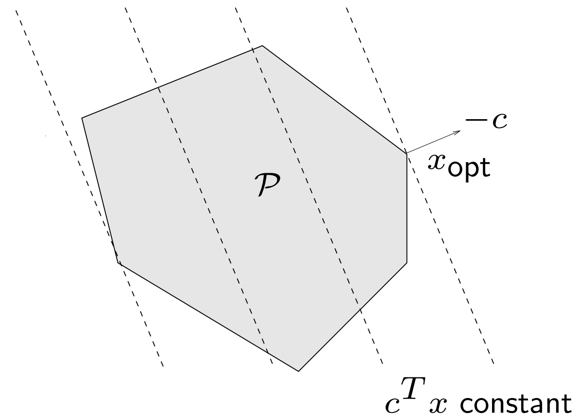
# Lecture 4: LP, QP, SOCP, SDP, and GP

- linear programming
- linear fractional programming
- (linearly constrained) quadratic programming
- (quadratically constrained) quadratic programming
- second-order cone programming
- semi-definite programming
- geometric programming

# Linear program (LP)




**linear program:**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h, \;\; Ax = b \end{array}$$

'standard' form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \;\; x \succeq 0 \end{array}$$

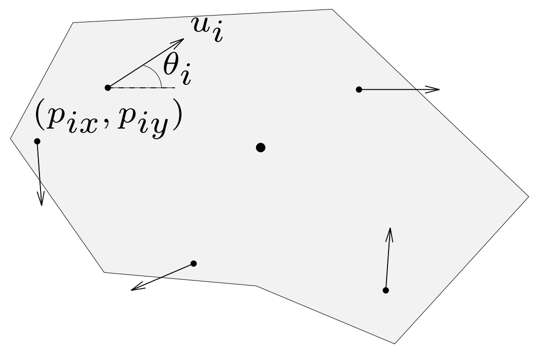
(widely used in LP literature & software)

**variations:** *e.g.*,

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

## force/moment generation with thrusters

- rigid body with center of mass at origin $p = 0 \in \mathbf{R}^2$
- $n$ forces with magnitude $u_i$, acting at $p_i = (p_{ix}, p_{iy})$, in direction $\theta_i$




- resulting horizontal force: $F_x = \sum_{i=1}^n u_i \cos\theta_i$
- resulting vertical force: $F_y = \sum_{i=1}^n u_i \sin\theta_i$
- resulting torque: $T = \sum_{i=1}^n (p_{iy} u_i \cos\theta_i - p_{ix} u_i \sin\theta_i)$
- force limits: $0 \le u_i \le 1$ (thrusters)
- fuel usage: $u_1 + \cdots + u_n$

**problem:** find $u_i$ that yield given desired forces and torques and minimize fuel usage can be expressed as LP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Fu = f^{\text{des}} \\ & 0 \leq u_i \leq 1, \ i = 1, \ldots, n \end{array}$$

where

$$F = \left[ \begin{array}{ccc} \cos\theta_1 & \cdots & \cos\theta_n \\ \sin\theta_1 & \cdots & \sin\theta_n \\ p_{1y}\cos\theta_1 - p_{1x}\sin\theta_1 & \cdots & p_{ny}\cos\theta_n - p_{nx}\sin\theta_n \end{array} \right],$$

$$f^{\text{des}} = (F_x^{\text{des}}, \ F_y^{\text{des}}, \ T^{\text{des}}), \quad \mathbf{1} = (\ 1, \ 1, \ \cdots \ 1\ )$$

# Converting LP to 'standard' form

- inequality constraints: write $a_i^T x \leq b_i$ as

$$a_i^T x + s_i = b_i, \quad s_i \geq 0$$

$s_i$ is called *slack variable* associated with $a_i^T x \leq b_i$

- unconstrained variables: write $x_i \in \mathbf{R}$ as

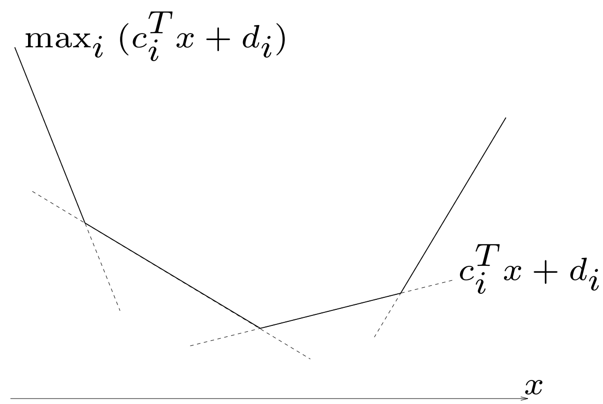
$$x_i = x_i^+ - x_i^-, \quad x_i^+, x_i^- \geq 0$$

