

Chapter 4: Dynamic Programming

Objectives of this chapter:

- Overview of a collection of classical solution methods for MDPs known as dynamic programming (DP)
- Show how DP can be used to compute value functions, and hence, optimal policies
- Discuss efficiency of DP

Policy Evaluation

Policy Evaluation: for a given policy π , compute the state-value function v_π

Recall: **State-value function for policy π**

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s \right]$$

Recall: **Bellman equation for v_π**

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a) [r + \gamma v_\pi(s')]$$

— a system of $|S|$ simultaneous equations

Iterative Methods

$$v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k \rightarrow v_{k+1} \rightarrow \dots \rightarrow v_\pi$$

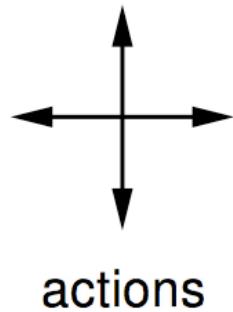
a “sweep” 

A sweep consists of applying a **backup operation** to each state.

A **full policy-evaluation backup**:

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a) [r + \gamma v_k(s')] \quad \forall s \in \mathcal{S}$$

A Small Gridworld



	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$R = -1$
on all transitions

$$\gamma = 1$$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . . , 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is -1 until the terminal state is reached

Iterative Policy Eval for the Small Gridworld

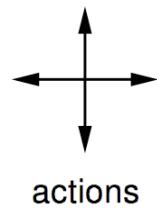
V_k for the
Random Policy

π = equiprobable random action choices

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$



	1	2	3
4	5	6	7
8	9	10	11
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$R = -1$
on all transitions

$\gamma = 1$

$k = 2$

$k = 3$

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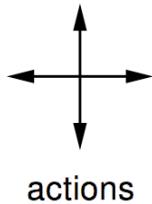
$k = 10$

$k = \infty$

Iterative Policy Eval for the Small Gridworld

V_k for the
Random Policy

π = equiprobable random action choices



	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$R = -1$
on all transitions

$\gamma = 1$

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

$k = 3$

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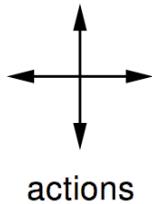
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$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

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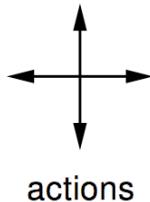
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$k = \infty$

Iterative Policy Eval for the Small Gridworld

V_k for the
Random Policy

π = equiprobable random action choices



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on all transitions

$\gamma = 1$

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

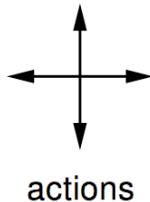
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$k = \infty$

Iterative Policy Eval for the Small Gridworld

V_k for the
Random Policy

π = equiprobable random action choices



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8		10	11
12	13	14	

$R = -1$
on all transitions

$\gamma = 1$

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Iterative Policy Evaluation – One array version

Input π , the policy to be evaluated

Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$

Repeat

$$\Delta \leftarrow 0$$

For each $s \in \mathcal{S}$:

$$v \leftarrow V(s)$$

$$V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a) [r + \gamma V(s')]$$

$$\Delta \leftarrow \max(\Delta, |v - V(s)|)$$

until $\Delta < \theta$ (a small positive number)

Output $V \approx v_\pi$

Policy Improvement

Suppose we have computed v_π for a deterministic policy π .

For a given state s ,
would it be better to do an action $a \neq \pi(s)$?

It is better to switch to action a for state s if and only if

$$q_\pi(s, a) > v_\pi(s)$$

And, we can compute $q_\pi(s, a)$ from v_π by:

$$\begin{aligned} q_\pi(s, a) &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \sum_{s', r} p(s', r | s, a) [r + \gamma v_\pi(s')]. \end{aligned}$$

Policy Improvement Cont.

Do this for all states to get a new policy $\pi' \geq \pi$ that is **greedy** with respect to v_π :

$$\begin{aligned}\pi'(s) &= \arg \max_a q_\pi(s, a) \\ &= \arg \max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \arg \max_a \sum_{s', r} p(s', r | s, a) [r + \gamma v_\pi(s')],\end{aligned}$$

What if the policy is unchanged by this?
Then the policy must be optimal!

