MAT 226B Large Scale Matrix Computation
Homework 4

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March, 12th 2020

Problem 1:

(a) The breakdown in nonsymmetric Lanczos process occurs

- when v_k or w_k is a zero vector. This is the case where V spans $K_k\{A, r\}$ or W spans $K_k\left\{A^T, c\right\}$. The is a termination condition and occurs when k reaches $d(A, r)$ or $d(A^T, c)$. Since, $d(A, r) \geq 2$ and $d(A^T, c) \geq 2$, then this is not the case.
- when $w_k^T v_k = 0$. Since v_k and w_k is the normalized r and c respectively for $k = 1$. Then, $w_k^T v_k = 0$ for $k = 1$ and breakdown will occur since $w_k^T v_k$ is used as the denominator to compute α_k leading to division by zero.

(b) At step $k = 2$, a breakdown will occur if

$$\delta_2 = w_2^T v_2 = 0$$

Assuming that r and c are normalized (i.e., $\| r \|_2 = \| c \|_2 = 1$), we can re-write the above condition as

$$w_2^T v_2 = (A^T w_1 - \alpha_1 w_1)^T (A v_1 - \alpha_1 v_1) = 0$$

That could happens if $(A^T w_1 - \alpha_1 w_1)^T = 0$ or $A v_1 = \alpha_1 v_1$. Thus,

$$\begin{gathered} A v_1 = \alpha_1 v_1 \\ A r = \alpha_1 r \\ A r = \frac{c^T A r}{c^T r} r \\ (c^T r)(A r) = (c^T A r)(r) \\ (c^T r)(A) = (c^T A r) \\ (c^T r)(c^T A^2 r) = (c^T A r)^2 \end{gathered}$$

Problem 2:

The function `arnoldi_process` in `problem_2.m` implements Arnoldi process. It takes matrix A, initial r vector, and `KMAX` value and outputs matrices H_k and V_K. We test the function with $k = 5, 10,$ and 20. Table 2 shows the eignevalues of H_k for different values of k. Table 1 shows the eignevalues of A. To compare between the eigenvalue of H_{20} and A's eigenvalues, we computed $\| A\_eig - H_{20}\_eig \| = 2.022960706546113\text{e}{-14}$

Eigenvalues
$-1.246365732148965\text{e}{+00}+3.879484437687162\text{e}{+00}i$
$1.251730623491239\text{e}{-01}-3.705917327586323\text{e}{+00}i$
$2.594180629560029\text{e}{+00}-2.284902242884256\text{e}{+00}i$
$-1.983514535906377\text{e}{+00}+2.443522984313265\text{e}{+00}i$
$3.000394334727376\text{e}{+00}-9.028020730501798\text{e}{-01}i$
$-8.152087361523785\text{e}{-01}-2.850024263184167\text{e}{+00}i$

