Math 228B - HW4

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1 Problem No.1

1.1 Problem Description:

Write programs to solve the advection equation

$$u_t + au_x = 0,$$

on $[0,1]$ with periodic boundary conditions using upwinding and Lax-Wendroff. For smooth solutions we expect upwinding to be first-order accurate and Lax-Wendroff to be second-order accurate, but it is not clear what accuracy to expect for nonsmooth solution.

1. Let $a = 1$ and solve the problem up to time $t = 1$. Perform a refinement study for both upwinding and Lax-Wendroff with $\Delta t = 0.9a\Delta x$ with a smooth initial condition. Compute the rate of converge in the 1-norm and 2-norm, and max-norm. Note that the exact solution at time $t = 1$ is the initial condition, and so computing the error is easy.

2. Repeat the previous problem with the discontinuous initial condition

$$u(x,0) = \begin{cases} 1 & if \ \ |x-\frac{1}{2}| < \frac{1}{4} \\ 0 & otherwise \end{cases}$$

1.2 Solution:

Under upwinding scheme, the advection equation is discretized as:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

For which we can define *Courant* number ν as $\nu = a\Delta t/\Delta x$. For Lax-Wendroff scheme, the discretization would be

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - a^2\Delta t\frac{u_n^{j-1} - 2u_j^n + u_{j+1}^n}{2\Delta x^2} = 0$$

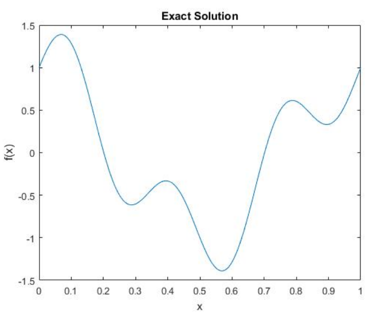
$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2\Delta t^2}{2\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

For smooth initial conditions we use $cos(2.0 * x * PI) + sin(6.0 * x * PI)/2.0$ as initial conditions which is smooth continuous function for the periodic domain [0,1]. The periodic boundary conditions mean that the value of the last grid point is equal to the value of the first gird point. We can consider this as solving over a circle.

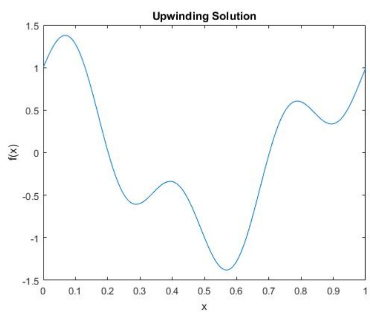
Since the problem asking for solving such that $\Delta t = 0.9a\Delta x$ and $a = 1.0$, thus $\nu = 0.9$ and the numerical scheme is stable. This applies for upwinding and Lax-Wendroff.

The error was calculated for 1-norm as $||e||_1 = h\sum_j |e_j|$, where e_j is the absolute difference (error) between the numerical solution and exact solution at grid point j. For 2-norm, the error is $||e||_2 = \sqrt{h\sum_j |e_j|^2}$. For max-norm, the error becomes $||e||_{max} = max_j|e_j|$.

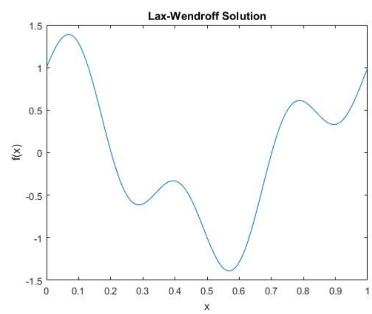
The exact and numerical solutions for both the smooth and discontinuous initial conditions are shown in Figure 1.



