4.21 *Some simple QCQPs.* Give an explicit solution of each of the following QCQPs.

(a) *Minimizing a linear function over an ellipsoid centered at the origin.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x^T A x \leq 1, \end{array}$$

where $A \in \mathbf{S}^n_{++}$ and $c \neq 0$. What is the solution if the problem is not convex ($A \notin \mathbf{S}^n_{+}$)?

Solution. If $A \succ 0$, the solution is

$$x^\star = -\frac{1}{\|A^{-1/2}c\|_2} A^{-1} \quad c, \qquad p^\star = -\|A^{-1/2}c\|_2 = -\sqrt{c^T A^{-1} c}.$$

This can be shown as follows. We make a change of variables $y = A^{1/2}x$, and write $\tilde{c} = A^{-1/2}c$. With this new variable the optimization problem becomes

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T y \\ \text{subject to} & y^T y \leq 1, \end{array}$$

i.e., we minimize a linear function over the unit ball. The answer is $y^\star = -\tilde{c}/\|\tilde{c}\|_2$.

In the general case, we can make a change of variables based on the eigenvalue decomposition

$$A = Q\,\mathbf{diag}(\lambda)Q^T = \sum_{i=1}^n \lambda_i q_i q_i^T.$$

We define $y = Qx$, $b = Qc$, and express the problem as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n b_i y_i \\ \text{subject to} & \sum_{i=1}^n \lambda_i y_i^2 \leq 1. \end{array}$$

If $\lambda_i > 0$ for all i, the problem reduces to the case we already discussed. Otherwise, we can distinguish several cases.

- $\lambda_n < 0$. The problem is unbounded below. By letting $y_n \to \pm\infty$, we can make any point feasible.
- $\lambda_n = 0$. If for some i, $b_i \neq 0$ and $\lambda_i = 0$, the problem is unbounded below.
- $\lambda_n = 0$, and $b_i = 0$ for all i with $\lambda_i = 0$. In this case we can reduce the problem to a smaller one with all $\lambda_i > 0$.

(b) *Minimizing a linear function over an ellipsoid.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (x - x_c)^T A (x - x_c) \leq 1, \end{array}$$

where $A \in \mathbf{S}^n_{++}$ and $c \neq 0$.

Solution. We make a change of variables

$$y = A^{1/2}(x - x_c), \qquad x = A^{-1/2} y + x_c,$$

and consider the problem

$$\begin{array}{ll} \text{minimize} & c^T A^{-1/2} y + c^T x_c \\ \text{subject to} & y^T y \leq 1. \end{array}$$

The solution is

$$y^\star = -(1/\|A^{-1/2}c\|_2) A^{-1/2} c, \qquad x^\star = x_c - (1/\|A^{-1/2}c\|_2) A^{-1} c.$$

(c) *Minimizing a quadratic form over an ellipsoid centered at the origin.*

$$\begin{array}{ll} \text{minimize} & x^T B x \\ \text{subject to} & x^T A x \leq 1, \end{array}$$

where $A \in \mathbf{S}^n_{++}$ and $B \in \mathbf{S}^n_{+}$. Also consider the nonconvex extension with $B \notin \mathbf{S}^n_{+}$. (See §B.1.)

Solution. If $B \succeq 0$, then the optimal value is obviously zero (since $x^T B x \geq 0$ for all x, with equality if $x = 0$).

In the general case, we use the following fact from linear algebra. The smallest eigenvalue of $B \in \mathbf{S}^n$, can be characterized as

$$\lambda_{\min}(B) = \inf_{x^T x = 1} x^T B x.$$

To solve the optimization problem

$$\begin{array}{ll} \text{minimize} & x^T B x \\ \text{subject to} & x^T A x \leq 1, \end{array}$$

with $A \succ 0$, we make a change of variables $y = A^{1/2} x$. This is possible since $A \succ 0$, so $A^{1/2}$ is defined and nonsingular. In the new variables the problem becomes

$$\begin{array}{ll} \text{minimize} & y^T A^{-1/2} B A^{-1/2} y \\ \text{subject to} & y^T y \leq 1. \end{array}$$

If the constraint $y^T y \leq 1$ is active at the optimum ($y^T y = 1$), then the optimal value is

$$\lambda_{\min}(A^{-1/2} B A^{-1/2}),$$

by the result mentioned above. If $y^T y < 1$ at the optimum, then it must be at a point where the gradient of the objective function vanishes, *i.e.*, $\boxed{By = 0}$. In that case the optimal value is zero.

