Lecture 1: Convex sets

- convex set, convex cone, subspace, affine set
- simple examples and properties
- combination and hulls
- polyhedra, norm balls
- affine and projective transformations
- ellipsoids
- separating hyperplanes
- generalized inequalities

Convex sets

$S \subseteq \mathbf{R}^n$ is a **convex set** if

$$x, y \in S, \quad \lambda, \mu \geq 0, \quad \lambda + \mu = 1 \Longrightarrow \lambda x + \mu y \in S$$

geometrically: $x, y \in S \Rightarrow$ segment $[x, y] \subseteq S$ . . . many representations

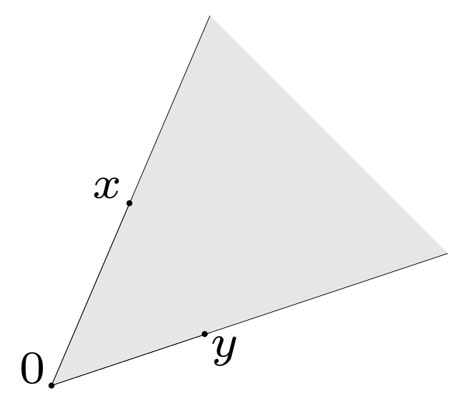
$S \subseteq \mathbf{R}^n$ is a **convex cone** if

$$x, y \in S, \quad \lambda, \mu \geq 0, \quad \Longrightarrow \lambda x + \mu y \in S$$

geometrically:

$$x, y \in S \Rightarrow \text{'pie slice' between } x, y \subseteq S$$

. . . many representations

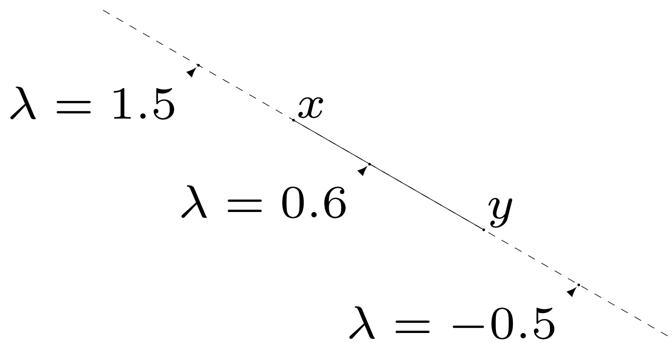


Affine sets

$S \subseteq \mathbf{R}^n$ is *affine* if

$$x, y \in S, \quad \lambda, \mu \in \mathbf{R}, \quad \lambda + \mu = 1 \Longrightarrow \lambda x + \mu y \in S$$

geometrically: $x, y \in S \Rightarrow$ line through $x, y \subseteq S$



representations: range of affine function

$$S = \{Az + b \mid z \in \mathbf{R}^q\}$$

via linear equalities

$$S = \{x \mid b_1^T x = d_1, \ldots, b_p^T x = d_p\} = \{x \mid Bx = d\}$$

Subspaces

$S \subseteq \mathbf{R}^n$ is a *subspace* if

$$x, y \in S, \quad \lambda, \mu \in \mathbf{R} \implies \lambda x + \mu y \in S$$

geometrically: $x, y \in S \Rightarrow$ plane through $0, x, y \subseteq S$

representations

$$\begin{aligned} \operatorname{range}(A) &= \{Aw \mid w \in \mathbf{R}^q\} \\ &= \{w_1 a_1 + \cdots + w_q a_q \mid w_i \in \mathbf{R}\} \\ &= \operatorname{span}(a_1, a_2, \ldots, a_q) \end{aligned}$$

where $A = \begin{bmatrix} a_1 & \cdots & a_q \end{bmatrix}$

$$\operatorname{nullspace}(B) = \{x \mid Bx = 0\} = \{x \mid b_1^T x = 0, \ldots, b_p^T x = 0\}$$

where $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_p^T \end{bmatrix}$

Combinations and hulls

$y = \theta_1 x_1 + \cdots + \theta_k x_k$ is a

- *linear combination* of $x_1, \ldots, x_k$
- *affine combination* if $\sum_i \theta_i = 1$
- *convex combination* if $\sum_i \theta_i = 1$, $\theta_i \geq 0$
- *conic combination* if $\theta_i \geq 0$

(linear,. . .) **hull** of S: set of all (linear, . . .) combinations from S

linear hull: $\mathrm{span}(S)$, affine hull: $\mathbf{Aff}(S)$,
convex hull: $\mathbf{Co}(S)$, conic hull: $\mathbf{Cone}(S)$.

$$\mathbf{Co}(S) = \bigcap \{G \mid S \subseteq G,\ G \text{ convex}\}, \ldots$$

example. $S = \{(1,0,0), (0,1,0), (0,0,1)\}$
what is linear, affine, . . . , hull?