**Example.** thruster problem in 'standard' form

$$\begin{array}{ll} \text{minimize} & \begin{bmatrix} \mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} u \\ s \end{bmatrix} \succeq 0, \quad \begin{bmatrix} F & 0 \\ I & I \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} f^{\text{des}} \\ \mathbf{1} \end{bmatrix} \end{array}$$

# Piecewise-linear minimization

$$\begin{array}{ll}\text{minimize} & \max_i (c_i^T x + d_i) \\ \text{subject to} & Ax \preceq b\end{array}$$




express as

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & c_i^T x + d_i \leq t, \quad Ax \preceq b\end{array}$$

an LP in variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$

# $\ell_\infty$- and $\ell_1$-norm approximation

**constrained $\ell_\infty$- (Chebychev) approximation**

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_\infty \\ \text{subject to} & Fx \preceq g \end{array}$$

write as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1}, \quad Ax - b \succeq -t\mathbf{1} \\ & Fx \preceq g \end{array}$$

**constrained $\ell_1$-approximation**

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_1 \\ \text{subject to} & Fx \preceq g \end{array}$$

write as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & Ax - b \preceq y, \quad Ax - b \succeq -y \\ & Fx \preceq g \end{array}$$

# Extensions of thruster problem

- opposing thruster pairs

$$\begin{array}{ll} \text{minimize} & \sum_i |u_i| \\ \text{subject to} & Fu = f^{\text{des}} \\ & |u_i| \leq 1, \quad i = 1, \ldots, n \end{array}$$

  can express as LP

- given $f^{\text{des}}$,

$$\begin{array}{ll} \text{minimize} & \|Fu - f^{\text{des}}\|_\infty \\ \text{subject to} & 0 \leq u_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

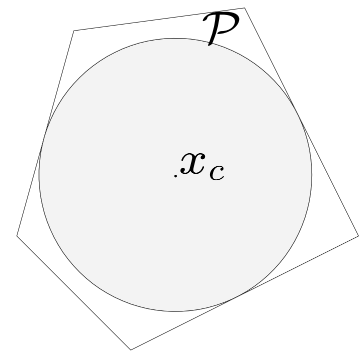
  can express as LP

- given $f^{\text{des}}$,

$$\begin{array}{ll} \text{minimize} & \#\text{ thrusters on} \\ \text{subject to} & Fu = f^{\text{des}} \\ & 0 \leq u_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

  **can't** express as LP (# thrusters on is quasiconcave!)

# Design centering




find largest ball inside a polyhedron

$$\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$$

center is called *Chebychev center*

ball $\{x_c + u \mid \|u\| \leq r\}$ lies in $\mathcal{P}$ if and only if

$$\sup\{a_i^T x_c + a_i^T u \mid \|u\| \leq r\} \leq b_i, \quad i = 1, \ldots, m,$$

$i.e.$,

$$a_i^T x_c + r\|a_i\| \leq b_i,\ i = 1, \ldots, m$$

hence, finding Chebychev center is an LP:

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\| \leq b_i, \quad i = 1, \ldots, m \end{array}$$

# Linear fractional program

$$\begin{array}{ll} \text{minimize} & \dfrac{c^T x + d}{f^T x + g} \\ \text{subject to} & Ax \preceq b, \quad f^T x + g > 0 \end{array}$$

- objective function is quasiconvex
- sublevel sets are polyhedra
- like LP, can be solved very efficiently

