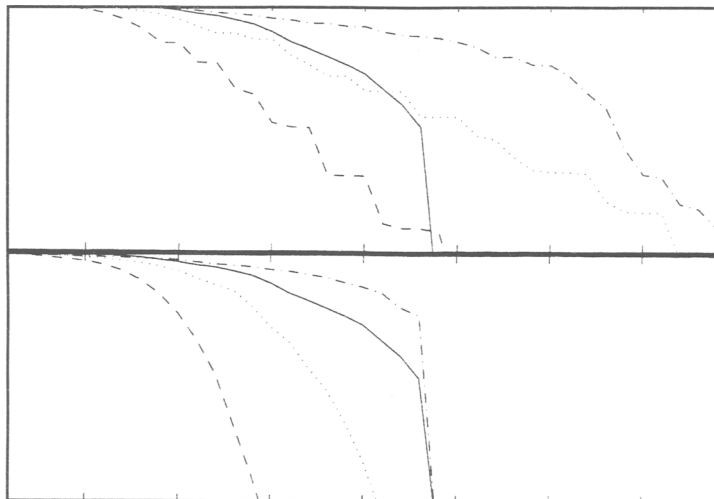
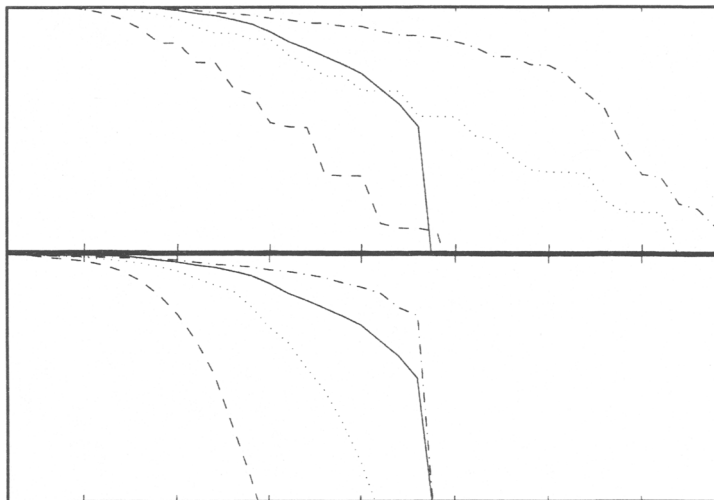


Iterative Methods for Solving Linear Systems



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Contents

List of Algorithms	xi
Preface	xiii
CHAPTER 1. Introduction	1
1.1 Brief Overview of the State of the Art	3
1.1.1 Hermitian Matrices	3
1.1.2 Non-Hermitian Matrices	5
1.1.3 Preconditioners	6
1.2 Notation	6
1.3 Review of Relevant Linear Algebra	7
1.3.1 Vector Norms and Inner Products	7
1.3.2 Orthogonality	8
1.3.3 Matrix Norms	9
1.3.4 The Spectral Radius	11
1.3.5 Canonical Forms and Decompositions	13
1.3.6 Eigenvalues and the Field of Values	16
 I Krylov Subspace Approximations	 23
CHAPTER 2. Some Iteration Methods	25
2.1 Simple Iteration	25
2.2 Orthomin(1) and Steepest Descent	29
2.3 Orthomin(2) and CG	33
2.4 Orthodir, MINRES, and GMRES	37
2.5 Derivation of MINRES and CG from the Lanczos Algorithm . .	41
 CHAPTER 3. Error Bounds for CG, MINRES, and GMRES	 49
3.1 Hermitian Problems—CG and MINRES	49
3.2 Non-Hermitian Problems—GMRES	54

