Image	Pixel center sampling	4x4 subpixel center sampling	Integration over pixel area	Convolution with two-pixel wide filter
nearly vertical polygon edge	jagged edge	partly anti-aliased edge, with 5 colors	smoothly anti-aliased edge	smoothly anti-aliased edge
vertical infinitely thin line	visible with probability zero	visible with probablity zero	visible in one column	visible in two columns
moving vertical infinitely thin line	visible with probability zero	visible with probability zero	jumps suddenly from one column to the next	fades out from one column, and fades in to the next.
moving nearly vertical infinitely thin line	visible with probability zero	visible with probability zero	corners crawl up or down the line	smoothly anti-aliased edge moves smoothly between columns
nearly vertical line of width 1/10 of a pixel	visible with probability 1/10	visible with probability about $\frac{4}{10}$	visible mostly in one column, with short transition regions in two	smoothly anti-aliased line, usually two pixels wide, and sometimes three
infinitely small dot	visible with probability zero	visible with probability zero	present in one pixel jumps when moving	present in four pixels moves smoothly
object of area $\frac{1}{10}$ of a pixel	visible with probability 1/10	usually visible, but may not be, depending on its shape	object always visible, usually in a single pixel	object always visible, usually in four pixels
moving object of area $\frac{1}{10}$ of a pixel	object flases on in one frame out of 10	object usually visible but may flash off &on, jumps when moving	object usually in a single pixel, so it jumps when moving	present usually in four pixels, and moves smoothly

Fourier Transforms

If $f(x)$ is a function defined for all real x, then its fourier transform, $F(u) = \mathcal{F}(f(x))$ is given by

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i u x}\, dx$$

Since $e^{-2\pi i u x} = \cos(2\pi u x) - i \sin(2\pi u x)$,

$$F(u) = \int_{-\infty}^{\infty} f(x) \cos(2\pi u x)\, dx - i \int_{-\infty}^{\infty} f(x) \sin(2\pi u x)\, dx.$$

The inverse fourier transform $\mathcal{F}^{-1}(F(u)) = f(x)$

is given by

$$f(x) = \int_{-\infty}^{\infty} F(u)\, e^{2\pi i u x}\, du.$$

This gives back the original function $f(x)$, as can be shown formally with double integrals.

The δ-function, or "impulse function" has the property that

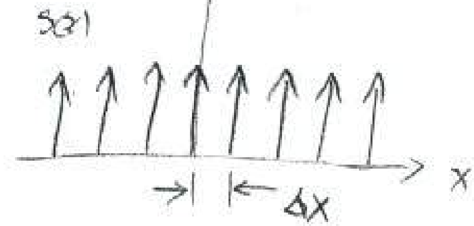
$$\int_{-\infty}^{\infty} f(u)\, \delta(x-u) = f(x).$$

The convolution of two functions $f(x)$ and $g(x)$ is

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du.$$

Facts.

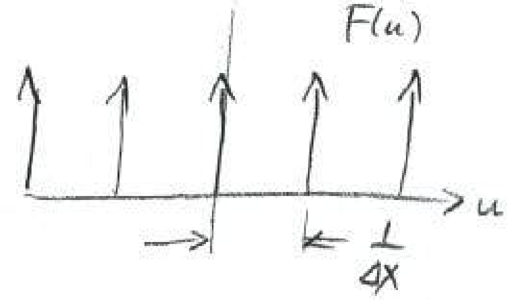
1) Let $s(x) = \sum_{-\infty}^{\infty} \delta(x - j\Delta x)$



be a train of impulse functions, with uniform spacing ΔX.

