Math 228B - HW5

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1 Problem No.1

1.1 Problem Description:

In one spatial dimension the linearized equations of acoustics (sound waves) are

$$p_t + Ku_x = 0$$
$$\rho u_t + p_x = 0$$

where u is the velocity and p is the pressure, ρ is the density, and K is the bulk modulus.

1. Show that this system is hyperbolic and find the wave speeds.

2. Write a program to solve this system using Lax-Wendroff in original variables on (0,1) using a cell centered grid $x_j = (j - \frac{1}{2})\Delta x$ for $j = 1...N$. Write the code to use ghost cells, so that different boundary conditions can be changed by simply changing the values in the ghost cells. Ghost cells are cells outside the domain whore values can be set at the beginning of a time step so that code for updating cells adjacent to the boundary is identical to the code for interior cells. Set the ghost cells at the left by

   $$p_0^n = p_1^n$$
   $$u_0^n = -u_1^n$$

   and set the ghost cells on the right by

   $$p_{N+1}^n = \frac{1}{2}\left(p_N^n + u_N^n\sqrt{K\rho}\right)$$
   $$u_{N+1}^n = \frac{1}{2}\left(\frac{p_N^n}{\sqrt{K\rho}} + u_N^n\right)$$

   Run simulations with different initial conditions. Explain what happens at the left and right boundaries.

3. Give a physical interpretation and a mathematical explanation of these boundary conditions

1.2 Solution:

Part 1: To prove that the system in hyperbolic, we start by writing the system of equation in matrix. Thus,

$$q_t + Aq_x = 0$$

$$\begin{bmatrix} p \\ u \end{bmatrix}\dot{}t + \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}\dot{}x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For the above system, we need to prove that A is diagonalizable and its eigenvalues are real and distinct (strictly hyperbolic). Since the transpose of A is a diagonal matrix, then A is diagonalizable. The eigenvalues of A can be obtain by solving the following

$$Det(A - \lambda I) = 0$$

$$\lambda^2 = \frac{K}{\rho}$$

$$\lambda_{1,2} = \pm\sqrt{\frac{K}{\rho}}$$

K is the bulk modulus which measure how incompressible a substance is which always positive (ration of infinitesimal pressure increase to resulting relative decrease in volume). ρ is density which always positive. Thus, the λ is real and distinct which makes the eigenvalues of A real and distinct. Thus, the system is (strictly) hyperbolic.

Part 2: The system of equations above can be discretized using Lax-Wendroff and it become

$$p_j^{n+1} = p_j^n - \frac{K\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{K^2\Delta t^2}{2\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

$$u_j^{n+1} = u_j^n - \frac{1/\rho\Delta t}{2\Delta x}(p_{j+1}^n - p_{j-1}^n) + \frac{(1/\rho)^2\Delta t^2}{2\Delta x^2}(p_{j-1}^n - 2p_j^n + p_{j+1}^n)$$

We used Courant number (ν) of 0.8 from which we can compute the time step Δt such that $\Delta t = \nu\Delta x/(|\lambda|)$, where λ is the speed of sound (according to Newton-Laplace).

We used four different initial conditions; sine wave, triangular function, mixed sine wave and triangular function and exponentially decaying sine wave. Figures 1, 2, 3 and 4 show the obtained solution at different time intervals up to time = 1 second.

For the run tests with different initial conditions, we observed that the initial conditions starts by splitting in two parts (waves) one moves to the right and the other moves to the left (as shown in Figures 1, 2, 3 and 4). Due to the given boundary conditions, the right side absorbs the incoming wave while the left side reflects/bounces the incoming wave which travel all the way to the right side which in turn absorbs it.

Part 3: The boundary conditions given on the left wall has zero pressure gradient and the velocity is zero (since the right and left cells has the same value with opposite signs). This means that the momentum of the sound wave is preserved and thus it bounces back and no sound it transmitted through the wall. On the right boundary, since the velocity and pressure has positive values and due to their momentum, they will continue travel in this same direction (and not bounce back).