CHAPTER 4. Effects of Finite Precision Arithmetic	61
4.1 Some Numerical Examples	62
4.2 The Lanczos Algorithm	63
4.3 A Hypothetical MINRES/CG Implementation	64
4.4 A Matrix Completion Problem	66
4.4.1 Paige's Theorem	67
4.4.2 A Different Matrix Completion	68
4.5 Orthogonal Polynomials	71
CHAPTER 5. BiCG and Related Methods	77
5.1 The Two-Sided Lanczos Algorithm	77
5.2 The Biconjugate Gradient Algorithm	79
5.3 The Quasi-Minimal Residual Algorithm	80
5.4 Relation Between BiCG and QMR	84
5.5 The Conjugate Gradient Squared Algorithm	88
5.6 The BiCGSTAB Algorithm	90
5.7 Which Method Should I Use?	92
CHAPTER 6. Is There a Short Recurrence for a Near-Optimal Approximation?	97
6.1 The Faber and Manteuffel Result	97
6.2 Implications	102
CHAPTER 7. Miscellaneous Issues	105
7.1 Symmetrizing the Problem	105
7.2 Error Estimation and Stopping Criteria	107
7.3 Attainable Accuracy	109
7.4 Multiple Right-Hand Sides and Block Methods	113
7.5 Computer Implementation	115
II Preconditioners	117
CHAPTER 8. Overview and Preconditioned Algorithms	119
CHAPTER 9. Two Example Problems	125
9.1 The Diffusion Equation	125
9.1.1 Poisson's Equation	129
9.2 The Transport Equation	134
CHAPTER 10. Comparison of Preconditioners	147
10.1 Jacobi, Gauss-Seidel, SOR	147
10.1.1 Analysis of SOR	149
10.2 The Perron-Frobenius Theorem	156
10.3 Comparison of Regular Splittings	160
10.4 Regular Splittings Used with the CG Algorithm	163

10.5 Optimal Diagonal and Block Diagonal Preconditioners	165
CHAPTER 11. Incomplete Decompositions	171
11.1 Incomplete Cholesky Decomposition	171
11.2 Modified Incomplete Cholesky Decomposition	175
CHAPTER 12. Multigrid and Domain Decomposition Meth-	
ods	183
12.1 Multigrid Methods	183
12.1.1 Aggregation Methods	184
12.1.2 Analysis of a Two-Grid Method for the Model Problem.	187
12.1.3 Extension to More General Finite Element Equations. .	193
12.1.4 Multigrid Methods	193
12.1.5 Multigrid as a Preconditioner for Krylov Subspace Meth-	
ods	197
12.2 Basic Ideas of Domain Decomposition Methods	197
12.2.1 Alternating Schwarz Method	198
12.2.2 Many Subdomains and the Use of Coarse Grids	201
12.2.3 Nonoverlapping Subdomains	203
References	205
Index	213

List of Algorithms

Algorithm 1.	Simple Iteration.	26
Algorithm 2.	Conjugate Gradient Method (CG)	35
Algorithm 3.	Generalized Minimal Residual Algorithm (GMRES) .	41
Algorithm 4.	Minimal Residual Algorithm (MINRES)	44
Algorithm 5.	Quasi-Minimal Residual Method (QMR).	83
Algorithm 6.	BiCGSTAB	91
Algorithm 7.	CG for the Normal Equations (CGNR and CGNE) . .	105
Algorithm 8.	Block Conjugate Gradient Method (Block CG)	114
Algorithm 2P.	Preconditioned Conjugate Gradient Method (PCG). .	121
Algorithm 4P.	Preconditioned Minimal Residual Algorithm (PMINRES)	122

Preface

In recent years much research has focused on the efficient solution of large sparse or structured linear systems using iterative methods. A language full of acronyms for a thousand different algorithms has developed, and it is often difficult for the nonspecialist (or sometimes even the specialist) to identify the basic principles involved. With this book, I hope to discuss a few of the most useful algorithms and the mathematical principles behind their derivation and analysis. The book does not constitute a complete survey. Instead I have tried to include the most *useful* algorithms from a practical point of view and the most *interesting* analysis from both a practical and mathematical point of view.