Figure 1: A Eigenvalues

K	Eigenvalues
5	$-1.667643118967301\text{e}{+00}+3.322698092379202\text{e}{+00}i$ $1.122189729586331\text{e}{+00}-2.935868201795706\text{e}{+00}i$ $-2.454291098365838\text{e}{+00}-1.034091783975195\text{e}{-01}i$ $1.901983203411739\text{e}{+00}+2.938354695246812\text{e}{-02}i$ $-1.488418331702275\text{e}{-01}+4.856460403443859\text{e}{-03}i$
10	$2.481430445455568\text{e}{-01}-3.792088063628674\text{e}{+00}i$ $-1.284055777449226\text{e}{+00}+3.812474971159054\text{e}{+00}i$ $-9.594670448581000\text{e}{-01}-2.797385858005139\text{e}{+00}i$ $-2.769085996727914\text{e}{+00}-4.783604953147819\text{e}{-01}i$ $-1.865849533089461\text{e}{+00}+2.203044871147163\text{e}{+00}i$ $-1.983244640080843\text{e}{+00}+6.985561973441846\text{e}{-01}i$ $2.484678533435320\text{e}{+00}-2.158804713951516\text{e}{+00}i$ $2.525082205768530\text{e}{+00}-9.804387029418838\text{e}{-01}i$ $1.801564391235925\text{e}{+00}+1.650150378773440\text{e}{+00}i$ $9.603199929809421\text{e}{-01}+1.532501604030885\text{e}{+00}i$
20	$-1.246365732148964\text{e}{+00}+3.879484437687161\text{e}{+00}i$ $2.594180629560022\text{e}{+00}-2.284902242884260\text{e}{+00}i$ $-1.983514535906369\text{e}{+00}+2.443522984313272\text{e}{+00}i$ $3.000394334727364\text{e}{+00}-9.028020730501802\text{e}{-01}i$ $-2.816575470153527\text{e}{+00}-4.470235401498117\text{e}{-01}i$ $-2.335649345045595\text{e}{+00}+1.118762976889542\text{e}{+00}i$ $4.970325872397225\text{e}{-01}+2.729519259822235\text{e}{+00}i$ $2.304844231638446\text{e}{+00}+1.654017825159756\text{e}{+00}i$ $1.282928255975394\text{e}{+00}+2.519903189853356\text{e}{+00}i$ $1.251730623491198\text{e}{-01}-3.705917327586333\text{e}{+00}i$ $-8.152087361523789\text{e}{-01}-2.850024263184166\text{e}{+00}i$ $-5.433637042721347\text{e}{-02}-2.946384869537709\text{e}{+00}i$ $1.564476048921009\text{e}{+00}+1.768834378823156\text{e}{+00}i$ $-1.643042886688547\text{e}{+00}-1.212344348158080\text{e}{+00}i$ $1.943119133724999\text{e}{+00}+8.605877308058085\text{e}{-02}i$ $1.486930904640139\text{e}{-01}-1.861577918840294\text{e}{+00}i$ $-4.328984355639898\text{e}{-01}+1.008036318652936\text{e}{+00}i$ $-1.013702946373261\text{e}{+00}-2.443176161667247\text{e}{-01}i$ $6.261926774054364\text{e}{-01}-5.762496449081369\text{e}{-01}i$ $4.487381899523373\text{e}{-01}-2.877609613824833\text{e}{-01}i$

Figure 2: H_k Eigenvalues for different values of K

Problem 3:

(a) Function `hermitian_lanczos` implements the Hermitian Lanczos process where it takes the matrix A, vector r, and `KMAX` parameter and output T_k tridiagonal matrix in sparse

format. We used this function to approximate eigenvalues of A from `make_3d_laplacian` function using $k = 7$. Table 3 shows the results where the first two columns show the exact eigenvalues of A (computed from the provided function) along its multiplicity and last column shows the approximate eigenvalues. We can see that the approximate solution is able to capture all the eigenvalues with very high accuracy.

Exact Eigenvalues	Multiplicity	Approximate Eigenvalues
1.757359312880715e+00	1	1.757359312880715e+00
3.171572875253810e+00	3	3.171572875253810e+00
4.585786437626905e+00	6	4.585786437626905e+00
6.000000000000000e+00	7	6.000000000000002e+00
7.414213562373095e+00	6	7.414213562373095e+00
8.828427124746190e+00	3	8.828427124746192e+00
1.024264068711928e+01	1	1.024264068711929e+01

Figure 3: Exact and approximate eigenvalues using Hermitian Lanczos process

(b) We used `hermitian_lanczos` to compute the approximate eigenvalues for the 262144×262144 matrix from `make_3d_laplacian(64)`. Table 4, Table 5, Table 6, Table 7, Table 8, Table 9, and Table 10 show the 10 smallest and 10 largest approximated eigenvalues. Table 11 shows the 10 smallest and 10 largest exact eigenvalues. Due to the multiplicity in exact eigenvalues and round of error in approximate one, it is harder to check the accuracy due to repeated eigenvalues.

Smallest Eigenvalues	Largest Eigenvalues
1.422787575075167e−02	1.170222368514153e+01
2.522292134967839e−02	1.175506886438500e+01
3.720946521956244e−02	1.180683457939747e+01
6.015341020592301e−02	1.184896392339356e+01
8.407846117839080e−02	1.188540931980204e+01
1.168498457907536e−01	1.191749739736500e+01
1.581734700307590e−01	1.194638621492721e+01
1.995732825474702e−01	1.196512279467138e+01
2.466691832816189e−01	1.198055273500129e+01
3.011244810109077e−01	1.199293004521624e+01