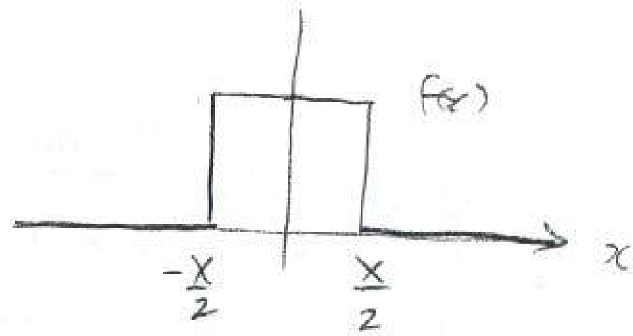
Then its fourier transform is

$$S(u) = \sum_{-\infty}^{\infty} \delta\left(u - \frac{k}{\Delta X}\right),$$



a train of impulse functions with uniform spacing $\frac{1}{\Delta X}$

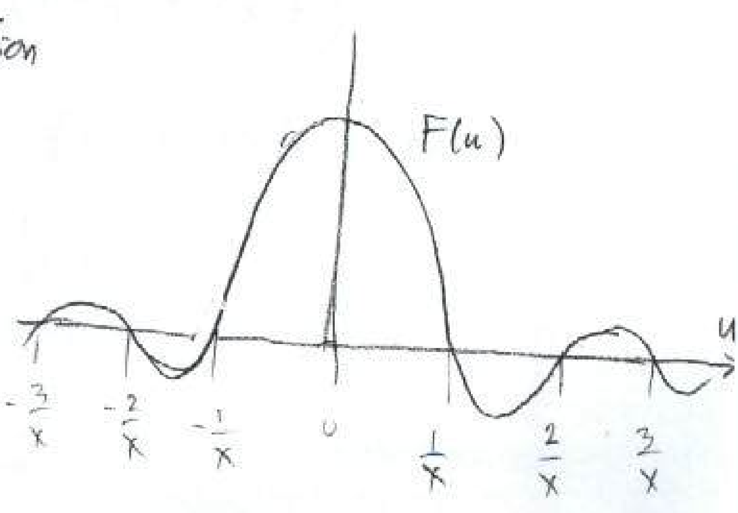
2) Let $f(x) = \begin{cases} 1 & \text{if } -\frac{X}{2} \le x \le \frac{X}{2} \\ 0 & \text{otherwise} \end{cases}$



be a pulse of width X, centered at the origin.

Then the fourier transform $F(u)$ of $f(x)$ is the "sinc" function

$$F(u) = \frac{\sin \pi u X}{\pi u}$$



Calculation

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi u x}$$

this term is zero because sine is an odd function

$$= \int_{-\frac{X}{2}}^{\frac{X}{2}} \cos(2\pi u x)\,dx \quad + i \int_{-\frac{X}{2}}^{\frac{X}{2}} \sin(2\pi u x)\,dx$$

$$= \left. \frac{\sin 2\pi u x}{2\pi u} \right]_{-\frac{X}{2}}^{\frac{X}{2}}$$

$$= \frac{\sin 2\pi u \frac{X}{2} - \sin\left(-2\pi \frac{X}{2}\right)}{2\pi u} = \frac{\sin \pi u X}{\pi u}$$

3)

Convolution Theorem

If $f(x)$ and $g(x)$ are functions, with fourier transforms $F(u)$ and $G(u)$, then

$$\mathcal{F}\left(f(x) * g(x)\right) = F(u)\,G(u) \qquad \text{and}$$

$$\mathcal{F}\left(f(x)\, g(x)\right) = F(u) * G(u)$$

Formal verification, with no attention to convergence

$$\mathcal{F}\left(f(x) * g(x)\right) =$$

$$\mathcal{F}\left(\int_{-\infty}^{\infty} f(v)\, g(x-v)\, dv\right) =$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\, g(x-v)\, dv\right) e^{-2\pi i u x}\, dx =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, g(x-v)\, e^{-2\pi i u\,(v+(x-v))}\, dv\, dx =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{-2\pi i u v}\; g(x-v)\, e^{-2\pi i u (x-v)}\, dv\, dx =$$

$$\int_{-\infty}^{\infty} f(v)\, e^{-2\pi i u v}\, dv \int_{-\infty}^{\infty} g(z)\, e^{-2\pi i u z}\, dz =$$

