Lecture 6

Non-binary BCH Codes

So far, we have only considered block codes with symbols from binary field GF(2). Block codes with symbols from non-binary fields can be constructed in exactly the same manner as for constructing binary block codes. Block codes with code symbols from GF(q) where q is a power of prime are called q-ary block codes (or block codes over GF(q)). A q-ary (n, k) block code has length n and q^k code words. A message for a q-ary (n, k) block code consists of k information symbols from GF(q). Non-binary codes are effective in combating mixed types of errors caused by the channel noise and interferences.

6.1 Introduction

- **Definition 6.1**: A q-ary (n, k) block code of length n with q^k code words is called a q-ary (n, k) linear block code if and only if its q^k code words form a k-dimensional subspace of the vector space of all q^n n-tuples over GF(q).
- All the fundamental concepts and structural properties developed for binary linear block codes (including cyclic codes) in the previous lectures apply to q-ary linear block codes with few modifications. We simply replace GF(2) with GF(q).
- A q-ary (n, k) linear block code is specified by either a $k \times n$ generator matrix $\mathbf{G}$ or an $(n - k) \times n$ parity-check matrix $\mathbf{H}$ over GF(q). Generator and parity-check matrices of a q-ary (n, k) linear block codes in systematic form are exactly the same forms as given by (3.9) and (3.14), except that the entries are from GF(q).

- Encoding and decoding of q-ary linear block codes are the same as for binary codes, except that operations and computations are performed over GF(q).

- A q-ary (n, k) cyclic code C is generated by a monic polynomial of degree $n-k$ over GF(q),

$$\mathbf{g}(X) = g_0 + g_1 X + \cdots + g_{n-k-1} X^{n-k-1} + X^{n-k},$$

  where $g_0 \neq 0$ and $g_i \in$ GF(q). This generator polynomial $\mathbf{g}(X)$ is a factor of $X^{q^m-1} - 1$. A polynomial $\mathbf{v}(X)$ of degree $n-1$ or less over GF(q) is a code polynomial if and only if $\mathbf{v}(X)$ is divisible by the generator polynomial $\mathbf{g}(X)$.

6.2. Non-Binary Primitive BCH Codes

- Let GF(q^m) be an extension field of GF(q) and α be a primitive element of GF(q^m). A q-ary t-symbol-error-correction primitive BCH code $C_{q,bch,t}$ of length $q^m - 1$ over GF(q) is a cyclic code generated by the smallest-degree polynomial $\mathbf{g}(X)$ over GF(q) that has $\alpha, \alpha^2, \ldots, \alpha^{2t}$ and their conjugates as roots. For $1 \leq i \leq 2t$, let $\phi_i(X)$ be the (monic) minimal polynomial of α^i over GF(q). Then

$$\mathbf{g}(x) = LCM\{\phi_1(X), \phi_2(X), \ldots, \phi_{2t}(X)\}. \tag{6.1}$$

  where $\phi_1(X)$ is a primitive polynomial.

- Since the degree of the minimal polynomial of an element in GF(q^m) is at most m, the degree of g(X) is at most $2mt$ and g(X) divides $X^{q^m-1} - 1$. The q-ary t-symbol-correction primitive BCH code $C_{q,bch,t}$ has length $q^m - 1$ with dimension at least $q^m - 2mt - 1$.

- A q-ary t-symbol-error-correcting BCH code can be characterized by a theorem similar to Theorem 5.1 that characterizes a binary t-error-correcting BCH code.

- **Theorem 6.1**: Let $n = q^m - 1$ and α be a primitive element of GF(q^m). A polynomial

$$\mathbf{v}(X) = v_0 + v_1 X + v_2 X^2 + \cdots + v_{n-1} X^{n-1}$$

over GF(q) is a code polynomial if and only if it has $\alpha, \alpha^2, \ldots, \alpha^{2t}$ as roots.