Figure 4: Approximate Eigenvalues using $K = 100$

Smallest Eigenvalues	Largest Eigenvalues
7.006926067695284e−03	1.191194541966088e+01
1.400816296330363e−02	1.192367820060319e+01
2.104749654401042e−02	1.193815033138952e+01
2.644416419643677e−02	1.194943725586547e+01
3.252921177956383e−02	1.195823115974695e+01
4.248144987518579e−02	1.196722905407029e+01
5.158180541911261e−02	1.197279443138257e+01
6.175958581860198e−02	1.197897280487774e+01
7.373377938324976e−02	1.198599186623671e+01
8.527634111033579e−02	1.199299336087879e+01

Figure 5: Approximate Eigenvalues using $K = 200$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006043782e−03	1.195105825040107e+01
1.400782323540659e−02	1.195568924396476e+01
2.100900746475762e−02	1.195806895929420e+01
2.565829376859278e−02	1.196033932970904e+01
2.801019169415529e−02	1.196734052200119e+01
3.265947799840193e−02	1.197198980830585e+01
3.966069944862968e−02	1.197434170623140e+01
4.193108831261764e−02	1.197899099253526e+01
4.431079048315387e−02	1.198599217676459e+01
4.897191625334762e−02	1.199299336099394e+01

Figure 6: Approximate Eigenvalues using $K = 400$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006040011e−03	1.197198980830589e+01
7.006639006045063e−03	1.197434170555307e+01
1.400782323539451e−02	1.197434170623140e+01
1.400782323540438e−02	1.197899099253525e+01
2.100900746475027e−02	1.197899099253527e+01
2.100900746475296e−02	1.198599217676461e+01
2.565829371816093e−02	1.198599217676461e+01
2.565829376859185e−02	1.199293916949044e+01
2.801018851772900e−02	1.199299336099396e+01
2.801019169410766e−02	1.199299336099397e+01

Figure 7: Approximate Eigenvalues using $K = 800$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006034983e−03	1.197899099253524e+01
7.006639006042774e−03	1.197899099253525e+01
7.006639006050910e−03	1.197899099253525e+01
7.006642433917978e−03	1.198599217676460e+01
1.400782323538701e−02	1.198599217676461e+01
1.400782323539747e−02	1.198599217676462e+01
1.400782323539999e−02	1.199299336099394e+01
2.100900746475157e−02	1.199299336099396e+01
2.100900746475692e−02	1.199299336099397e+01
2.100900746476310e−02	1.199299336099399e+01

Figure 8: Approximate Eigenvalues using $K = 1200$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006034724e−03	1.197899099253525e+01
7.006639006037083e−03	1.198599217676456e+01
7.006639006037985e−03	1.198599217676460e+01
7.006639006044512e−03	1.198599217676460e+01
7.006639006047808e−03	1.198599217676461e+01
1.400782323539137e−02	1.199299336099392e+01
1.400782323539370e−02	1.199299336099395e+01
1.400782323539474e−02	1.199299336099396e+01
1.400782323540451e−02	1.199299336099397e+01
1.412781215190661e−02	1.199299336099398e+01

Figure 9: Approximate Eigenvalues using $K = 1600$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006032009e−03	1.198599217676460e+01
7.006639006034958e−03	1.198599217676460e+01
7.006639006035550e−03	1.198599217676461e+01
7.006639006037528e−03	1.198599260901220e+01
7.006639006037725e−03	1.199299336099394e+01
7.006639006042897e−03	1.199299336099395e+01
1.400782323536894e−02	1.199299336099396e+01
1.400782323539092e−02	1.199299336099396e+01
1.400782323539872e−02	1.199299336099396e+01
1.400782323540259e−02	1.199299336099397e+01

Figure 10: Approximate Eigenvalues using $K = 2000$

Smallest Eigenvalues	Largest Eigenvalues
7.006639006040594e−03	1.197434170623141e+01
1.400782323539662e−02	1.197434170623141e+01
1.400782323539662e−02	1.197434170623141e+01
1.400782323539662e−02	1.197899099253525e+01
2.100900746475265e−02	1.197899099253525e+01
2.100900746475265e−02	1.197899099253525e+01
2.100900746475265e−02	1.198599217676460e+01
2.565829376859075e−02	1.198599217676460e+01
2.565829376859163e−02	1.198599217676460e+01
2.565829376859163e−02	1.199299336099396e+01

Figure 11: Exact Eigenvalues

Problem 4:

(a) The function `nonsymmetric_lanczos` implements the nonsymmetric Lanczos process where it takes a function that implements Av and another function that implements $A^T v$ along with the initial r and c vectors. It outputs the T_k tridiagonal matrix is sparse format. We used this function to compute T_k for $K = 5, 10,$ and 20 and computed the eigenvalues of T_k. To compare between the eigenvalue of T_{20} and A's eigenvalues, we computed $\| A\_eig - T_{20}\_eig \| = 6.448717\text{e}{-13}$

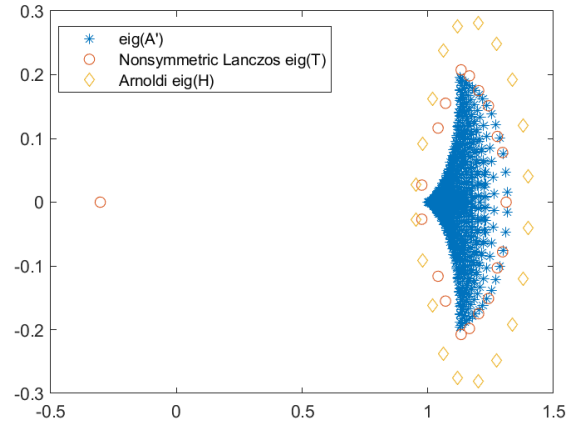
(b) we used our `nonsymmetric_lanczos` and `arnoldi_process` to find the converged eigenvalues for the matrix in file `HW4_P4b.mat` for increasing values of k. We note that in out implementation we used the efficient matrix-vector multiplication routine developed in homework 3. It easy to use it to also implement transposed-matrix-vector multiplication needed for Lanczos process.

It was hard to track down numerically what are the converged eigenvalues for both method. However, plotting the eigenvalues of A' along with the eigenvalues of T_k from the nonsymmetric Lanczos process and H_k from Arnoldi process, we can easily see the converged eigenvalues. Figure 13 shows how eigenvalues converges for increasing values of K. We notice that for small values of K, nonsymmetric Lanczos has a better chance to get eigenvalues that are true eigenvalues.

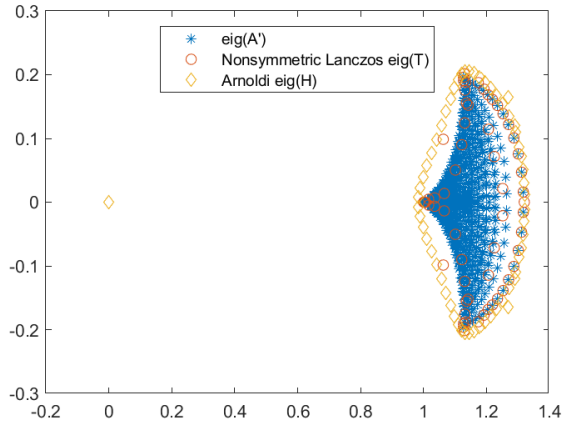
(c) We used the same function and efficient matrix-vector multiplication to carry on this part. However, on personal laptop, nonsymmetric Lanczos runs of memory quickly even for small K on the matrix provided on `HW4_P4c.mat`. Thus, we used the same matrix as part (b) to show the correctness of our method for increasing K. Figure 15 shows the eigenvalues obtained for different K values. We also computed the norm of difference between each consecutive step we took (after sorting the eigenvalues). Table 14 shows these norm. While Figure 15 shows that the eigenvalues converges, the differences shown in 14 does not correlate with this as the difference increases with increasing K. We think this is due to some outlier eigenvalues that can be seen along the x-axis.

