Problem 2.4

+	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	0
2	2	3	4	5	6	7	8	9	10	0	1
3	3	4	5	6	7	8	9	10	0	1	2
4	4	5	6	7	8	9	10	0	1	2	3
5	5	6	7	8	9	10	0	1	2	3	4
6	6	7	8	9	10	0	1	2	3	4	5
7	7	8	9	10	0	1	2	3	4	5	6
8	8	9	10	0	1	2	3	4	5	6	7
9	9	10	0	1	2	3	4	5	6	7	8
10	10	0	1	2	3	4	5	6	7	8	9

·	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	1	3	5	7	9
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	6	10	3	7
5	0	5	10	4	9	3	8	2	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5
7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	1
9	0	9	7	5	3	1	10	8	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

element	powers	order
1	1	1
2	$2, 2^2=4, 2^3=8, 2^4=5, 2^5=10, 2^6=9, 2^7=7, 2^8=3, 2^9=6, 2^{10}=1$	10
3	$3, 3^2=9, 3^3=5, 3^4=4, 3^5=1$	5
4	$4, 4^2=5, 4^3=9, 4^4=3, 4^5=1$	5
5	$5, 5^2=3, 5^3=4, 5^4=9, 5^5=1$	5
6	$6, 6^2=3, 6^3=7, 6^4=9, 6^5=10, 6^6=5, 6^7=8, 6^8=4, 6^9=2, 6^{10}=1$	10
7	$7, 7^2=5, 7^3=2, 7^4=3, 7^5=10, 7^6=4, 7^7=6, 7^8=9, 7^9=8, 7^{10}=1$	10
8	$8, 8^2=9, 8^3=6, 8^4=4, 8^5=10, 8^6=3, 8^7=2, 8^8=5, 8^9=7, 8^{10}=1$	10
9	$9, 9^2=4, 9^3=3, 9^4=5, 9^5=1$	5
10	$10, 10^2=1$	2

The primitive elements are 2, 6, 7, 8.

Problem 2.14

power	polynomial	5-tuple	power	polynomial	5-tuple
0	0	0 0 0 0 0	α^{15}	$1+\alpha+\alpha^2+\alpha^3+\alpha^4$	1 1 1 1 1
1	1	1 0 0 0 0	α^{16}	$1+\alpha+\alpha^3+\alpha^4$	1 1 0 1 1
α	α	0 1 0 0 0	α^{17}	$1+\alpha+\alpha^4$	1 1 0 0 1
α^2	α^2	0 0 1 0 0	α^{18}	$1+\alpha$	1 1 0 0 0
α^3	α^3	0 0 0 1 0	α^{19}	$\alpha+\alpha^2$	0 1 1 0 0
α^4	α^4	0 0 0 0 1	α^{20}	$\alpha^2+\alpha^3$	0 0 1 1 0
α^5	$1+\alpha^2$	1 0 1 0 0	α^{21}	$\alpha^3+\alpha^4$	0 0 0 1 1
α^6	$\alpha+\alpha^3$	0 1 0 1 0	α^{22}	$1+\alpha^2+\alpha^4$	1 0 1 0 1
α^7	$\alpha^2+\alpha^4$	0 0 1 0 1	α^{23}	$1+\alpha+\alpha^2+\alpha^3$	1 1 1 1 0
α^8	$1+\alpha^2+\alpha^3$	1 0 1 1 0	α^{24}	$\alpha+\alpha^2+\alpha^3+\alpha^4$	0 1 1 1 1
α^9	$\alpha+\alpha^3+\alpha^4$	0 1 0 1 1	α^{25}	$1+\alpha^3+\alpha^4$	1 0 0 1 1
α^{10}	$1+\alpha^4$	1 0 0 0 1	α^{26}	$1+\alpha+\alpha^2+\alpha^4$	1 1 1 0 1
α^{11}	$1+\alpha+\alpha^2$	1 1 1 0 0	α^{27}	$1+\alpha+\alpha^3$	1 1 0 1 0
α^{12}	$\alpha+\alpha^2+\alpha^3$	0 1 1 1 0	α^{28}	$\alpha+\alpha^2+\alpha^4$	0 1 1 0 1
α^{13}	$\alpha^2+\alpha^3+\alpha^4$	0 0 1 1 1	α^{29}	$1+\alpha^3$	1 0 0 1 0
α^{14}	$1+\alpha^2+\alpha^3+\alpha^4$	1 0 1 1 1	α^{30}	$\alpha+\alpha^4$	0 1 0 0 1