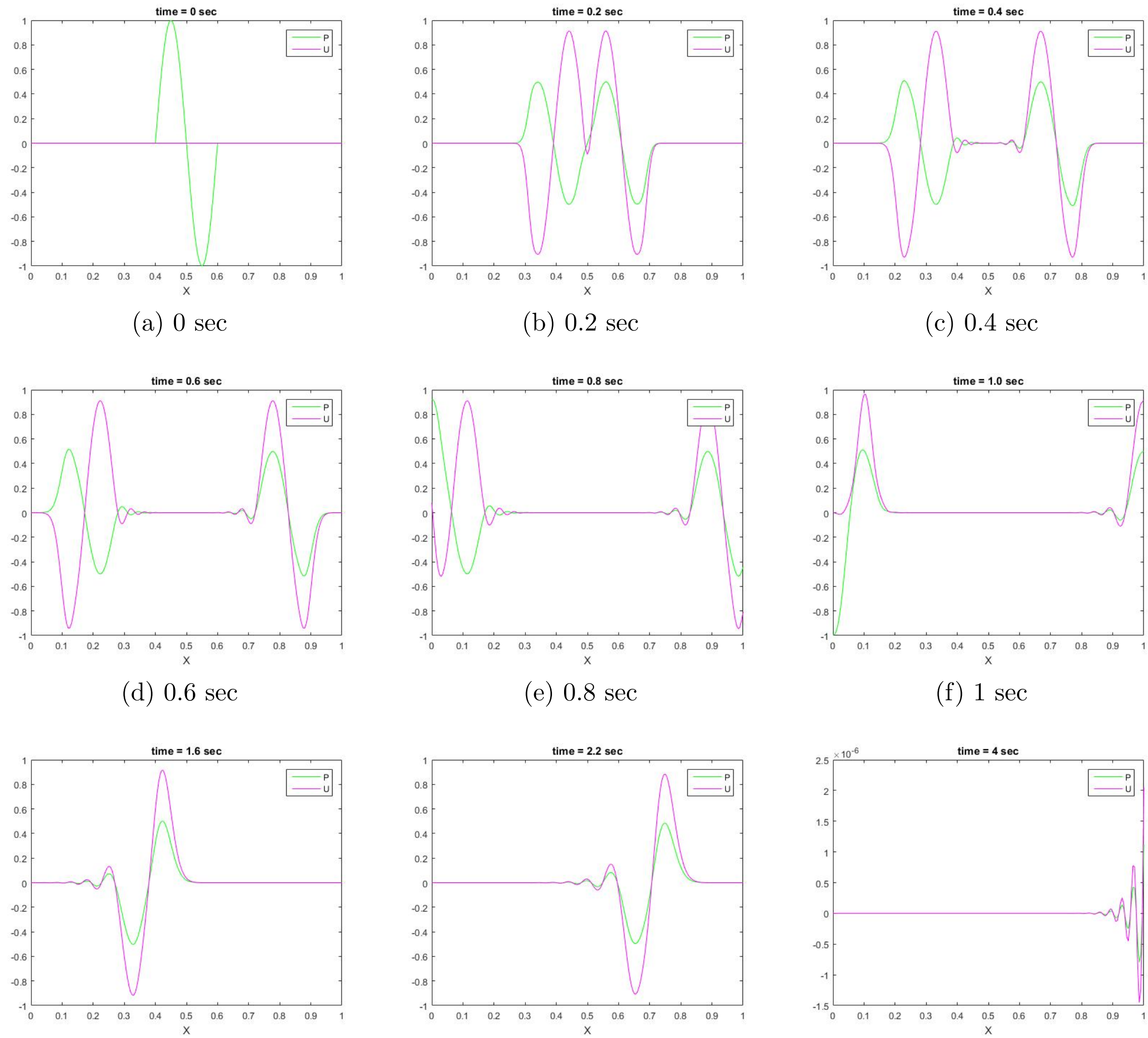

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Figure 1: Solution of the linearized equations of acoustics using Lax-Wendroff method on cell-centered grid with sine wave as initial condition for the pressure and zero initial condition for the speed at different time intervals.