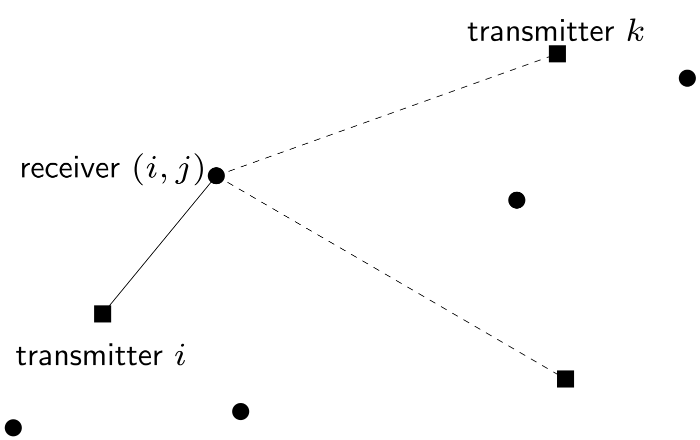
**extension:**

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,K} \dfrac{c_i^T x + d_i}{f_i^T x + g_i} \\ \text{subject to} & Ax \preceq b, \quad f_i^T x + g_i > 0, \ i = 1, \ldots, K \end{array}$$

- objective function is quasiconvex
- sublevel sets are polyhedra

# Optimal transmitter power allocation

- $m$ transmitters, $mn$ receivers all at same frequency
- transmitter $i$ wants to transmit to $n$ receivers labeled $(i, j)$, $j = 1, \ldots, n$




- $A_{ijk}$ is path gain from transmitter $k$ to receiver $(i, j)$
- $N_{ij}$ is (self) noise power of receiver $(i, j)$
- variables: transmitter powers $p_k$, $k = 1, \ldots, m$

at receiver $(i, j)$:

- signal power:

$$S_{ij} = A_{iji} p_i$$

- noise plus interference power:

$$I_{ij} = \sum_{k \neq i} A_{ijk} p_k + N_{ij}$$

- signal to interference/noise ratio (SINR): $S_{ij}/I_{ij}$

**problem:** choose $p_i$ to maximize smallest SINR:

$$\begin{array}{ll} \text{maximize} & \displaystyle\min_{i,j} \frac{A_{iji} p_i}{\sum_{k \neq i} A_{ijk} p_k + N_{ij}} \\ \text{subject to} & 0 \leq p_i \leq p_{\mathsf{max}} \end{array}$$

. . . a (generalized) linear fractional program

# Nonconvex extensions of LP

**Boolean LP** or **zero-one LP**:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Fx = g \\ & x_i \in \{0, 1\} \end{array}$$

**integer LP**:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Fx = g \\ & x_i \in \mathbf{Z} \end{array}$$

these are in general

- not convex problems
- **extremely difficult** to solve

# Quadratic functions and forms

- quadratic function

$$
\begin{aligned}
f(x) &= x^T P x + 2q^T x + r \\
&= \left[\begin{array}{c} x \\ 1 \end{array}\right]^T \left[\begin{array}{cc} P & q \\ q^T & r \end{array}\right] \left[\begin{array}{c} x \\ 1 \end{array}\right]
\end{aligned}
$$

  convex if and only if $P \succeq 0$

- quadratic form $f(x) = x^T P x$
  convex if and only if $P \succeq 0$

- Euclidean norm $f(x) = \|Ax + b\|$
  ($f^2$ is a convex quadratic function . . . )

# Minimizing a quadratic function

$$\text{minimize } f(x) = x^T P x + 2q^T x + r$$

**nonconvex** case ($P \not\succeq 0$): unbounded below

proof: take $x = tv$, $t \to \infty$, where $Pv = \lambda v$, $\lambda < 0$

**convex** case ($P \succeq 0$): $x$ is optimal if and only if

$$\nabla f(x) = 2Px + 2q = 0$$

two cases:

- $q \in \mathrm{range}(P)$: $f^\star > -\infty$
- $q \notin \mathrm{range}(P)$: unbounded below

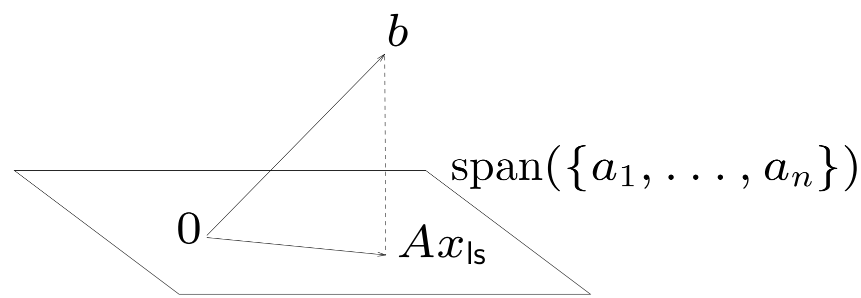
important special case, $P \succ 0$:
unique optimal point $x_{\text{opt}} = -P^{-1}q$; $f^\star = r - q^T P^{-1} q$

