MAT 226B Large Scale Matrix Computation
Homework 4

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Problem 1:

(a) The breakdown in nonsymmetric Lanczos process occurs

- when v_k or w_k is a zero vector. This is the case where V spans $K_k\{A, r\}$ or W spans $K_k\left\{A^T, c\right\}$. The is a termination condition and occurs when k reaches $d(A, r)$ or $d(A^T, c)$. Since, $d(A, r) \geq 2$ and $d(A^T, c) \geq 2$, then this is not the case.
- when $w_k^T v_k = 0$. Since v_k and w_k is the normalized r and c respectively for $k = 1$. Then, $w_k^T v_k = 0$ for $k = 1$ and breakdown will occur since $w_k^T v_k$ is used as the denominator to compute α_k leading to division by zero.

(b) At step $k = 2$, a breakdown will occur if

$$\delta_2 = w_2^T v_2 = 0$$

Assuming that r and c are normalized (i.e., $\| r \|_2 = \| c \|_2 = 1$), we can re-write the above condition as

$$w_2^T v_2 = (A^T w_1 - \alpha_1 w_1)^T (A v_1 - \alpha_1 v_1) = 0$$

That could happens if $(A^T w_1 - \alpha_1 w_1)^T = 0$ or $A v_1 = \alpha_1 v_1$. Thus,

$$\begin{gathered} A v_1 = \alpha_1 v_1 \\ A r = \alpha_1 r \\ A r = \frac{c^T A r}{c^T r} r \\ (c^T r)(A r) = (c^T A r)(r) \\ (c^T r)(A) = (c^T A r) \\ (c^T r)(c^T A^2 r) = (c^T A r)^2 \end{gathered}$$

Problem 2:

The function `arnoldi_process` in `problem_2.m` implements Arnoldi process. It takes matrix A, initial r vector, and `KMAX` value and outputs matrices H_k and V_K. We test the function with $k = 5, 10,$ and 20. Table 2 shows the eignevalues of H_k for different values of k. Table 1 shows the eignevalues of A. To compare between the eigenvalue of H_{20} and A's eigenvalues, we computed $\| A\_eig - H_{20}\_eig \| = 2.022960706546113\text{e}{-14}$

Eigenvalues
$-1.246365732148965\text{e}{+00}+3.879484437687162\text{e}{+00}i$
$1.251730623491239\text{e}{-01}-3.705917327586323\text{e}{+00}i$
$2.594180629560029\text{e}{+00}-2.284902242884256\text{e}{+00}i$
$-1.983514535906377\text{e}{+00}+2.443522984313265\text{e}{+00}i$
$3.000394334727376\text{e}{+00}-9.028020730501798\text{e}{-01}i$
$-8.152087361523785\text{e}{-01}-2.850024263184167\text{e}{+00}i$