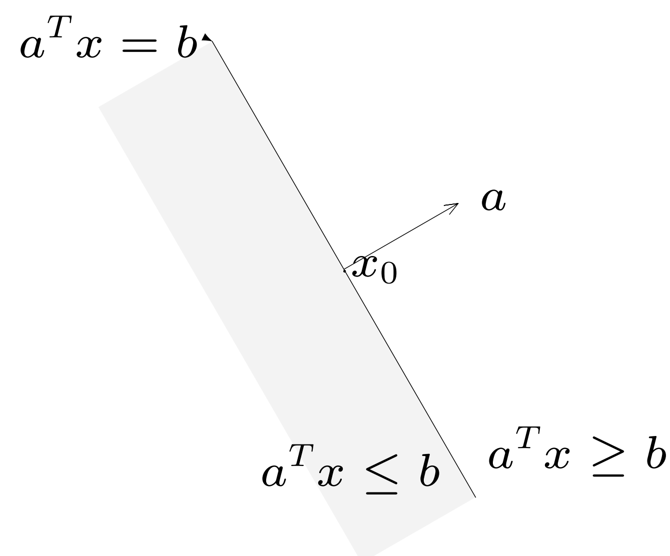
Hyperplanes and halfspaces

hyperplane: $\{x \mid a^T x = b\}$ $(a \neq 0)$

affine; subspace if $b = 0$

useful representation: $\{x \mid a^T(x - x_0) = 0\}$

a is *normal vector*; x_0 lies on hyperplane



halfspace: $\{x \mid a^T x \leq b\}$ $(a \neq 0)$ convex;

convex cone if $b = 0$

useful representation: $\{x \mid a^T(x - x_0) \leq 0\}$

a is *(outward) normal vector*; x_0 lies on boundary

Intersections

$$S_\alpha \text{ is } \left(\begin{array}{l} \textbf{subspace} \\ \textbf{affine} \\ \textbf{convex} \\ \textbf{convex cone} \end{array} \right) \text{ for } \alpha \in \mathcal{A} \Longrightarrow \bigcap_{\alpha \in \mathcal{A}} S_\alpha \text{ is } \left(\begin{array}{l} \textbf{subspace} \\ \textbf{affine} \\ \textbf{convex} \\ \textbf{convex cone} \end{array} \right)$$

example: *polyhedron* is intersection of a finite number of halfspaces

$$\begin{aligned} \mathcal{P} &= \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, k\} \\ &= \{x \mid Ax \preceq b\} \end{aligned}$$

($\preceq$ means componentwise)

a bounded polyhedron is called a *polytope*

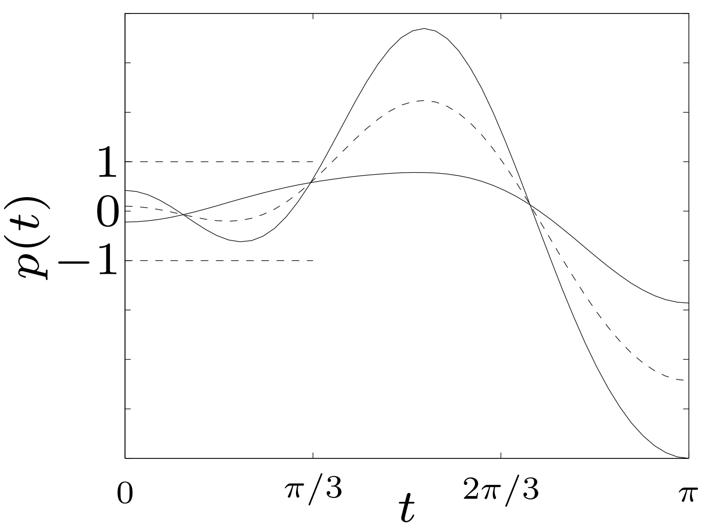
in fact, *every* closed convex set S is (usually infinite) intersection of halfspaces:

