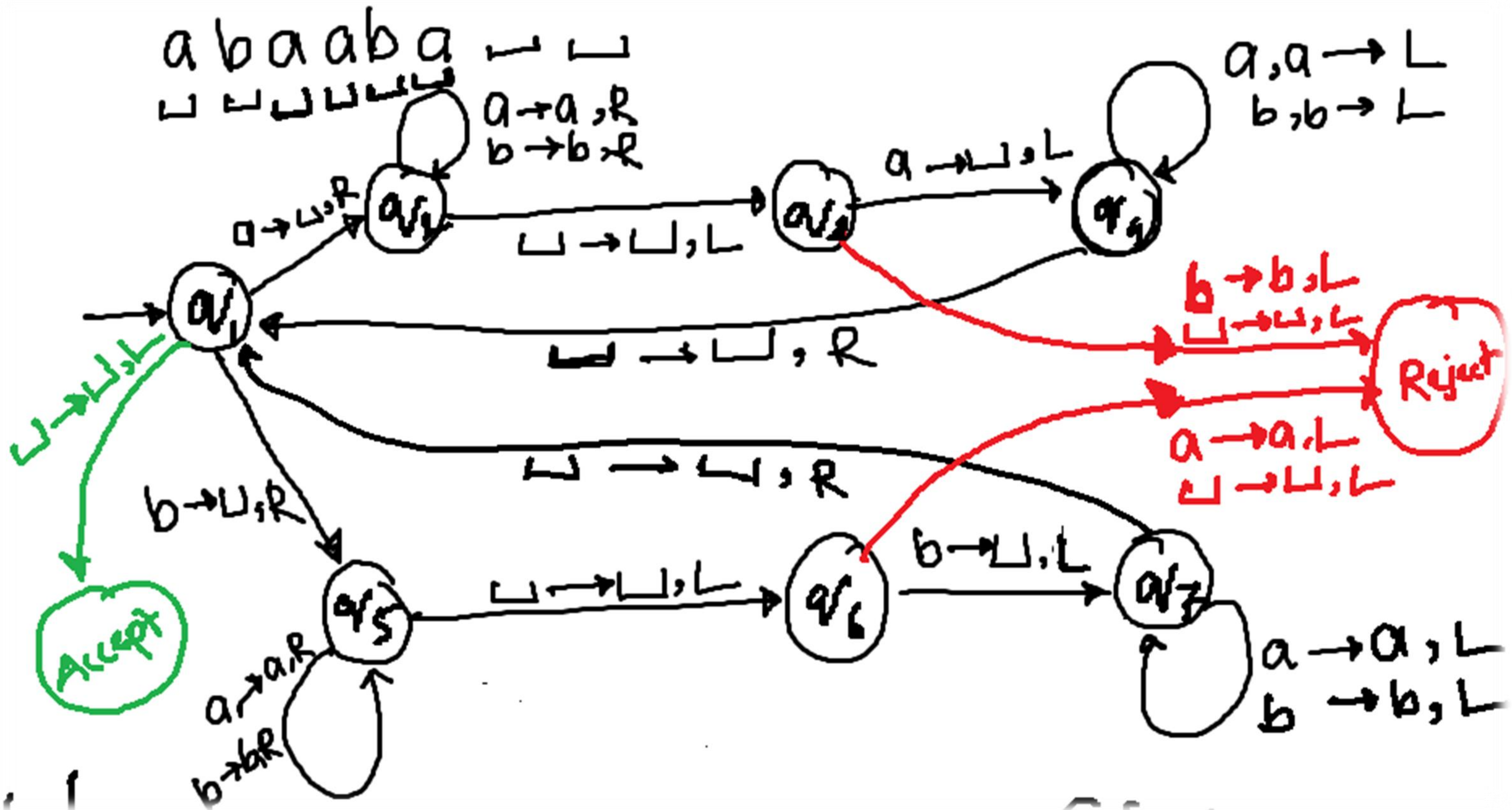


① Algorithm:

$\boxed{b}$ ✓ $\boxed{a \text{ or } b}$

$\boxed{a}$

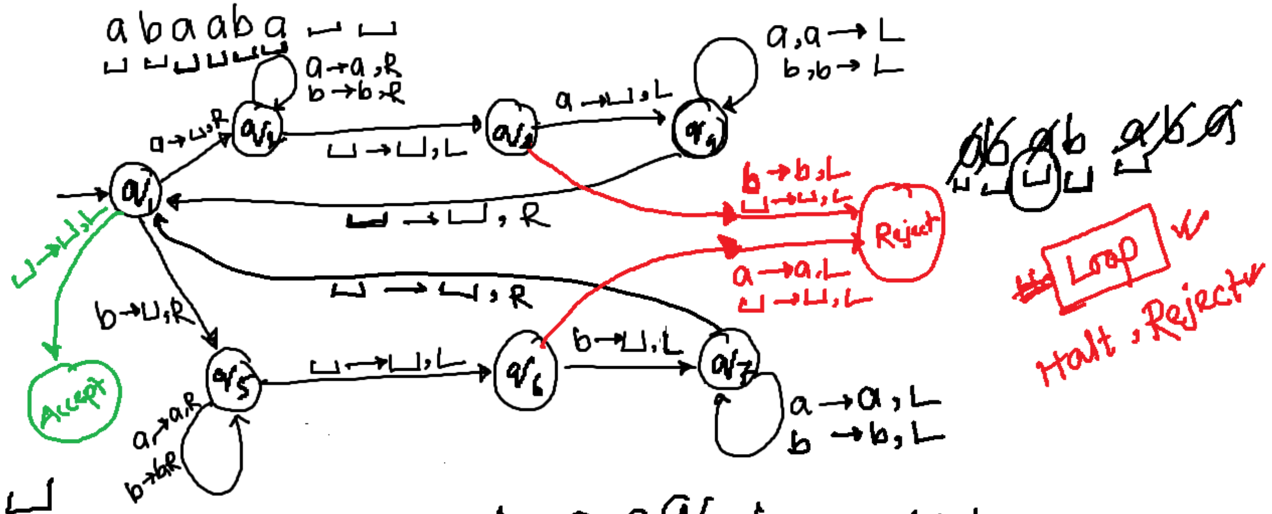
1. a → ⊔
2. b, a → ignore : ⊔ → Left Move : a ?
3. If "a" → ⊔ : Else REJECT
4. Left Traversal until Find '⊔' → Right Move


