

MAT 226B Large Scale Matrix Computation

Homework 3

Ahmed Mahmoud

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Problem 1:

(a) The structure of A looks as following

$$A = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Krylov subspace of $K_k(A, r_0), \forall k = 1, 2, \dots, d(A, r_0)$ where $r_0 = e_n$ is the n -th unit vector is

$$K_k(A, r_0) = \text{span}\{r_0, Ar_0, A^2r_0, \dots, A^{k-1}r_0\}$$

which can be determined using the following observation. Since $r_0 = e_n$, then $Ar_0 = \alpha_{n-1}e_{n-1}$, $Ae_{n-1} = \alpha_{n-1}e_{n-2}$, and so on where $\alpha \in \mathbb{R}$ is some factor depends on the non-zero values in D . Thus, $K_k(A, r_0) = \text{span}\{e_1, e_2, \dots, e_k\}$.

To show that $d(A, r_0) = n$, we use the following observation that $Ae_1 = \alpha e_1$. Thus, after the first n unit vector, the vectors produced by multiplication by A are no longer linearly independent and thus the $d(A, r_0) = n$

(b) In exact arithmetic, the number of iteration needed by MR method with starting residual vector r_0 is n .

(c) The sparsity structure of A^T

$$A^T = \begin{bmatrix} * & * & * & \dots & \dots & \dots & * & 1 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \end{bmatrix}$$

And the sparsity structure of $A^T A$

$$A^T A = \begin{bmatrix} * & * & * & \dots & \dots & \dots & * & * \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} =$$

$$\begin{bmatrix} * & * & * & \dots & \dots & \dots & * & * \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

The first matrix is a rank 2 matrix which has at most two distinct eigenvalue while the second matrix is a diagonal (identity) matrix that has only one distinct eigenvalue. Thus, $A^T A$ can have at most three eigenvalues.

(d)

Problem 3:

(a) We can derive the formula for A' as follows

$$\begin{aligned}
 A' &= M_1^{-1} A M_2^{-1} \\
 A' &= (D - F)^{-1} A [D^{-1}(D - G)]^{-1} \\
 A' &= D(D - F)^{-1}(D_0 - F - G)(D - G)^{-1} \\
 A' &= D(D - F)^{-1}(D_1 + 2D - F - G)(D - G)^{-1} \\
 A' &= D(D - F)^{-1}[D_1 + (D - F) + (D - G)](D - G)^{-1} \\
 A' &= D[(D - F)^{-1}(D - F)(D - G)^{-1} + (D - F)^{-1}((D - G)(D - G)^{-1} + D_1(D - G)^{-1})] \\
 A' &= D[(D - G)^{-1} + (D - F)^{-1}(I + D_1(D - G)^{-1})]
 \end{aligned}$$

We used the fact that $(D^{-1})^{-1} = D$, $D_0 = D_1 + 2D$, and $(D - F)^{-1}(D - F) = (D - G)^{-1}(D - G) = I$ in the derivation of the above formula.

(b) We first expand q' such that

$$\begin{aligned}
 q' &= A'v' \\
 q' &= D[(D - G)^{-1} + (D - F)^{-1}(I + D_1(D - G)^{-1})]v' \\
 q' &= D \left[\underbrace{(D - G)^{-1}v'}_{Q_1} + (D - F)^{-1} \underbrace{\left(Iv' + \underbrace{D_1 \underbrace{(D - G)^{-1}v'}_{Q_3}}_{Q_2} \right)}_{Q_4} \right] \\
 &\quad \underbrace{\hspace{15em}}_{Q_5}
 \end{aligned}$$

- $Q_1 = (D - G)^{-1}v'$ is one triangular solve in Q_1
- $Q_2 = D_1 Q_3$ is one multiplication with the diagonal entries of D_1
- $Q_3 = Iv' + Q_2$ is a SAXPY
- $Q_4 = (D - F)^{-1}Q_3$ is one triangular solve in Q_4
- $Q_5 = Q_1 + Q_4$ is a SAXPY
- $q' = DQ_5$ is one multiplication with the diagonal entries of D

- (c) Computing Q_2 and the final q' each requires only n flops. Computing Q_3 and Q_4 each requires $2n$ flops. Here we assume that Q_5 will be implemented using some standard routine for SAXPY such that Q_1 (or Q_4) is multiplied by 1. The triangular solvers each requires multiplying by the diagonal entries (i.e., n flops) and $2m$ flops to multiply and add the off-diagonal entries. **Thus, the total number of flops is $8n + 4m$ flops.**