Sept. 30th, 2024

Session 10: Interpreting Linear Optimization

BUAD 311 Operations Management, Fall 2024

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Last Time: Solving Linear Optimization

- How to solve linear and integer optimization problems
  - Excel solver

Today: Sensitivity Analysis

- How do optimal objective value and decisions variables change if we have
  1. Changes in constraints
  2. Changes in objective function

$$\begin{aligned} \max \quad & 55\mathrm{M} + 89\mathrm{H} \\ \text{s.t.} \quad & 4\mathrm{M} + 18\mathrm{H} \leq 1296 \\ & 12\mathrm{M} + 6\mathrm{H} \leq 1824 \\ & \mathrm{M} \geq 0, \mathrm{H} \geq 0 \end{aligned}$$

- Sensitivity Analysis tests the ***robustness*** of the decision
- Today's exercises are again available in an in-class activity workbook

Microbrewers Problem

- Microbrewers Incorporated makes four beers:
  - Light, Dark, Ale, and Premium
- Main ingredients: malt, hops, and yeast
- Here is how much of each ingredient is required in the production of 1 gallon of each product, the amounts of each resource available, and the revenue from 1 gallon of each product:

	Light	Dark	Ale	Premium	Available
Malt	1	1	0	3	50 lb
Hops	2	1	2	1	150 lb
Yeast	1	1	1	4	80 lb
Revenue	$6	$5	$3	$7	

obj. function:

max. revenue

6L + 5D + 3A + 7P

objective coeff.

Decision variables:

L = # light beers

D = # dark

A = # Ale

P = # prem.

- You would like to optimize their product mix (how many to make of each type)

optimal obj. function:

$6(40) + 5(10) + 3(30) + 7(0) = \$380$

Microbrewers Problem - Solving with Solver

- Solve with Solver → Answer report, Sensitivity report
  - What is optimal solution?

    $L = 40,\ D = 10,\ A = 30,\ P = 0$
  - What is optimal objective function value?

    value of obj. funct. when you plug in optimal solution
  - What are the binding constraints?

    do the same for yeast

    malt → $L + D + 3P = 40 + 10 + 3(0) = 50$ = RHS of malt const.

    ⇒ it is binding

    hops → $2L + D + 2A + P = 2(40) + 10 + 2(30) + 0 = 150$ = RHS of hops constraint ⇒ it is binding
  - How would you describe the solution in plain terms?

LHS — constraints: — RHS of const.

malt: $L + D + 3P \leq 50$

hops: $2L + D + 2A + P \leq 150$

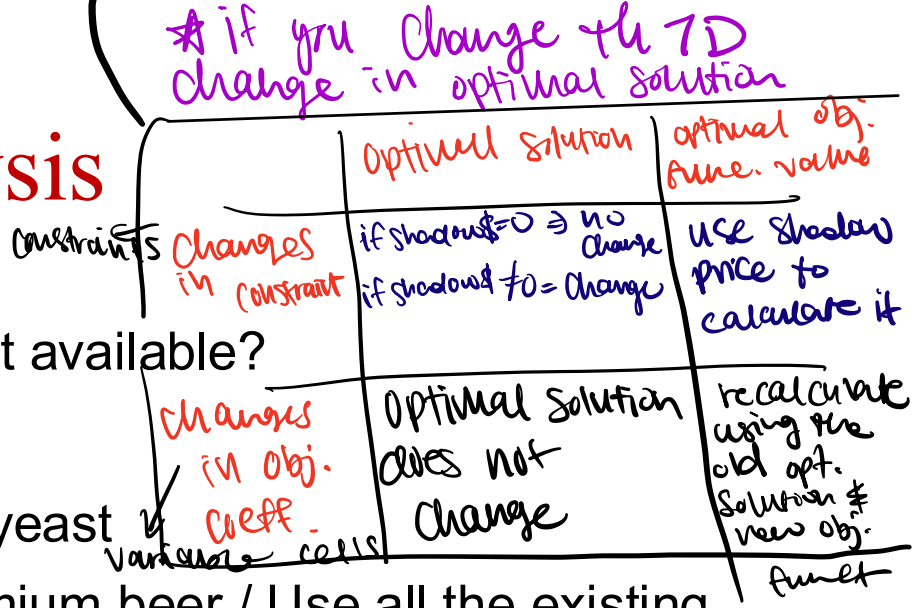
yeast: $L + D + 4P \leq 80$

$L \geq 0,\ D \geq 0,\ A \geq 0,\ P \geq 0$

refer to slide #13 for chart details

Need for Sensitivity Analysis

- Inputs to optimization frequently estimated
  - Are there EXACTLY 80 pounds of yeast available?
- Re-solve assuming there is now 85 lbs of yeast
  - Same constraints are binding / No premium beer / Use all the existing resources