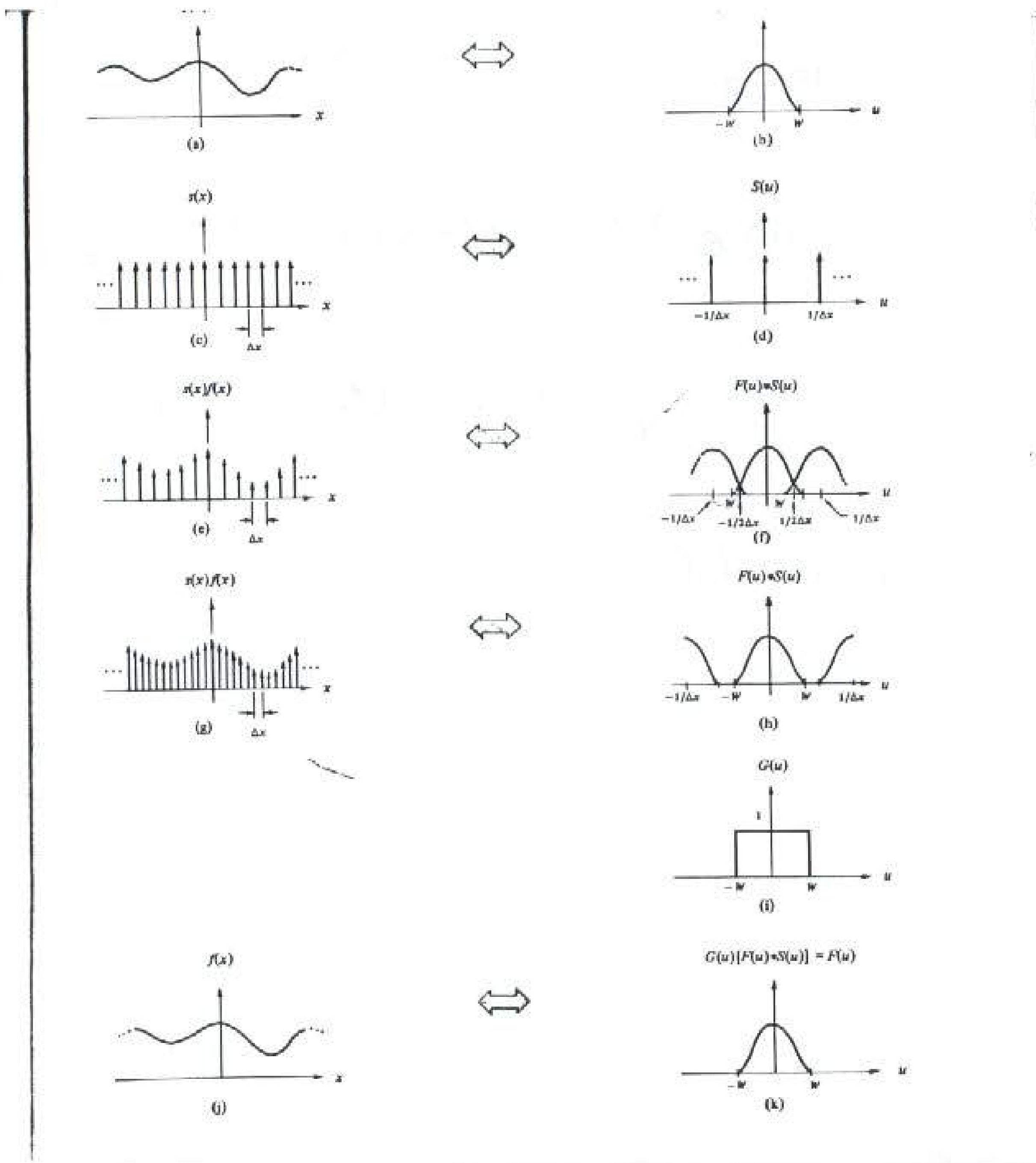
$$F(u)\; G(u)$$

4) Shannon Whittaker Sampling Theorem

If $f(x)$ has no frequency components of frequency greater than W, ie is "band limited" so that

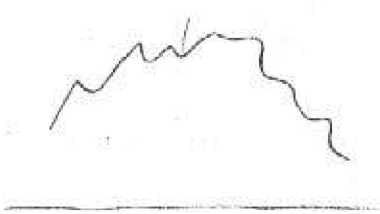
$$F(u) = 0 \quad \text{if} \quad |u| > W$$

then $f(x)$ can be reconstructed from discrete samples at spacing Δx, whenever $\Delta x \le \frac{1}{2W}$