The material should be accessible to anyone with graduate-level knowledge of linear algebra and some experience with numerical computing. The relevant linear algebra concepts are reviewed in a separate section and are restated as they are used, but it is expected that the reader will already be familiar with most of this material. In particular, it may be appropriate to review the QR decomposition using the modified Gram–Schmidt algorithm or Givens rotations, since these form the basis for a number of algorithms described here.

Part I of the book, entitled *Krylov Subspace Approximations*, deals with general linear systems, although it is noted that the methods described are most often useful for very large sparse or structured matrices, for which direct methods are too costly in terms of computer time and/or storage. No specific applications are mentioned there. Part II of the book deals with *Preconditioners*, and here applications must be described in order to define and analyze some of the most efficient preconditioners, e.g., multigrid methods. It is assumed that the reader is acquainted with the concept of finite difference approximations, but no detailed knowledge of finite difference or finite element methods is assumed. This means that the analysis of preconditioners must generally be limited to model problems, but, in most cases, the proof techniques carry over easily to more general equations. It is appropriate to separate the study of iterative methods into these two parts because, as the reader will see, the tools of analysis for Krylov space methods and for preconditioners are really quite different. The field of preconditioners is a much broader one, since

the derivation of the preconditioner can rely on knowledge of an underlying problem from which the linear system arose.

This book arose out of a one-semester graduate seminar in Iterative Methods for Solving Linear Systems that I taught at Cornell University during the fall of 1994. When iterative methods are covered as part of a broader course on numerical linear algebra or numerical solution of partial differential equations, I usually cover the overview in section 1.1, sections 2.1–2.4 and 3.1, and some material from Chapter 5 and from Part II on preconditioners.

The book has a number of features that may be different from other books on this subject. I hope that these may attract the interest of graduate students (since a number of interesting open problems are discussed), of mathematicians from other fields (since I have attempted to relate the problems that arise in analyzing iterative methods to those that arise in other areas of mathematics), and also of specialists in this field. These features include:

- A brief overview of the state of the art in section 1.1. This gives the reader an understanding of what has been accomplished and what open problems remain in this field, without going into the details of any particular algorithm.
- Analysis of the effect of rounding errors on the convergence rate of the conjugate gradient method in Chapter 4 and discussion of how this problem relates to some other areas of mathematics. In particular, the analysis is presented as a matrix completion result or as a result about orthogonal polynomials.
- Discussion of open problems involving error bounds for GMRES in section 3.2, along with exercises in which some recently proved results are derived (with many hints included).
- Discussion of the transport equation as an example problem in section 9.2. This important equation has received far less attention from numerical analysts than the more commonly studied diffusion equation of section 9.1, yet it serves to illustrate many of the principles of non-Hermitian matrix iterations.
- Inclusion of multigrid methods in the part of the book on preconditioners (Chapter 12). Multigrid methods have proved extremely effective for solving the linear systems that arise from differential equations, and they should not be omitted from a book on iterative methods. Other recent books on iterative methods have also included this topic; see, e.g., [77].
- A small set of recommended algorithms and implementations. These are enclosed in boxes throughout the text.

This last item should prove helpful to those interested in solving particular problems as well as those more interested in general properties of iterative

methods. Most of these algorithms have been implemented in the *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods* [10], and the reader is encouraged to experiment with these or other iterative routines for solving linear systems. This book could serve as a supplement to the *Templates* documentation, providing a deeper look at the theory behind these algorithms.

I would like to thank the graduate students and faculty at Cornell University who attended my seminar on iterative methods during the fall of 1994 for their many helpful questions and comments. I also wish to thank a number of people who read through earlier drafts or sections of this manuscript and made important suggestions for improvement. This list includes Michele Benzi, Jim Ferguson, Martin Gutknecht, Paul Holloway, Zdeněk Strakoš, and Nick Trefethen.

Finally, I wish to thank the Courant Institute for providing me the opportunity for many years of uninterrupted research, without which this book might not have developed. I look forward to further work at the University of Washington, where I have recently joined the Mathematics Department.

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