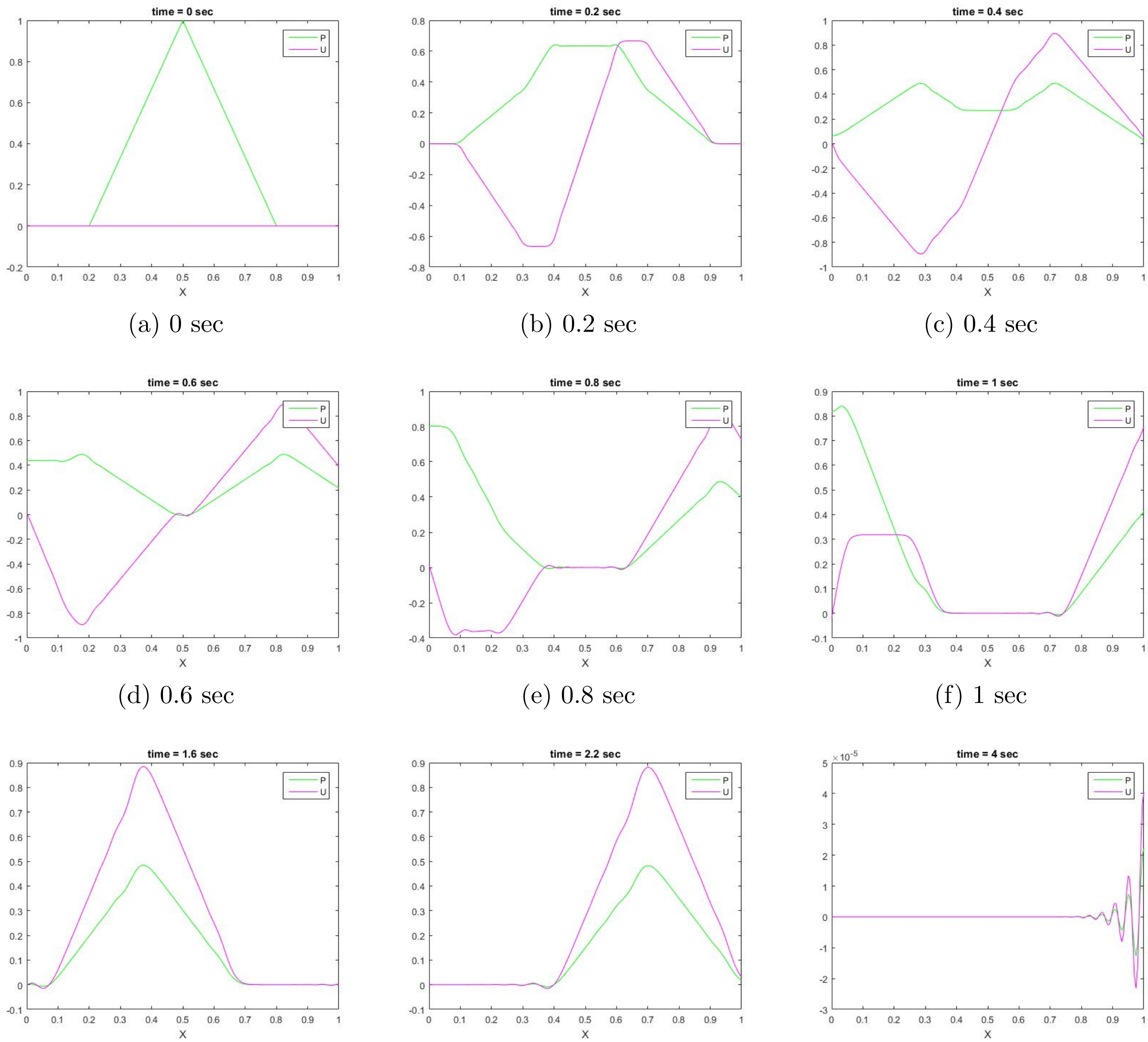

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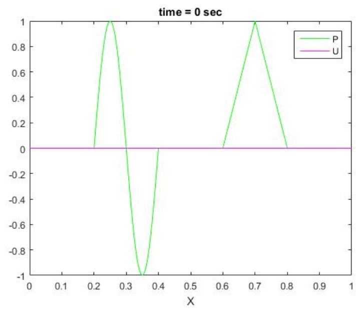
Figure 2: Solution of the linearized equations of acoustics using Lax-Wendroff method on cell-centered grid triangular function as initial condition for the pressure and zero initial condition for the speed at different time intervals.