a b a a b a ⊔ ⊔
a → a, R
b → b, R
a → ⊔, R
⊔ → ⊔, L
a → ⊔, L
a, a → L
b, b → L
b → b, L
⊔ → ⊔, L
Reject
a → a, L
⊔ → ⊔, L
⊔ → ⊔, R
⊔ → ⊔, R
⊔ → ⊔, L
Accept
b → ⊔, R
⊔ → ⊔, L
b → ⊔, L
a → a, R
b → b, R
a → a, L
b → b, L

q_1 a b b a a b b a ⊔
⊔ q_2 b b a a b b a ⊔
⊔ b q_2 b a a b b a ⊔
⊔ b b q_2 a a b b a ⊔
⊔ b b a q_2 a b b a ⊔
⊔ b b a a q_2 b b a ⊔
⊔ b b a a b q_2 b a ⊔
⊔ b b a a b b q_2 a ⊔
⊔ b b a a b b a q_2 ⊔
⊔ b b a a b b q_3 a ⊔

⊔ b b a a b q_4 b ⊔ ⊔
⊔ b b a a q_4 b b ⊔ ⊔
⋮
⊔ q_4 b b a a b b ⊔ ⊔
q_4 ⊔ b b a a b b ⊔ ⊔
⊔ q_1 b b a a b b ⊔ ⊔
⊔ ⊔ q_5 b a a b b ⊔ ⊔
⋮
⊔ ⊔ b a a b q_5 b ⊔ ⊔
⊔ ⊔ b a a b b q_5 ⊔ ⊔
⊔ ⊔ b a a b q_6 b ⊔ ⊔

⊔ ⊔ b a a q_7 b ⊔ ⊔ ⊔
⋮
⊔ ⊔ q_7 b a a b ⊔ ⊔ ⊔
⊔ q_7 ⊔ b a a b ⊔ ⊔ ⊔
⊔ ⊔ q_1 b a a b ⊔ ⊔ ⊔



Turing Machine (Continuation…)

- Turing Machine $\mathbf{M_1}$

$$B = \{w\#w \mid w \in \{0,1\}^*\}$$

0101 # 0101

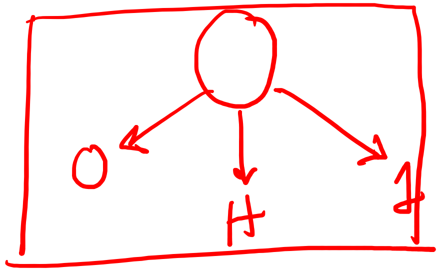
011 # 011

011 # 110

011 # 011

x x x # x x x

Accept



Turing Machine (Continuation…)

- **M_1** algorithm is as follows:

M_1 = "On input string w:

1. Zig-zag across the tape to corresponding positions on either side of the # symbol to check whether these positions contain the same symbol. If they do not, or if no # is found, *reject*. Cross off symbols as they are checked to keep track of which symbols correspond.
2. When all symbols to the left of the # have been crossed off, check for any remaining symbols to the right of the #. If any symbols remain, *reject*; otherwise, *accept*."

Turing Machine (Continuation…)