➢This is typical of small perturbations of the data: Insights drawn from the binding constraints remain true even if the data is (a little) corrupt

Sensitivity Analysis: Changes in Available Resources (1)

- Could we have determined how much additional revenue the additional 5 lbs of yeast would generate ***without*** re-solving?

- To answer this question, you need to know the **shadow price** and **allowable increase** of the constraint for yeast.
  - Key idea: for “small” changes, the binding constraints stay the same

Sensitivity Analysis: Changes in Available Resources (2)

- **Shadow price**: added value to our objective for each unit increase in the resource

- **Allowable Increase/Decrease**: the amount the resource could be increased/decreased while the shadow price is still valid

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C15	Malt Used	50	3	50	30	40
C16	Hops Used	150	1	150	10	40
C17	Yeast Used	80	1	80	20	5

Sensitivity Analysis: Changes in Available Resources (3)

- Could we have determined how much additional revenue the additional 5 lbs of yeast would generate ***without*** re-solving?

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C15	Malt Used	50	3	50	30	40
C16	Hops Used	150	1	150	10	40
C17	Yeast Used	80	1	80	20	5

Sensitivity Analysis: Changes in Available Resources (4)

- Could we have determined how much additional revenue the additional 5 lbs of yeast would generate ***without*** re-solving?

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C15	Malt Used	50	3	50	30	40
C16	Hops Used	150	1	150	10	40
C17	Yeast Used	80	1	80	20	5

Sensitivity Analysis: Changes in Objective Function (0)

$$
\begin{aligned}
\text{max} \quad & 6L + 5D + 3A + 7P \\
\text{s.t.} \quad & 1L + 1D + 3P \leq 50 \\
& 2L + 1D + 2A + 1P \leq 150 \\
& 1L + 1D + 1A + 4P \leq 80 \\
& L, D, A, P \geq 0
\end{aligned}
$$

Sensitivity Analysis: Changes in Objective Function (1)

Marketing suggests we can increase the price of ale from $3 to $4 without affecting sales

- Now what is the optimal solution?
- Do we need to use Excel to re-solve?

Sensitivity Analysis: Changes in Objective Function (2)

go to slide #6

* Constraint that is non-binding ⇒ shadow price is 0.

optimal solution

current obj. coeff

range of change in which the table can still use

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C9	Decision var L	40	0	6	0.5	1
D9	Decision var D	10	0	5	1	0.5
E9	Decision var A	30	0	3 → 4	3	1
F9	Decision var P	0	-7	7 → 8 → 7 → 14	7	1E+30

= ∞

amt w/ which costs need to go down to produce this in optimal solution

range of changes for which we can use this table

new opt. obj. funct. 380 + 5 = $385

What if we had 85 lbs of yeast?
- New opt. obj. function value?

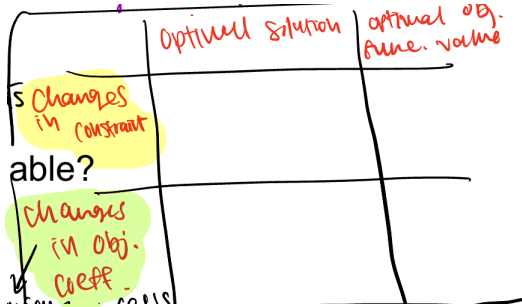
1.) 85 − 80 = 5 < 20 (within allowable inc) ⇒ will shadow price of yeast ⇒ 1 × 5 = $5 additional

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C15	Malt Used	50	3	50	30	40
C16	Hops Used	150	1	150	10	40
C17	Yeast Used	80	1	80	20	5

LHS of our constraints evaluated at the optimal solution

RHS of our constraints



per unit value of this resource

Sensitivity Analysis: Changes in Objective Function (3)

- **Reduced cost**: how much the coefficient would have to change before a zero-valued decision variable becomes non-zero

- **Allowable Increase/Decrease**: the amount the coefficient could be increased/decreased while the optimal solution does not change