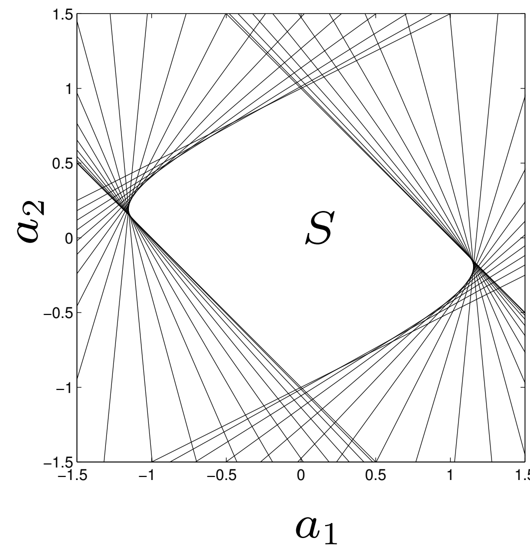
$$S = \bigcap \{\mathcal{H} \mid \mathcal{H} \text{ halfspace}, \ S \subseteq \mathcal{H}\}$$

example:

$$S = \{a \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

with $p(t) = \sum_{k=1}^{m} a_k \cos kt$

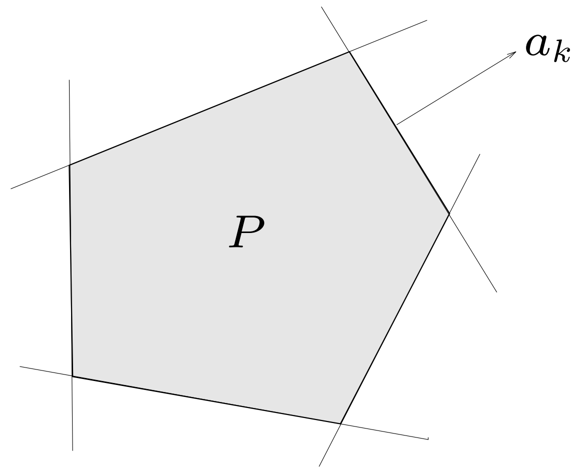




can express S as intersection of slabs: $S = \bigcap_{|t| \leq \pi/3} S_t$,

$$S_t = \left\{a \;\middle|\; -1 \leq \begin{bmatrix} \cos t & \cdots & \cos mt \end{bmatrix} a \leq 1\right\}.$$

Polyhedra



Examples

- nonnegative orthant $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x \succeq 0\}$
- *k-simplex* $\mathbf{Co}\{x_0, \ldots, x_k\}$ with $x_0, \ldots, x_k$ affinely independent, $i.e.$,

$$\mathbf{Rank}\left(\begin{bmatrix} x_0 & x_1 & \cdots & x_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}\right) = k+1,$$

  or equivalently, $x_1 - x_0, \ldots, x_k - x_0$ lin. indep.
- *probability simplex* $\{x \in \mathbf{R}^n \mid x \succeq 0,\ \sum_i x_i = 1\}$

Norm balls & cones

$f : \mathbf{R}^n \to \mathbf{R}$ is a *norm* if for all $x, y \in \mathbf{R}^n$, $t \in \mathbf{R}$,

1. $f(x) \geq 0$; $f(x) = 0 \Longrightarrow x = 0$
2. $f(tx) = |t| f(x)$
3. $f(x + y) \leq f(x) + f(y)$

$f(x)$ usually denoted $\|x\|_{\mathrm{mark}}$ (subscript identifies norm)

if f is a norm,

- the *norm ball* $\mathrm{B} = \{x \mid f(x - x_c) \leq 1\}$ is convex
- the *norm cone* $C = \{(x, t) \mid f(x) \leq t\}$ is a convex cone

ℓ_p norms

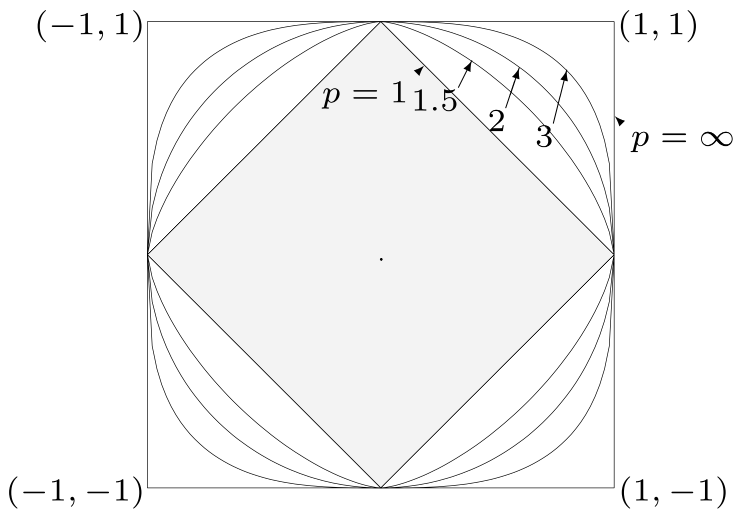
ℓ_p norms on $\mathbf{R}^n$: for $p \geq 1$,