Figure 1: A Eigenvalues

K	Eigenvalues
5	$-1.667643118967301\text{e}{+00}+3.322698092379202\text{e}{+00}i$ $1.122189729586331\text{e}{+00}-2.935868201795706\text{e}{+00}i$ $-2.454291098365838\text{e}{+00}-1.034091783975195\text{e}{-01}i$ $1.901983203411739\text{e}{+00}+2.938354695246812\text{e}{-02}i$ $-1.488418331702275\text{e}{-01}+4.856460403443859\text{e}{-03}i$
10	$2.481430445455568\text{e}{-01}-3.792088063628674\text{e}{+00}i$ $-1.284055777449226\text{e}{+00}+3.812474971159054\text{e}{+00}i$ $-9.594670448581000\text{e}{-01}-2.797385858005139\text{e}{+00}i$ $-2.769085996727914\text{e}{+00}-4.783604953147819\text{e}{-01}i$ $-1.865849533089461\text{e}{+00}+2.203044871147163\text{e}{+00}i$ $-1.983244640080843\text{e}{+00}+6.985561973441846\text{e}{-01}i$ $2.484678533435320\text{e}{+00}-2.158804713951516\text{e}{+00}i$ $2.525082205768530\text{e}{+00}-9.804387029418838\text{e}{-01}i$ $1.801564391235925\text{e}{+00}+1.650150378773440\text{e}{+00}i$ $9.603199929809421\text{e}{-01}+1.532501604030885\text{e}{+00}i$
20	$-1.246365732148964\text{e}{+00}+3.879484437687161\text{e}{+00}i$ $2.594180629560022\text{e}{+00}-2.284902242884260\text{e}{+00}i$ $-1.983514535906369\text{e}{+00}+2.443522984313272\text{e}{+00}i$ $3.000394334727364\text{e}{+00}-9.028020730501802\text{e}{-01}i$ $-2.816575470153527\text{e}{+00}-4.470235401498117\text{e}{-01}i$ $-2.335649345045595\text{e}{+00}+1.118762976889542\text{e}{+00}i$ $4.970325872397225\text{e}{-01}+2.729519259822235\text{e}{+00}i$ $2.304844231638446\text{e}{+00}+1.654017825159756\text{e}{+00}i$ $1.282928255975394\text{e}{+00}+2.519903189853356\text{e}{+00}i$ $1.251730623491198\text{e}{-01}-3.705917327586333\text{e}{+00}i$ $-8.152087361523789\text{e}{-01}-2.850024263184166\text{e}{+00}i$ $-5.433637042721347\text{e}{-02}-2.946384869537709\text{e}{+00}i$ $1.564476048921009\text{e}{+00}+1.768834378823156\text{e}{+00}i$ $-1.643042886688547\text{e}{+00}-1.212344348158080\text{e}{+00}i$ $1.943119133724999\text{e}{+00}+8.605877308058085\text{e}{-02}i$ $1.486930904640139\text{e}{-01}-1.861577918840294\text{e}{+00}i$ $-4.328984355639898\text{e}{-01}+1.008036318652936\text{e}{+00}i$ $-1.013702946373261\text{e}{+00}-2.443176161667247\text{e}{-01}i$ $6.261926774054364\text{e}{-01}-5.762496449081369\text{e}{-01}i$ $4.487381899523373\text{e}{-01}-2.877609613824833\text{e}{-01}i$

Figure 2: H_k Eigenvalues for different values of K

Problem 3:

(a) Function `hermitian_lanczos` implements the Hermitian Lanczos process where it takes the matrix A, vector r, and `KMAX` parameter and output T_k tridiagonal matrix in sparse

format. We used this function to approximate eigenvalues of A from `make_3d_laplacian` function using $k = 7$. Table 3 shows the results where the first two columns show the exact eigenvalues of A (computed from the provided function) along its multiplicity and last column shows the approximate eigenvalues. We can see that the approximate solution is able to capture all the eigenvalues with very high accuracy.

Exact Eigenvalues	Multiplicity	Approximate Eigenvalues
1.757359312880715e+00	1	1.757359312880715e+00
3.171572875253810e+00	3	3.171572875253810e+00
4.585786437626905e+00	6	4.585786437626905e+00
6.000000000000000e+00	7	6.000000000000002e+00
7.414213562373095e+00	6	7.414213562373095e+00
8.828427124746190e+00	3	8.828427124746192e+00
1.024264068711928e+01	1	1.024264068711929e+01

Figure 3: Exact and approximate eigenvalues using Hermitian Lanczos process

(b) We used `hermitian_lanczos` to compute the approximate eigenvalues for the 262144×262144 matrix from `make_3d_laplacian(64)`. Table 4, Table 5, Table 6, Table 7, Table 8, Table 9, and Table 10 show the 10 smallest and 10 largest approximated eigenvalues. Table 11 shows the 10 smallest and 10 largest exact eigenvalues. Due to the multiplicity in exact eigenvalues and round of error in approximate one, it is harder to check the accuracy due to repeated eigenvalues.