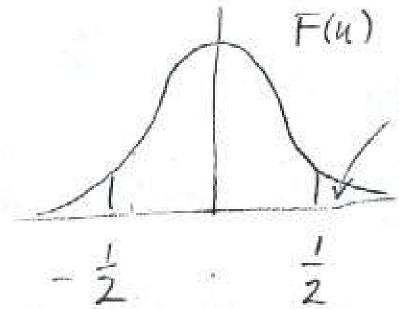


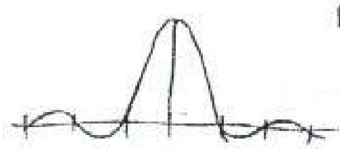
Anti-Aliasing

Suppose pixels are at spacings of one screen unit, and centered at integers. Then the sample spacing Δx is 1, and the sampled picture must contain no frequencies greater than $\frac{1}{2}$, if it is to be reconstructed from the samples. If it does, these frequencies must be removed by a "pre-filter" before sampling.

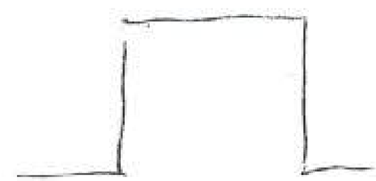


$f(x)$



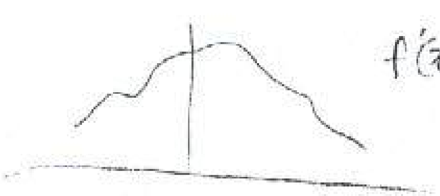


$$r(x) = \frac{\sin \pi x}{\pi x}$$

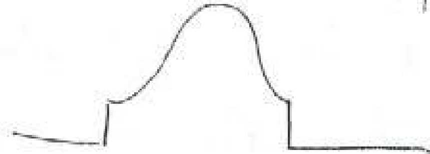


filter

$$R(u) = \begin{cases} 1 & \text{if } |u| < \\ 0 & \text{otherwise} \end{cases}$$



$$f'(x) = r(x) * f(x)$$



product

$R(u)\,F(u)$

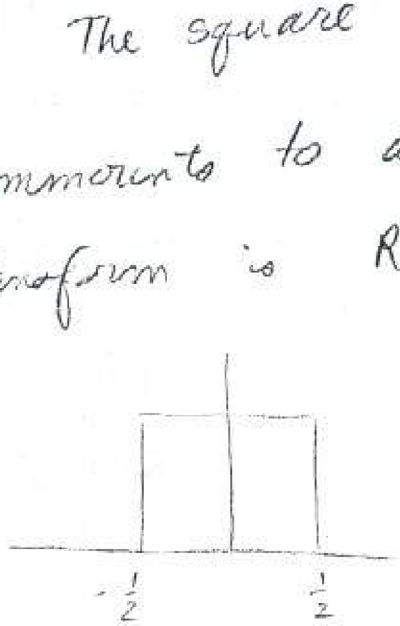
This filtering can be accomplished by taking the convolution of $f(x)$ with a smoothing function $r(x)$, to get a new function $f'(x)$, which now has all frequencies $\leq \frac{1}{2}$. The new function $f'(x)$ can now be reconstructed from its samples, by the use of an appropriate "post filter."

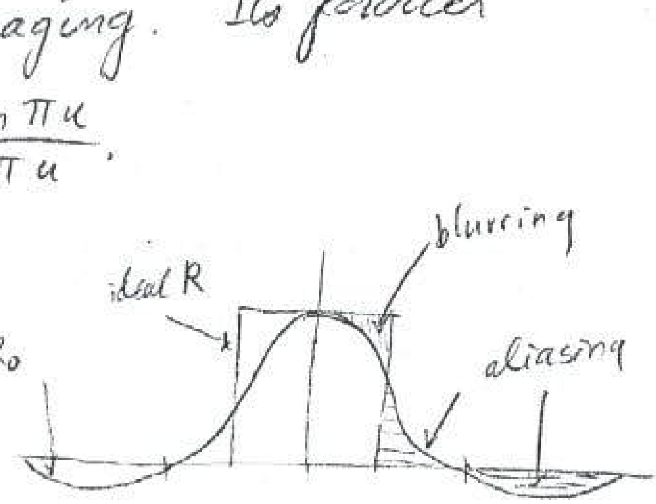
The ideal function $r(x) = \frac{\sin \pi x}{\pi x}$ is difficult to use in practice because it has negative components, and infinte extent.