$$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p},$$

for $p = \infty$, $\|x\|_\infty = \max_i |x_i|$

- ℓ_2 norm is Euclidean norm $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$
- ℓ_1 norm is sum-abs-values $\|x\|_1 = \sum_i |x_i|$
- ℓ_∞ norm is max-abs-value $\|x\|_\infty = \max_i |x_i|$

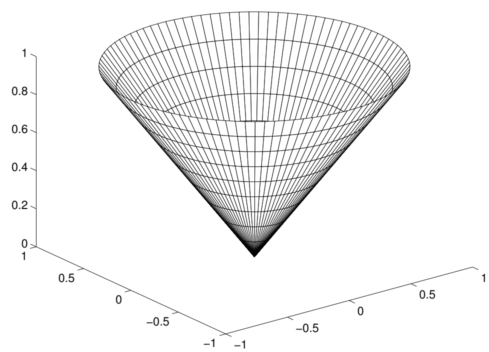
Figure shows corresponding norm balls (in $\mathbf{R}^2$)



Second-order cone

norm cone associated with Euclidean norm is *second-order cone*
(also called *quadratic* or *Lorentz* cone)

$$
\begin{aligned}
S &= \{(x,t) \mid \sqrt{x^T x} \le t\} \\
&= \left\{ (x,t) \;\middle|\; \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le 0,\; t \ge 0 \right\}
\end{aligned}
$$



Affine transformations

suppose f is affine, $i.e.$, linear plus constant:

$$f(x) = Ax + b$$

if S, T convex, then so are

$$\begin{aligned} f^{-1}(S) &= \{x \mid Ax + b \in S\} \\ f(T) &= \{Ax + b \mid x \in T\} \end{aligned}$$

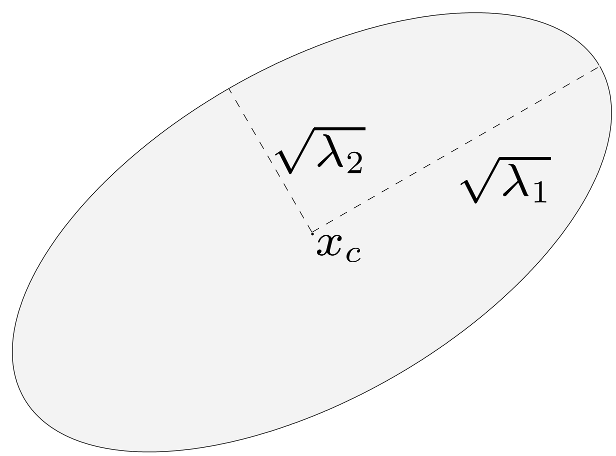
example: coordinate projection

$$\left\{ x \,\middle|\, \begin{bmatrix} x \\ y \end{bmatrix} \in S \text{ for some } y \right\}$$

Ellipsoids

$$\mathcal{E} = \{x \mid (x - x_c)^T A^{-1} (x - x_c) \leq 1\}$$

($A = A^T \succ 0$; $x_c \in \mathbf{R}^n$ center)



- semiaxis lengths: $\sqrt{\lambda_i}$; λ_i eigenvalues of A
- semiaxis directions: eigenvectors of A
- volume: $\alpha_n \left(\prod \lambda_i\right)^{1/2} = \alpha_n \left(\det A\right)^{1/2}$

other descriptions

- $\mathcal{E} = \{Bu + x_c \mid \|u\| \le 1\}$ ($\|u\| = \sqrt{u^T u}$)
- $\mathcal{E} = \{x \mid f(x) \le 0\}$

$$
\begin{aligned}
f(x) &= x^T C x + 2d^T x + e \\
&= \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} C & d \\ d^T & e \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}
\end{aligned}
$$

