2.10 *Solution set of a quadratic inequality.* Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbf{R}^n \mid x^T Ax + b^T x + c \leq 0\},$$

with $A \in \mathbf{S}^n$, $b \in \mathbf{R}^n$, and $c \in \mathbf{R}$.

(a) Show that C is convex if $A \succeq 0$.

(b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbf{R}$.

Are the converses of these statements true?

Solution:

(a) Prove convexity of $S = \{x | x^T Ax + b^T + C \leq 0, A \geq 0\}$

A set is convex if its intersection with arbitrary lines is convex, i.e. need to show

$(x_0 + tv)^T A(x_0 + tv) + b^T(x_0 + tv) + C \leq 0$ is convex set over t, i.e.,

$\alpha t^2 + \beta t + \gamma \leq 0$, where $\alpha = v^T Av, \beta = b^T v + 2x_0^T Av, \gamma = C + b^T x_0 + x_0^T Ax_0$.

To prove $\alpha t^2 + \beta t + \gamma \leq 0$ is convex set over t, take $t_1, t_2, 0 \leq \theta \leq 1$

$$\alpha(\theta t_1 + (1-\theta)t_2)^2 + \beta(\theta t_1 + (1-\theta)t_2) + \gamma$$

$$= \alpha(\theta^2 t_1^2 + 2\theta(1-\theta)t_1 t_2 + (1-\theta)^2 t_2^2) + \beta\theta t_1 + \beta(1-\theta)t_2 + \theta\gamma + (1-\theta)\gamma$$

$$= \theta(\alpha\theta t_1^2 + \beta t_1 + \gamma) + (1-\theta)(\alpha(1-\theta)t_2^2 + \beta t_2 + \gamma) + 2\alpha\theta(1-\theta)t_1 t_2$$

$$= \theta(\alpha t_1^2 + \beta t_1 + \gamma) - \theta(\alpha(1-\theta)t_1^2) + (1-\theta)(\alpha t_2^2 + \beta t_2 + \gamma) - (1-\theta)(\alpha\theta t_2^2) + 2\alpha\theta(1-\theta)t_1 t_2$$

Let $A = \theta(\alpha t_1^2 + \beta t_1 + \gamma), B = (1-\theta)(\alpha t_2^2 + \beta t_2 + \gamma)$, then the above equation can be written as

$$A + B - 2\alpha\theta(1-\theta)(t_1 - t_2)^2$$

Since $A \leq 0, B \leq 0$ and $-2\alpha\theta(1-\theta)(t_1 - t_2)^2 \leq 0$ when $\alpha \geq 0$ we have

$A + B - 2\alpha\theta(1-\theta)(t_1 - t_2)^2 \leq 0$. Therefore, when $\alpha \geq 0, \alpha t^2 + \beta t + \gamma \leq 0$ give a convex set over t.

This is true for any v that $\alpha = v^T Av \geq 0$, when $A \succcurlyeq 0$

The converse does not hold; for example, take $A = -1$, $b = 0$, $c = -1$. Then $A \not\succeq 0$, but $C = \mathbf{R}$ is convex.

2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

Solution.

(a) A slab is an intersection of two halfspaces, hence it is a convex set and a polyhedron.

(b) As in part (a), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex and a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.

(d) This set is convex because it can be expressed as

$$\bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\},$$

i.e., an intersection of halfspaces. (Recall from exercise 2.9 that, for fixed y, the set

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

is a halfspace.)

(e) In general this set is not convex, as the following example in $\mathbf{R}$ shows. With $S = \{-1, 1\}$ and $T = \{0\}$, we have

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\} = \{x \in \mathbf{R} \mid x \leq -1/2 \text{ or } x \geq 1/2\}$$

which clearly is not convex.

(f) This set is convex. $x + S_2 \subseteq S_1$ if $x + y \in S_1$ for all $y \in S_2$. Therefore

$$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcap_{y \in S_2} (S_1 - y),$$

the intersection of convex sets $S_1 - y$.

(g) The set is convex, in fact a ball.

$$\begin{aligned} & \{x \mid \|x - a\|_2 \le \theta \|x - b\|_2\} \\ = & \quad \{x \mid \|x - a\|_2^2 \le \theta^2 \|x - b\|_2^2\} \\ = & \quad \{x \mid (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x + (a^T a - \theta^2 b^T b) \le 0\} \end{aligned}$$

2.16 Show that if S_1 and S_2 are convex sets in $\mathbf{R}^{m \times n}$, then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbf{R}^m,\ y_1,\ y_2 \in \mathbf{R}^n, (x, y_1) \in S_1,\ (x, y_2) \in S_2\}.$$

Solution. We consider two points $(\bar{x}, \bar{y}_1 + \bar{y}_2)$, $(\tilde{x}, \tilde{y}_1 + \tilde{y}_2) \in S$, *i.e.*, with

$$(\bar{x}, \bar{y}_1) \in S_1, \qquad (\bar{x}, \bar{y}_2) \in S_2, \qquad (\tilde{x}, \tilde{y}_1) \in S_1, \qquad (\tilde{x}, \tilde{y}_2) \in S_2.$$

For $0 \le \theta \le 1$,

$$\theta(\bar{x}, \bar{y}_1 + \bar{y}_2) + (1 - \theta)(\tilde{x}, \tilde{y}_1 + \tilde{y}_2) = (\theta \bar{x} + (1 - \theta)\tilde{x}, (\theta \bar{y}_1 + (1 - \theta)\tilde{y}_1) + (\theta \bar{y}_2 + (1 - \theta)\tilde{y}_2))$$

is in S because, by convexity of S_1 and S_2,

$$(\theta \bar{x} + (1 - \theta)\tilde{x}, \theta \bar{y}_1 + (1 - \theta)\tilde{y}_1) \in S_1, \qquad (\theta \bar{x} + (1 - \theta)\tilde{x}, \theta \bar{y}_2 + (1 - \theta)\tilde{y}_2) \in S_2.$$

2.31 *Properties of dual cones.* Let K^* be the dual cone of a convex cone K, as defined in (2.19). Prove the following.

(a) K^* is indeed a convex cone.

Solution. K^* is the intersection of a set of homogeneous halfspaces (meaning, halfspaces that include the origin as a boundary point). Hence it is a closed convex cone.

(b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.

Solution. $y \in K_2^*$ means $x^T y \ge 0$ for all $x \in K_2$, which is includes K_1, therefore $x^T y \ge 0$ for all $x \in K_1$.

2.32 Find the dual cone of $\{Ax \mid x \succeq 0\}$, where $A \in \mathbf{R}^{m \times n}$.

Solution. $K^* = \{y \mid A^T y \succeq 0\}$.