Practical Pre Filters

1) The square filter $r_0(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

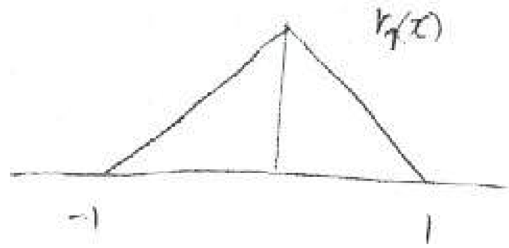
ammounts to area averaging. Its fourier transform is $R_0(u) = \frac{\sin \pi u}{\pi u}$.

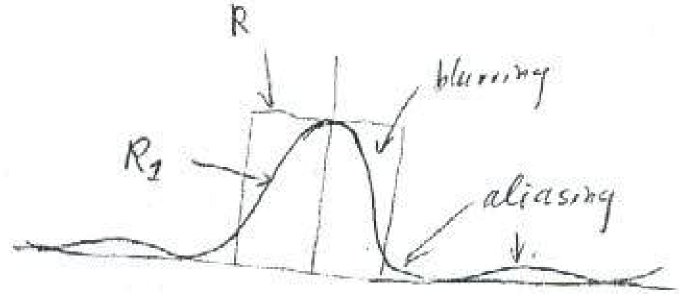




It can be seen that this filter does not give perfect anti-aliasing; frequencies between $\frac{1}{2}$ and 1 still appear, and the frequency 1.5 comes out negative, at power $1/(1.5\pi)$. Also frequecies less than $\frac{1}{2}$ are reproduced with less than full power, causing blurring.

2) The triangular filter $r_1(x) = \begin{cases} 1-|x| & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$



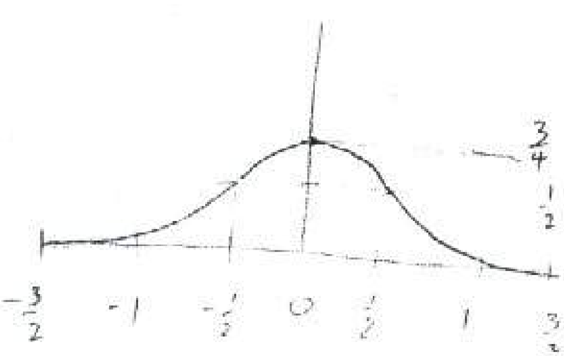


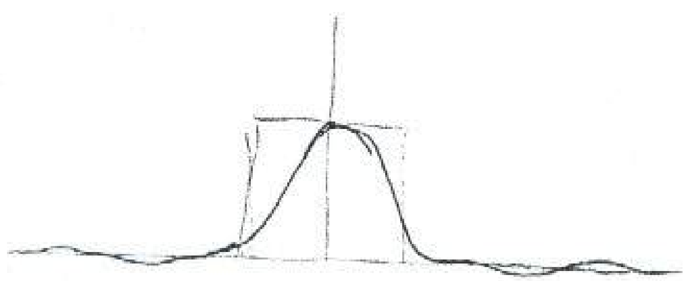
Since $r_1(x) = r_0(x) * r_0(x)$, the convolution of $r_0(x)$ with itself, the transform $R_1(u)$ is the product of $r_0(x)$ with itself, i.e.

$$R_1(u) = R_0(u)^2 = \left(\frac{\sin(\pi u)}{\pi u}\right)^2$$

This gives greater blurring, but also less aliasing.

3) A filter made up of pieces of parabolas





$$r_2(x) = r_0(x) * r_1(x) = \begin{cases} \frac{1}{2}\left(x+\frac{3}{2}\right)^2 & \text{if } -\frac{3}{2} \le x \le -\frac{1}{2} \\ \frac{3}{4} - x^2 & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{1}{2}\left(x-\frac{3}{2}\right)^2 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$