- The parity-check matrix $\mathbf{H}$ of a q-ary BCH code in terms of its roots is exactly the same as that given by (5.11),

$$\mathbf{H} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^2 & (\alpha^2)^2 & \cdots & (\alpha^2)^{n-1} \\ 1 & \alpha^3 & (\alpha^3)^2 & \cdots & (\alpha^3)^{n-1} \\ \vdots & & & & \vdots \\ 1 & \alpha^{2t} & (\alpha^{2t})^2 & \cdots & (\alpha^{2t})^{n-1} \end{bmatrix} \tag{6.2}$$

- In the same manner, we can prove that no $2t$ or fewer columns of $\mathbf{H}$ can be added to a zero column vector. Hence, the code has minimum distance at least $2t+1$ (BCH bound) and is capable of correcting t or fewer random symbol errors over a span of $q^m - 1$ symbol positions.

- We can characterize a q-ary t-symbol-error-correcting BCH code $C_{q,bch,t}$ in terms of the parity-check matrix given by (6.2).

- **Theorem 6.2**: Let $n = q^m - 1$. An n-tuple $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$ over GF(q) is a code word in the q-ary t-symbol-error-correcting BCH code $C_{q,bch,t}$ if and only if

$$\mathbf{v}\mathbf{H}^T = \mathbf{0}, \tag{6.3}$$

where $\mathbf{0} = (0, 0, \ldots, 0)$ is a zero $2t$-tuple.

- For a given field GF(q^m), a family of q-ary BCH codes can be constructed.

Decoding of q-ary BCH Codes

- Suppose a code **polynomial**

$$\mathbf{v}(X) = v_0 + v_1X + v_2X^2 + \cdots + v_{n-1}X^{n-1}$$

in a q-ary BCH code $C_{q,bch,t}$ is transmitted and

$$\mathbf{r}(X) = r_0 + r_1X + r_2X^2 + \cdots + r_{n-1}X^{n-1}$$

is the corresponding received polynomial.

- Both $\mathbf{v}(X)$ and $\mathbf{r}(X)$ are polynomials over GF(q). The difference between them

$$\begin{aligned} \mathbf{e}(X) &= \mathbf{r}(X) - \mathbf{v}(X) \\ &= \mathbf{e}(X) = e_0 + e_1X + e_2X^2 + \cdots + e_{n-1}X^{n-1} \end{aligned} \quad (6.4)$$

is defined as the error polynomial, where $e_0, e_1, \ldots, e_{n-1}$ are elements in GF(q).

- Decoding a q-ary t-symbol-error-correcting BCH code can be accomplished in a manner similar to decoding of a binary t-error-correcting BCH code. However, an additional step is needed to determine the values of errors at the error locations.

- Decoding consists of following steps:

  (1) Compute the syndrome of the received polynomial $\mathbf{r}(X)$.

  (2) Find the error-location polynomial $\sigma(X)$.

  (3) Determine the error locations.

  (4) Compute the values of errors at the error locations.

  (5) Perform error correction.

- The Berlekamp-Massey iterative algorithm presented in Section 5.8 of Lecture 5 can be used to find the error-location polynomial $\sigma(X)$, but $2t$ steps are needed.

6.4 Syndrome and Error Pattern

- The syndrome of a received polynomial $\mathbf{r}(X)$ is given by a $2t$-tuple over GF(q^m),

$$\mathbf{S} = (S_1, S_2, \ldots, S_{2t})$$

with

$$S_i = \mathbf{r}(\alpha^i) = r_0 + r_1\alpha^i + \cdots + r_{n-1}\alpha^{(n-1)i}, \tag{6.5}$$

for $1 \leq i \leq 2t$, where addition and multiplication are carried out in GF(q^m).

- Suppose the error polynomial $\mathbf{e}(X)$ contains ν errors at the locations X^{j_1}, $X^{j_2}, \cdots, X^{j_\nu}$. Then

$$\mathbf{e}(X) = e_{j_1}X^{j_1} + e_{j_2}X^{j_2} + \cdots + e_{j_\nu}X^{j_\nu} \tag{6.6}$$

where $e_{j_1}, e_{j_2}, \cdots, e_{j_\nu}$ are the values of errors at the locations, X^{j_1}, X^{j_2}, $\ldots, X^{j_\nu}$. These error values are elements of GF(q).