Equivalently

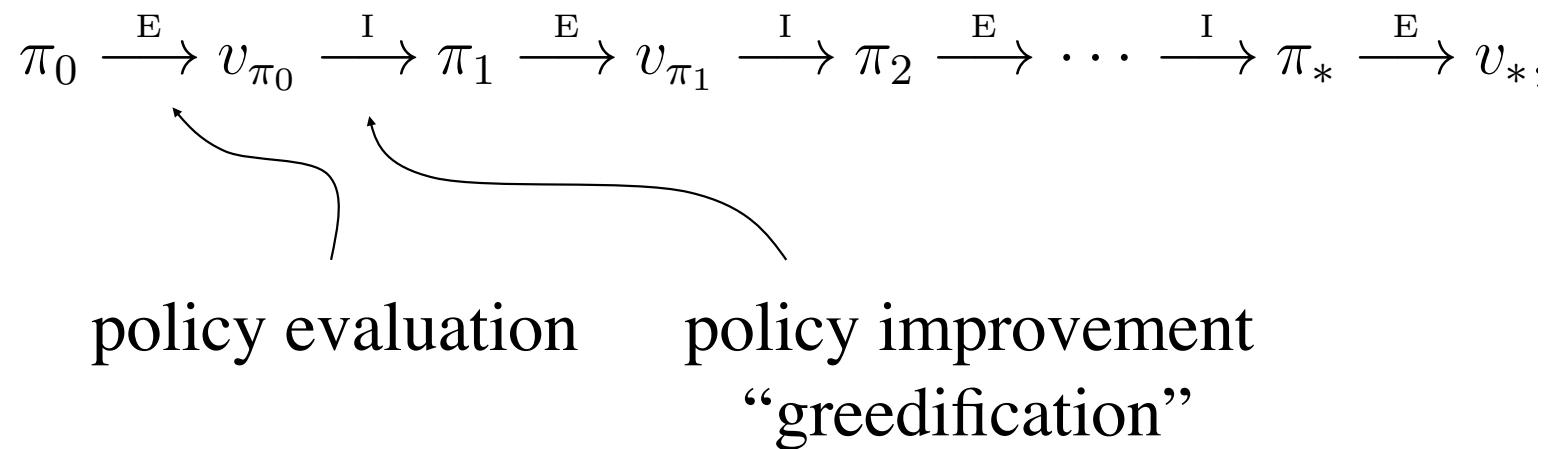
What if $v_{\pi'} = v_{\pi}$?

i.e., for all $s \in \mathcal{S}$, $v_{\pi}(s) = \max_a \sum_{s'} p(s' | s, a) [r(s, a, s') + \gamma v_{\pi}(s')] ?$

But this is the Bellman Optimality Equation.

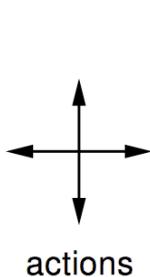
So $v_{\pi} = v_*$ and both π and π' are optimal policies.

Policy Iteration



Iterative Policy Eval for the Small Gridworld

π = equiprobable random action choices



	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

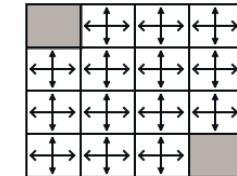
$R = -1$
on all transitions

$\gamma = 1$

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

Greedy Policy
w.r.t. V_k



random policy

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

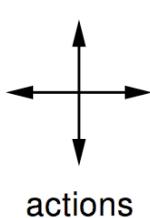
0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Iterative Policy Eval for the Small Gridworld

π = equiprobable random action choices



	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$R = -1$
on all transitions

$\gamma = 1$

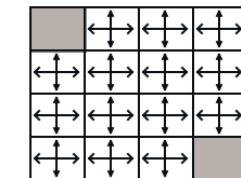
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V_k for the
Random Policy

$k = 0$

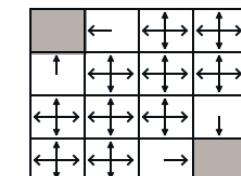
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

Greedy Policy
w.r.t. V_k



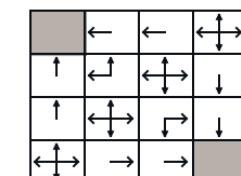
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



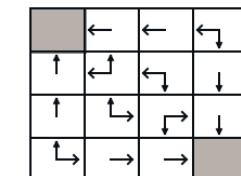
$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



