Math 228B - HW1

Ahmed H. Mahmoud

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1 Problem No.1

1.1 Problem Description:

Consider the advection equation

$$u_t + au_x = 0$$

on the interval $[0, 1)$ with periodic boundary conditions. Space is discretized as $x_j = j\Delta x$ for $j = 0 ... N-1$, so that $\Delta x = 1/N$. Descretize the spatial derivative with the second-order centered difference operator.

1. For simplicity, assume N is odd. The eigenvectors of the centered difference operator are

   $$v_j^k = exp(2\pi jkx_j)$$

   for $k = (-(N-1)/2)...((N-1)/2)$. Compute the eigenvalues.

2. Derive a time step restriction on a method-of-lines approach which uses classical fourth-order Runge-Kutta for time stepping.

1.2 Solution:

Eigenvalues: The periodic boundary condition is expressed as

$$u(0, t) = u(1, t)$$

which physically means that whatever flows out at the outflow boundary flows back in at the inflow boundary (assuming $a > 0$)[1]. Thus we have $u_0(t) = u_{n-1}(t)$, which contributes as a single unknown($u_{n-1}(t)$). Using the centered difference operator for spatial derivative, we get

$$u_j'(t) = \frac{-a}{2\Delta x}(u_{j+1}(t) - u_{j-1}(t))$$

$$\text{For } j = 1 \longrightarrow u_1'(t) = (-a/2\Delta x)(u_2(t) - u_0(t))$$
$$\text{or } j = 1 \longrightarrow u_1'(t) = (-a/2\Delta x)(u_2(t) - u_{n-1}(t))$$

For $j = 2 \longrightarrow u_2'(t) = (-a/2\Delta x)(u_3(t) - u_1(t))$

And so forth, till we reach $i = n - 1$
For $i = n \longrightarrow u_{n-1}'(t) = (-a/2\Delta x)(u_1(t) - u_{n-2}(t))$

The system of linear equations can be written in the following form

$$\begin{vmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \\ ... \\ ... \\ ... \\ u_{n-2}'(t) \\ u_{n-1}'(t) \\ \end{vmatrix} = \begin{vmatrix} 0 & \alpha & 0 & 0 & ... & ... & 0 & -\alpha \\ -\alpha & 0 & \alpha & 0 & ... & ... & 0 & 0 \\ 0 & -\alpha & 0 & \alpha & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 0 & -\alpha & 0 & \alpha \\ \alpha & 0 & ... & ... & 0 & 0 & -\alpha & 0 \\ \end{vmatrix} \begin{vmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ ... \\ ... \\ ... \\ u_{n-2}(t) \\ u_{n-1}(t) \\ \end{vmatrix}$$

where $\alpha = -a/2\Delta x$. We notice that A is *skew-symmetric* since $A_T = -A$, and thus its eigenvalues are pure imaginary [2]. Additionally, the nonzero eigenvalues come in pairs, each the negation of the other. The eigenvalues of the matrix can be derived by assuming that $u_j(t)$ take the form $e^{i\zeta x_j}$, where ζ is the wave number and $i = \sqrt{-1}$. By substituting into the discretized equation, it becomes

$$u_j'(t) = \frac{-a}{2\Delta x}(e^{i\zeta(x_j+\Delta x)} - e^{i\zeta(x_j-\Delta x)}) = \frac{-a}{2\Delta x}e^{i\zeta x_j}(e^{i\zeta\Delta x} - e^{-i\zeta\Delta x})$$

$$u_j'(t) = \frac{-a}{2\Delta x}e^{i\zeta x_j}2isin(\zeta\Delta x) = \frac{-a}{\Delta x}e^{i\zeta x_j}isin(\zeta\Delta x)$$

The wave number can be written as $\zeta = 2\pi k$. Thus,

$$u_j'(t) = \frac{-ia}{\Delta x}sin(2\pi k\Delta x)u_j(t)$$

Thus, the eigenvalues of the matrix are

$$\lambda_k = -\frac{ia}{\Delta x}sin(2\pi k\Delta x), \quad k = 1, 2, ...n - 1$$

Stability Restrictions: To derive the stability restriction, we start by expressing the advection equation as $u_\prime(t) = \lambda u(t)$, where λ is the eigenvalue. After applying the descritization method (fourth-order Runge-Kutta), we obtain $U^{n+1} = R(z)U^n$, such that $R(z)$ is some function of $z = \lambda\Delta t$. $R(z)$ is a polynimal for explicit methods and rational for implicit methods [2]. Then, the range of absolute stability will be
$\mathcal{S} = z \in \mathbb{C} : |R(z)| \leq 1$