- Since $\mathbf{r}(X) = \mathbf{v}(X) + \mathbf{e}(X)$, then for $1 \leq i \leq 2t$, the i-th component S_i of the syndrome $\mathbf{S} = (S_1, S_2, \ldots, S_{2t})$ of the received polynomial $\mathbf{r}(X)$ is related to the error pattern as follows:

$$\begin{aligned} S_i &= \mathbf{v}(\alpha^i) + \mathbf{e}(\alpha^i) \\ &= \mathbf{e}(\alpha^i). \end{aligned} \tag{6.7}$$

- From (6.6) and (6.7), we obtain following equalities that relate the error locations and values to the computed syndrome:

$$\begin{aligned} S_1 &= e_{j_1}\alpha^{j_1} + e_{j_2}\alpha^{j_2} + \cdots + e_{j_\nu}\alpha^{j_\nu} \\ S_2 &= e_{j_1}\alpha^{2j_1} + e_{j_2}\alpha^{2j_2} + \cdots + e_{j_\nu}\alpha^{2j_\nu} \\ &\vdots \\ S_{2t} &= e_{j_1}\alpha^{2tj_1} + e_{j_2}\alpha^{2tj_2} + \cdots + e_{j_\nu}\alpha^{2tj_\nu}. \end{aligned} \tag{6.8}$$

- For $1 \leq i \leq \nu$, let

$$\beta_i = \alpha^{j_i}, \qquad \delta_i = e_{j_i}. \tag{6.9}$$

- The elements, $\beta_1, \beta_2, \ldots, \beta_\nu$, and $\delta_1, \delta_2, \ldots, \delta_\nu$ called the **error-location numbers** and **error-values**, respectively.

- With the above definitions of β_i and δ_i, the equalities of (6.8) can be simplified as follows:

$$\begin{aligned} S_1 &= \delta_1\beta_1 + \delta_2\beta_2 + \cdots + \delta_\nu\beta_\nu \\ S_2 &= \delta_1\beta_1^2 + \delta_2\beta_2^2 + \cdots + \delta_\nu\beta_\nu^2 \\ &\vdots \\ S_{2t} &= \delta_1\beta_1^{2t} + \delta_2\beta_2^{2t} + \cdots + \delta_\nu\beta_\nu^{2t}. \end{aligned} \tag{6.10}$$

6.5 Error-Location Polynomial

- The error-location polynomial is defined as follows:

$$\begin{aligned}\sigma(X) &= (1-\beta_1 X)(1-\beta_2 X)\cdots(1-\beta_\nu X) \\ &= \sigma_0 + \sigma_1 X + \sigma_2 X^2 + \cdots + \sigma_\nu X^\nu \end{aligned} \tag{6.11}$$

where

$$\begin{aligned}\sigma_0 &= 1 \\ \sigma_1 &= -(\beta_1 + \beta_2 + \cdots + \beta_\nu) \\ \sigma_2 &= (-1)^2(\beta_1\beta_2 + \beta_1\beta_3 + \cdots + \beta_{\nu-1}\beta_\nu) \\ &\vdots \\ \sigma_\nu &= (-1)^\nu \beta_1\beta_2\cdots\beta_\nu \end{aligned} \tag{6.12}$$

- We readily see that $\beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_\nu^{-1}$ are the roots of the error-location polynomial $\sigma(X)$ and their inverses, $\beta_1, \beta_2, \ldots, \beta_\nu$ are the location numbers.

- From (6.10) and (6.12), it is possible (see Appendix 6-A) to obtain the following set of equalities that relate the coefficients of the error-location polynomial $\sigma(X)$ and the computed syndrome components:

$$
\begin{aligned}
&S_{\nu+1} + \sigma_1 S_\nu + \sigma_2 S_{\nu-1} + \cdots + \sigma_\nu S_1 = 0 \\
&S_{\nu+2} + \sigma_1 S_{\nu+1} + \sigma_2 S_\nu + \cdots + \sigma_\nu S_2 = 0 \\
&\vdots \\
&S_{2t} + \sigma_1 S_{2t-1} + \sigma_2 S_{2t-2} + \cdots + \sigma_\nu S_{2t-\nu} = 0.
\end{aligned}
\tag{6.13}
$$

- The above $2t - \nu$ equalities are called **generalized Newton's identities**.

- Our objective is to find the minimum-degree polynomial $\sigma(X)$ whose coefficients satisfy the $2t - \nu$ generalized Newton's identities.

- This can be accomplished with the Berlekamp-Massey iterative algorithm with $2t$ step as described in Sections (5.9) and (5.10) of Lecture 5.