# Least-squares

**minimize Euclidean norm** (squared) ($A = [a_1 \cdots a_n]$ full rank, skinny)

$$\text{minimize } \|Ax - b\|^2 = x^T(A^TA)x - 2b^TAx + b^Tb$$

geometrically: project $b$ on $\mathrm{span}(\{a_1, \ldots, a_n\})$




**solution:** set gradient equal to zero

$$x_{\mathsf{ls}} = (A^TA)^{-1}A^Tb$$

general solution, without rank assumption:

$$x_{\mathsf{ls}} = A^{\dagger}b + v$$

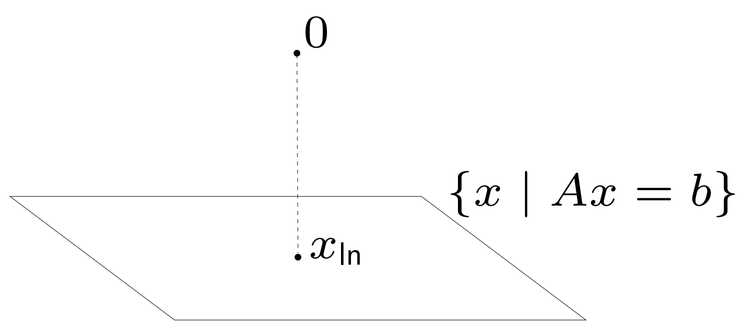
$A^{\dagger}$ is Moore-Penrose inverse of $A$, $v \in \mathcal{N}(A)$

# Least-norm solution of linear equations

($A$ full rank, fat)

$$\begin{array}{ll} \text{minimize} & \|x\|^2 \\ \text{subject to} & Ax = b \end{array}$$

feasible if $b \in \mathcal{R}(A)$ geometrically: project $0$ on $\{x \mid Ax = b\}$




solution: $x_{\mathrm{ln}} = A^T(AA^T)^{-1}b$

general solution, without rank assumption: $x_{\mathrm{ln}} = A^\dagger b$

## Extension: linearly constrained least-squares

$$\begin{array}{ll} \text{minimize} & \|Ax - b\| \\ \text{subject to} & Cx = d \end{array}$$

($C \in \mathbf{R}^{r \times s}$ full rank, fat)

can be solved by elimination: write

$$\{x \mid Cx = d\} = \{Fw + g \mid w \in \mathbf{R}^{s-r}\}$$

- $\text{span}(F) = \text{nullspace}(C)$
- $Cg = d$ (any solution)

and solve

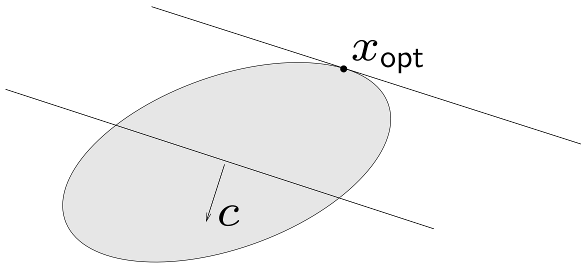
$$\text{minimize } \|AFw + Ag - b\|$$

(may or may not be a good idea in practice)

# Minimizing a linear function with quadratic constraint

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x^T A x \leq 1 \end{array}$$

$(A = A^T \succ 0)$




$$x_{\text{opt}} = -A^{-1}c/\sqrt{c^T A^{-1} c}$$

**proof.** Change of variables $y = A^{1/2}x$, $\tilde{c} = A^{-1/2}c$

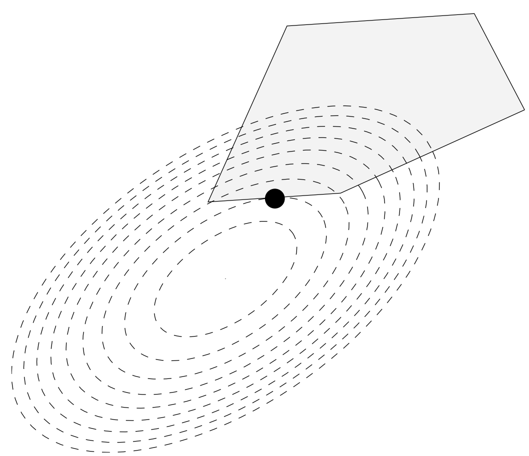
$$\begin{array}{ll} \text{minimize} & \tilde{c}^T y \\ \text{subject to} & y^T y \leq 1 \end{array}$$