Smallest Eigenvalues	Largest Eigenvalues
1.422787575075167e−02	1.170222368514153e+01
2.522292134967839e−02	1.175506886438500e+01
3.720946521956244e−02	1.180683457939747e+01
6.015341020592301e−02	1.184896392339356e+01
8.407846117839080e−02	1.188540931980204e+01
1.168498457907536e−01	1.191749739736500e+01
1.581734700307590e−01	1.194638621492721e+01
1.995732825474702e−01	1.196512279467138e+01
2.466691832816189e−01	1.198055273500129e+01
3.011244810109077e−01	1.199293004521624e+01

Figure 4: Approximate Eigenvalues using $K = 100$

Smallest Eigenvalues	Largest Eigenvalues
7.006926067695284e−03	1.191194541966088e+01
1.400816296330363e−02	1.192367820060319e+01
2.104749654401042e−02	1.193815033138952e+01
2.644416419643677e−02	1.194943725586547e+01
3.252921177956383e−02	1.195823115974695e+01
4.248144987518579e−02	1.196722905407029e+01
5.158180541911261e−02	1.197279443138257e+01
6.175958581860198e−02	1.197897280487774e+01
7.373377938324976e−02	1.198599186623671e+01
8.527634111033579e−02	1.199299336087879e+01

Figure 5: Approximate Eigenvalues using $K = 200$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006043782e−03	1.195105825040107e+01
1.400782323540659e−02	1.195568924396476e+01
2.100900746475762e−02	1.195806895929420e+01
2.565829376859278e−02	1.196033932970904e+01
2.801019169415529e−02	1.196734052200119e+01
3.265947799840193e−02	1.197198980830585e+01
3.966069944862968e−02	1.197434170623140e+01
4.193108831261764e−02	1.197899099253526e+01
4.431079048315387e−02	1.198599217676459e+01
4.897191625334762e−02	1.199299336099394e+01

Figure 6: Approximate Eigenvalues using $K = 400$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006040011e−03	1.197198980830589e+01
7.006639006045063e−03	1.197434170555307e+01
1.400782323539451e−02	1.197434170623140e+01
1.400782323540438e−02	1.197899099253525e+01
2.100900746475027e−02	1.197899099253527e+01
2.100900746475296e−02	1.198599217676461e+01
2.565829371816093e−02	1.198599217676461e+01
2.565829376859185e−02	1.199293916949044e+01
2.801018851772900e−02	1.199299336099396e+01
2.801019169410766e−02	1.199299336099397e+01

Figure 7: Approximate Eigenvalues using $K = 800$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006034983e−03	1.197899099253524e+01
7.006639006042774e−03	1.197899099253525e+01
7.006639006050910e−03	1.197899099253525e+01
7.006642433917978e−03	1.198599217676460e+01
1.400782323538701e−02	1.198599217676461e+01
1.400782323539747e−02	1.198599217676462e+01
1.400782323539999e−02	1.199299336099394e+01
2.100900746475157e−02	1.199299336099396e+01
2.100900746475692e−02	1.199299336099397e+01
2.100900746476310e−02	1.199299336099399e+01

Figure 8: Approximate Eigenvalues using $K = 1200$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006034724e−03	1.197899099253525e+01
7.006639006037083e−03	1.198599217676456e+01
7.006639006037985e−03	1.198599217676460e+01
7.006639006044512e−03	1.198599217676460e+01
7.006639006047808e−03	1.198599217676461e+01
1.400782323539137e−02	1.199299336099392e+01
1.400782323539370e−02	1.199299336099395e+01
1.400782323539474e−02	1.199299336099396e+01
1.400782323540451e−02	1.199299336099397e+01
1.412781215190661e−02	1.199299336099398e+01

Figure 9: Approximate Eigenvalues using $K = 1600$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006032009e−03	1.198599217676460e+01
7.006639006034958e−03	1.198599217676460e+01
7.006639006035550e−03	1.198599217676461e+01
7.006639006037528e−03	1.198599260901220e+01
7.006639006037725e−03	1.199299336099394e+01
7.006639006042897e−03	1.199299336099395e+01
1.400782323536894e−02	1.199299336099396e+01
1.400782323539092e−02	1.199299336099396e+01
1.400782323539872e−02	1.199299336099396e+01
1.400782323540259e−02	1.199299336099397e+01

Figure 10: Approximate Eigenvalues using $K = 2000$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006040594e−03	1.197434170623141e+01
1.400782323539662e−02	1.197434170623141e+01
1.400782323539662e−02	1.197434170623141e+01
1.400782323539662e−02	1.197899099253525e+01
2.100900746475265e−02	1.197899099253525e+01
2.100900746475265e−02	1.197899099253525e+01
2.100900746475265e−02	1.198599217676460e+01
2.565829376859075e−02	1.198599217676460e+01
2.565829376859163e−02	1.198599217676460e+01
2.565829376859163e−02	1.199299336099396e+01