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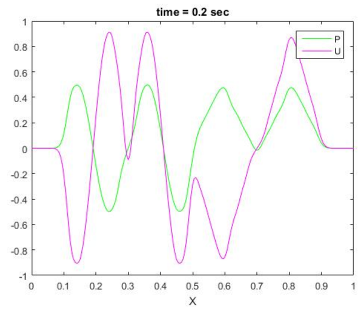

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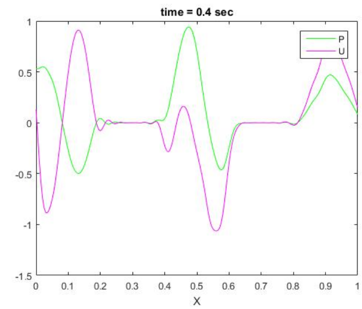

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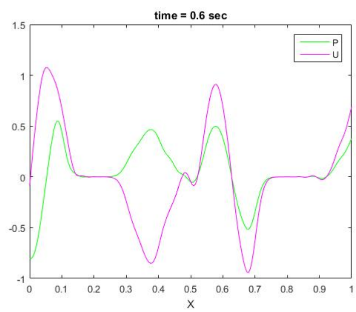

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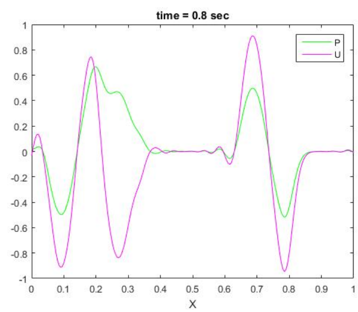

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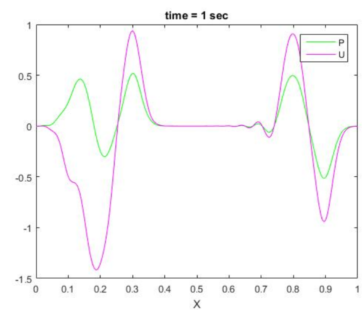

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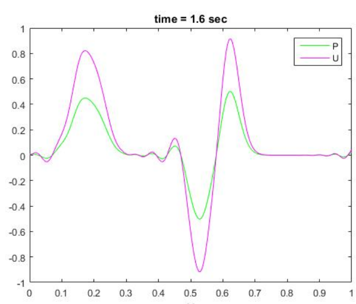

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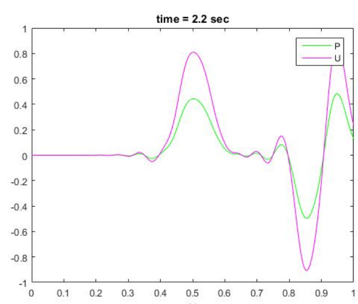

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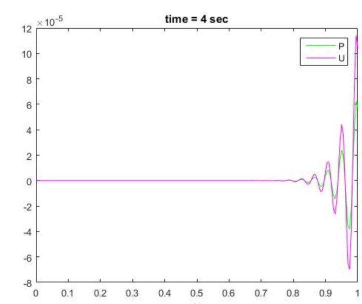

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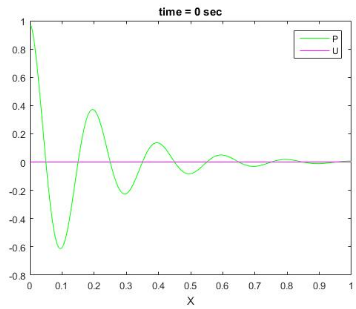
Figure 3: Solution of the linearized equations of acoustics using Lax-Wendroff method on cell-centered grid with mixed sine wave and triangular function as initial condition for the pressure and zero initial condition for the speed at different time intervals.