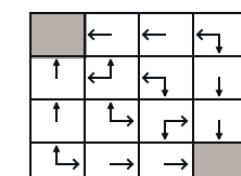
$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0



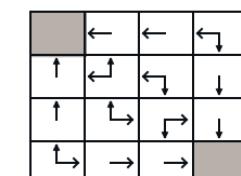
$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0



$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



Policy Iteration – One array version (+ policy)

1. Initialization

$V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation

Repeat

$$\Delta \leftarrow 0$$

For each $s \in \mathcal{S}$:

$$v \leftarrow V(s)$$

$$V(s) \leftarrow \sum_{s',r} p(s', r | s, \pi(s)) [r + \gamma V(s')]$$

$$\Delta \leftarrow \max(\Delta, |v - V(s)|)$$

until $\Delta < \theta$ (a small positive number)

3. Policy Improvement

policy-stable $\leftarrow true$

For each $s \in \mathcal{S}$:

$$a \leftarrow \pi(s)$$

$$\pi(s) \leftarrow \arg \max_a \sum_{s',r} p(s', r | s, a) [r + \gamma V(s')]$$

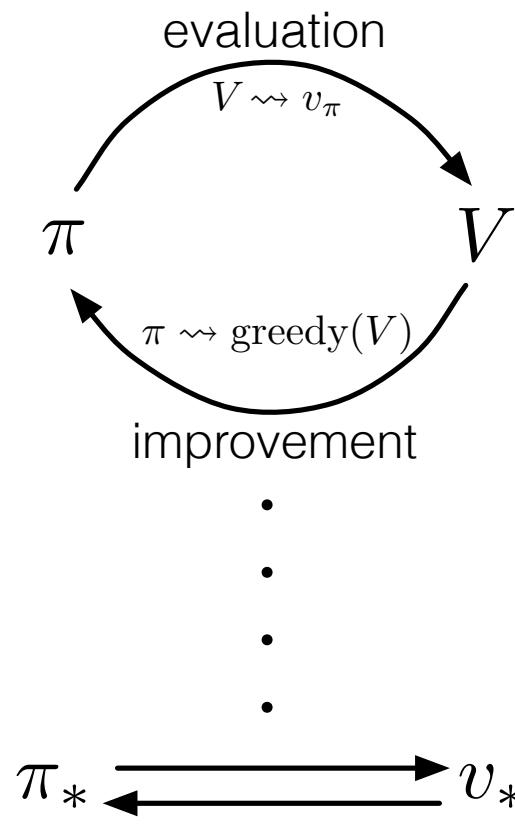
If $a \neq \pi(s)$, then *policy-stable* $\leftarrow false$

If *policy-stable*, then stop and return V and π ; else go to 2

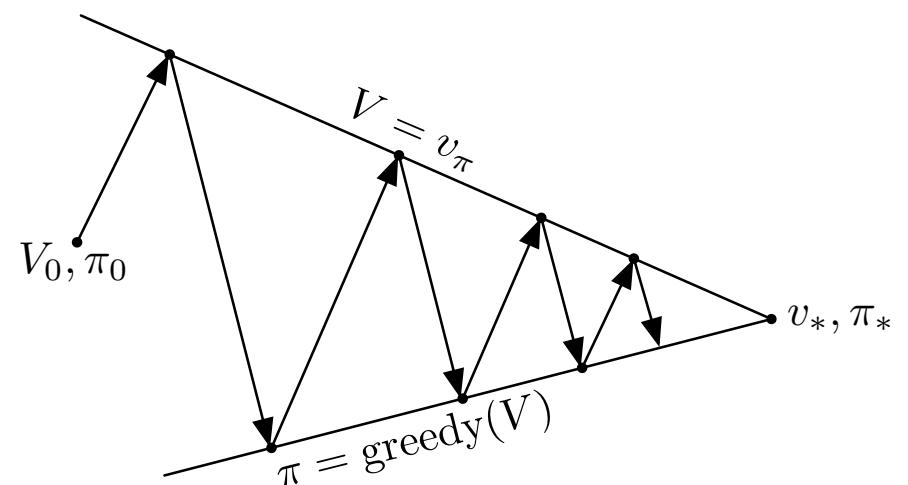
Generalized Policy Iteration

Generalized Policy Iteration (GPI):

any interaction of policy evaluation and policy improvement,
independent of their granularity.