- At the μ-th step, determine a polynomial of minimum-degree

$$\sigma^{(\mu)}(X) = \sigma_0^{(\mu)} + \sigma_1^{(\mu)} X + \cdots + \sigma_{l_\mu}^{(\mu)} X^{l_\mu} \tag{6.14}$$

whose coefficients satisfy the following $\mu - l_\mu$ generalized Newton's identities:

$$\begin{aligned} &S_{l_\mu+1} + \sigma_1^{(\mu)} S_{l_\mu} + \cdots + \sigma_{l_\mu}^{(\mu)} S_1 = 0 \\ &S_{l_\mu+2} + \sigma_1^{(\mu)} S_{l_\mu+1} + \cdots + \sigma_{l_\mu}^{(\mu)} S_2 = 0 \\ &\vdots \\ &S_\mu + \sigma_1^{(\mu)} S_{\mu-1} + \cdots + \sigma_{l_\mu}^{(\mu)} S_{\mu-l_\mu} = 0. \end{aligned} \tag{6.15}$$

- To find the solution $\sigma^{(\mu+1)}(X)$ at the $(\mu+1)$-th step, we check whether the coefficients of $\sigma^{(\mu)}(X)$ satisfy the next generalized Newton's identity. To do this, we compute the **discrepancy**

$$d_\mu = S_{\mu+1} + \sigma_1^{(\mu)} S_\mu + \cdots + \sigma_{l_\mu}^{(\mu)} S_{\mu+1-l_\mu}. \tag{6.16}$$

- If $d_\mu = 0$, the coefficients of the current solution $\sigma^{(\mu)}(X)$ satisfy the $(\mu+1-l_\mu)$-th identity, i.e.,

$$S_{\mu+1} + \sigma_1^{(\mu)} S_\mu + \cdots + \sigma_{l_\mu}^{(\mu)} S_{\mu+1-l_\mu} = 0.$$

- In this case, we set

$$\sigma^{(\mu+1)}(X) = \sigma^{(\mu)}(X).$$

- If $d_\mu \neq 0$, a correction term with minimum-degree is added to $\sigma^{(\mu)}(X)$ to obtain the solution for the $(\mu+1)$-th step,

$$\sigma^{(\mu+1)}(X) = \sigma_0^{(\mu+1)} + \sigma_1^{(\mu+1)} X + \cdots + \sigma_{l_{\mu+1}}^{(\mu+1)} X^{l_{\mu+1}} \tag{6.17}$$

whose coefficients satisfy the first $\mu + 1 - l_\mu$ generalized Newton's identities of (6.11),

$$\begin{aligned} & S_{l_{\mu+1}+1} + \sigma_1^{(\mu+1)} S_{l_{\mu+1}} + \cdots + \sigma_{l_{\mu+1}}^{(\mu+1)} S_1 = 0 \\ & S_{l_{\mu+1}+2} + \sigma_1^{(\mu+1)} S_{l_{\mu+1}+1} + \cdots + \sigma_{l_{\mu+1}}^{(\mu+1)} S_2 = 0 \\ & \vdots \\ & S_{\mu+1} + \sigma_1^{(\mu+1)} S_\mu + \cdots + \sigma_{l_{\mu+1}}^{(\mu+1)} S_{\mu+1-l_{\mu+1}} = 0. \end{aligned} \tag{6.18}$$

- **Correction**: Go back to the steps prior to the μ-th step and determine a polynomial $\sigma^{(\rho)}(X)$ such that $d_\rho \neq 0$ and $\rho - l_\rho$ has the largest value, where l_ρ is the degree of $\sigma^{(\rho)}(X)$. Then

$$\sigma^{(\mu+1)}(X) = \sigma^{(\mu)}(X) - d_\mu d_\rho^{-1} X^{(\mu-\rho)} \tag{6.19}$$

  is the solution at the $(\mu + 1)$-th step of the iteration process.

- Continue the above iterative process until $2t$ steps have been completed. At the $2t$-th step, we have

$$\sigma(X) = \sigma^{(2t)}(X), \tag{6.20}$$

  which is the minimum-degree polynomial whose coefficients satisfy the $2t - \nu$ generalized Newton's identities given by (6.13).

- If $\nu \leq t$ (the designed error-correcting capability), $\sigma^{(2t)}$ is unique and the true error-location polynomial with all its roots in GF(q^m).