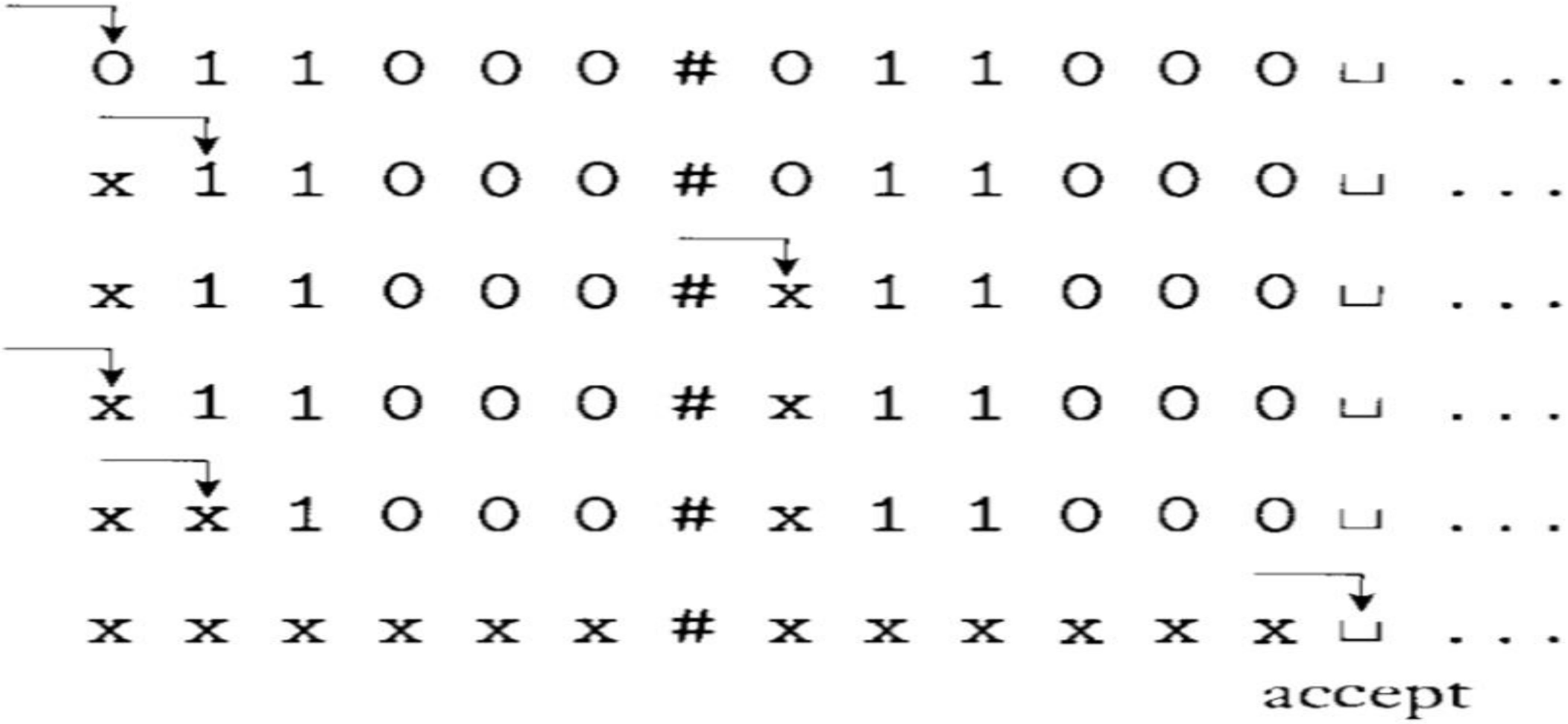


FIGURE 3.2
Snapshots of Turing machine M_1 computing on input 011000#011000

Formal Definition of a Turing Machine

DEFINITION 3.3

A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta: Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{reject}} \neq q_{\text{accept}}$.

Configuration of a Turing Machine

- As a Turing Machine computes, changes occur in
  - Current state
  - Current tape contents
  - Current head location
- A setting of these three items called a **configuration** of a Turing Machine
- For a state q and two strings u and v over the tape alphabet, we write uqv for the configuration where
  - the current state is q
  - the current tape contents is uv and
  - the current head location is the first symbol of v
- The tape contains only blanks following the last symbol of v

Configuration of a Turing Machine (Continuation…)

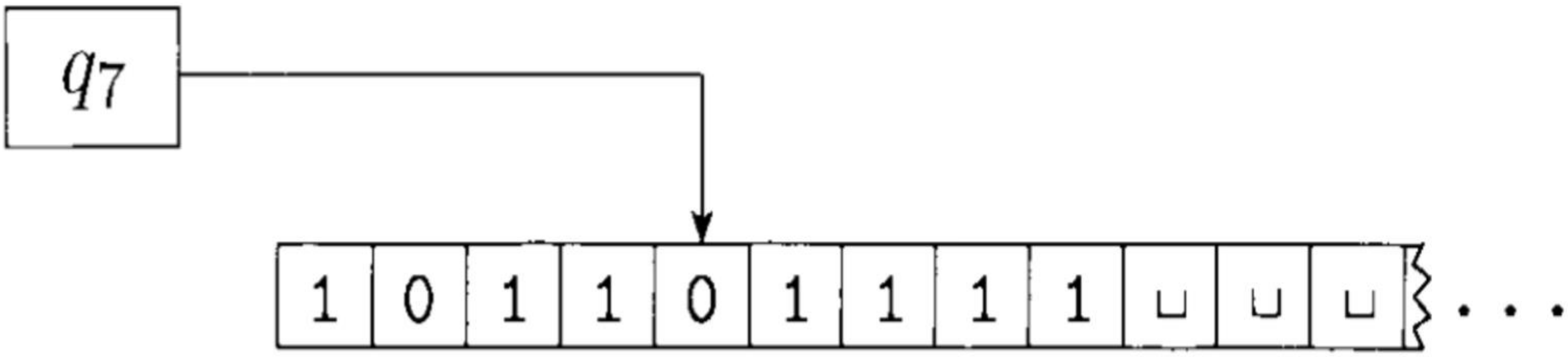


FIGURE 3.4
A Turing machine with configuration $1011q_701111$

Configuration of a Turing Machine (Continuation…)

- Configuration **C_1 yields configuration C_2** if TM can legally go from C_1 to C_2 in a single step

Suppose that we have a, b, and c in Γ, as well as u and v in Γ^* and states q_i and q_j. In that case $ua\,q_i\,bv$ and $u\,q_j\,acv$ are two configurations. Say that

$$ua\,q_i\,bv \quad \text{yields} \quad u\,q_j\,acv$$

if in the transition function $\delta(q_i, b) = (q_j, c, \mathrm{L})$. That handles the case where the Turing machine moves leftward. For a rightward move, say that

$$ua\,q_i\,bv \quad \text{yields} \quad uac\,q_j\,v$$

if $\delta(q_i, b) = (q_j, c, \mathrm{R})$.

