4.1 Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$.

(b) $f_0(x_1, x_2) = -x_1 - x_2$.

(c) $f_0(x_1, x_2) = x_1$.

(d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$.

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

Solution. The feasible set is the convex hull of $(0, \infty)$, $(0, 1)$, $(2/5, 1/5)$, $(1, 0)$, $(\infty, 0)$.

(a) $x^\star = (2/5, 1/5)$.

(b) Unbounded below.

(c) $X_{\text{opt}} = \{(0, x_2) \mid x_2 \geq 1\}$.

(d) $x^\star = (1/3, 1/3)$.

(e) $x^\star = (1/2, 1/6)$. This is optimal because it satisfies $2x_1 + x_2 = 7/6 > 1$, $x_1 + 3x_2 = 1$, and

$$\nabla f_0(x^\star) = (1, 3)$$

is perpendicular to the line $x_1 + 3x_2 = 1$.

4.3 Prove that $x^\star = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & -1 \leq x_i \leq 1, \quad i = 1, 2, 3, \end{array}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \qquad q = \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix}, \qquad r = 1.$$

Solution. We verify that $x^\star$ satisfies the optimality condition (4.21). The gradient of the objective function at $x^\star$ is

$$\nabla f_0(x^\star) = (-1, 0, 2).$$

Therefore the optimality condition is that

$$\nabla f_0(x^\star)^T (y - x) = -1(y_1 - 1) + 2(y_2 + 1) \geq 0$$

for all y satisfying $-1 \leq y_i \leq 1$, which is clearly true.

4.8 *Some simple LPs.* Give an explicit solution of each of the following LPs.

(a) *Minimizing a linear function over an affine set.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b. \end{array}$$

Solution. We distinguish three possibilities.

- The problem is infeasible ($b \notin \mathcal{R}(A)$). The optimal value is ∞.

- The problem is feasible, and c is orthogonal to the nullspace of A. We can decompose c as
$$c = A^T\lambda + \hat{c}, \qquad A\hat{c} = 0.$$
($\hat{c}$ is the component in the nullspace of A; $A^T\lambda$ is orthogonal to the nullspace.) If $\hat{c} = 0$, then on the feasible set the objective function reduces to a constant:
$$c^Tx = \lambda^T Ax + \hat{c}^Tx = \lambda^T b.$$
The optimal value is λ^Tb. All feasible solutions are optimal.
- The problem is feasible, and c is not in the range of A^T ($\hat{c} \neq 0$). The problem is unbounded ($p^\star = -\infty$). To verify this, note that $x = x_0 - t\hat{c}$ is feasible for all t; as t goes to infinity, the objective value decreases unboundedly.

In summary,
$$p^\star = \begin{cases} +\infty & b \notin \mathcal{R}(A) \\ \lambda^Tb & c = A^T\lambda \text{ for some } \lambda \\ -\infty & \text{otherwise.} \end{cases}$$

(b) *Minimizing a linear function over a halfspace.*
$$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & a^Tx \leq b, \end{array}$$
where $a \neq 0$.

Solution. This problem is always feasible. The vector c can be decomposed into a component parallel to a and a component orthogonal to a:
$$c = a\lambda + \hat{c},$$
with $a^T\hat{c} = 0$.

- If $\lambda > 0$, the problem is unbounded below. Choose $x = -ta$, and let t go to infinity:
$$c^Tx = -tc^Ta = -t\lambda a^Ta \to -\infty$$
and
$$a^Tx - b = -ta^Ta - b \leq 0$$
for large t, so x is feasible for large t. Intuitively, by going very far in the direction $-a$, we find feasible points with arbitrarily negative objective values.
- If $\hat{c} \neq 0$, the problem is unbounded below. Choose $x = ba - t\hat{c}$ and let t go to infinity.
- If $c = a\lambda$ for some $\lambda \leq 0$, the optimal value is $c^Tab = \lambda b$.