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C9	Decision var L	40	0	6	0.5	1
D9	Decision var D	10	0	5	1	0.5
E9	Decision var A	30	0	3	3	1
F9	Decision var P	0	-7	7	7	1E+30

Sensitivity Analysis: Summary

- How much will the optimal value change? Two different types of changes:
  - **Changes in constraints**
    - **Shadow price**: change to optimal value for each unit increase in the resource
    - **Allowable increase/decrease**: amount the resource could be increased/decreased while the shadow price is still valid. (Note: The optimal solution may change.)
  - **Changes in objective function coefficients**
    - **Allowable increase/decrease**: how much the objective coefficient could change so there would not be a change in the optimal solution

Sensitivity Analysis: Discussion

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C9	Decision var L	40	0	6	0.5	1
D9	Decision var D	10	0	5	1	0.5
E9	Decision var A	30	0	3	3	1
F9	Decision var P	0	-7	7	7	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C15	Malt Used	50	3	50	30	40
C16	Hops Used	150	1	150	10	40
C17	Yeast Used	80	1	80	20	5

You can answer a lot of business questions through sensitivity analysis and binding constraints:

1. An alternate supplier wants to sell you additional 20 lbs of malt. How much would you be willing to pay?

2. What would the price of Premium beer need to be for you to consider making it? Can you estimate?

3. Sales argues they'd sell more Premium beer if they drop the price. Do you agree?

Break

Example 2

- Consider the following linear optimization problem:

$$\begin{aligned}
&\text{Maximize} && 3x_1 + 4x_2 + 6x_3 + 10x_4 \\
&\text{Subject to:} && x_1 + x_2 + 3x_4 \le 120 \\
& && x_1 + x_2 + 3x_3 + x_4 \le 150 \\
& && x_1 + 2x_2 + x_3 + 2x_4 \le 100 \\
& && x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0
\end{aligned}$$

Example 2 (2)

- After solving the problem using Excel solver, the following sensitivity repost was generated:

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C3	x1	0	-2.2	3	2.2	1E+30
C4	x2	0	-6	4	6	1E+30
C5	x3	40	0	6	24	1
C6	x4	30	0	10	2	5.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C11	1st	90	0	120	1E+30	30
C12	2nd	150	0.4	150	150	50
C13	3rd	100	4.8	100	16.66666667	50

Example 2 (3)

a) What is the optimal solution?

b) What is the optimal objective function value?

c) Suppose the first constraint instead was: $x_1 + x_2 + 3x_4 \leq 100$. What is the new optimal objective function value?

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C3	x1	0	-2.2	3	2.2	1E+30
C4	x2	0	-6	4	6	1E+30
C5	x3	40	0	6	24	1
C6	x4	30	0	10	2	5.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C11	1st	90	0	120	1E+30	30
C12	2nd	150	0.4	150	150	50
C13	3rd	100	4.8	100	16.66666667	50

Example 2 (4)

d) Suppose the second constraint instead was: $x_1 + x_2 + 3x_3 + x_4 \leq 190$. What is the new optimal objective function value?

e) Suppose the coefficient multiplying x_3 in the objective function was 10 instead of 6. What is the new optimal objective function value?

f) Suppose the coefficient multiplying x_4 in the objective function was 13 instead of 10. What is the new optimal objective function value?

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C3	x1	0	-2.2	3	2.2	1E+30
C4	x2	0	-6	4	6	1E+30
C5	x3	40	0	6	24	1
C6	x4	30	0	10	2	5.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C11	1st	90	0	120	1E+30	30
C12	2nd	150	0.4	150	150	50
C13	3rd	100	4.8	100	16.66666667	50

Recap: What did we do today?

Wrap-Up

in class ws:

d) 100 is outside your allowable increase range from 20 of yeast so no need to resolve.

f) 4 − 3 = 1 ≤ 3 (allowable inc. range) ⇒ optimal solution does not change

→ L=40, D=10, A=30, P=0

g) 20 ≤ 30 (within allowable inverse range) ⇒ shadow price

20 × 3 = $60

h) only produce when past the allowable increase range, if within range the optimal price solution doesn't change, therefore will change if P ↑ by more than 7 we may consider making it but not before. ⇒ 7+7=14

i) allowable dec. is ∞ ⇒ reducing prices will not change the optimal solution & P=D will still be at the case for the optimal solution