Configuration of a Turing Machine (Continuation…)

- Special cases occur when the head is at one of the ends of the configuration
- For the left-hand end, the configuration $\mathbf{q_i bv}$ yields $\mathbf{q_j cv}$ if the transition is left moving and it yields $\mathbf{cq_j v}$ for the right moving transition
- For the right-hand end , the configuration $\mathbf{uaq_i}$ is equivalent $\mathbf{uaq_i}$ ␣

Configuration of a Turing Machine (Continuation…)

- Start Configuration
- Accepting Configuration
- Rejecting Configuration
- Halting Configuration
- A TM M accepts input w if a sequence of configurations $\mathbf{C_1, C_2, \ldots, C_k}$ exists, where

1. C_1 is the start configuration of M on input w,
2. each C_i yields C_{i+1}, and
3. C_k is an accepting configuration.

Turing Recognizable and Turing Decidable Language

DEFINITION 3.5

Call a language ***Turing-recognizable*** if some Turing machine recognizes it.[1]

DEFINITION 3.6

Call a language ***Turing-decidable*** or simply ***decidable*** if some Turing machine decides it.[2]

Examples of Turing Machine

0^{2^n} ✓.

$1, 2, 4, 8, 16 \ldots$

EXAMPLE 3.7

Here we describe a Turing machine (TM) M_2 that decides $A = \{0^{2^n} \mid n \geq 0\}$, the language consisting of all strings of 0s whose length is a power of 2.

$0^{2^0} = 0^1 = 0$ ✓

$0^{2^1} = 0^2 = 00$ ✓

$0^{2^2} = 0^4 = 0000$ ✓.

(i) Even number or 1 ✓

(ii) 6, 10 $\longrightarrow$

(iii) $\circ \times \circ \times \circ \times$? Edge

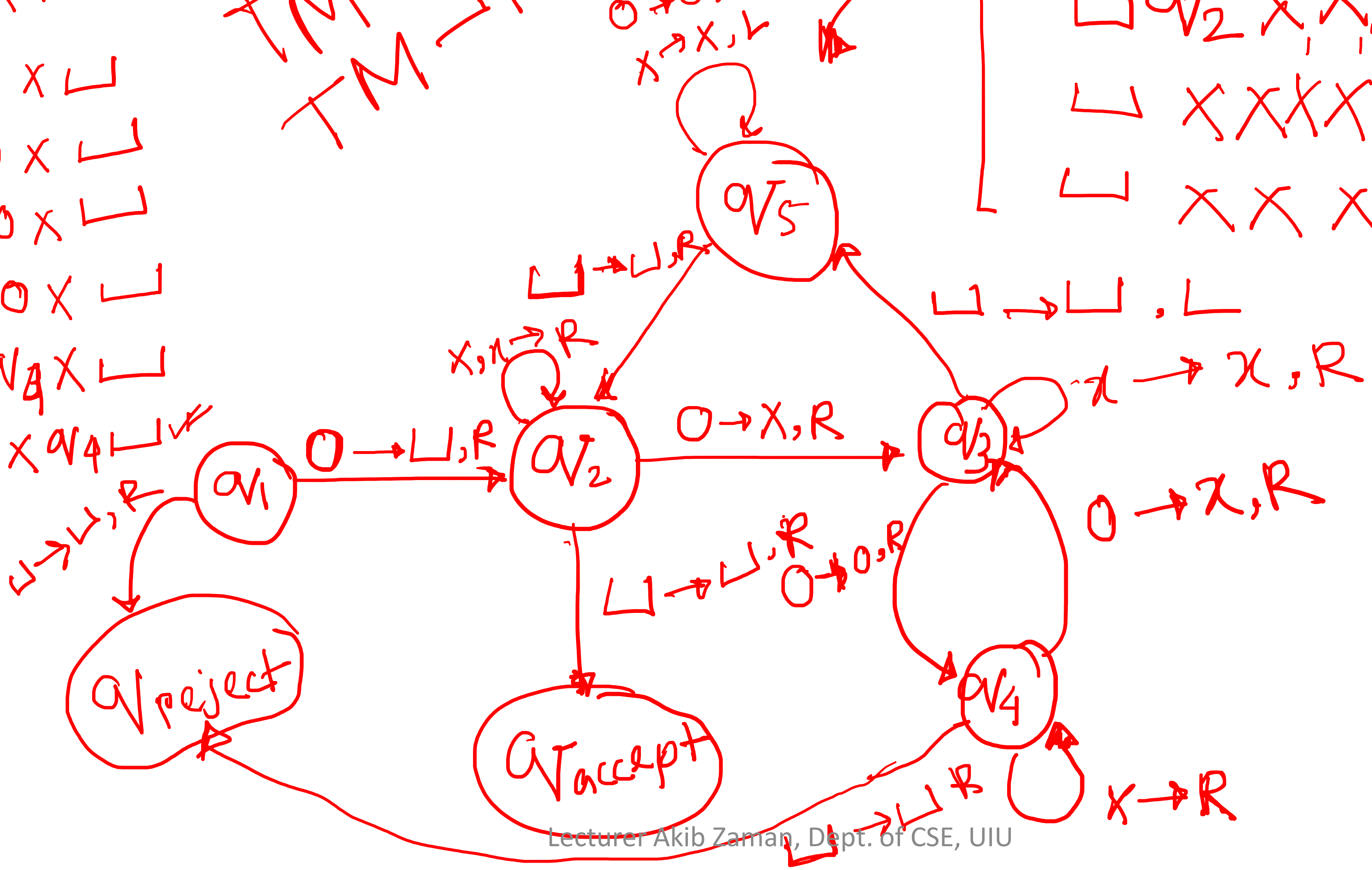
(iv) parallel Tape Traversal.