A^(-1/2)BA^(1/2)y=0

To summarize, the optimal value is

$$p^\star = \begin{cases} \lambda_{\min}(A^{-1/2} B A^{-1/2}) & \lambda_{\min}(A^{-1/2} B A^{-1/2}) \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In the first case any (normalized) eigenvector of $A^{-1/2} B A^{-1/2}$ corresponding to the smallest eigenvalue is an optimal y. In the second case $y = 0$ is optimal.

4.40 *LPs, QPs, QCQPs, and SOCPs as SDPs.* Express the following problems as SDPs.

(a) The LP (4.27).

Solution.

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b. \end{array}$$

(b) The QP (4.34), the QCQP (4.35) and the SOCP (4.36). *Hint.* Suppose $A \in \mathbf{S}^r_{++}$, $C \in \mathbf{S}^s$, and $B \in \mathbf{R}^{r \times s}$. Then

$$\left[\begin{array}{cc} A & B \\ B^T & C \end{array} \right] \succeq 0 \iff C - B^T A^{-1} B \succeq 0.$$

For a more complete statement, which applies also to singular A, and a proof, see §A.5.5.

Solution.

(a) QP. Express $P = WW^T$ with $W \in \mathbf{R}^{n \times r}$.

$$\begin{array}{ll} \text{minimize} & t + 2q^T x + r \\ \text{subject to} & \left[\begin{array}{cc} I & W^T x \\ x^T W & tI \end{array} \right] \succeq 0 \\ & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b, \end{array}$$

with variables x, $t \in \mathbf{R}$.

(b) QCQP. Express $P_i = W_i W_i^T$ with $W_i \in \mathbf{R}^{n \times r_i}$.

$$\begin{array}{ll} \text{minimize} & t_0 + 2q_0^T x + r_0 \\ \text{subject to} & t_i + 2q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & \left[\begin{array}{cc} I & W_i^T x \\ x^T W_i & t_i I \end{array} \right] \succeq 0, \quad i = 0, 1, \ldots, m \\ & Ax = b, \end{array}$$

with variables x, $t_i \in \mathbf{R}$.

(c) SOCP.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \left[\begin{array}{cc} (c_i^T x + d_i)I & A_i x + b_i \\ (Ax_i + b_i)^T & (c_i^T x + d_i)I \end{array} \right] \succeq 0, \quad i = 1, \dots, N \\ & Fx = g. \end{array}$$

By the result in the hint, the constraint is equivalent with $\|A_i x + b_i\|_2 < c_i^T x + d_i$ when $c_i^T x + d_i > 0$. We have to check the case $c_i^T x + d_i = 0$ separately. In this case, the LMI constraint means $A_i x + b_i = 0$, so we can conclude that the LMI constraint and the SOC constraint are equivalent.

(c) The matrix fractional optimization problem

$$\text{minimize} \quad (Ax + b)^T F(x)^{-1} (Ax + b)$$

where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$,

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n,$$

with $F_i \in \mathbf{S}^m$, and we take the domain of the objective to be $\{x \mid F(x) \succ 0\}$. You can assume the problem is feasible (there exists at least one x with $F(x) \succ 0$).

Solution.

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} F(x) & Ax + b \\ (Ax + b)^T & t \end{array} \right] \succeq 0 \end{array}$$

with variables x, $t \in \mathbf{R}$. The LMI constraint is equivalent to

$$(Ax + b)^T F(x)^{-1} (Ax + b) \leq t$$

if $F(x) \succ 0$.

