ECS 275 Fall 2011 Midterm Solutions

1) (a) Radiosity is the radiant power leaving a surface per unit surface area.

(b) Radiance is the radiant power per unit area normal to the beam, per unit solid angle.

(c) To find the radiosity we need to integrate the radiance leaving the surface over the hemisphere of solid angles above the surface, multiplied by the $\cos\theta$ factor to spread out flux per unit area normal to the beam into flux per unit area on the surface.

$$B = \int_{\Omega_x} L \cos\theta \, d\omega$$

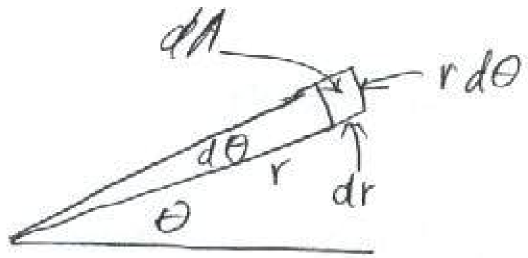
$$= \int_0^{2\pi} d\varphi \int_0^{\pi/2} L \cos\theta \sin\theta \, d\theta \qquad (\text{since } d\omega = \sin\theta \, d\theta \, d\varphi)$$

$$= 2\pi L \int_0^{\frac{\pi}{2}} \frac{d}{d\theta}\left(-\frac{\cos^2\theta}{2}\right) d\theta$$

$$= 2\pi L \left(0 - \left(-\tfrac{1}{2}\right)\right) = 2\pi L \times \tfrac{1}{2} = \pi L$$

(Full credit will be given for the correct answer πL, even without this derivation.)

2) (a) Choose the point according to polar coordinates (r, θ), with $x = r\cos\theta$ and $y = r\sin\theta$. Because of circular symmetry, the distribution of θ is uniform. To find the distribution of r, we use the cumulative distribution function for $p(r)$. We need the conversion from polar coordinates differentials to area differentials on the annulus:



$$dA = r\,dr\,d\theta$$

Thus, assuming the uniform distribution has a constant probability density k per unit area, the cumulative distribution in the annulus $S(R)$ from radius R_1 up to radius R is

$$
\begin{aligned}
CDF(R) &= \int_{S(R)} p(r,\theta)\,dA \\
&= \int_0^{2\pi}\int_{R_1}^{R} k\, r\,dr\,d\theta \\
&= 2\pi k \int_{R_1}^{R} r\,dr = 2\pi k \left(\frac{r^2}{2}\right)\Big|_{R_1}^{R} \\
&= 2\pi k\,(R^2/2 - R_1^2/2) \\
&= \pi k\,(R^2 - R_1^2)
\end{aligned}
$$

This must equal 1 when $R = R_2$, since a pdf must integrate to 1, so

$$\pi k\,(R_2^2 - R_1^2) = 1$$

$$k = \frac{1}{\pi(R_2^2 - R_1^2)}$$

Then $$CDF(R) = \frac{\pi(R^2 - R_1^2)}{\pi(R_2^2 - R_1^2)} = \frac{R^2 - R_1^2}{R_2^2 - R_1^2}$$

Thus to choose an appropriately distributed r, we choose a random u in [0,1], say using drand48(), and then choose r so that

$$\frac{r^2 - R_1^2}{R_2^2 - R_1^2} = u$$

$$r^2 - R_1^2 = u(R_2^2 - R_1^2)$$

$$r^2 = u(R_2^2 - R_1^2) + R_1^2$$

$$r = \sqrt{u(R_2^2 - R_1^2) + R_1^2}$$

Then we choose θ uniformly in $(0, 2\pi)$, say by letting v = drand48() be a random number uniformly distributed in [0,1), and letting $\theta = 2\pi v$.

An alternate answer could use rejection sampling: Try a random point uniformly chosen in the square around the annulus, and reject it if it is outside the larger circle, or inside the smaller one:

```
while(true){
  x = 2*R2* drand48() - R2;
  y = 2*R2* drand48() - R2;
  if ( x^2 + y^2 >= R1^2 &&
       x^2 + y^2 <= R2^2) return point(x,y);
}
```

(b) Stratified sampling

```
For (i=0; i<4; ++i)        // ring
  For (j=0; j<4; ++j){     // sector
    u = (i + drand48())/ 4.;
    v = (j + drand48())/4.;
    r = sqrt(u * (R2*R2 - R1*R1) + R1*R1);
    theta = 2 * PI * v;
    x = r * cos(theta);
    y = r * sin(theta);
    sample at (x, y);
  }
```

3) The radiosity of the LEDs is 100 watts per meter squared, so, by part(c) of question (1), the emitted radiance $L_e = (100/\pi)$ watts/m^2. The BRDF for a perfectly diffuse surface is $f_r = \rho/\pi$, where ρ is the hemispherical reflectivity (the fraction of incident flux reflected) which is 80% = 0.8 in our case, so $f_r = 0.8/\pi$. Then the reflected radiance from direct illumination is

$$\int_{\Omega_x} f_r L_e \cos\Theta \, d\omega = \pi f_r L = \pi (0.8/\pi) L = 0.8 L_e$$

(using the integral from question 1)

Thus since this is constant on all surfaces, the second bounce of the direct illumination is also constant, for the same reason, and so is the third bounce, and every other bounce. Thus the equation for the final radiance L is

$$L = L_e + \int_{\Omega_x} f_r L \cos\theta \, d\omega$$

$$= L_e + .8 L \qquad \text{(same integral as before)}$$

$$.2 L = L_e$$

$$L = 5 L_e = \frac{500}{\pi} \text{ watts}/(m^2 \times \text{steradian})$$

This is actually the answer to part (b), so the answer to part a) is $B = \pi L = 500$ watts$/m^2$.

It would also be possible to solve the radiosity equations for B, using, instead of the fact that the cosine weighted solid angle adds up to π, the fact that the sum of the form factors in a closed environment adds up to 1.

$$B_i = B_{e_i} + \rho \sum_j F_{ij} B_j$$

By symmetry of the cube, all B_i's are equal, say to B. Then, since $\rho = .8$ and $\sum_j F_{ij} = 1$, we have

$$B = B_e + .8 B$$

$$.2 B = B_e$$

$$B = 5 B_e = 500 \text{ watts}/m^2$$

Then $L = B/\pi = \frac{500}{\pi}$ watts$/(m^2 \times$ steradian$)$