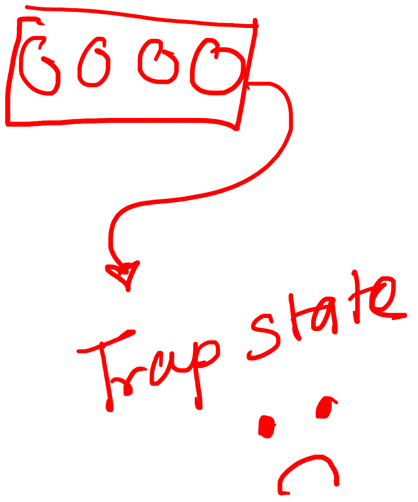
q_1 0 0 0 0 0 0 ␣
␣ q_2 0 0 0 0 0 ␣
␣ x q_3 0 0 0 0 ␣
␣ x 0 x q_3 0 0 ␣
␣ x 0 x 0 x q_3 ␣
q_5 ␣ x 0 x 0 x ␣
␣ q_2 x 0 x 0 x ␣
␣ x q_2 0 x 0 x ␣
␣ x x q_3 x 0 x ␣
␣ x x x q_3 0 x ␣
␣ x x x 0 q_4 x ␣
␣ x x x 0 x q_4 ␣

6 10

TM - Decidable
TM - Recognizable

q_1 0 0 0 0 0 0 0 0 0 ␣
␣ x 0 x 0 x 0 x q_3 ␣
␣ x x x 0 x x x q_3 ␣
␣ x x x x x x x q_3 ␣
␣ q_2 x x x x x x x ␣
␣ x x x x x x x q_2 ␣
␣ x x x x x x x ␣ q_{accept}





QUESTION:

- Difference Between TM and PDA.
- TM Recognizable and TM-Decidable Machine.

Examples of Turing Machine (Continuation…)

$M_2 =$ "On input string w:

1. Sweep left to right across the tape, crossing off every other 0.
2. If in stage 1 the tape contained a single 0, *accept*.
3. If in stage 1 the tape contained more than a single 0 and the number of 0s was odd, *reject*.
4. Return the head to the left-hand end of the tape.
5. Go to stage 1."

Examples of Turing Machine (Continuation…)