optimal solution: $y_{\text{opt}} = -\tilde{c}/\|\tilde{c}\|$

# Quadratic program (QP)

quadratic objective, linear inequalities & equalities

$$
\begin{array}{ll}
\text{minimize} & x^T P x + 2q^T x + r \\
\text{subject to} & Ax \preceq b, \quad Fx = g
\end{array}
$$



convex optimization problem if $P \succeq 0$

**very hard problem** if $P \not\succeq 0$

# QCQP and SOCP

**quadratically constrained quadratic programming (QCQP):**

$$\begin{array}{ll}\text{minimize} & x^T P_0 x + 2q_0^T x + r_0 \\ \text{subject to} & x^T P_i x + 2q_i^T x + r_i \leq 0, \quad i = 1, \ldots, L\end{array}$$

- convex if $P_i \succeq 0$, $i = 0, \ldots, L$
- nonconvex QCQP **very difficult**

**second-order cone programming (SOCP):**

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & \|A_i x + b_i\| \leq e_i^T x + d_i, \quad i = 1, \ldots, L\end{array}$$

includes QCQP (QP, LP)

# Robust linear program

**linear program**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

suppose $a_i$ are uncertain

$$a_i \in \mathcal{E}_i = \{\overline{a}_i + F_i u \mid \|u\| \leq 1\}$$

**robust linear program**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \ \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

Note

$$a_i^T x \leq b_i \ \forall a_i \in \mathcal{E}_i \ \Leftrightarrow \ \overline{a}_i^T x + u^T F_i^T x \leq b_i \ \forall \|u\| \leq 1 \ \Leftrightarrow \ \overline{a}_i^T x + \|F_i^T x\| \leq b_i$$

hence, **robust LP is SOCP**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \overline{a}_i^T x + \|F_i^T x\| \leq b_i, \quad i = 1, \ldots, m \end{array}$$

# Semidefinite programming (SDP)

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n \preceq 0 \\
& Ax = b
\end{array}
$$

where $F_i = F_i^T \in \mathbf{R}^{p \times p}$

- SDP is cvx opt problem in standard form with generalized (matrix) inequality
- multiple LMIs can be combined into one (block diagonal) LMI
- many nonlinear cvx problems can be cast as SDPs

# Examples

**LP as SDP**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

can be expressed as SDP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{diag}(Ax - b) \preceq 0 \end{array}$$

since $Ax - b \preceq 0 \iff \mathbf{diag}(Ax - b) \preceq 0$

**maximum eigenvalue minimization**

$$\text{minimize}_x \ \lambda_{\max}(A(x))$$

$A(x) = A_0 + x_1 A_1 + \cdots + x_m A_m$, $A_i = A_i^T$

SDP with variables $x \in \mathbf{R}^m$ and $t \in \mathbf{R}$:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) - tI \preceq 0 \end{array}$$

# Schur complements

$$X = X^T = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

- $S = C - B^T A^{-1} B$ is the **Schur complement** of $A$ in $X$ (provided $\det A \neq 0$)
- trade dimension with nonlinearity; useful to represent nonlinear convex constraints as LMIs

**facts:** (exercise)

- $X \succ 0$ if and only if $A \succ 0$ and $S \succ 0$
- if $A \succ 0$, then $X \succeq 0$ if and only if $S \succeq 0$

**example.** (convex) quadratic inequality can be cast as an LMIT

$$(Ax+b)^T(Ax+b) - c^T x - d \leq 0 \iff \begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^T x + d \end{bmatrix} \succeq 0$$

# QCQP as SDP

the quadratically constrained quadratic program

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, L \end{array}$$

where $f_i(x) \triangleq (A_i x + b)^T (A_i x + b) - c_i^T x - d_i$

can be expressed as SDP (in $x$ and $t$)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[ \begin{array}{cc} I & A_0 x + b_0 \\ (A_0 x + b_0)^T & c_0^T x + d_0 + t \end{array} \right] \succeq 0, \\ & \\ & \left[ \begin{array}{cc} I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{array} \right] \succeq 0, \quad i = 1, \ldots, L \end{array}$$

extends to problems over second-order cone:

$$\|Ax + b\| \leq e^T x + d \iff \left[ \begin{array}{cc} (e^T x + d) I & Ax + b \\ (Ax + b)^T & e^T x + d \end{array} \right] \succeq 0$$

