UNIT-2

Fourier Series

Euler's formula.

$f(x) = f(x+2l) = \cdots$ is a periodic function in $[c, c+2l]$

Fourier series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum \left\{ a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x \right\}$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$.

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi}{l}x\,dx.$$

1. Expand following functions into Fourier series in the given intervals.

(i) $f(x) = e^{-x}$ in $0 \le x \le 2\pi$.

$$f(x) = a_0 + \sum (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$.

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx.$$

Notes

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx.$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi} \left. \frac{e^{-x}}{-1} \right|_0^{2\pi} = \frac{1 - e^{-2\pi}}{2\pi}.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx\,dx = \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\cos nx + n \sin nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)} \left[e^{-2\pi} (-\cos 2n\pi + n \sin 2n\pi) - (-\cos 0 + 0) \right]$$

$$a_n = \frac{1}{\pi}\left[e^{-2\pi}(-1)+1\right]\frac{1}{1+n^2}$$

$$= \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx \, dx.$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}(-\sin 2n\pi - n\cos 2n\pi)-(-0-n)}{1+n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}(-n)+n}{1+n^2}\right] = \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}.$$

$$= \cancel{n(e^{-2\pi}} \quad f(x) = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{4}+\sum\frac{1}{1+n^2}(\cos nx + n\sin nx)\right]$$

(ii) $f(x) = x - x^2$ in $[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \cancel{\frac{(-\pi)^2}{2}} + \frac{(-\pi)^3}{3} \right] = \frac{-2\pi^3}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx \, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[\cancel{(x-x^2)\frac{\sin nx}{n}}^{\,0} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (1-2x)\frac{\sin nx}{n}\,dx \right]$$

$$= \frac{1}{\pi}\left[(1-2x)\frac{\cos nx}{n^2} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (-2)\frac{\cos nx}{n^2}\,dx \right]$$

$$= \frac{1}{\pi}\left[\frac{(1-2\pi)\cos n\pi}{n^2} - (1+2\pi)\frac{\cos n\pi}{n^2} + 2/n^2\, \cancel{\frac{\sin nx}{n}}^{\,0} \Big|_{-\pi}^{\pi} \right]$$

$$= \frac{1}{\pi n^2}\left[\cancel{\cos n\pi} - 2\pi\cos n\pi - \cancel{\cos n\pi} - 2\pi\cos n\pi \right]$$

$$= -\frac{4}{n^2}(-1)^n.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[-(x-x^2)\frac{\cos nx}{n}\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1-2x)\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\Bigg[-\frac{(\pi-\pi^2)}{n}\cos nx + \left(\frac{-\pi-\pi^2}{n}\right)\cos nx.$$

$$+\ \frac{1}{n}\left\{\cancel{(1-2x)\overset{0}{\frac{\sin nx}{n^2}}}\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(-2)\frac{\sin nx}{n^2}\,dx\right\}\Bigg]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos nx + \frac{2}{n^2}(-\cos nx)\Bigg|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos nx - \frac{2}{n^2}\left(\cancel{\cos n\pi} - \cancel{\cos n\pi}\right)\right]$$

$$= -\frac{2}{n}(-1)^n.$$

$$f(x) = -\frac{\pi^2}{3} + -\sum \frac{4}{n}(-1)^n\cos nx + \frac{2}{n}(-1)^n \sin nx.$$

iii) $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx.$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} (-\pi)\, dx + \int_{0}^{\pi} x\, dx \right]$$

$$= -\pi x \Big|_{-\pi}^{0} + \frac{x^2}{2} \Big|_{0}^{\pi} = -\pi^2 + \pi^2/2. = \frac{-\cancel{3}\pi^2}{2}.$$

2024

శనివారము/SATURDAY

March 9

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \cdot = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx + \int^{\pi} x\cos nx\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{-\pi\,\cancel{\sin nx}^{\,0}}{n}\Big|_{-\pi}^{0} + \frac{x\sin nx}{n}\Big|_{0}^{\pi\ 0} - \int_{0}^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\sin n\pi}{n} - 0 + \frac{\cos nx}{n^2}\Big|_{0}^{\pi}\right]$$

