MAT 226B Large Scale Matrix Computation
Homework 3

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Problem 1:

(a) The structure of A looks as following

$$A = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Krylov subspace of $K_k(A, r_0), \forall k = 1, 2, \cdots d(A, r_0)$ where $r_0 = e_n$ is the $n-$th unit vector is

$$K_k(A, r_0) = span\{r_0, Ar_0, A^2r_0, \cdots, A^{k-1}r_0\}$$

which can be determined using the following observation. Since $r_0 = e_n$, then $Ar_0 = \alpha_{n-1}e_{n-1}$, $Ae_{n-1} = \alpha_{n-1}e_{n-2}$, and so on where $\alpha \in \mathbb{R}$ is some factor depends on the non-zero values in D. Thus, $K_k(A, r_0) = span\{e_1, e_2, \cdots, e_k\}$.

To show that $d(A, r_0) = n$, we use the following observation that $Ae_1 = \alpha e_1$. Thus, after the first n unit vector, the vectors produced by multiplication by A are no longer linearly independent and thus the $d(A, r_0) = n$

(b) In exact arithmetic, the number of iteration needed by MR method with starting residual vector r_0 is n. We proved in the lecture that $x^* = A^{-1}b \in x_0 + K_d(A, r_0)$ and $d = d(A, r_0)$ is the minimum such value. Thus, the number of iterations of MR can not be less than n.

(c) The sparsity structure of A^T

$$A^T = \begin{bmatrix} * & * & * & \cdots & \cdots & \cdots & * & 1 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \end{bmatrix}$$

And the sparsity structure of A^TA

$$A^TA = \begin{bmatrix} * & * & * & \cdots & \cdots & \cdots & * & * \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} =$$

$$\begin{bmatrix} * & * & * & \cdots & \cdots & \cdots & * & * \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

The first matrix is a rank 2 matrix which has at most two distinct eigenvalue while the second matrix is a diagonal (identity) matrix that has only one distinct eigenvalue. Thus, A^TA can have at most three eigenvalues.

(d)

Problem 2:

(a) If $A \in \mathbb{R}^{n\times n}$ is a skew-symmetric, then $x^T Ax = 0$ since the diagonal elements of the A by definition are zero. We can show that for A^{2j+1} for $j = 0, 1, 2, \ldots$ (i.e., raising A to an odd power) will result into a skew-symmetric.

For a skew-symmetric matrix A (i.e., $A^T = -A$), we have

$$
\begin{aligned}
&A^m = (-A^T)^m \\
&A.A.A\cdots = (-1)^m.A^T.A^T.A^T\cdots = (-1)^m A^m
\end{aligned}
$$

Thus, if m is odd, then $A^m = -(A^T)^m = -(A^m)^T$ and thus the resulting matrix is skew-symmetric. Since any skew-symmetric matrix has zero diagonal elements, we can deduce that

$$x^T A^{2j+1} x = 0 \quad \forall j = 0, 1, 2\ldots \text{ and } x \in \mathbb{R}^n$$

(b) Find the eigenvalues of A can be done by solving the following for λ

$$Ax = \lambda x$$

From (a), we can multiply the above by x^T to get

$$x^T Ax = \lambda x = x^T \lambda x = \lambda \parallel x \parallel^2 = 0$$

One solution is $\lambda = 0$. Since x is a (non-trivial) eigenvector, then $\parallel x \parallel^2 \neq 0$. However, if we consider λ as an imaginary, then we can write the above as

$$
\begin{aligned}
&(\lambda_{RE} + i\lambda_{IM}) \parallel x \parallel^2 = 0 \\
&-\lambda_{RE} \parallel x \parallel^2 = i\lambda_{IM} \parallel x \parallel^2 \\
&(\lambda_{RE})^2 = (-1)(\lambda_{IM})^2
\end{aligned}
$$

where λ_{RE} is the real part and λ_{IM} is the imaginary part of λ. Besides the zero eigenvalue, the above shows that the eigenvalues λ are purely imaginary since λ_{IM} can not be zero. Also, the eigenvalues comes in as conjugate pairs since A is a real matrix.

(c) From the lectures notes, we have the following fact about Krylov subspace. If A is diagonalizable, then $d(A, r_0)$ is the minimum number of eigenvectors in the eigendecomposition of A. From (b), since the eigenvalues comes in conjugate pairs, then the number of eigenvectors is even and thus $d(A, r_0)$ is even.

Now we need to prove that A is diagonalizable. Since A is a normal matrix i.e., $A^T A = AA^T = -AA$, then it must be diagonalizable (following the Spectral theorem).

Problem 3:

(a) We can derive the formula for A' as follows

$$
\begin{aligned}
&A' = M_1^{-1} A M_2^{-1} \\
&A' = (D-F)^{-1} A \left[D^{-1}(D-G)\right]^{-1} \\
&A' = D(D-F)^{-1}(D_0 - F - G)(D-G)^{-1} \\
&A' = D(D-F)^{-1}\left(D_1 + 2D - F - G\right)(D-G)^{-1} \\
&A' = D(D-F)^{-1}\left[D_1 + (D-F) + (D-G)\right](D-G)^{-1} \\
&A' = D\left[(D-F)^{-1}(D-F)(D-G)^{-1} + (D-F)^{-1}\left((D-G)(D-G)^{-1} + D_1(D-G)^{-1}\right)\right] \\
&A' = D\left[(D-G)^{-1} + (D-F)^{-1}\left(I + D_1(D-G)^{-1}\right)\right]
\end{aligned}
$$

We used the fact that $\left(D^{-1}\right)^{-1} = D$, $D_0 = D_1 + 2D$, and $(D-F)^{-1}(D-F) = (D-G)^{-1}(D-G) = I$ in the derivation of the above formula.

(b) We first expand q' such that

$$
\begin{aligned}
&q' = A'v' \\
&q' = D\left[(D-G)^{-1} + (D-F)^{-1}\left(I + D_1(D-G)^{-1}\right)\right]v' \\
&q' = D\underbrace{\left[\underbrace{(D-G)^{-1}v'}_{Q_1} + \underbrace{(D-F)^{-1}\left(\overbrace{Iv' + \overbrace{D_1 \overbrace{\underset{Q_3}{(D-G)^{-1}v'}}^{Q_1}}^{Q_2}}^{Q_4}\right)}\right]}_{Q_5}
\end{aligned}
$$

- $Q_1 = (D-G)^{-1}v'$ is one triangular solve in $Q1$
- $Q_2 = D1Q_2$ is one multiplication with the diagonal entries of D_1
- $Q_3 = Iv' + Q_2$ is a SAXPY
- $Q_4 = (D-F)^{-1}Q_3$ is one triangular solve in Q_4
- $Q_5 = Q_1 + Q_4$ is a SAXPY
- $q' = DQ_5$ is one multiplication with the diagonal entries of D

(c) Computing Q_2 and the final q' each requires only n flops. Computing Q_3 and Q_4 each requires $2n$ flops. Here we assume that Q_5 will be implemented using some standard routine for SAXPY such that Q_1 (or $Q4$) is multiplied by 1. The triangular solvers each requires multiplying by the diagonal entries (i.e., n flops) and $2m$ flops to multiply and add the off-diagonal entries. **Thus, the total number of flops is** $8n + 4m$ **flops**.