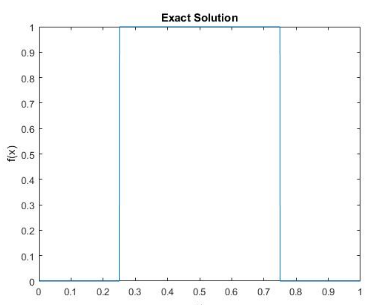
(a) Smooth exact solution



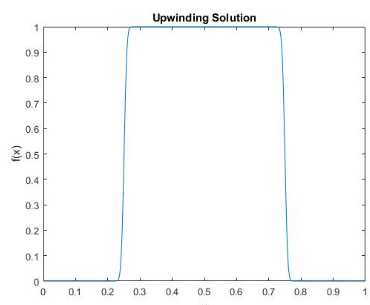
(b) Smooth Upwinding solution



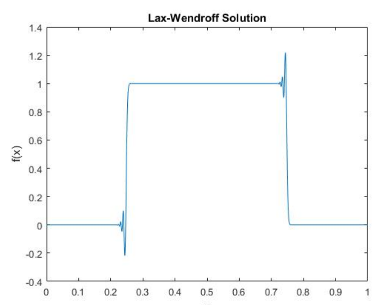
(c) Smooth LW solution



(d) Discont. exact solution



(e) Discont. Upwinding solution



(f) Discont. LW solution

Figure 1: The exact solution (first column), upwinding solution (second column) and Lax-Wendroff solution (third column) for both smooth (first row) and discontinuous (second row) initial conditions for the 1d advection equation with $\Delta x = \frac{1}{2048}$.

A refinement study is carried out to investigate the rate of convergence. Figure 2 (end of the report) shows the convergence rate (in log-log scale) in 1-norm, 2-norm and max-norm for continuous and discontinuous initial condition for both upwinding and Lax-Wendroff schemes. With continuous initial conditions, the slope of the line for upwinding is 1.2 for continuous initial conditions and 1.85 for Lax-Wendroff which indicates that the scheme is first-order accurate and second-order accurate respectively. This is for the three tested norms. For discontinuous data, both method convergences in 1-norm and 2-norm. This is clear from the solution plot in Figure 1. For max-norm, non of the two scheme convergence since for Lax-Wendroff the wiggles moves closer to the discontinuity but it has the same magnitude (Gibbs phenomenon). For upwinding, there is always smearing near the sharp discontinuity which prevents the max-norm from converging.

2 Problem No.2

2.1 Problem Description:

For solving the heat equation we frequently use Crank-Nicolson, which is trapezoidal rule time integration with a second-order space discretization. The analogous scheme for the linear advection equation is

$$u_j^{n+1} - u_j^n + \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = 0,$$

where $\nu = a\Delta t/\Delta x$

1. Use von Neumann analysis to show that this scheme is unconditionally stable and that $||u^n||_2 = ||u^0||_2$. This scheme is said to be non-dissipative - i.e. there is no amplitude error. This seems reasonable because this is a property of the PDE.

2. Solve the advection equation on the periodic domain $[0, 1]$ with the initial condition from problem 1 (part 2). Show the solution and comment on your results.

3. Compute the relative phase error as $arg(g(\theta))/(-\nu\theta)$, where g is the amplification factor and $\theta = \zeta\Delta x$, and plot it for $\theta \in [0, \pi]$. How does the relative phase error and lack of amplitude error relate to the numerical solutions you observed in part 2.

2.2 Solution:

Part 1: To show that the scheme is unconditionally stable using von Neumann analysis, we start by assuming that the solution is of the form

$$u_j^n = e^{i\zeta x_j}$$

$$u_j^{n+1} = g(\zeta)e^{i\zeta x_j}$$

where $g(\zeta)$ is the amplification factor and ζ is the wave number. In order to show the scheme is stable, $|g(\zeta)| \leq 1$. We start by plugging in the solution above in advection equation, the result will be

$$g(\zeta)e^{i\zeta x_j} - e^{i\zeta x_j} + \frac{\nu}{4}(e^{i\zeta(x_j+\Delta x)} - e^{i\zeta(x_j-\Delta x)}) + \frac{\nu}{4}(g(\zeta)e^{i\zeta(x_j+\Delta x)} - g(\zeta)e^{i\zeta(x_j-\Delta x)}) = 0$$

Diving by $e^{i\zeta x_j}$ and arranging, the equation becomes

$$g(\zeta) - 1 + \frac{\nu}{4}(e^{i\zeta\Delta x} - e^{-i\zeta\Delta x}) + \frac{\nu}{4}(g(\zeta)e^{i\zeta\Delta x} - g(\zeta)e^{-i\zeta\Delta x}) = 0$$

$$g(\zeta) = \frac{1 - \frac{\nu}{4}(e^{i\zeta\Delta x} - e^{-i\zeta\Delta x})}{1 + \frac{\nu}{4}(e^{i\zeta\Delta x} - e^{-i\zeta\Delta x})}$$

$$g(\zeta) = \frac{1 - i\frac{\nu}{2}sin(\zeta\Delta x)}{1 + i\frac{\nu}{2}sin(\zeta\Delta x)}$$