- To execute the above algorithm for finding the error-location polynomial, we set up and fill a table as given below;

Table 6.1 Berlekamp-Massey algorithm for finding the error-location polynomial of a q-ary BCH code

step	Partial solution	Discrepancy	Degree	Step/degree difference
μ	$\sigma^{(\mu)}(X)$	d_μ	l_μ	$\mu - l_\mu$
-1	1	1	0	-1
0	1	S_1	0	0
1	1-S_1X			
2				
$\vdots$				
$2t$				

6.6 Error-Value Evaluator

- Once the error-location polynomial $\sigma(X) = \sigma_0 + \sigma_1 X + \cdots + \sigma_\nu X^\nu$ is found. We determine its roots $\beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_\nu^{-1}$ and error-location numbers $\beta_1, \beta_2, \ldots, \beta_\nu$.

- Define a polynomial from the syndrome components and the coefficients of the error-location polynomial as follow:

$$\begin{aligned} \mathbf{Z}_0(X) &= S_1 + (S_2 + \sigma_1 S_1)X + (S_3 + \sigma_1 S_2 + \sigma_2 S_1)X^2 \\ &\quad + \cdots + (S_\nu + \sigma_1 S_{\nu-1} + \cdots + \sigma_{\nu-1} S_1)X^{\nu-1}. \end{aligned} \tag{6.21}$$

- We can show that

$$\mathbf{Z}_0(X) = \sum_{l=1}^{\nu} \delta_l \beta_l \prod_{i=1, i \neq l}^{\nu} (1 - \beta_i X). \tag{6.22}$$

- Substituting the variable X of $\mathbf{Z}_0(X)$ with β_k^{-1} for $1 \leq k \leq \nu$, we have

$$\begin{aligned}
\mathbf{Z}_0(\beta_k^{-1}) &= \sum_{l=1}^{\nu} \delta_l \beta_l \prod_{i=1, i\neq l}^{\nu} (1 - \beta_i \beta_k^{-1}) \\
&= \delta_k \beta_k \prod_{i=1, i\neq k}^{\nu} (1 - \beta_i \beta_k^{-1}). \qquad (6.23)
\end{aligned}$$

- Consider the derivative of the error-location polynomial $\sigma(X)$ given by (6.11),

$$\begin{aligned}
\sigma'(X) &= \frac{d}{dX} \prod_{i=1}^{\nu} (1 - \beta_i X) \\
&= -\sum_{l=1}^{\nu} \beta_l \prod_{i=1, i\neq l}^{\nu} (1 - \beta_i X). \qquad (6.24)
\end{aligned}$$

- Substituting X of $\sigma'(X)$ with β_k^{-1}, we have

$$\sigma'(\beta_k^{-1}) = -\beta_k \prod_{i=1, i\neq k}^{\nu} (1 - \beta_i \beta_k^{-1}). \tag{6.25}$$

- From (6.20) and (6.22), we obtain the value δ_k of error at the location β_k,

$$\delta_k = \frac{-\mathbf{Z}_0(\beta_k^{-1})}{\sigma'(\beta_k^{-1})}.$$

- $\mathbf{Z}_0$ is called **error-value evaluator**.

6.7 Decoding Procedure for a q-ary BCH Code

1. Compute the syndrome $\mathbf{S} = (S_1, S_2, \ldots, S_{2t})$ of the received polynomial $\mathbf{r}(X)$.
2. Determine the error-location polynomial $\sigma(X)$.
3. Determine the error-value evaluator $\mathbf{Z}_0(X)$.
4. Determine the error-location numbers and evaluate the error values at the locations of errors.
5. Perform error correction by subtracting the error-pattern $\mathbf{e}(X)$ from the received polynomial $\mathbf{r}(X)$.

6.8 Finding the Roots of $\sigma(X)$

- The roots of $\sigma(X)$ in GF(q^m) can be determined by substituting the elements $\alpha^0, \alpha, \ldots, \alpha^{q^m-2}$ of GF(q^m) into $\sigma(X)$ in turn. If $\sigma(\alpha^i) = 0$, then α^i is a root of $\sigma(X)$ and
$$\alpha^{-i} = \alpha^{q^m-1-i}$$
is an error-location number.
- Then the decoded symbol at the location $q^m - 1 - i$ is
$$v_{q^m-1-i} = r_{q^m-1-i} - e_{q^m-1-i}.$$
- A general organization of a q-ary BCH decoder is shown in Figure 6.1.