ఆదివారము/SUNDAY

10

$$= \frac{1}{\pi}\left[\frac{\cos n\pi - 1}{n^2}\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi\frac{\cos nx}{n}\Big|_{-\pi}^{0} - \frac{x\cos nx}{n}\Big|_{0}^{\pi} - \cancel{\int_{0}^{\pi}\frac{\cos nx}{n^2}\,dx}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi\cos n\pi}{n}\right] = \frac{1}{n}\cdot(1-2\cos n\pi)$$

$$f(x) = \frac{-\pi^2}{4} + \frac{1}{\pi}\sum \frac{(-1)^n - 1}{n^2}\cos nx$$

$$+ 2\sum \frac{1-2(-1)^n}{n}\sin nx.$$

(iv) $$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\pi/2 \\ 0 & -\pi/2 < x < \pi/2 \\ 1 & \pi/2 < t < \pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum\left(a_n \cos nx + b_n \sin nx\right)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)\,dx + 0\int_{\pi/2}^{\pi} 1\,dx\right]$$

$$a_0 = \frac{1}{\pi}\left[-x \Big|_{-\pi}^{-\pi/2} + x \Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\pi/2 - \pi + \pi - \pi/2 \right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos nx \, dx + 0 + \int_{\pi/2}^{\pi} \cos nx \, dx \right]$$

$$= \frac{1}{\pi}\left[-\frac{\sin nx}{n} \Big|_{-\pi}^{-\pi/2} + \frac{\sin nx}{n} \Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi n}\left[+\sin n\pi/2 - \sin n\pi + \sin n\pi - \sin n\pi/2 \right]$$

$$= 0$$

$$a_1 = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos x \, dx + 0 + \int_{\pi/2}^{\pi} \cos x \, dx \right]$$

March
15

2024

శుక్రవారము/FRIDAY

$$a_1 = \frac{1}{\pi}\left[\sin x\Big|_{-\pi}^{-\pi/2} + \sin x\Big|_{\pi/2}^{\pi}\right] = \frac{1}{\pi}[-1-1] = \frac{-2}{\pi}$$

$$= \frac{1}{\pi}\cancel{\left[-\cos x\Big|_{-\pi}^{-\pi/2} - \cos x\Big|_{\pi/2}^{\pi}\right]}$$

$$= \frac{1}{\pi}\left[\cancel{-\cos\pi/2 + \cos\pi - \cos\pi + \cos\pi/2}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} \cancel{f(x)} \sin nx\, dx$$

16

శనివారము/SATURDAY

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\sin nx\, dx + \int_{\pi/2}^{\pi} \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\Big|_{-\pi}^{-\pi/2} - \frac{\cos nx}{n}\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi/2}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{\cos n\pi/2}{n}\right]$$

$$= \frac{2}{n\pi}\left[\cos n\pi/2 - (-1)^n\right]$$

$$f(x) = -\frac{2}{\pi}\cos x + \frac{2}{\pi}\sum\left\{\cos n\pi/2 - (-1)^n\right\}\sin nx$$

(V) $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2. \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

$$a_0 = \frac{1}{l}\int^{c+2l} f(x)\,dx \qquad l = 1$$

$$a_n = \frac{1}{l}\int_{l}^{c+2l} f(x)\cos\frac{n\pi}{l}x\,dx.$$

$$b_n = \frac{1}{l}\int_{l}^{c+2l} f(x)\sin\frac{n\pi}{l}x\,dx$$

$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\, x^2/2\Big|_0^1 + \pi\left(2x - x^2/2\right)\Big|_1^2$$

$$a_0 = \pi/2 + \pi\left[4 - 2 - 2 + 1/2\right] = \pi$$

$$a_n = \int_0^1 \pi x \cos n\pi x \, dx + \int_1^2 \pi(2-x) \cos n\pi x \, dx$$

$$= \pi x \cancel{\frac{\sin n\pi x}{n\pi}}^{\,0}\Bigg|_0^1 - \int_0^1 \cancel{\pi} \frac{\sin n\pi x}{n\cancel{\pi}} \, dx$$

$$+ \pi(2-x) \cancel{\frac{\sin n\pi x}{n\pi}}^{\,0}\Bigg|_1^2 - \int_0^1 \cancel{\pi}(-1) \frac{\sin n\pi x}{n\cancel{\pi}} \, dx.$$

$$= \frac{\cos n\pi x}{n^2\pi}\Bigg|_0^1 - \frac{\cos n\pi x}{n^2\pi}\Bigg|_1^2$$

$$= \frac{\cos n\pi - 1}{n^2\pi} - \frac{\cos 2n\pi - \cos n\pi}{n^2\pi}$$

$$= \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\}$$

2024

గురువారము/THURSDAY

March

21

$$b_n = \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x)\sin n\pi x \, dx .$$