Taking the absolute value of the above

$$|g(\zeta)|^2 = \frac{1 + \dfrac{\nu^2}{4} sin^2(\zeta \Delta x)}{1 + \dfrac{\nu^2}{4} sin^2(\zeta \Delta x)} = 1$$

Since the $|g(\zeta)| \leq 1$, then the scheme is unconditionally stable. $\square$

To prove that $||u^n||_2 = ||u^0||_2$, we start by arranging the advection equation so follows

$$u_j^{n+1} + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n - \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n)$$

Taking the sum squared of both sides

$$\sum_j \Big(u_j^{n+1} + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) \Big)^2 = \sum_j \Big(u_j^n - \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n) \Big)^2$$

$$\sum_j \Big((u_j^{n+1})^2 + \frac{\nu}{2} u_j^{n+1}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + \frac{\nu^2}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1})^2 \Big) =$$

$$\sum_j \Big((u_j^n)^2 - \frac{\nu}{2} u_j^n (u_{j+1}^n - u_{j-1}^n) + \frac{\nu^2}{4}(u_{j+1}^n - u_{j-1}^n)^2 \Big)$$

$$\sum_j (u_j^{n+1})^2 = \sum_j \Big((u_j^n)^2 - \frac{\nu}{2}(u_j^n(u_{j+1}^n - u_{j-1}^n) + u_j^{n+1}(u_{j+1}^{n+1} - u_{j-1}^{n+1})) +$$

$$\Big(\frac{\nu}{4}\Big)^2 ((u_{j+1}^n - u_{j-1}^n)^2 - (u_{j+1}^{n+1} - u_{j-1}^{n+1})^2) \Big)$$

Since it's proven that $u_j^{n+1} = |g(\zeta)| u_j^n = u_j^n$, then

$$\sum_j (u_j^{n+1})^2 = \sum_j \Big((u_j^n)^2 - \frac{\nu}{2}(u_j^n(u_{j+1}^n - u_{j-1}^n) + u_j^n(u_{j+1}^n - u_{j-1}^n)) +$$

$$\Big(\frac{\nu}{4}\Big)^2 ((u_{j+1}^n - u_{j-1}^n)^2 - (u_{j+1}^n - u_{j-1}^n)^2) \Big)$$

$$\sum_j (u_j^{n+1})^2 = \sum_j ((u_j^n)^2 - \nu u_j^n (u_{j+1}^n - u_{j-1}^n))$$

$$\sum_j (u_j^{n+1})^2 = \sum_j ((u_j^n)^2) + \nu \Big(\sum_j u_j^n u_{j-1}^n - \sum_j u_j^n u_{j+1}^n \Big)$$

The product at grid point j and the next point is the same as the product of j and the previous point since it is a domain with periodic boundary conditions and we can imagine this as moving over a circle.

$$\sum_j (u_j^{n+1})^2 = \sum_j ((u_j^n)^2)$$

$$||u^{n+1}||_2 = ||u^n||_2$$

Thus, $||u^n||_2 = ||u^{n-1}||_2 = ||u^{n-2}||_2 ..... = ||u^0||_2$ $\square$

Part 2: We can rewrite the equation as

$$Au_j^{n+1} + Bu_{j+1}^{n+1} + Cu_{j-1}^{n+1} = D_j^n$$

where

$$A = 1, \;\; B = \frac{\nu}{4}, \;\; C = \frac{-\nu}{4}, \;\; D_i^n = u_i^n - \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n)$$

The matrix formulation for the above equation for the periodic domain is

$$\begin{vmatrix} A & B & 0 & 0 & ... & ... & 0 & C \\ C & A & B & 0 & ... & ... & 0 & 0 \\ 0 & C & A & B & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 0 & C & A & B \\ B & 0 & ... & ... & 0 & 0 & C & A \\ \end{vmatrix} \begin{vmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ ... \\ ... \\ ... \\ u_{nx-2}^{n+1} \\ u_{nx-1}^{n+1} \\ \end{vmatrix} = \begin{vmatrix} D_0^n \\ D_1^n \\ D_2^n \\ ... \\ ... \\ ... \\ D_{nx-2}^n \\ D_{nx-1}^n \\ \end{vmatrix}$$

The above system was solved using QR decomposition and the result is shown in Figure 3 (b) for discontinuous initial conditions. The solution shows a lot of wiggles that are not just confined to where the discontinuity occurs but it spreads across the whole solution.