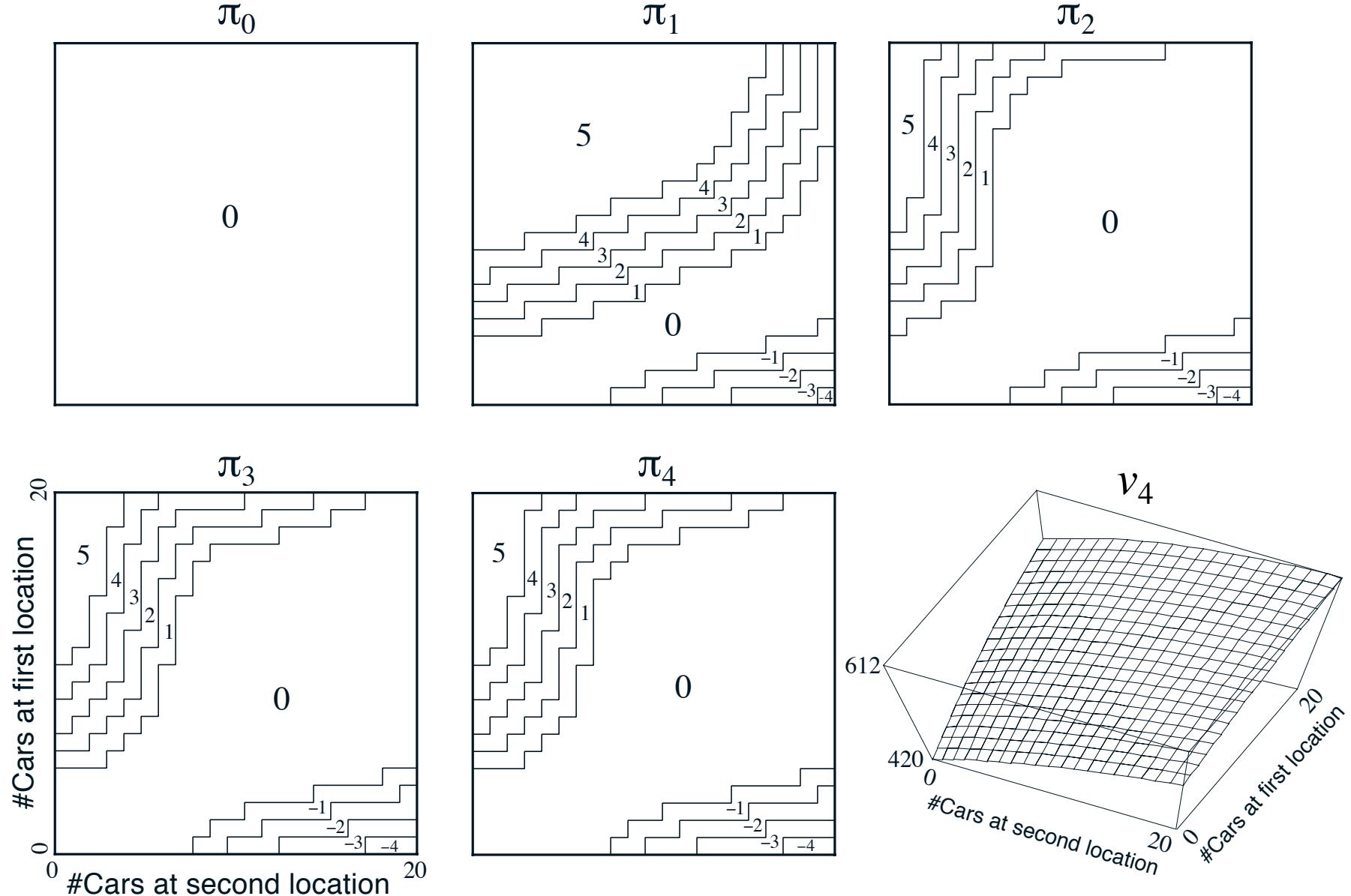
A geometric metaphor for convergence of GPI:



Jack's Car Rental

- ❑ \$10 for each car rented (must be available when request rec'd)
- ❑ Two locations, maximum of 20 cars at each
- ❑ Cars returned and requested randomly
 - Poisson distribution, n returns/requests with prob $\frac{\lambda^n}{n!} e^{-\lambda}$
 - 1st location: average requests = 3, average returns = 3
 - 2nd location: average requests = 4, average returns = 2
- ❑ Can move up to 5 cars between locations overnight
 - at a cost of \$2/car
- ❑ States, Actions, Rewards?
- ❑ Transition probabilities? Discounting? $\gamma = 0.9$

Jack's Car Rental



Jack's CR Exercise

- ❑ Suppose the first car moved is free
 - From 1st to 2nd location
 - Because an employee travels that way anyway (by bus)
- ❑ Suppose only 10 cars can be parked for free at each location
 - More than 10 cost \$4 for using an extra parking lot
- ❑ Such arbitrary nonlinearities are common in real problems

Value Iteration

Recall the **full policy-evaluation backup**:

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a) \left[r + \gamma v_k(s') \right] \quad \forall s \in \mathcal{S}$$

Here is the **full value-iteration backup**:

$$v_{k+1}(s) = \max_a \sum_{s',r} p(s',r|s,a) \left[r + \gamma v_k(s') \right] \quad \forall s \in \mathcal{S}$$

Value Iteration – One array version

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat

$$\Delta \leftarrow 0$$

For each $s \in \mathcal{S}$:

$$v \leftarrow V(s)$$

$$V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a) [r + \gamma V(s')]$$

$$\Delta \leftarrow \max(\Delta, |v - V(s)|)$$

until $\Delta < \theta$ (a small positive number)

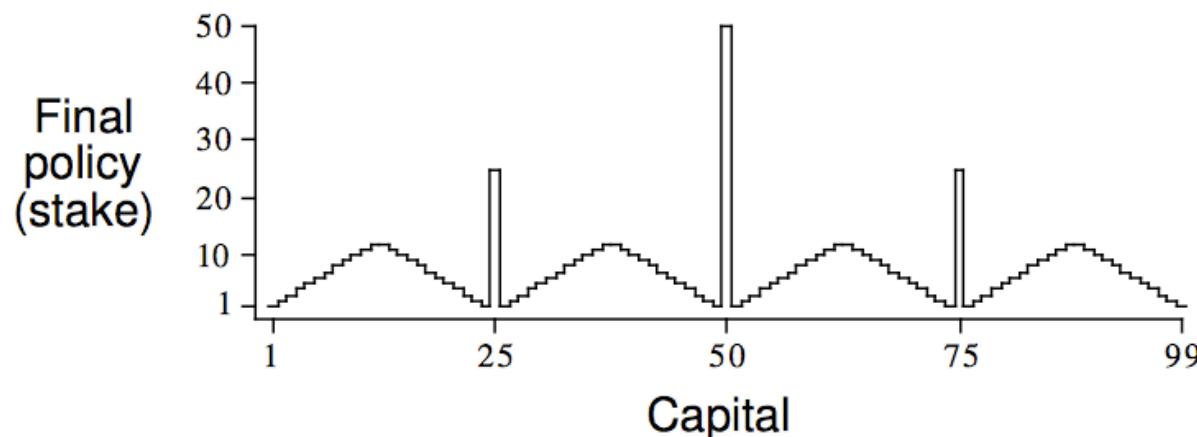
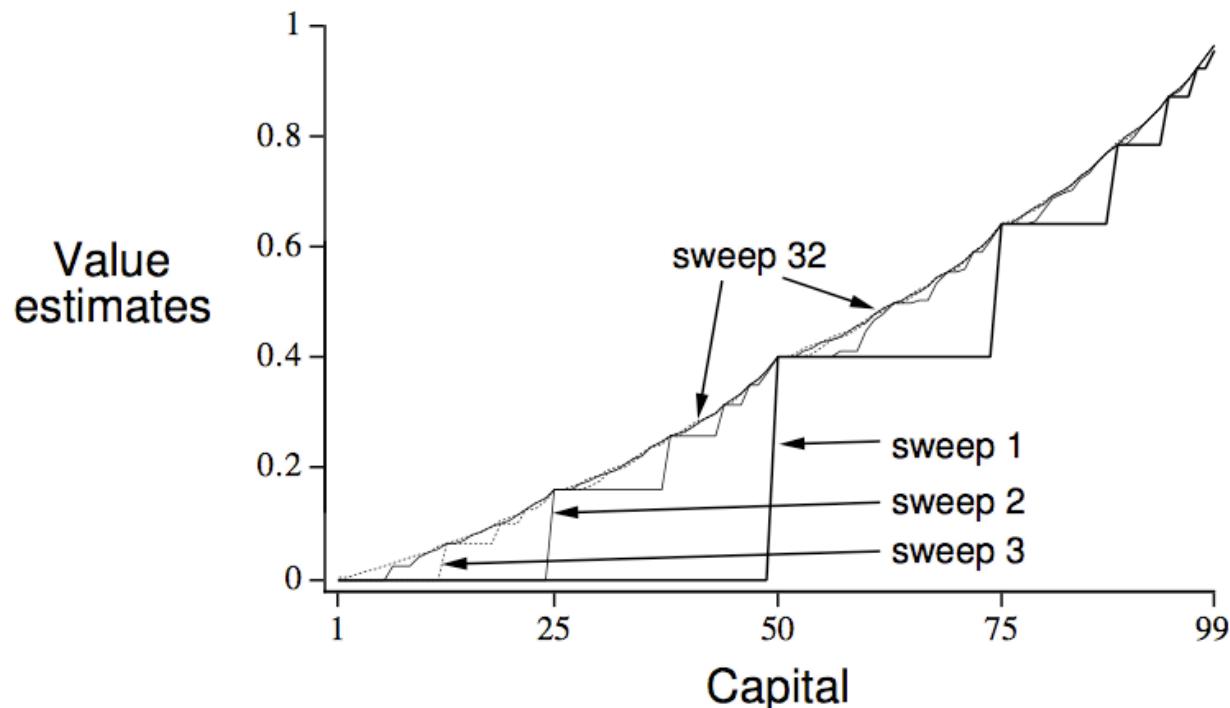
Output a deterministic policy, π , such that