In summary, the optimal value is
$$p^\star = \begin{cases} \lambda b & c = a\lambda \text{ for some } \lambda \leq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

(c) *Minimizing a linear function over a rectangle.*
$$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & l \preceq x \preceq u, \end{array}$$
where l and u satisfy $l \preceq u$.

Solution. The objective and the constraints are separable: The objective is a sum of terms c_ix_i, each dependent on one variable only; each constraint depends on only one variable. We can therefore solve the problem by minimizing over each component of x

independently. The optimal $x_i^\star$ minimizes $c_i x_i$ subject to the constraint $l_i \le x_i \le u_i$. If $c_i > 0$, then $x_i^\star = l_i$; if $c_i < 0$, then $x_i^\star = u_i$; if $c_i = 0$, then any x_i in the interval $[l_i, u_i]$ is optimal. Therefore, the optimal value of the problem is

$$p^\star = l^T c^+ + u^T c^-,$$

where $c_i^+ = \max\{c_i, 0\}$ and $c_i^- = \max\{-c_i, 0\}$.

(d) *Minimizing a linear function over the probability simplex.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0. \end{array}$$

What happens if the equality constraint is replaced by an inequality $\mathbf{1}^T x \le 1$?

We can interpret this LP as a simple portfolio optimization problem. The vector x represents the allocation of our total budget over different assets, with x_i the fraction invested in asset i. The return of each investment is fixed and given by $-c_i$, so our total return (which we want to maximize) is $-c^T x$. If we replace the budget constraint $\mathbf{1}^T x = 1$ with an inequality $\mathbf{1}^T x \le 1$, we have the option of not investing a portion of the total budget.

Solution. Suppose the components of c are sorted in increasing order with

$$c_1 = c_2 = \cdots = c_k < c_{k+1} \le \cdots \le c_n.$$

We have

$$c^T x \ge c_1(\mathbf{1}^T x) = c_{\min}$$

for all feasible x, with equality if and only if

$$x_1 + \cdots + x_k = 1, \qquad x_1 \ge 0, \ldots, x_k \ge 0, \qquad x_{k+1} = \cdots = x_n = 0.$$

We conclude that the optimal value is $p^\star = c_1 = c_{\min}$. In the investment interpretation this choice is quite obvious. If the returns are fixed and known, we invest our total budget in the investment with the highest return.

If we replace the equality with an inequality, the optimal value is equal to

$$p^\star = \min\{0, c_{\min}\}.$$

(If $c_{\min} \le 0$, we make the same choice for x as above. Otherwise, we choose $x = 0$.)

(e) *Minimizing a linear function over a unit box with a total budget constraint.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{1}^T x = \alpha, \quad 0 \preceq x \preceq \mathbf{1}, \end{array}$$

where α is an integer between 0 and n. What happens if α is not an integer (but satisfies $0 \le \alpha \le n$)? What if we change the equality to an inequality $\mathbf{1}^T x \le \alpha$?

Solution. We first consider the case of integer α. Suppose

$$c_1 \le \cdots \le c_{i-1} < c_i = \cdots = c_\alpha = \cdots = c_k < c_{k+1} \le \cdots \le c_n.$$

The optimal value is

$$c_1 + c_2 + \cdots + c_\alpha$$

i.e., the sum of the smallest α elements of c. x is optimal if and only if

$$x_1 = \cdots = x_{i-1} = 1, \qquad x_i + \cdots + x_k = \alpha - i + 1, \qquad x_{k+1} = \cdots = x_n = 0.$$

If α is not an integer, the optimal value is

$$p^\star = c_1 + c_2 + \cdots + c_{\lfloor \alpha \rfloor} + c_{1+\lfloor \alpha \rfloor}(\alpha - \lfloor \alpha \rfloor).$$

In the case of an inequality constraint $\mathbf{1}^T x \le \alpha$, with α an integer between 0 and n, the optimal value is the sum of the α smallest nonpositive coefficients of c.