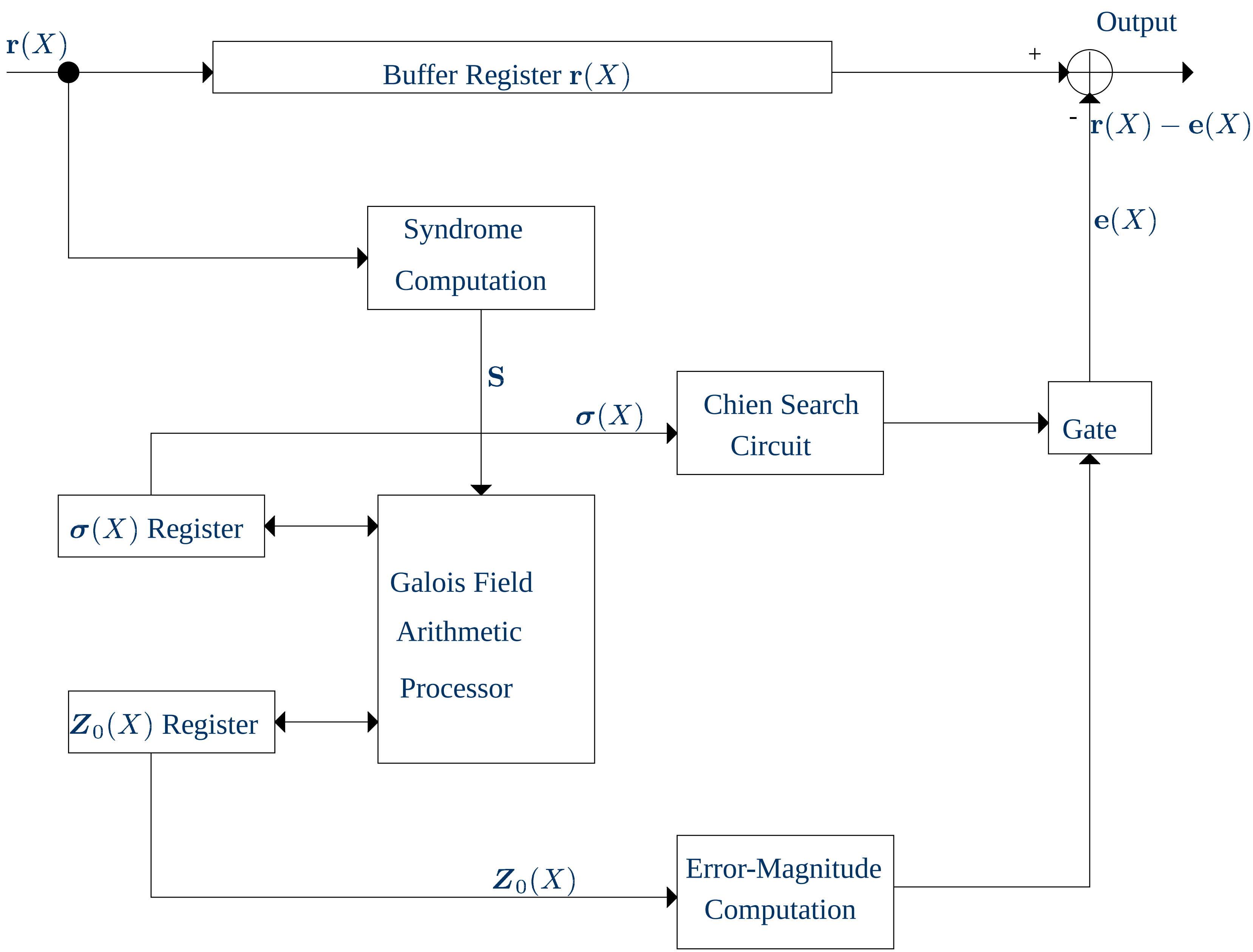

Figure 6.1: A general organization of a q-ary BCH decoder.

