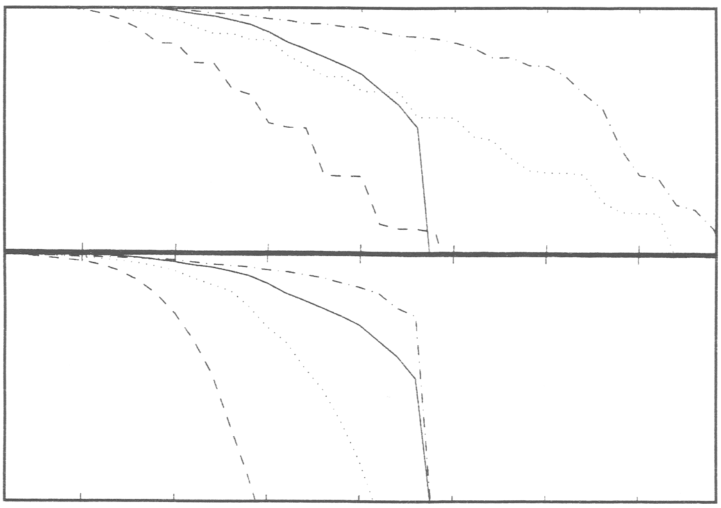
Iterative Methods for Solving Linear Systems



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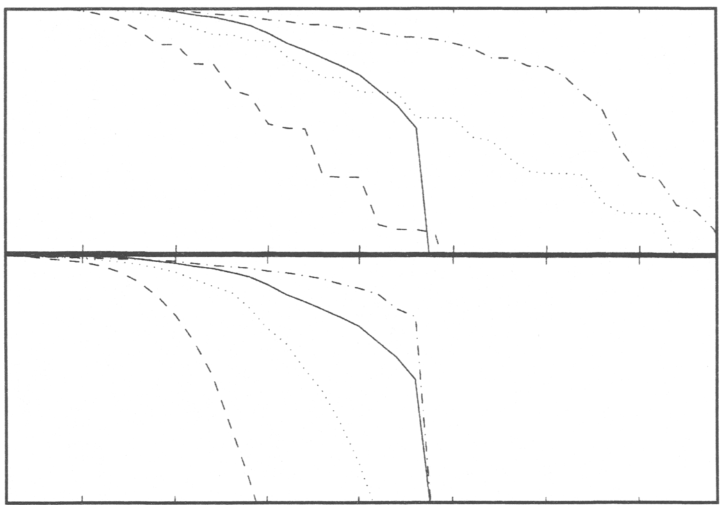

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Preface

In recent years much research has focused on the efficient solution of large sparse or structured linear systems using iterative methods. A language full of acronyms for a thousand different algorithms has developed, and it is often difficult for the nonspecialist (or sometimes even the specialist) to identify the basic principles involved. With this book, I hope to discuss a few of the most useful algorithms and the mathematical principles behind their derivation and analysis. The book does not constitute a complete survey. Instead I have tried to include the most *useful* algorithms from a practical point of view and the most *interesting* analysis from both a practical and mathematical point of view.

The material should be accessible to anyone with graduate-level knowledge of linear algebra and some experience with numerical computing. The relevant linear algebra concepts are reviewed in a separate section and are restated as they are used, but it is expected that the reader will already be familiar with most of this material. In particular, it may be appropriate to review the QR decomposition using the modified Gram–Schmidt algorithm or Givens rotations, since these form the basis for a number of algorithms described here.

Part I of the book, entitled *Krylov Subspace Approximations*, deals with general linear systems, although it is noted that the methods described are most often useful for very large sparse or structured matrices, for which direct methods are too costly in terms of computer time and/or storage. No specific applications are mentioned there. Part II of the book deals with *Preconditioners*, and here applications must be described in order to define and analyze some of the most efficient preconditioners, e.g., multigrid methods. It is assumed that the reader is acquainted with the concept of finite difference approximations, but no detailed knowledge of finite difference or finite element methods is assumed. This means that the analysis of preconditioners must generally be limited to model problems, but, in most cases, the proof techniques carry over easily to more general equations. It is appropriate to separate the study of iterative methods into these two parts because, as the reader will see, the tools of analysis for Krylov space methods and for preconditioners are really quite different. The field of preconditioners is a much broader one, since the derivation of the preconditioner can rely on knowledge of an underlying problem from which the linear system arose.

This book arose out of a one-semester graduate seminar in Iterative Methods for Solving Linear Systems that I taught at Cornell University during the fall of 1994. When iterative methods are covered as part of a broader course on numerical linear algebra or numerical solution of partial differential equations, I usually cover the overview in section 1.1, sections 2.1–2.4 and 3.1, and some material from Chapter 5 and from Part II on preconditioners.

The book has a number of features that may be different from other books on this subject. I hope that these may attract the interest of graduate students (since a number of interesting open problems are discussed), of mathematicians from other fields (since I have attempted to relate the problems that arise in analyzing iterative methods to those that arise in other areas of mathematics), and also of specialists in this field. These features include:

- A brief overview of the state of the art in section 1.1. This gives the reader an understanding of what has been accomplished and what open problems remain in this field, without going into the details of any particular algorithm.

- Analysis of the effect of rounding errors on the convergence rate of the conjugate gradient method in Chapter 4 and discussion of how this problem relates to some other areas of mathematics. In particular, the analysis is presented as a matrix completion result or as a result about orthogonal polynomials.

- Discussion of open problems involving error bounds for GMRES in section 3.2, along with exercises in which some recently proved results are derived (with many hints included).

- Discussion of the transport equation as an example problem in section 9.2. This important equation has received far less attention from numerical analysts than the more commonly studied diffusion equation of section 9.1, yet it serves to illustrate many of the principles of non-Hermitian matrix iterations.

- Inclusion of multigrid methods in the part of the book on preconditioners (Chapter 12). Multigrid methods have proved extremely effective for solving the linear systems that arise from differential equations, and they should not be omitted from a book on iterative methods. Other recent books on iterative methods have also included this topic; see, e.g., [77].

- A small set of recommended algorithms and implementations. These are enclosed in boxes throughout the text.

This last item should prove helpful to those interested in solving particular problems as well as those more interested in general properties of iterative methods. Most of these algorithms have been implemented in the *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods* [10], and the reader is encouraged to experiment with these or other iterative routines for solving linear systems. This book could serve as a supplement to the *Templates* documentation, providing a deeper look at the theory behind these algorithms.

I would like to thank the graduate students and faculty at Cornell University who attended my seminar on iterative methods during the fall of 1994 for their many helpful questions and comments. I also wish to thank a number of people who read through earlier drafts or sections of this manuscript and made important suggestions for improvement. This list includes Michele Benzi, Jim Ferguson, Martin Gutknecht, Paul Holloway, Zděnek Strakoš, and Nick Trefethen.

Finally, I wish to thank the Courant Institute for providing me the opportunity for many years of uninterrupted research, without which this book might not have developed. I look forward to further work at the University of Washington, where I have recently joined the Mathematics Department.

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