($C = C^T \succ 0$, $e - d^T C^{-1} d < 0$)

exercise: convert among representations; give center, semiaxes, volume

Linear matrix inequalities

- set of symmetric matrices $\mathbf{S}^n = \{X \in \mathbf{R}^{n\times n} \mid X = X^T\}$; subspace of $\mathbf{R}^{n\times n}$
- set of symmetric positive semidefinite (PSD) matrices

$$\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$$

is a convex cone

$$\mathbf{S}_+^n = \bigcap_{z\in\mathbf{R}^n} \left\{ X \in \mathbf{S}^n \;\middle|\; z^T X z = \sum_{i,j=1}^n z_i z_j X_{ij} \geq 0 \right\}$$

(intersection of infinite number of halfspaces in $\mathbf{S}^n$)

- hence, if $A_0,\ A_1, \ldots, A_m \in \mathbf{S}^n$, solution set of the *linear matrix inequality* (LMI)

$$A_0 + x_1 A_1 + \cdots + x_m A_m \succeq 0$$

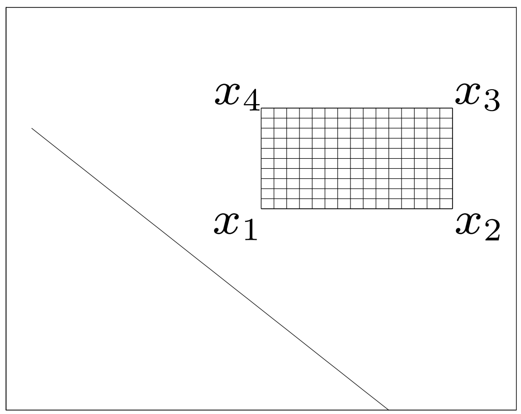
is convex

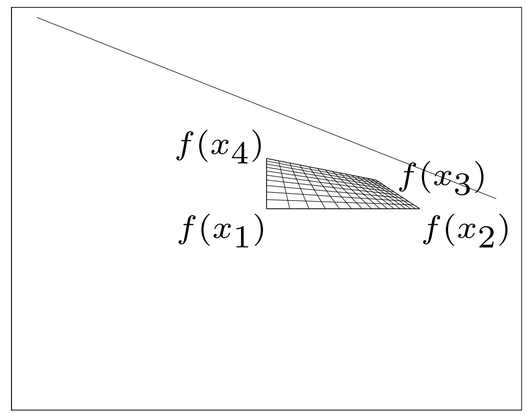
Linear-fractional transformation

linear-fractional (or projective) function $f : \mathbf{R}^m \to \mathbf{R}^n$,

$$f(x) = \frac{Ax + b}{c^T x + d}$$

on domain $\mathbf{dom}\, f = \mathcal{H} = \{x \mid c^T x + d > 0\}$





line segments preserved: for x, $y \in \mathcal{H}$,

$$f([x, y]) = [f(x), f(y)]$$

hence, if C convex, $C \subseteq \mathcal{H}$, then $f(C)$ convex

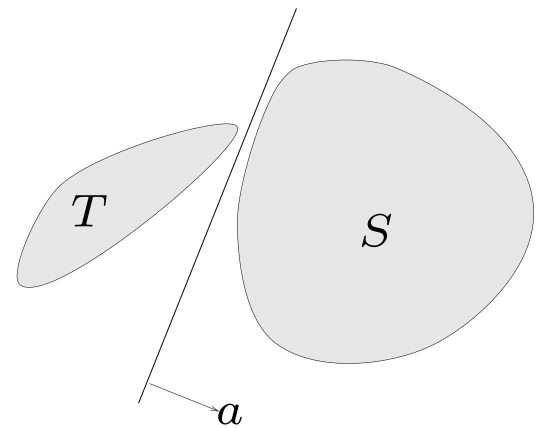
Separating hyperplanes

separating hyperplane theorem:

if $S,\ T \subseteq \mathbf{R}^n$ are convex and disjoint ($S \cap T = \emptyset$),

then, there are $a \neq 0,\ b$ such that

$$a^T x \geq b \text{ for } x \in S, \quad a^T x \leq b \text{ for } x \in T$$



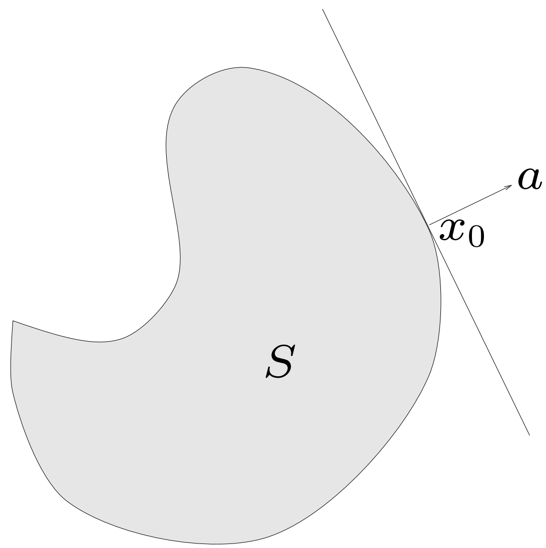
$i.e.$, hyperplane $\{x \mid a^T x - b = 0\}$ *separates* S, T