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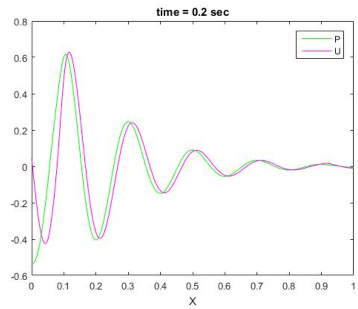

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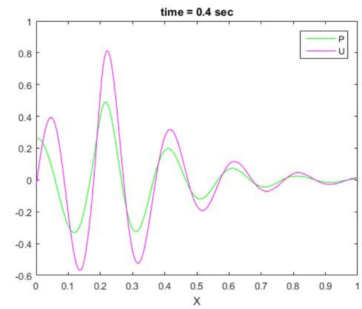

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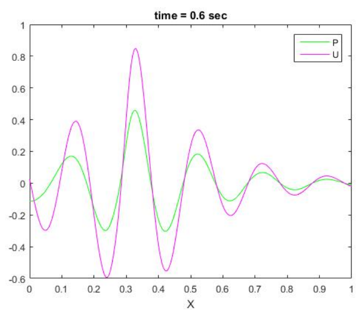

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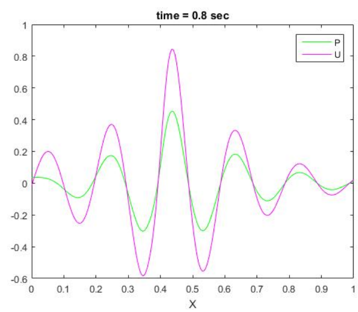

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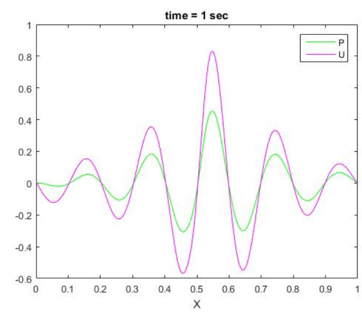

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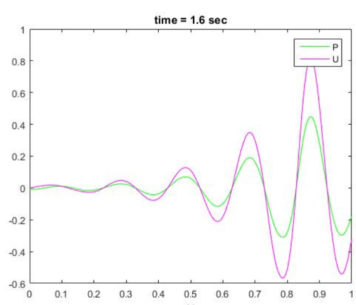

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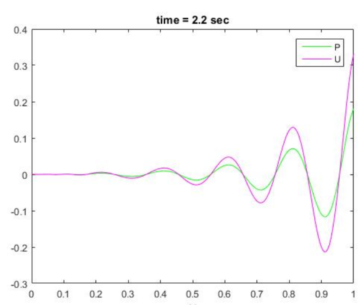

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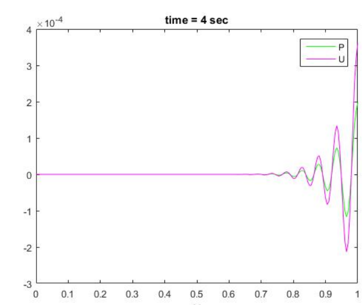

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Figure 4: Solution of the linearized equations of acoustics using Lax-Wendroff method on cell-centered grid with decaying exponential sine wave as initial condition for the pressure and zero initial condition for the speed at different time intervals.