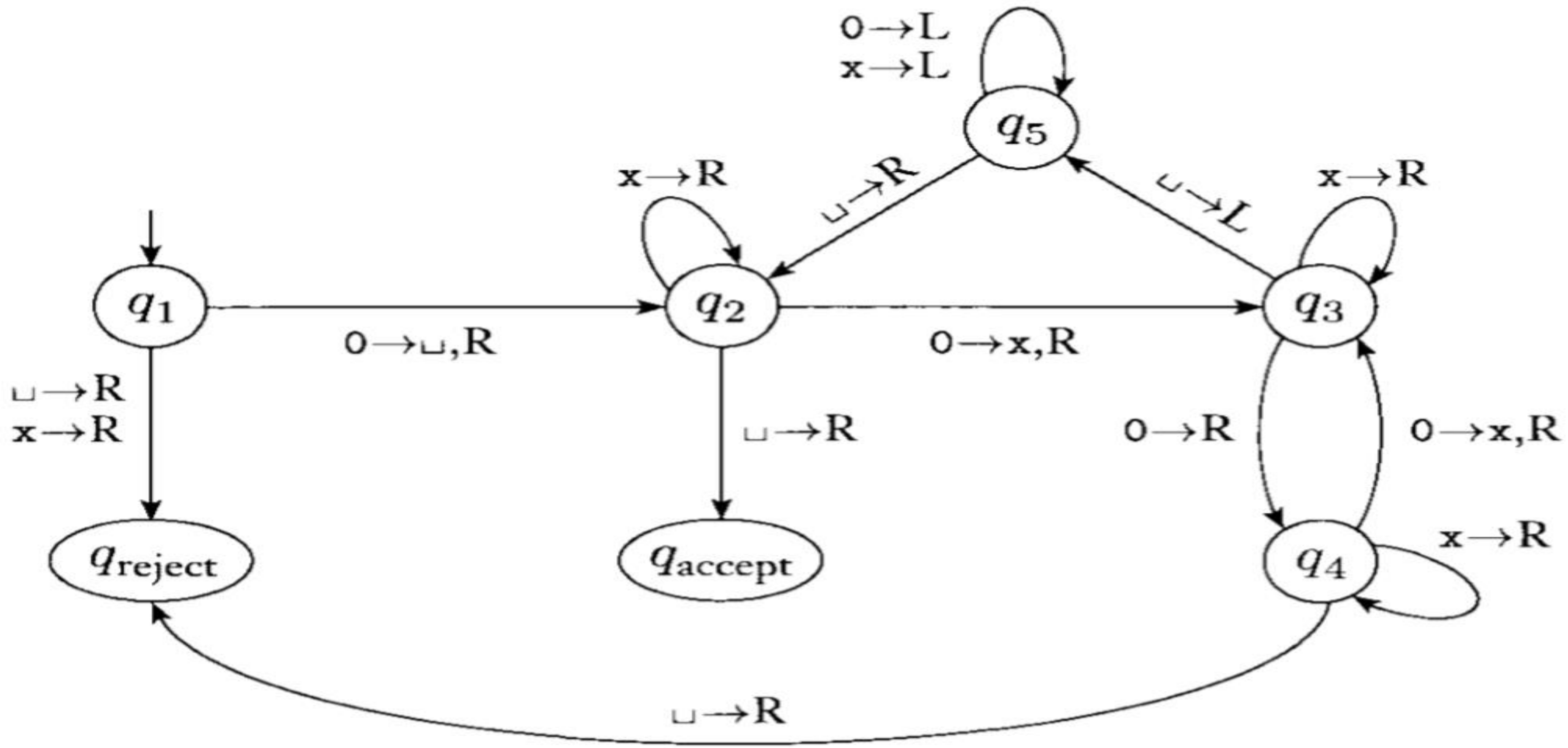


FIGURE 3.8
State diagram for Turing machine M_2

Examples of Turing Machine (Continuation…)

Now we give the formal description of $M_2 = (Q, \Sigma, \Gamma, \delta, q_1, q_{\text{accept}}, q_{\text{reject}})$:

- $Q = \{q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}}\}$,
- $\Sigma = \{0\}$, and
- $\Gamma = \{0, \text{x}, \sqcup\}$.
- We describe δ with a state diagram (see Figure 3.8).
- The start, accept, and reject states are q_1, q_{accept}, and q_{reject}.

Examples of Turing Machine (Continuation…)

- **q_1 :**
  - Blank the leftmost 0
  - If starts with a blank, **reject**
  - If starts with X, **reject**
- **q_2 :**
  - Move right so long as X’s are encountered
  - If blank encountered while moving to the right, **accept** (The only accepting condition)
  - 0 replaced with X

Examples of Turing Machine (Continuation…)

- **q_3 :**
  - Skip the 0, this is the next 0 after the last replaced 0 in **q_2**
  - Skip all the X's
  - When a blank is found, we have reached the right end of the string, move left
- **q_4:**
  - Skip all the X's
  - 0 replaced with X, this 0 is after the last skipped 0 in **q_3**
  - A blank here will mean an odd number of 0's, **reject** the string
- **q_5 :**
  - Skip to left all the X's and 0's
  - When a blank is found, we have reached the left end of the string, move right

Examples of Turing Machine (Continuation…)

$q_1 0000$

$\sqcup q_2 000$

$\sqcup \mathrm{x} q_3 00$

$\sqcup \mathrm{x} 0 q_4 0$

$\sqcup \mathrm{x} 0 \mathrm{x} q_3 \sqcup$

$\sqcup \mathrm{x} 0 q_5 \mathrm{x} \sqcup$

$\sqcup \mathrm{x} q_5 0 \mathrm{x} \sqcup$

$\sqcup q_5 \mathrm{x} 0 \mathrm{x} \sqcup$

$q_5 \sqcup \mathrm{x} 0 \mathrm{x} \sqcup$

$\sqcup q_2 \mathrm{x} 0 \mathrm{x} \sqcup$

$\sqcup \mathrm{x} q_2 0 \mathrm{x} \sqcup$

$\sqcup \mathrm{xx} q_3 \mathrm{x} \sqcup$

$\sqcup \mathrm{xxx} q_3 \sqcup$

$\sqcup \mathrm{xx} q_5 \mathrm{x} \sqcup$

$\sqcup \mathrm{x} q_5 \mathrm{xx} \sqcup$

$\sqcup q_5 \mathrm{xxx} \sqcup$

$q_5 \sqcup \mathrm{xxx} \sqcup$

$\sqcup q_2 \mathrm{xxx} \sqcup$

$\sqcup \mathrm{x} q_2 \mathrm{xx} \sqcup$

$\sqcup \mathrm{xx} q_2 \mathrm{x} \sqcup$

$\sqcup \mathrm{xxx} q_2 \sqcup$

$\sqcup \mathrm{xxx} \sqcup q_{\mathrm{accept}}$

Examples of Turing Machine (Continuation…)

- **Example 3.9**: Turing Machine $\mathbf{M_1}$

$$B = \{w\#w \mid w \in \{0,1\}^*\}$$

Examples of Turing Machine (Continuation…)

- Formal definition of TM $\mathbf{M_1}$

$M_1 = (Q, \Sigma, \Gamma, \delta, q_1, q_{accept}, q_{reject})$

$Q = \{q_1, \ldots, q_{14}, q_{accept}, q_{reject}\}$,

$\Sigma = \{0,1,\#\}$, and $\Gamma = \{0,1,\#,x,\sqcup\}$.

We describe δ with a state diagram (see the following figure).

The start, accept, and reject states are q_1, q_{accept}, and q_{reject}.