$$\pi(s) = \arg \max_a \sum_{s',r} p(s',r|s,a) [r + \gamma V(s')]$$

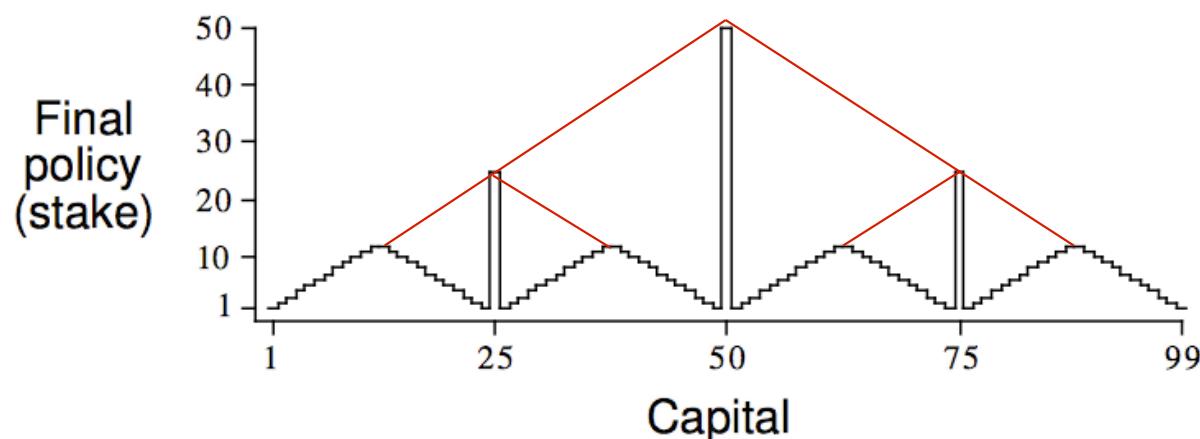
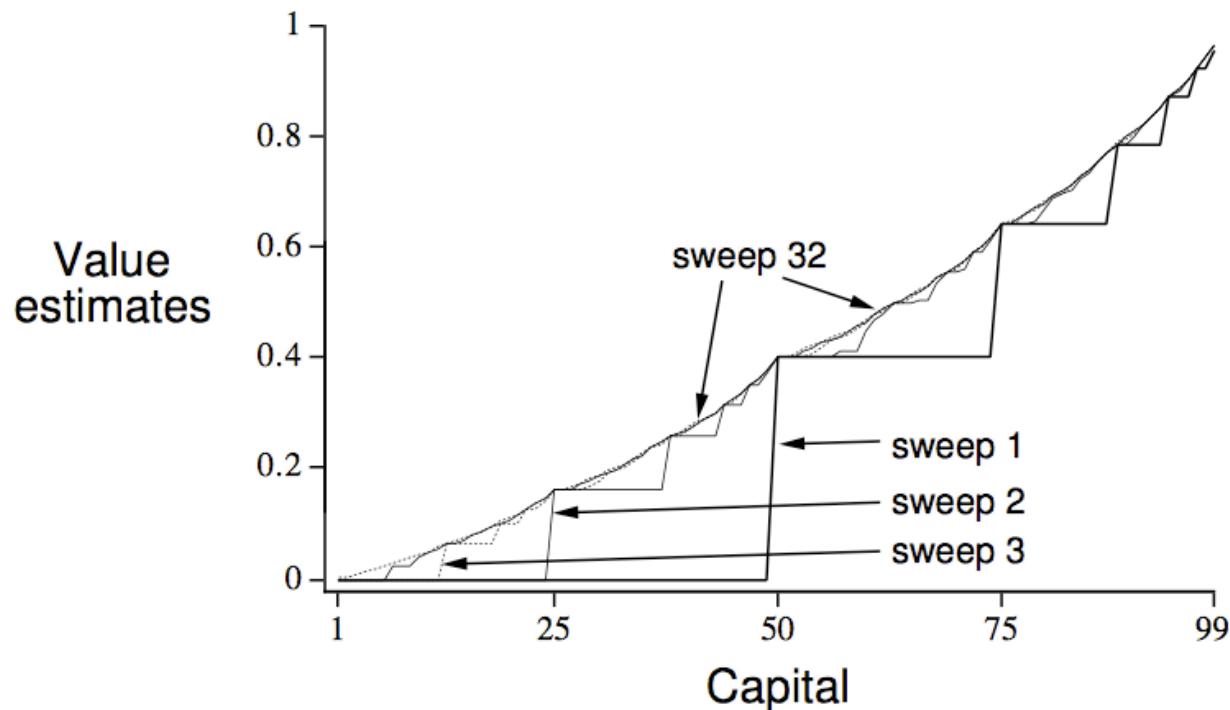
Gambler's Problem

- ❑ Gambler can repeatedly bet \$ on a coin flip
- ❑ Heads he wins his stake, tails he loses it
- ❑ Initial capital $\in \{\$1, \$2, \dots \$99\}$
- ❑ Gambler wins if his capital becomes \$100
loses if it becomes \$0
- ❑ Coin is unfair
 - Heads (gambler wins) with probability $p = .4$
- ❑ States, Actions, Rewards? Discounting?

Gambler's Problem Solution



Gambler's Problem Solution



Asynchronous DP

- ❑ All the DP methods described so far require exhaustive sweeps of the entire state set.
- ❑ Asynchronous DP does not use sweeps. Instead it works like this:
 - Repeat until convergence criterion is met:
 - Pick a state at random and apply the appropriate backup
- ❑ Still need lots of computation, but does not get locked into hopelessly long sweeps
- ❑ Can you select states to backup intelligently? YES: an agent's experience can act as a guide.

Efficiency of DP

- ❑ To find an optimal policy is polynomial in the number of states...
- ❑ BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called “the curse of dimensionality”).
- ❑ In practice, classical DP can be applied to problems with a few millions of states.
- ❑ Asynchronous DP can be applied to larger problems, and is appropriate for parallel computation.
- ❑ It is surprisingly easy to come up with MDPs for which DP methods are not practical.

Summary

- ❑ Policy evaluation: backups without a max
- ❑ Policy improvement: form a greedy policy, if only locally
- ❑ Policy iteration: alternate the above two processes
- ❑ Value iteration: backups with a max
- ❑ Full backups (to be contrasted later with sample backups)
- ❑ Generalized Policy Iteration (GPI)
- ❑ Asynchronous DP: a way to avoid exhaustive sweeps
- ❑ **Bootstrapping**: updating estimates based on other estimates
- ❑ Biggest limitation of DP is that it requires a *probability model* (as opposed to a generative or simulation model)