2 Problem No.2

2.1 Problem Description:

Write a program to solve the linear advection equation,

$$u_t + au_x = 0$$

on the unit interval using a finite volume method of the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}})$$

Use the numerical flux function

$$F_{j-\frac{1}{2}} = F^{up}_{j-\frac{1}{2}} + \frac{|a|}{2}\left(1 - |\frac{a\Delta t}{\Delta x}|\right)\delta_{j-\frac{1}{2}}$$

where $F^{up}_{j-\frac{1}{2}}$ is the upwinding flux,

$$F^{up}_{j-\frac{1}{2}} = \begin{cases} au_{j-1} & if \ \ a > 0 \\ au_j & if \ \ a < 0 \end{cases}$$

and $\delta_{j-\frac{1}{2}}$ i the limited difference. Let $\Delta u_{j-\frac{1}{2}} = u_j - u_{j-1}$ denote the jump in u across the edge at $x_{j-\frac{1}{2}}$. The limited difference is

$$\delta_{j-\frac{1}{2}} = \phi(\theta_{j-\frac{1}{2}})\Delta u_{j-\frac{1}{2}}$$

where

$$\theta_{j-\frac{1}{2}} = \frac{\Delta u_{J_{up}-\frac{1}{2}}}{\Delta u_{j-\frac{1}{2}}}$$

$$J_{up} = \begin{cases} j-1 & if \ \ a > 0 \\ j+1 & if \ \ a < 0 \end{cases}$$

Note that you will need two ghost cells on each end of the domain. Write your program so that you may choose from the different limiter functions listed below.
Upwinding $\phi(\theta) = 0$
Lax-Wendroff $\phi(\theta) = 1$
Beam-Warming $\phi(\theta) = \theta$
minmod $\phi(\theta) = minmod(1, \theta)$
superbee $\phi(\theta) = max(0, min(1, 2\theta), min(2, \theta))$
MC $\phi(\theta) = max(0, min((1+\theta)/2, 2, 2\theta))$
van Leer $\phi(\theta) = (\theta + |\theta|)/(1 + |\theta|)$
The first three are linear methods that we have already studied, and the last four are high resolution methods.

Solve the advection equation with $a = 1$ with periodic boundary conditions for the different initial conditions listed below until time $t = 5$ at Courant number 0.9.

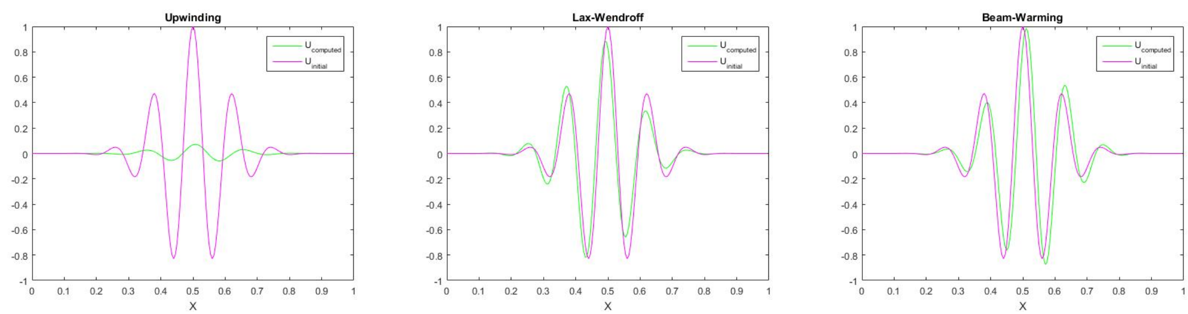
1. Wave packet: $u(x,0) = cos(16\pi x)exp(-50(x-0.5)^2)$
2. Smooth, low frequency: $u(x,0) = sin(2\pi x)sin(4\pi x)$
3. Step function:

$$u(x,0) \begin{cases} 1 & if |x - \frac{1}{2}| < \frac{1}{4} \\ 0 & otherwise \end{cases}$$