K	Eigenvalues
5	$-2.501781014189590\text{e}{+00}+5.900692650763462\text{e}{+00}i$ $-3.090333049113557\text{e}{+00}+4.825912738594896\text{e}{-01}i$ $5.172926282011777\text{e}{-01}-7.888954236341315\text{e}{-01}i$ $1.256199307483214\text{e}{+00}-3.493703520606647\text{e}{+00}i$ $1.274556869629201\text{e}{+01}-1.125556172280826\text{e}{+01}i$
10	$1.338852744862344\text{e}{-01}-3.681053207507567\text{e}{+00}i$ $2.508862893210253\text{e}{+00}-2.197729989474411\text{e}{+00}i$ $2.670866099884081\text{e}{+00}-1.314179550767412\text{e}{+00}i$ $-4.439295624831807\text{e}{-01}-2.347985483160573\text{e}{+00}i$ $-2.826197293664038\text{e}{+00}-2.998937801231081\text{e}{-01}i$ $-1.899586046943437\text{e}{+00}+7.856568851355709\text{e}{-02}i$ $-2.187569680623757\text{e}{+00}+2.082401070017189\text{e}{+00}i$ $-1.238832783943400\text{e}{+00}+3.769011471476989\text{e}{+00}i$ $1.936757547900978\text{e}{+00}+2.974410140200699\text{e}{+00}i$ $3.216278496510915\text{e}{+00}+5.676288164244975\text{e}{+00}i$
20	$-1.246365732148833\text{e}{+00}+3.879484437687169\text{e}{+00}i$ $2.594180629560308\text{e}{+00}-2.284902242884396\text{e}{+00}i$ $-1.983514535906585\text{e}{+00}+2.443522984313314\text{e}{+00}i$ $3.000394334727138\text{e}{+00}-9.028020730499668\text{e}{-01}i$ $-2.816575470153615\text{e}{+00}-4.470235401498108\text{e}{-01}i$ $2.304844231638672\text{e}{+00}+1.654017825159949\text{e}{+00}i$ $-2.335649345045510\text{e}{+00}+1.118762976889514\text{e}{+00}i$ $1.251730623491955\text{e}{-01}-3.705917327586384\text{e}{+00}i$ $-8.152087361523326\text{e}{-01}-2.850024263184126\text{e}{+00}i$ $4.970325872396055\text{e}{-01}+2.729519259822502\text{e}{+00}i$ $-5.433637042723353\text{e}{-02}-2.946384869537706\text{e}{+00}i$ $1.282928255975401\text{e}{+00}+2.519903189853295\text{e}{+00}i$ $1.564476048921003\text{e}{+00}+1.768834378822957\text{e}{+00}i$ $-1.643042886688555\text{e}{+00}-1.212344348158034\text{e}{+00}i$ $1.943119133724864\text{e}{+00}+8.605877308030484\text{e}{-02}i$ $1.486930904640041\text{e}{-01}-1.861577918840281\text{e}{+00}i$ $-1.013702946373266\text{e}{+00}-2.443176161667233\text{e}{-01}i$ $-4.328984355639987\text{e}{-01}+1.008036318652911\text{e}{+00}i$ $6.261926774054385\text{e}{-01}-5.762496449081425\text{e}{-01}i$ $4.487381899523429\text{e}{-01}-2.877609613824950\text{e}{-01}i$

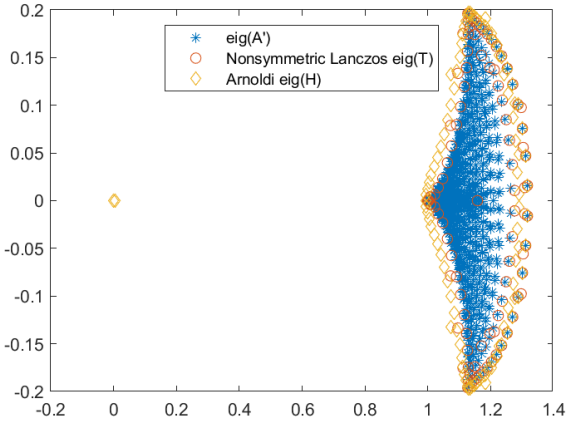
Figure 12: T_k Eigenvalues for different values of K