Part 3: We start by deriving the expression $arg(g(\theta))$, where $\theta = \zeta\Delta x$ form the amplification factor such that

$$g(\zeta) = \frac{1 - i\frac{\nu}{2}sin(\zeta\Delta x)}{1 + i\frac{\nu}{2}sin(\zeta\Delta x)}$$

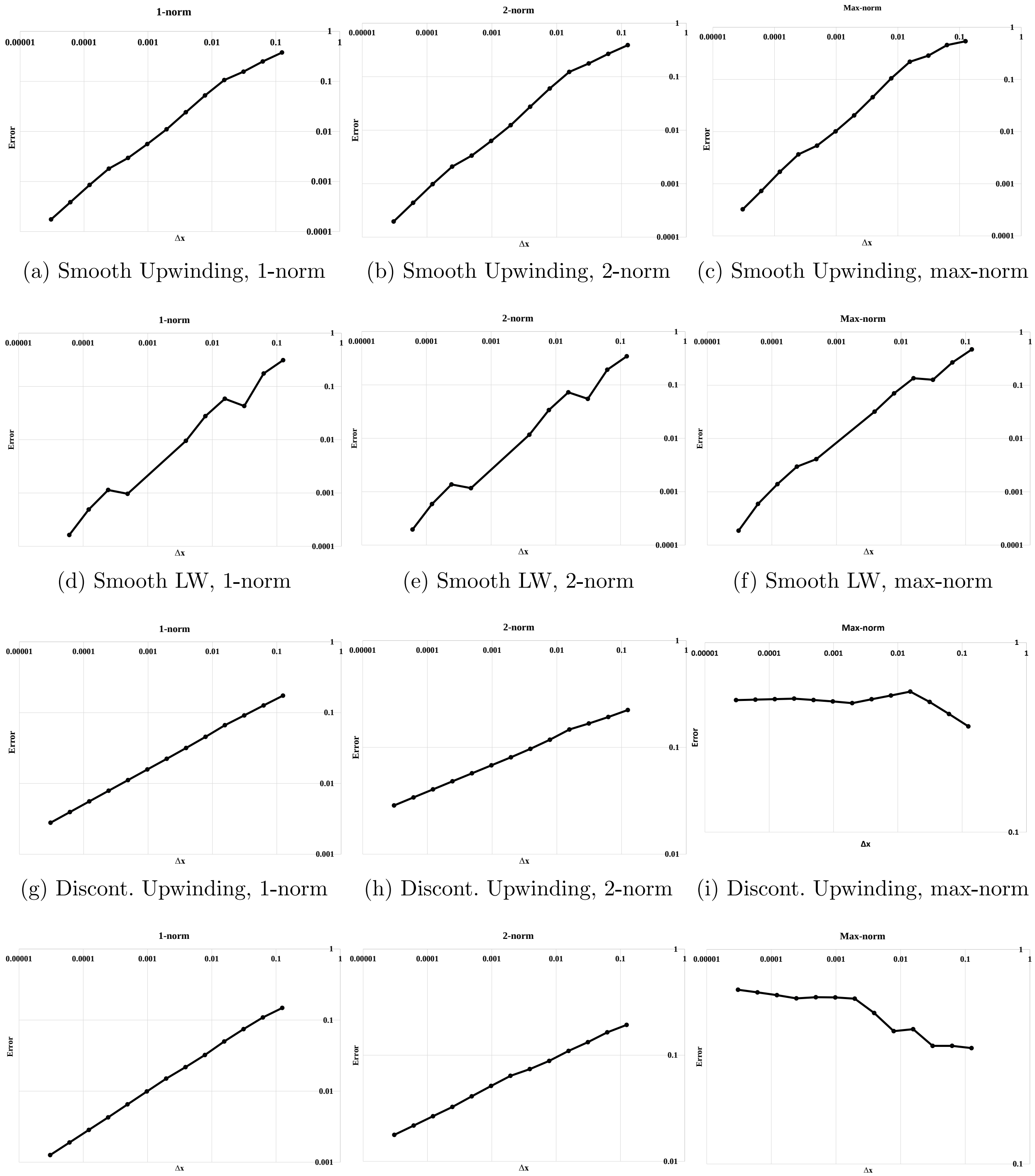
$$g(\zeta) = \frac{1 - \frac{\nu^2}{4}sin^2(\zeta\Delta x)}{1 + \frac{\nu^2}{4}sin^2(\zeta\Delta x)} - i\frac{\nu sin(\zeta\Delta x)}{1 + \frac{\nu^2}{4}sin^2(\zeta\Delta x)}$$