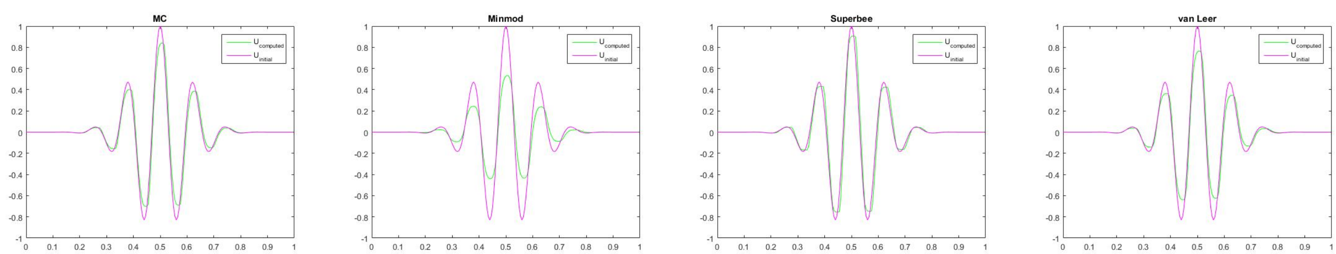
Compare the results with the exact solution, and comment on the solutions generated by the different methods. How do the different high-resolution methods perform in the different test? What high-resolution method would you choose to use in practice?

2.2 Solution:

Figure 5, 6 and 7 show the solution obtained from solving the above defined finite volume method and flux functions. The comparison between the computed solution and exact one is also shown.



(a) Upwinding (b) Lax-Wendroff (c) Beam-Warming

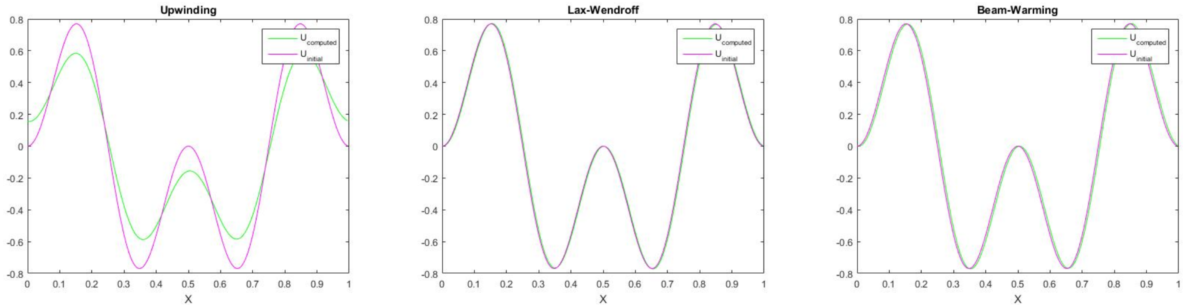


(d) MC (e) Minmod (f) Superbee (g) van Leer

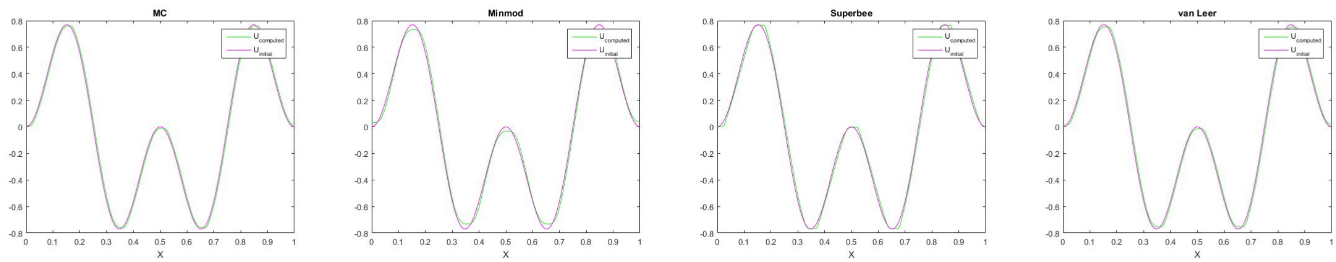
Figure 5: Solution using wave packet initial conditions with different limiter functions; green is the computed solution, magenta is the exact solution.

For the wave packet initial conditions, Upwinding method shows large diffusion but there is no shift in the solution. Lax-Wendroff shows a little diffusion plus a shift to the left while Beam-Warming has its shift to the right. MC, Superbee, Minmod and van Leer all have diffusion with no shift. Minmod has the largest diffuse while Superbee has the least diffuse. Thus, for smooth initial data, I would choose Superbee.