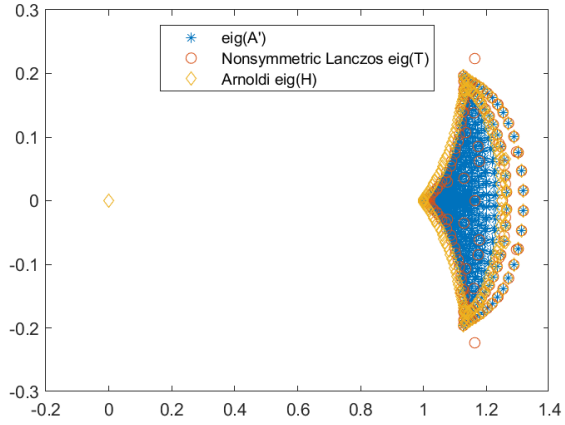
(a) K= 20



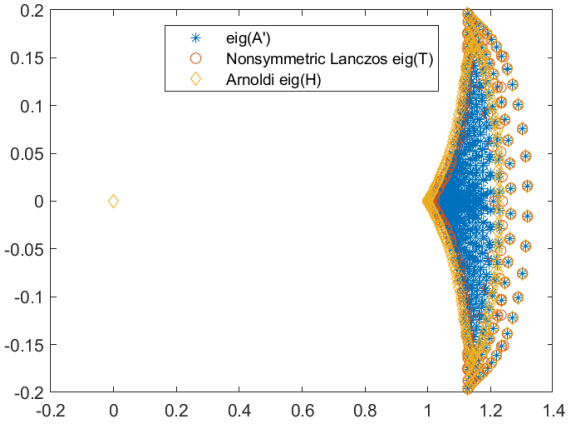
(b) K=60



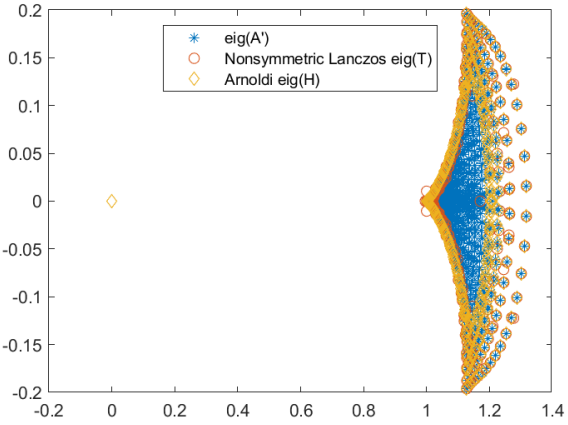
(c) K=100



(d) K=200



(e) K=300

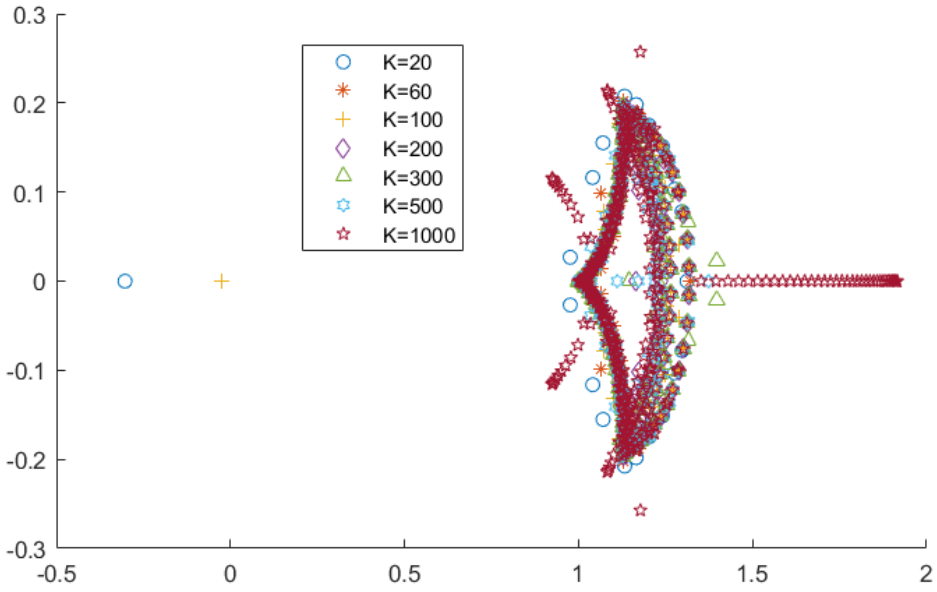


(f) K=500

Figure 13: Converged eigenvalues from nonsymmetric Lanczos and Arnoldi process.

K	Norm of the difference between eigenvalue of T_k and T_{k-1}
20	
60	1.478839
100	2.283969
200	1.590567
300	2.787656
500	4.120523
1000	3.833748

Figure 14: The norm of the difference between the eigenvalues at K and $K-1$ using for increasing values of k of nonsymmetric Lanczos process



(a) K= 20

Figure 15: Eigenvalues obtain from nonsymmetric Lanczos for increasing value of K