$$g(\zeta) = \frac{1 - \frac{\nu^2}{4}sin^2(\theta)}{1 + \frac{\nu^2}{4}sin^2(\theta)} - i\frac{\nu sin(\theta)}{1 + \frac{\nu^2}{4}sin^2(\theta)}$$

This gives the phase as

$$arg(g(\theta)) = tan^{-1}\left(-\frac{\nu sin(\theta)}{1 - \frac{\nu^2}{4}sin^2(\theta)}\right)$$

Figure 3 (b) shows the relative phase error. The plot suggests that at all frequencies, there will be always a phase error and it is not constant (each frequency will exhibit different phase error). This explains the results in (a) and suggests that this method is not suitable for solving the advection equation with discontinuous initial conditions.



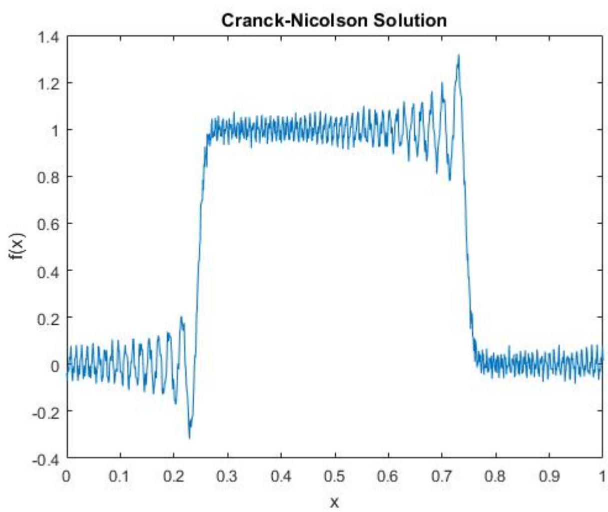
(a) Smooth Upwinding, 1-norm (b) Smooth Upwinding, 2-norm (c) Smooth Upwinding, max-norm

(d) Smooth LW, 1-norm (e) Smooth LW, 2-norm (f) Smooth LW, max-norm

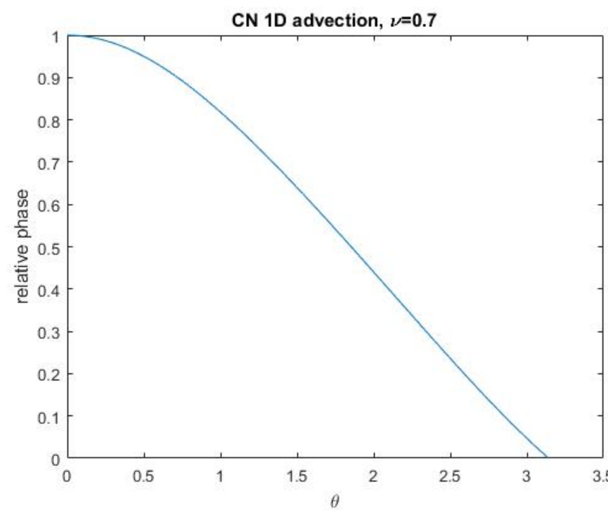
(g) Discont. Upwinding, 1-norm (h) Discont. Upwinding, 2-norm (i) Discont. Upwinding, max-norm

(j) Discont. LW, 1-norm (k) Discont. LW, 2-norm (l) Discont. LW, max-norm

Figure 2: The rate convergence (on log-log scale) in 1-norm (first column),2-norm (second column) and max-norm (third column) for smooth (first two rows) and discontinuous (last two rows) initial condition using upwinding and Lax-Wendroff scheme to solve the 1d advection equation.



(a) Solution



(b) Relative Phase

Figure 3: Solution for of 1D advection equation using CN method; (a) shows the solution with discontinuous initial conditions and (b) shows the the relative phase error.