Appendix 6-A: Derivation of Generalized Newton's

- Define a syndrome polynomial as follows:

$$\begin{aligned} \mathbf{S}(X) &= S_1 + S_2 X + \cdots + S_{2t} X^{2t-1} + S_{2t+1} X^{2t} + \cdots \\ &= \sum_{1}^{\infty} S_j X^{j-1}. \end{aligned} \tag{6-A.1}$$

Note that only the coefficients of the first $2t$ terms of $\mathbf{S}(X)$ are known.

- Recall that

$$S_j = \sum_{l=1}^{\nu} \delta_l \beta_l^j. \tag{6-A.2}$$

- Substituting S_j with the expression of (6-A.2), we have

$$\begin{aligned} \mathbf{S}(X) &= \sum_{j=1}^{\infty} X^{j-1} \sum_{l=1}^{\nu} \delta_l \beta_l^j \\ &= \sum_{l=1}^{\nu} \delta_l \beta_l \sum_{j=1}^{\infty} (\beta_l X)^{j-1}. \end{aligned} \tag{6-A.3}$$

- Note that

$$\frac{1}{(1-\beta_l X)} = \sum_{j=1}^{\infty} (\beta_l X)^{j-1}. \tag{6-A.4}$$

- Combining (6-A.3) and (6-A.4), we obtain

$$\mathbf{S}(X) = \sum_{l=1}^{\nu} \frac{\delta_l \beta_l}{1-\beta_l X}. \tag{6-A.5}$$

- Recall that the expression of the error-location polynomial $\sigma(X)$ given in (6.11),

$$\begin{aligned}\sigma(X) &= \prod_{i=1}^{\nu}(1-\beta_i X) \\ &= 1+\sigma_1 X+\cdots+\sigma_\nu X^\nu. \end{aligned} \tag{6-A.6}$$

- Consider the product $\sigma(X)S(X)$,

$$\begin{aligned}\sigma(X)\mathbf{S}(X) &= (1+\sigma_1 X+\cdots+\sigma_\nu X^\nu)\cdot(S_1+S_2X+S_3X^2+\cdots) \\ &= S_1+(S_2+\sigma_1 S_1)X+(S_3+\sigma_1 S_2+\sigma_2 S_1)X^2+\cdots+ \\ &\quad (S_{2t}+\sigma_1 S_{2t-1}+\cdots+\sigma_\nu S_{2t-\nu})X^{2t-1}+\cdots \end{aligned} \tag{6-A.7}$$

- However, if we use expression of $\mathbf{S}(X)$ given by (6-A.5) and the product expression of $\sigma(X)$ given in (6-A.6), the product $\sigma(X)S(X)$ can be expressed as follows:

$$\begin{aligned} \sigma(X)\mathbf{S}(X) &= \left\{\prod_{i=1}^{\nu}(1-\beta_i X)\right\} \cdot \left\{\sum_{l=1}^{\nu} \frac{\delta_l \beta_l}{1-\beta_l X}\right\} \\ &= \sum_{l=1}^{\nu} \frac{\delta_l \beta_l}{1-\beta_l X} \cdot \prod_{i=1}^{\nu}(1-\beta_i X) \\ &= \sum_{l=1}^{\nu} \delta_l \beta_l \prod_{i=1, i \neq l}^{\nu} (1-\beta_i X). \end{aligned} \tag{6-A.8}$$

- Note from the expression of (6-A.8), we see that $\sigma(X)S(X)$ is a polynomial of degree $\nu - 1$.

- Equating the two expressions of $\sigma(X)S(X)$ given by (6-A.7) and (6-A.8), we find the coefficients of X^{ν} to X^{2t-1} must be equal to zero, i.e.,

$$\begin{aligned}
&S_{\nu+1}+\sigma_1 S_{\nu}+\sigma_2 S_{\nu-1}+\cdots+\sigma_{\nu} S_1=0\\
&S_{\nu+2}+\sigma_1 S_{\nu+1}+\sigma_2 S_{\nu}+\cdots+\sigma_{\nu} S_2=0\\
&\vdots\\
&S_{2t}+\sigma_1 S_{2t-1}+\sigma_2 S_{2t-2}+\cdots+\sigma_{\nu} S_{2t-\nu}=0.
\end{aligned} \tag{6-A.9}$$

- The above equalities are the $2t-\nu$ generalized Newton's identities given by (6.13).