# Simple nonlinear example

**example:**

$$\text{minimize} \quad \frac{(c^T x)^2}{d^T x}, \quad \text{subject to} \quad Ax \preceq b$$

(assume $d^T x > 0$ whenever $Ax \preceq b$)

1. equivalent problem with linear objective:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax \preceq b, \quad t - \dfrac{(c^T x)^2}{d^T x} \geq 0 \end{array}$$

2. SDP (in $x$, $t$) using Schur complement:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[ \begin{array}{ccc} \mathbf{diag}(b - Ax) & 0 & 0 \\ 0 & t & c^T x \\ 0 & c^T x & d^T x \end{array} \right] \succeq 0 \end{array}$$

# Matrix norm minimization

**example:**

$$\text{minimize } \|A(x)\|$$

where

$$A(x) = A_0 + x_1A_1 + \cdots + x_mA_m, \quad A_i \in \mathbf{R}^{p\times q}$$

and $\|A\| = \sigma_1(A) = \left(\lambda_{\max}(A^TA)\right)^{1/2}$

can cast as SDP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{array}$$

# Convex Conic Optimization Problems

- **Second order cone program (SOCP):**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \|A_i x + b_i\| \leq d_i^T x + e_i \end{array}$$

includes LP, convex QP, and convex quadratically constrained QP as special cases.

- **Semi-definite program (SDP):**

$$\begin{array}{ll} \text{minimize} & \text{tr}(CX) \\ \text{subject to} & \text{tr}(A_i X) = b_i, \quad i = 1, .., m, \\ & X \succeq 0. \end{array}$$

includes LP (if $C$, $A_i$ are diagonal) and SOCP as special cases.

- **Model generality:** LP $<$ QP $<$ QCQP $<$ SOCP $<$ SDP.
- **Solution efficiency:** LP $>$ QP $>$ QCQP $>$ SOCP $>$ SDP. For example, it is much easier to exploit sparsity in LP than in SDP.

# Geometric programming

**monomial function:**

$$f(x) = cx_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$c \geq 0$, $\alpha_j \in \mathbf{R}$, $\mathbf{dom}\, f = \{x \mid x \succ 0\}$

**example:** $f(x) = 2x_1^{2.1} x_2^{-1.7} x_3^{0.5}$

**posynomial function:**

$$f(x) = \sum_{k=1}^{r} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}}$$

$c_k \geq 0$, $\alpha_{ik} \in \mathbf{R}$, $\mathbf{dom}\, f = \{x \mid x \succ 0\}$

**example:** $f(x_1, x_2, x_3) = x_1^2 x_2^{-1} x_3^{0.5} + 2x_1^{2.1} x_2^3$

**geometric program**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \ldots, m \\ & h_i(x) = 1, \quad i = 1, \ldots, p \end{array}$$

$f_0, \ldots, f_m$ posynomial, $h_1, \ldots, h_p$ monomial. **not** a convex problem

# GP in convex form

**transformation of variables:** $y_i = \log x_i$, $x_i = e^{y_i}$:

if $f(x) = c x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ is monomial then

$$\log f(e^{y_1}, \ldots, e^{y_n}) = \alpha_1 y_1 + \cdots + \alpha_n y_n + \beta$$

(where $\beta = \log c$) is ***affine*** in $y$;

if $f(x) = \sum_{k=1}^{r} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}}$ is posynomial then

$$\log f(e^{y_1}, \ldots, e^{y_n}) = \log \sum_{k=1}^{r} \exp(\alpha_{1k} y_1 + \cdots + \alpha_{nk} y_n + \beta_k)$$

(where $\beta_k = \log c_k$) is ***convex*** in $y$

**geometric program in convex form**

$$\begin{array}{ll} \text{minimize} & \log f_0(e^{y_1}, \ldots, e^{y_n}) \\ \text{subject to} & \log f_i(e^{y_1}, \ldots, e^{y_n}) \leq 0, \quad i = 1, \ldots, m \\ & \log h_i(e^{y_1}, \ldots, e^{y_n}) = 0, \quad i = 1, \ldots, p \end{array}$$