$$= \cancel{\pi} x\left(-\frac{\cos n\pi x}{n\cancel{\pi}}\right)\Big|_0^1 + \int_0^1 \cancel{\pi}\cdot\left(-\frac{\cos n\pi x}{n\cancel{\pi}}\right)dx$$

$$+ \cancel{\pi}(2-x)\left(-\frac{\cos n\pi x}{n\cancel{\pi}}\right)\Big|_1^2 - \int_1^2 \cancel{\pi}(-1)\left(-\frac{\cos n\pi x}{n\cancel{\pi}}\right)dx$$

$$= -\cancel{\frac{(-1)^n}{n}} + \frac{\sin n\pi x}{n^2\pi}\Big|_0^1 + \cancel{\frac{(-1)^n}{n}} - \frac{\sin n\pi x}{n^2\pi}\Big|_1^2$$

22

శుక్రవారము/FRIDAY

$$= 0.$$

$$= \frac{(-1)^n}{n}.$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum \frac{(-1)^n - 1}{n^2}\cos n\pi x .$$

iv) $f(t) = \begin{cases} 0 & -2 < t < -1 \\ 1+t & -1 < t < 0 \\ 1-t & 0 < t < 1 \\ 0 & 1 < t < 2. \end{cases}$ $l = 2$.

$$f(x) = \frac{a_0}{2} + \sum \left(a_n \cos \frac{n\pi}{2} t + b_n \sin \frac{n\pi}{2} t \right)$$

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(t)\,dt \quad , \quad a_n = \frac{1}{2} \int_{-2}^{2} f(t) \cos \frac{n\pi}{2} t \,dt$$

$$b_n = \frac{1}{2} \int_{-2}^{2} f(t) \sin \frac{n\pi}{2} t \,dt.$$

$$a_0 = \frac{1}{2} \left[\int_{-1}^{0} (1+t)\,dt + \int_{0}^{1} (1-t)\,dt \right]$$

$$= \frac{1}{2} \left[t + \frac{t^2}{2} \Big|_{-1}^{0} + t - \frac{t^2}{2} \Big|_{0}^{1} \right]$$

$$= \frac{1}{2} \left[0 - (-1 + \tfrac{1}{2}) + (1 - \tfrac{1}{2}) - 0 \right]$$

$$= +\frac{1}{2}.$$

$$a_n = \frac{1}{2}\left[\int_{-1}^{0}(1+t)\cos\frac{n\pi}{2}t\,dt + \int_{0}^{1}(1-t)\cos\frac{n\pi}{2}t\,dt\right]$$

$$= \frac{1}{2}\left[\cancel{(1+t)\frac{\sin\frac{n\pi}{2}t}{n\pi/2}}\Bigg|_{-1}^{0} - \int_{-1}^{0} 1\cdot\frac{\sin n\pi/2\,t}{n\pi/2}\,dt\right.$$

$$\left. + \cancel{(1-t)\frac{\sin n\pi/2\,t}{n\pi/2}}\Bigg|_{0}^{1} - \int_{0}^{1}(-1)\frac{\sin n\pi/2\,t}{n\pi/2}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{2}{n\pi}\frac{\cos n\pi/2\,t}{n\pi/2}\Bigg|_{-1}^{0} - \frac{2}{n\pi}\frac{\cos n\pi/2\,t}{n\pi/2}\Bigg|_{0}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left(\cos n\pi/2 - 1\right)\right]$$

$$= \frac{2}{n^2\pi^2}\left[1-\cos n\pi/2 - \cos n\pi/2 + 1\right]$$

$$= \frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{2}\left[\int_{-1}^{0}(1+t)\sin\frac{n\pi}{2}t\,dt + \int_{0}^{1}(1-t)\sin\frac{n\pi}{2}t\,dt\right]$$

$$= \frac{1}{2}\left[(1+t)\left(-\frac{\cos n\pi/2\,t}{n\pi/2}\right)\Bigg|_{-1}^{0} - \int_{-1}^{0}\frac{-\cos n\pi/2\,t}{n\pi/2}\,dt\right.$$

$$\left. + (1-t)\left(-\frac{\cos n\pi/2\,t}{n\pi/2}\right)\Bigg|_{0}^{1} - \int_{0}^{1}(-1)\left(-\frac{\cos n\pi/2\,t}{n\pi/2}\right)dt\right]$$