Minimal polynomial of $\alpha^3 = (x+\alpha^3)(x+\alpha^6)(x+\alpha^{12})(x+\alpha^{24})(x+\alpha^{17}) =$

$$x^5+(\alpha^3+\alpha^6+\alpha^{12}+\alpha^{24}+\alpha^{17})x^4+(\alpha^9+\alpha^{15}+\alpha^{27}+\alpha^{20}+\alpha^{18}+\alpha^{30}+\alpha^{23}+\alpha^5+\alpha^{29}+\alpha^{10})x^3$$

$$+(\alpha^{21}+\alpha^2+\alpha^{26}+\alpha^8+\alpha+\alpha^{13}+\alpha^{11}+\alpha^4+\alpha^{16}+\alpha^{22})x^2+(\alpha^{28}+\alpha^{25}+\alpha^{19}+\alpha^7+\alpha^{14})x+1$$

$$= x^5+x^4+x^3+x^2+1$$

Minimal polynomial of $x^7 = (x+\alpha^7)(x+\alpha^{14})(x+\alpha^{28})(x+\alpha^{25})(x+\alpha^{19}) =$

$$x^5+(\alpha^7+\alpha^{14}+\alpha^{28}+\alpha^{25}+\alpha^{19})x^4$$

$$+(\alpha^{21}+\alpha^4+\alpha+\alpha^{26}+\alpha^{11}+\alpha^8+\alpha^2+\alpha^{22}+\alpha^{16}+\alpha^{13})x^3$$

$$+(\alpha^{18}+\alpha^{15}+\alpha^9+\alpha^{29}+\alpha^{23}+\alpha^{20}+\alpha^5+\alpha^{30}+\alpha^{27}+\alpha^{10})x^2$$

$$+(\alpha^{12}+\alpha^6+\alpha^3+\alpha^{17}+\alpha^{24})x+1$$

$$= x^5+x^3+x^2+x+1.$$

Problem 2.17

	x	$f(x) = x^3 + \alpha^6 x^2 + \alpha^9 x + \alpha^9$
	0	$0 + 0 + 0 + \alpha^9 = \alpha^9$
	1	$1 + \alpha^6 + \alpha^9 + \alpha^9 = \alpha^{13}$
root $\longrightarrow$	α	$\alpha^3 + \alpha^8 + \alpha^{10} + \alpha^9 = 0$
	α^2	$\alpha^6 + \alpha^{10} + \alpha^{11} + \alpha^9 = \alpha^{12}$
root $\longrightarrow$	α^3	$\alpha^9 + \alpha^{12} + \alpha^{12} + \alpha^9 = 0$
	α^4	$\alpha^{12} + \alpha^{14} + \alpha^{13} + \alpha^9 = 1$
root $\longrightarrow$	α^5	$1 + \alpha + \alpha^{14} + \alpha^9 = 0$

Since $f(x)$ has degree 3, it has 3 roots.

Therefore, the roots of $f(x)$ are $\alpha, \alpha^3, \alpha^5$.

It is easy to check that $(x+\alpha)(x+\alpha^3)(x+\alpha^5) = f(x)$.

Problem 2.19

$$X + \alpha^5 Y + Z = \alpha^7 \quad \text{——— } ①$$
$$X + \alpha Y + \alpha^7 Z = \alpha^9 \quad \text{——— } ②$$
$$\alpha^2 X + Y + \alpha^6 Z = \alpha \quad \text{——— } ③$$

Adding ① and ②: $(\alpha^5 + \alpha)Y + (1 + \alpha^7)Z = \alpha^7 + \alpha^9$

i.e., $\alpha^2 Y + \alpha^9 Z = 1$ ——— ④

Multiplying ① by α^2 and adding to ③: $(\alpha^7 + 1)Y + (\alpha^2 + \alpha^6)Z = \alpha^9 + \alpha$

i.e., $\alpha^9 Y + \alpha^3 Z = \alpha^3$ ——— ⑤

Multiplying ④ by α^7 and adding to ⑤: $(\alpha^{16} + \alpha^3)Z = \alpha^7 + \alpha^3$

i.e., $(\alpha + \alpha^3)Z = \alpha^7 + \alpha^3 \Rightarrow \alpha^9 Z = \alpha^4 \Rightarrow \underline{\underline{Z = \alpha^{-5} = \alpha^{10}}}$ ——— ⑥

Substituting ⑥ in ④: $\alpha^2 Y + \alpha^4 = 1$.

i.e., $\alpha^2 Y = \alpha^4 + 1 = \alpha \Rightarrow \underline{\underline{Y = \alpha^{-1} = \alpha^{14}}}$ ——— ⑦

Substituting ⑥ and ⑦ in ① gives:

$X + \alpha^4 + \alpha^{10} = \alpha^7 \Rightarrow \underline{\underline{X = \alpha^7 + \alpha^4 + \alpha^{10} = \alpha^{12}}}$ ——— ⑧

In summary:

$X = \alpha^{12}$, $Y = \alpha^{14}$, and $Z = \alpha^{10}$