More generally, let

$$f_0(x) = (Ax + b)^T F(x)^{-1} (Ax + b), \qquad \mathbf{dom}\, f_0(x) = \{x \mid F(x) \succ 0\}.$$

We have

$$\mathbf{epi}\, f_0 = \left\{ (x, t) \;\middle|\; F(x) \succ 0, \left[\begin{array}{cc} F(x) & Ax + b \\ (Ax + b)^T & t \end{array} \right] \succeq 0 \right\}.$$

Then $\mathbf{cl}(\mathbf{epi}\, f_0) = \mathbf{epi}\, g$ where g is defined by

$$\begin{array}{rcl} \mathbf{epi}\, g & = & \left\{ (x, t) \;\middle|\; \left[\begin{array}{cc} F(x) & Ax + b \\ (Ax + b)^T & t \end{array} \right] \succeq 0 \right\} \\ g(x) & = & \inf \left\{ t \;\middle|\; \left[\begin{array}{cc} F(x) & Ax + b \\ (Ax + b)^T & t \end{array} \right] \succeq 0 \right\}. \end{array}$$

We conclude that both problems have the same optimal values. An optimal solution for the matrix fractional problem is optimal for the SDP. An optimal solution for the SDP, with $F(x) \succ 0$, is optimal for the matrix fractional problem. If $F(x)$ is singular at the optimal solution of the SDP, then the optimum for the matrix fractional problem is not attained.

4.42 *Complex LMIs and SDPs.* A complex LMI has the form

$$x_1F_1 + \cdots + x_nF_n + G \preceq 0$$

where $F_1, \ldots, F_n$, G are complex $n \times n$ Hermitian matrices, *i.e.*, $F_i^H = F_i$, $G^H = G$, and $x \in \mathbf{R}^n$ is a real variable. A complex SDP is the problem of minimizing a (real) linear function of x subject to a complex LMI constraint.

Complex LMIs and SDPs can be transformed to real LMIs and SDPs, using the fact that

$$X \succeq 0 \iff \begin{bmatrix} \Re X & -\Im X \\ \Im X & \Re X \end{bmatrix} \succeq 0,$$

where $\Re X \in \mathbf{R}^{n\times n}$ is the real part of the complex Hermitian matrix X, and $\Im X \in \mathbf{R}^{n\times n}$ is the imaginary part of X.

Verify this result, and show how to pose a complex SDP as a real SDP.

Solution. For a Hermitian matrix $\Re X = (\Re X)^T$ and $\Im X = -\Im X^T$. Now let $z = u + iv$, where u, v are real vectors, and $i = \sqrt{-1}$. We have

$$\begin{aligned} z^HXz &= (u - iv)^T(\Re X + i\Im X)(u + iv) \\ &= u^T\Re Xu + v^T\Re Xv - u^T\Im Xv + v^T\Im Xu \\ &= \begin{bmatrix} u^T & v^T \end{bmatrix} \begin{bmatrix} \Re X & -\Im X \\ \Im X & \Re X \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \end{aligned}$$

Therefore $z^HXz \geq 0$ for all z if and only if the $2n \times 2n$ real (symmetric) matrix above is positive semidefinite.

Thus, we can convert a complex LMI into a real LMI with twice the size. The conversion is linear, a complex LMI becomes a real LMI, of twice the size.

4.50 *Bi-criterion optimization.* Figure 4.11 shows the optimal trade-off curve and the set of achievable values for the bi-criterion optimization problem

$$\text{minimize (w.r.t. } \mathbf{R}_+^2) \quad (\|Ax - b\|^2, \|x\|_2^2),$$

for some $A \in \mathbf{R}^{100\times 10}$, $b \in \mathbf{R}^{100}$. Answer the following questions using information from the plot. We denote by $x_{\rm ls}$ the solution of the least-squares problem

$$\text{minimize} \quad \|Ax - b\|_2^2.$$

(a) What is $\|x_{\rm ls}\|_2$?

(b) What is $\|Ax_{\rm ls} - b\|_2$?

(c) What is $\|b\|_2$?

(d) Give the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & \|x\|_2^2 = 1. \end{array}$$

(e) Give the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & \|x\|_2^2 \leq 1. \end{array}$$

(f) Give the optimal value of the problem

$$\text{minimize } \|Ax - b\|_2^2 + \|x\|_2^2.$$

(g) What is the rank of A?

Solution.

(a) $\|x_{\rm ls}\|_2 = 3$.

(b) $\|Ax_{\rm ls} - b\|_2^2 = 2$.

(c) $\|b\|_2 = \sqrt{10}$.

(d) About 5.

(e) About 5.

(f) About $3 + 4$.

(g) $\mathbf{rank}\, A = 10$, since the LS solution is unique.