$$= \frac{1}{2}\left[\frac{2}{n\pi}(-1+0) + \frac{2}{n\pi}\,\frac{\sin n\pi/2\,t}{n\pi/2}\Bigg|_{-1}^{0}\right.$$

$$\left. \frac{2}{n\pi}(0+1) - \frac{2}{n\pi}\,\frac{\sin n\pi/2\,t}{n\pi/2}\Bigg|_{0}^{1}\right]$$

$$= \frac{1}{2}\left[\cancel{-\frac{2}{n\pi}} + \frac{4}{n^2\pi^2}\left(0 + \cancel{\sin\frac{n\pi}{2}}\right) + \cancel{\frac{2}{n\pi}} - \cancel{\frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - 0\right)}\right]$$

$$= \cancel{\frac{4}{n\pi}} \sin \qquad = 0.$$

$$f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum \frac{1-\cos n\pi/2}{n^2}\cos\frac{n\pi}{2}t.$$

Even and odd function

If $f(x)$ is even in $[-l, l]$ then

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi}{l}x \quad , \quad b_n = 0$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}x\,dx$.

If $f(x)$ is odd in $[-l, l]$

$a_0 = a_n = 0$ $\quad f(x) = \sum b_n \sin\frac{n\pi}{l}x$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx.$$

1. Expand $f(x) = x^2$ into fourier series in $(-l, l)$

b $f(x) = x^2$ is even ; $b_n = 0$.

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi}{l}x$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{\cancel{l}}\cdot\frac{x^3}{3}\Big|_0^l$$

$$= \frac{2l^2}{3}.$$

$$a_n = \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi}{l}x\,dx.$$

$$= \frac{2}{l}\left[\cancel{\frac{x^2 \sin n\pi/l\,x}{n\pi/l}}^{\,0}\Big|_0^l - \int_0^l 2x.\frac{\sin n\pi/l\,x}{n\pi/l}\,dx\right]$$

Notes

$$= \frac{2}{l}\left[+\frac{2l}{n\pi}\left\{x\frac{\cos n\pi/l\,x}{n\pi/l}\Big|_0^l - \int_0^l 1.\frac{\cos n\pi/l\,x}{n\pi/l}\,dx\right\}\right]$$

$$= \frac{2}{\cancel{l}}\left[\frac{2l}{n\pi}\left\{\frac{l\cos n\pi}{n\pi/l}\right\} - \cancel{\frac{\sin n\pi/l\,x}{(n\pi/l)^2}}^{\,0}\Big|_0^l\right]$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n.$$

$$f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos\frac{n\pi}{l}x.$$

2) $f(x) = |x|$ in $(-\pi, \pi)$.

$|x|$ is even. $b_n = 0$. $f(x) = \frac{a_0}{2} + \sum a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left.\frac{x^2}{2}\right|_0^{\pi} = \pi$

$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[\left. \cancel{x\frac{\sin nx}{n}}\right|_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx\right]$

$= \frac{2}{\pi}\left[\left.\frac{\cos nx}{n^2}\right|_0^{\pi}\right] = \frac{2}{n^2\pi}\{(-1)^n - 1\}$.

$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum \frac{(-1)^n - 1}{n^2}\cos nx$

3) $f(x) = x\cos x$ in $(-\pi, \pi)$.

$f(x)$ is odd $f^{\underline{n}}$ $a_0 = a_n = 0$.

$f(x) = \sum b_n \sin nx$ $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$.

$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin x\, dx$.

$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\, dx = \frac{1}{\pi}\left[\left.-x\frac{\cos 2x}{2}\right|_0^{\pi} - \int_0^{\pi}\frac{\cos 2x}{2}dx\right]$

$= \frac{1}{\cancel{\pi}}\left[-\cancel{\pi}\cdot\frac{1}{2} - \left.\frac{\sin 2x}{2}\right|_0^{\pi}\right] = -\frac{1}{2}$.

$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\, dx$.

$= \frac{1}{\pi}\int_0^{\pi} x\{\sin(n+1)x + \sin(n-1)x\}dx$.

$$= \frac{1}{\pi}\left[-x\left\{\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}\Big|_0^{\pi}\right.$$

$$\left. - \int_0^{\pi}\left(\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right)dx\right.$$

$$= \frac{1}{\cancel{\pi}}\left[-\cancel{\pi}\left\{\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - 0\right]$$

$$= \frac{2n(-1)^n}{n^2-1}$$

$$f(x) = \frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2-1}\sin nx.$$