(f) *Minimizing a linear function over a unit box with a weighted budget constraint.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & d^T x = \alpha, \quad 0 \preceq x \preceq \mathbf{1}, \end{array}$$

with $d \succ 0$, and $0 \le \alpha \le \mathbf{1}^T d$.

Solution. We make a change of variables $y_i = d_i x_i$, and consider the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n (c_i/d_i) y_i \\ \text{subject to} & \mathbf{1}^T x = \alpha, \quad 0 \preceq y \preceq d. \end{array}$$

Suppose the ratios c_i/d_i have been sorted in increasing order:

$$\frac{c_1}{d_1} \le \frac{c_2}{d_2} \le \cdots \le \frac{c_n}{d_n}.$$

To minimize the objective, we choose

$$y_1 = d_1, \quad y_2 = d_2, \quad \ldots, \quad y_k = d_k,$$

$$y_{k+1} = \alpha - (d_1 + \cdots + d_k), \qquad y_{k+2} = \cdots = y_n = 0,$$

where $k = \max\{i \in \{1, \ldots, n\} \mid d_1 + \cdots + d_i \le \alpha\}$ (and $k = 0$ if $d_1 > \alpha$). In terms of the original variables,

$$x_1 = \cdots = x_k = 1, \qquad x_{k+1} = (\alpha - (d_1 + \cdots + d_k))/d_{k+1}, \qquad x_{k+2} = \cdots = x_n = 0.$$

4.11 *Problems involving ℓ_1- and ℓ_∞-norms.* Formulate the following problems as LPs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP.

(a) Minimize $\|Ax - b\|_\infty$ (ℓ_∞-norm approximation).

(b) Minimize $\|Ax - b\|_1$ (ℓ_1-norm approximation).

(c) Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \le 1$.

(d) Minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \le 1$.

(e) Minimize $\|Ax - b\|_1 + \|x\|_\infty$.

In each problem, $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$ are given. (See §6.1 for more problems involving approximation and constrained approximation.)

Solution.

(a) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & Ax - b \succeq -t\mathbf{1}. \end{array}$$

in the variables x, t. To see the equivalence, assume x is fixed in this problem, and we optimize only over t. The constraints say that

$$-t \le a_k^T x - b_k \le t$$

for each k, *i.e.*, $t \ge |a_k^T x - b_k|$, *i.e.*,

$$t \ge \max_k |a_k^T x - b_k| = \|Ax - b\|_\infty.$$

Clearly, if x is fixed, the optimal value of the LP is $p^\star(x) = \|Ax - b\|_\infty$. Therefore optimizing over t and x simultaneously is equivalent to the original problem.

(b) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & Ax - b \preceq s \\ & Ax - b \succeq -s. \end{array}$$

Assume x is fixed in this problem, and we optimize only over s. The constraints say that

$$-s_k \le a_k^T x - b_k \le s_k$$

for each k, *i.e.*, $s_k \ge |a_k^T x - b_k|$. The objective function of the LP is separable, so we achieve the optimum over s by choosing

$$s_k = |a_k^T x - b_k|,$$

and obtain the optimal value $p^\star(x) = \|Ax - b\|_1$. Therefore optimizing over t and s simultaneously is equivalent to the original problem.

(c) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -\mathbf{1} \le x \le \mathbf{1}, \end{array}$$

with variables $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

(d) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \le x \le y \\ & -\mathbf{1} \le Ax - b \le \mathbf{1} \end{array}$$

with variables x and y.

Another good solution is to write x as the difference of two nonnegative vectors $x = x^+ - x^-$, and to express the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x^+ + \mathbf{1}^T x^- \\ \text{subject to} & -\mathbf{1} \preceq Ax^+ - Ax^- - b \preceq \mathbf{1} \\ & x^+ \succeq 0, \quad x^- \succeq 0, \end{array}$$

with variables $x^+ \in \mathbf{R}^n$ and $x^- \in \mathbf{R}^n$.

(e) Equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y + t \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -t\mathbf{1} \preceq x \preceq t\mathbf{1}, \end{array}$$

with variables x, y, and t.