(stronger forms use strict inequality, require more conditions on S, T)

Supporting hyperplane

hyperplane $\{x \mid a^T x = a^T x_0\}$ *supports* S at $x_0 \in \partial S$ if

$$x \in S \Rightarrow a^T x \leq a^T x_0$$



halfspace $\{x \mid a^T x \leq b\}$ contains S for $b = a^T x_0$ but not for smaller b

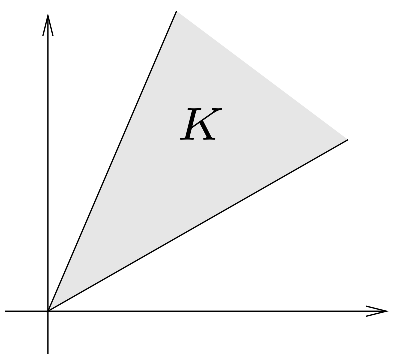
$$S \text{ convex } \Rightarrow \exists \text{ supporting hyperplane for each } x_0 \in \partial S$$

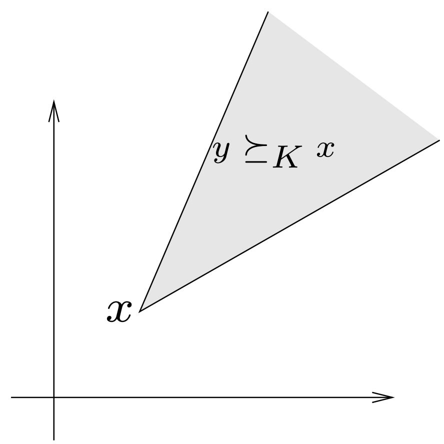
Generalized inequalities

convex cone $K \subseteq \mathbf{R}^n$ is *proper* if it

- is closed
- has nonempty interior
- is *pointed*: there is no line in K

a proper cone K defines a *generalized inequality* $\preceq_K$ in $\mathbf{R}^n$: $x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K$
strict version: $x \prec_K y \quad \Longleftrightarrow \quad y - x \in \textbf{int}\, K$





examples:

- $K = \mathbf{R}^n_+$: $x \preceq_K y$ means $x_i \leq y_i$ (componentwise vector inequality)
- K is PSD cone in $\{X \in \mathbf{R}^{n\times n} | X = X^T\}$: $X \preceq_K Y$ means $Y - X$ is PSD

(these are so common we drop K)

many properties of $\preceq_K$ similar to $\leq$ on $\mathbf{R}$, $e.g.$,

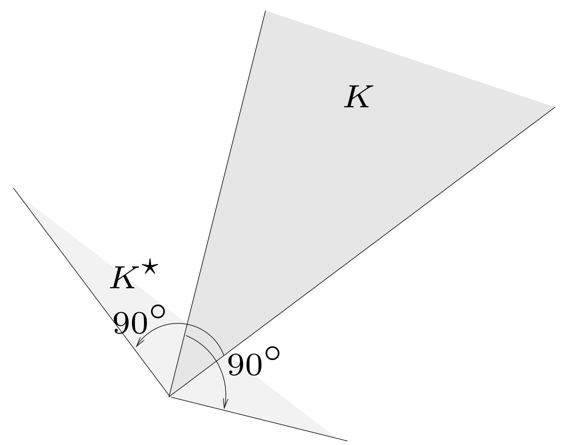
- $x \preceq_K y, \; u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$
- $x \preceq_K y, \; y \preceq_K x \quad \Longrightarrow \quad x = y$

unlike $\leq$, $\preceq_K$ is not in general a *linear ordering*

Dual cones and inequalities

if K is a cone, *dual cone* is defined as

$$K^\star = \{ \; y \mid x^T y \geq 0 \text{ for all } x \in K \; \}$$



- for $K = \mathbf{R}^n_+$, $K^\star = K$, since

$$\sum_i x_i y_i \geq 0 \text{ for all } x_i \geq 0 \quad \Longleftrightarrow \quad y_i \geq 0$$

- for $K =$ PSD cone, $K^\star = K$; (called *self-dual* cones)