Figure 11: Exact Eigenvalues

Problem 4:

(a) The function `nonsymmetric_lanczos` implements the nonsymmetric Lanczos process where it takes a function that implements Av and another function that implements $A^T v$ along with the initial r and c vectors. It outputs the T_k tridiagonal matrix is sparse format. We used this function to compute T_k for $K = 5, 10,$ and 20 and computed the eigenvalues of T_k. To compare between the eigenvalue of T_{20} and A's eigenvalues, we computed $\| A\_eig - T_{20}\_eig \| = 6.448717\text{e}{-13}$

K	Eigenvalues
5	$-2.501781014189590\text{e}{+00}+5.900692650763462\text{e}{+00}i$ $-3.090333049113557\text{e}{+00}+4.825912738594896\text{e}{-01}i$ $5.172926282011777\text{e}{-01}-7.888954236341315\text{e}{-01}i$ $1.256199307483214\text{e}{+00}-3.493703520606647\text{e}{+00}i$ $1.274556869629201\text{e}{+01}-1.125556172280826\text{e}{+01}i$
10	$1.338852744862344\text{e}{-01}-3.681053207507567\text{e}{+00}i$ $2.508862893210253\text{e}{+00}-2.197729989474411\text{e}{+00}i$ $2.670866099884081\text{e}{+00}-1.314179550767412\text{e}{+00}i$ $-4.439295624831807\text{e}{-01}-2.347985483160573\text{e}{+00}i$ $-2.826197293664038\text{e}{+00}-2.998937801231081\text{e}{-01}i$ $-1.899586046943437\text{e}{+00}+7.856568851355709\text{e}{-02}i$ $-2.187569680623757\text{e}{+00}+2.082401070017189\text{e}{+00}i$ $-1.238832783943400\text{e}{+00}+3.769011471476989\text{e}{+00}i$ $1.936757547900978\text{e}{+00}+2.974410140200699\text{e}{+00}i$ $3.216278496510915\text{e}{+00}+5.676288164244975\text{e}{+00}i$
20	$-1.246365732148833\text{e}{+00}+3.879484437687169\text{e}{+00}i$ $2.594180629560308\text{e}{+00}-2.284902242884396\text{e}{+00}i$ $-1.983514535906585\text{e}{+00}+2.443522984313314\text{e}{+00}i$ $3.000394334727138\text{e}{+00}-9.028020730499668\text{e}{-01}i$ $-2.816575470153615\text{e}{+00}-4.470235401498108\text{e}{-01}i$ $2.304844231638672\text{e}{+00}+1.654017825159949\text{e}{+00}i$ $-2.335649345045510\text{e}{+00}+1.118762976889514\text{e}{+00}i$ $1.251730623491955\text{e}{-01}-3.705917327586384\text{e}{+00}i$ $-8.152087361523326\text{e}{-01}-2.850024263184126\text{e}{+00}i$ $4.970325872396055\text{e}{-01}+2.729519259822502\text{e}{+00}i$ $-5.433637042723353\text{e}{-02}-2.946384869537706\text{e}{+00}i$ $1.282928255975401\text{e}{+00}+2.519903189853295\text{e}{+00}i$ $1.564476048921003\text{e}{+00}+1.768834378822957\text{e}{+00}i$ $-1.643042886688555\text{e}{+00}-1.212344348158034\text{e}{+00}i$ $1.943119133724864\text{e}{+00}+8.605877308030484\text{e}{-02}i$ $1.486930904640041\text{e}{-01}-1.861577918840281\text{e}{+00}i$ $-1.013702946373266\text{e}{+00}-2.443176161667233\text{e}{-01}i$ $-4.328984355639987\text{e}{-01}+1.008036318652911\text{e}{+00}i$ $6.261926774054385\text{e}{-01}-5.762496449081425\text{e}{-01}i$ $4.487381899523429\text{e}{-01}-2.877609613824950\text{e}{-01}i$

Figure 12: T_k Eigenvalues for different values of K