Examples of Turing Machine (Continuation…)

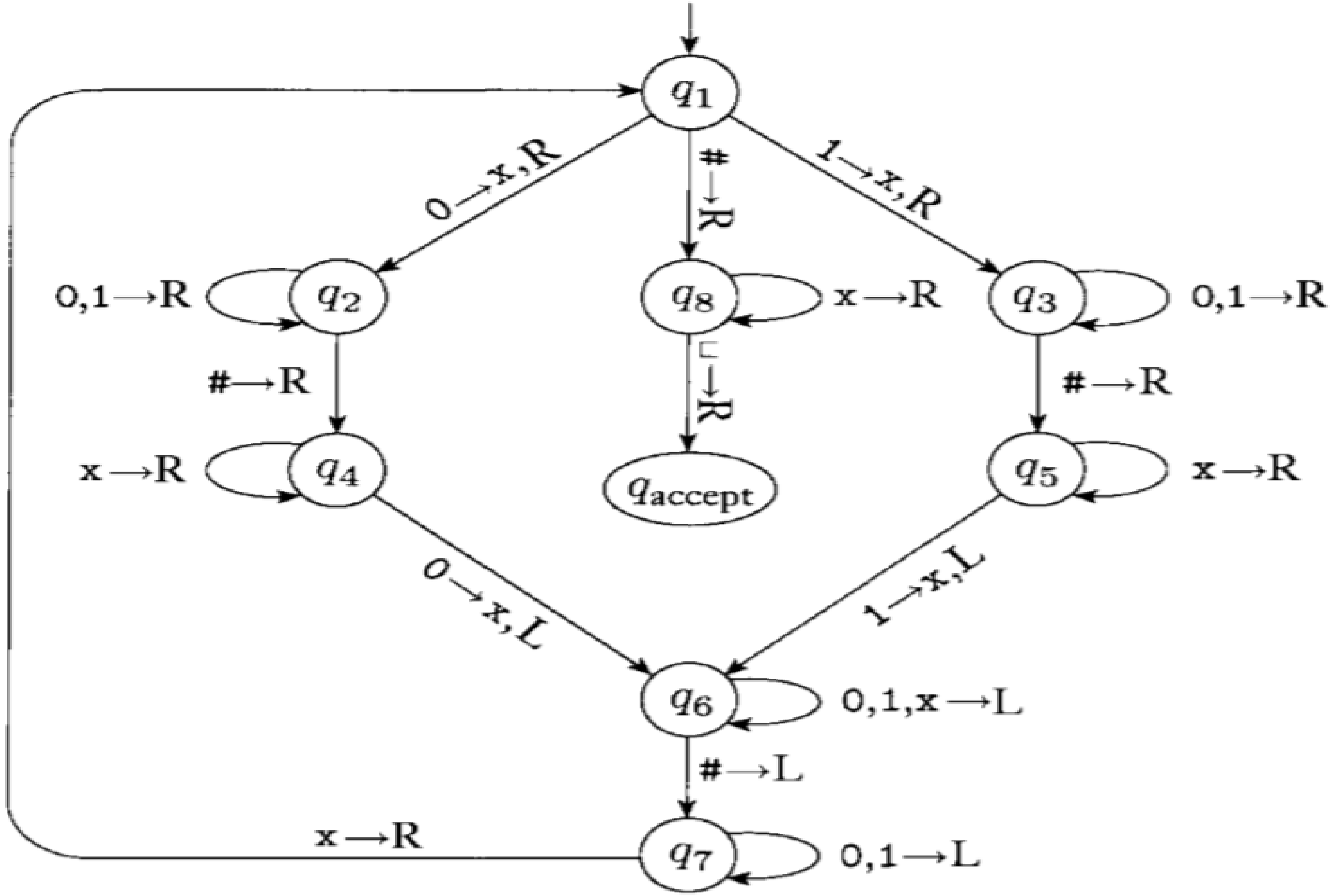


FIGURE 3.10
State diagram for Turing machine M_1

Examples of Turing Machine (Continuation…)

- **Example 3.11**: Turing Machine $\mathbf{M_3}$

$$C = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i \times j = k \text{ and } i, j, k \geq 1\}$$

Examples of Turing Machine (Continuation…)

$M_3 =$ "On input string w:

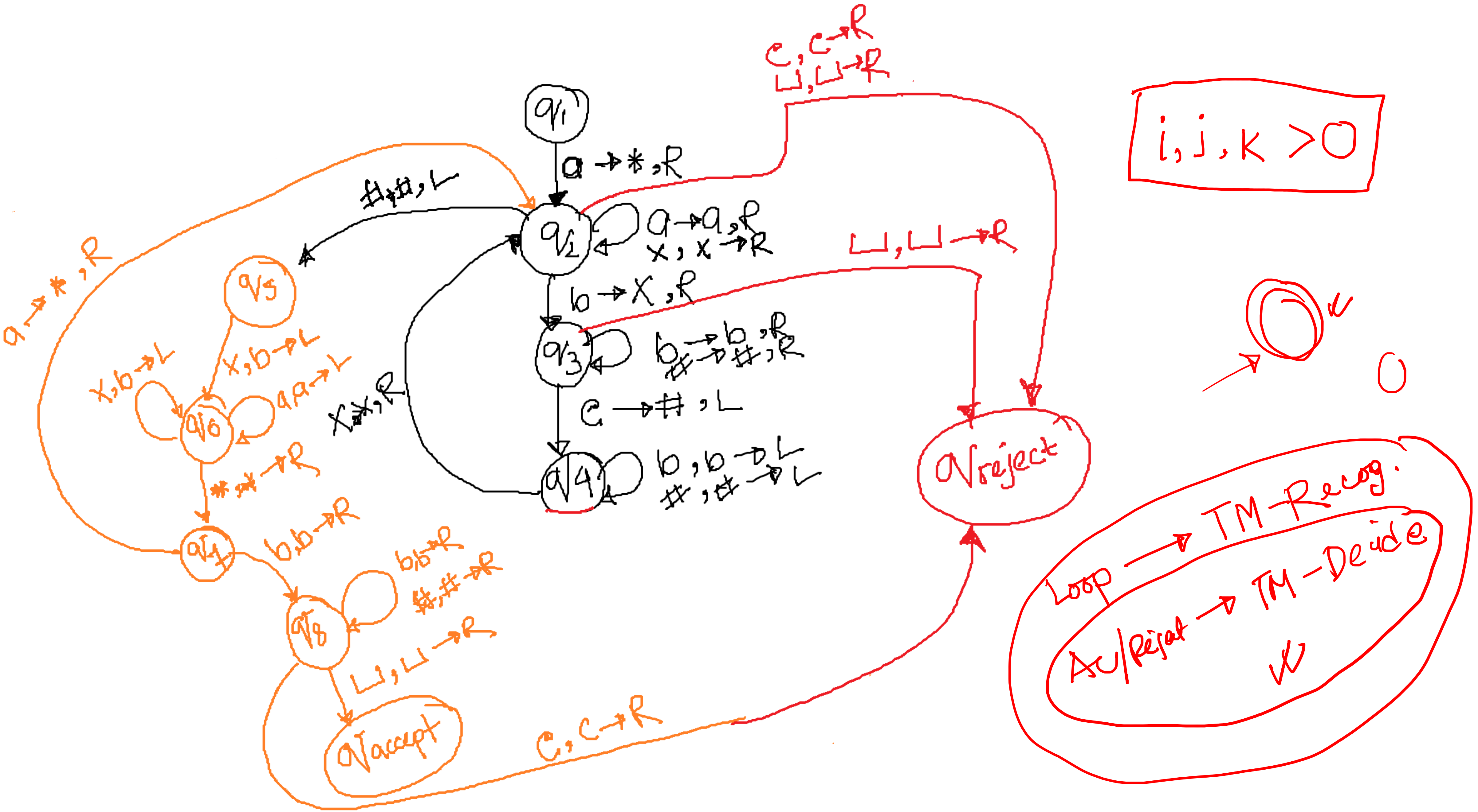
1. Scan the input from left to right to determine whether it is a member of $a^+b^+c^+$ and *reject* if it isn't.
2. Return the head to the left-hand end of the tape.
3. Cross off an a and scan to the right until a b occurs. Shuttle between the b's and the c's, crossing off one of each until all b's are gone. If all c's have been crossed off and some b's remain, *reject*.
4. Restore the crossed off b's and repeat stage 3 if there is another a to cross off. If all a's have been crossed off, determine whether all c's also have been crossed off. If yes, *accept*; otherwise, *reject*."

Examples of Turing Machine (Continuation…)

- Modification of $\mathbf{M_3}$ algorithm:
  - Mark left hand end of the input by $\coprod$ symbol
  - Use three different symbols for marking a, b and c on the tape
    - Use of ***** symbol for marking **a**
    - Use of **X** symbol for marking **b**
    - Use of **#** symbol for marking **c**

Examples of Turing Machine (Continuation…)

Fig: State Diagram for TM M3 (modified)


q1
a → *, R
q2
a → a, R
x, x → R
#, #, L
b → X, R
q3
b → b, R
→ #, R
c → #, L
q4
b, b → L
#, # → L
X, X, R
q5
X, b → L
X, b → L
q6
a, a → L
*, * → R
q7
b, b → R
q8
b, b → R
#, # → R
⊔, ⊔ → R
q accept
c, c → R
a → *, R
q reject
c, c → R
⊔, ⊔ → R
⊔, ⊔ → R
i, j, k > 0
Loop → TM-Recog.
Acc/Reject → TM-Decide

Examples of Turing Machine (Continuation…)

EXAMPLE 3.12

Here, a TM M_4 is solving what is called the *element distinctness problem*. It is given a list of strings over $\{0,1\}$ separated by #s and its job is to accept if all the strings are different. The language is

$$E = \{\#x_1\#x_2\#\cdots\#x_l \mid \text{each } x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for each } i \neq j\}.$$

Machine M_4 works by comparing x_1 with x_2 through x_l, then by comparing x_2 with x_3 through x_l, and so on. An informal description of the TM M_4 deciding this language follows.

Examples of Turing Machine (Continuation…)

M_4 = "On input w:

1. Place a mark on top of the leftmost tape symbol. If that symbol was a blank, *accept*. If that symbol was a #, continue with the next stage. Otherwise, *reject*.
2. Scan right to the next # and place a second mark on top of it. If no # is encountered before a blank symbol, only x_1 was present, so *accept*.
3. By zig-zagging, compare the two strings to the right of the marked #s. If they are equal, *reject*.
4. Move the rightmost of the two marks to the next # symbol to the right. If no # symbol is encountered before a blank symbol, move the leftmost mark to the next # to its right and the rightmost mark to the # after that. This time, if no # is available for the rightmost mark, all the strings have been compared, so *accept*.
5. Go to Stage 3."