For the smooth, low frequency initial conditions, Upwinding shows the same diffuse while Lax-Wendroff and Beam-Warming have a barely noticeable shift (to left and to right respectively). No shift

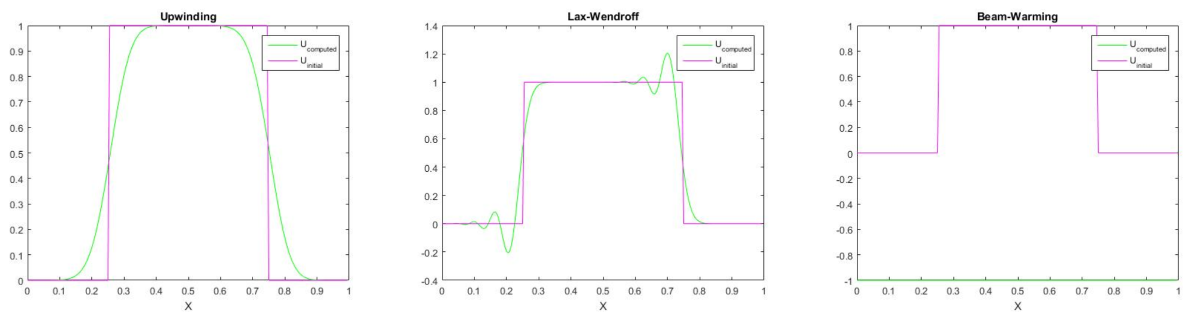


(a) Upwinding (b) Lax-Wendroff (c) Beam-Warming

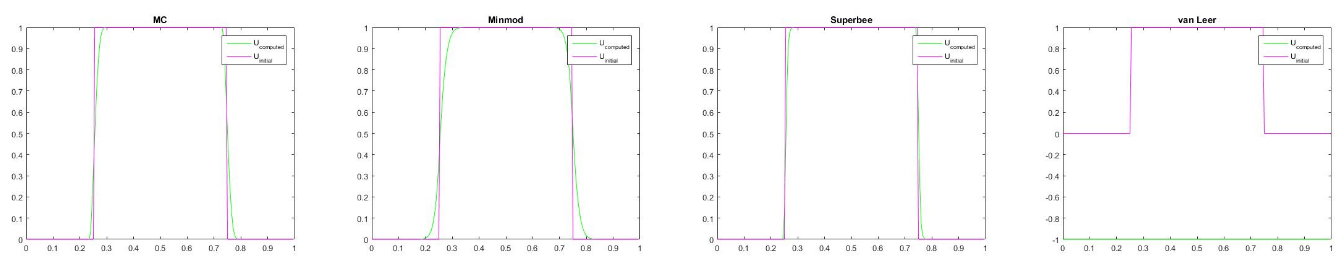


(d) MC (e) Minmod (f) Superbee (g) van Leer

Figure 6: Solution using smooth, low frequency initial conditions with different limiter functions; green is the computed solution, magenta is the exact solution.



(a) Upwinding (b) Lax-Wendroff (c) Beam-Warming



(d) MC (e) Minmod (f) Superbee (g) van Leer

Figure 7: Solution using step function initial conditions with different limiter functions; green is the computed solution, magenta is the exact solution.

exist for the high-resolution methods. For Minmod, small amount of diffusion was observed near the crests. For Superbee, the solution is less smooth at the crests. Thus, I would choose MC or van Leer for this initial conditions as a high-resolution methods. If non-high resolution method was to be picked, I would pick Lax-Wendroff or Beam-Warming.

For the step function, there sounds to be an error in the solution of both Beam-Warming and van Leer. It was expected that Lax-Wendroff to have ripples that are larger than that for Lax-Wendroff. For van Leer, it was expected to be similar to MC for such step function. Upwinding has no ripples but the function got diffused. Lax-Wendroff has ripples around the corners but less diffusive than Upwinding. MC has no ripples and it matches closely the exact solution. Same thing goes for Superbee where the solution is even closer to the exact one. Minmode has also no ripples but it is no better than Superbee. Thus, for step/discontinuous initial conditions, I would pick Superbee.