We start with applying the fourth-order Runge-Kutta method on the advection equation. We obtain

$$U^{n+1} = U^n + \frac{\Delta t}{6}\Big[f(Y_1, t_n) + 2f\Big(Y_2, t_n + \frac{\Delta t}{2}\Big) + 2f\Big(Y_3, t_n + \frac{\Delta t}{2}\Big) + f(Y_4, t_n + \Delta t)\Big]$$

such that

$$
\begin{aligned}
Y_1 &= U^n \\
Y_2 &= U^n + \frac{1}{2}\,\Delta t f(Y_1, t_n) \\
Y_3 &= U^n + \frac{1}{2}\,\Delta t f\Big(Y_2, t_n + \frac{\Delta t}{2}\Big) \\
Y_4 &= U^n + \Delta t f\Big(Y_3, t_n + \frac{\Delta t}{2}\Big)
\end{aligned}
$$

The full derivation can be found in the Appendix. We obtain at the end

$$U^{n+1} = U^n[1 + z + \tfrac{1}{2}\,z^2 + \tfrac{1}{6}\,z^3 + \tfrac{1}{24}\,z^4]$$

From the definition of absolute stability introduced earlier, we have the following restriction

$$|1 + z + \tfrac{1}{2}\,z^2 + \tfrac{1}{6}\,z^3 + \tfrac{1}{24}\,z^4| \leq 1$$

The above equation should give the restriction over the time step by substituing $z = \lambda\Delta t$.

2 Problem No.2

2.1 Problem Description:

Consider the following PDE:

$$u_t = 0.01u_{xx} + 1 - exp(-t), \quad 0 < x < 1$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

$$u(x,0) = 0$$

Write a program to solve the problem using Crank-Nicolson up to time $t = 1$, and perform a refinement study that demonstrates that the method is second-order accurate in space and time.

2.2 Solution:

The discretized PDE under Crank-Nicolson scheme is[1]:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\Big(0.01\Big(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\Big) - exp(-n\Delta t)(1 + exp(\Delta t))\Big)$$

where Δt is the time spacing and Δx is the space spacing. The left side is central difference with time-step $\Delta t/2$, and the right side is the average of two central difference second order derivative. Thus the scheme is second-order accurate in space and time.

The equation can be rearranged as follows

$$\Big(1 + \Delta t\frac{0.01}{(\Delta x)^2}\Big)u_i^{n+1} - \frac{\Delta t}{2}\frac{0.01}{(\Delta x)^2}u_{i+1}^{n+1} - \frac{\Delta t}{2}\frac{0.01}{(\Delta x)^2}u_{i-1}^{n+1} =$$

$$\Big(1 - \Delta t\frac{0.01}{(\Delta x)^2}\Big)u_i^n + \frac{\Delta t}{2}\frac{0.01}{(\Delta x)^2}(u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{2}(2 - exp(-n\Delta t) - exp(-(n+1)\Delta t))$$

or

$$Au_i^{n+1} + Bu_{i+1}^{n+1} + Cu_{i-1}^{n+1} = D_i^n$$

where $A = (1 + \Delta t(0.01/(\Delta x)^2))$, $B = C = -(\Delta t/2)(0.01/(\Delta x)^2)$ and

$$D_i^n = \Big(1 - \Delta t\frac{0.01}{(\Delta x)^2}\Big)u_i^n + \frac{\Delta t}{2}\frac{0.01}{(\Delta x)^2}(u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{2}(2 - exp(-n\Delta t) - exp(-(n+1)\Delta t))$$

Knowing the boundary and initial condition, the formula can be used to solve the internal gird points i.e., $0 < i < nx-1$, and $0 < n < ny$ where nx is the number of grid points in the space direction and ny is the number of gird points in time direction.

For $i = 1 \longrightarrow Au_1^{n+1} + Bu_2^{n+1} + Cu_0^{n+1} = D_1^n$

or $Au_1^{n+1} + Bu_2^{n+1} = D_1^n - Cu_0^{n+1}$

For $i = 2 \longrightarrow Au_2^{n+1} + Bu_3^{n+1} + Cu_1^{n+1} = D_2^n$

For $i = 3 \longrightarrow Au_3^{n+1} + Bu_4^{n+1} + Cu_2^{n+1} = D_3^n$

And so forth, till we reach $i = nx - 2$

For $i = nx - 2 \longrightarrow Au_{nx-2}^{n+1} + Bu_{nx-1}^{n+1} + Cu_{nx-3}^{n+1} = D_{nx-2}^{n}$

or $Au_{nx-2}^{n+1} + Cu_{nx-3}^{n+1} = D_{nx-2}^{n} - Bu_{nx-1}^{n+1}$

This will give a system of equation that can be represented by a tri-diagonal matrix as follows

$$\begin{vmatrix} A & B & 0 & 0 & \dots & \dots & 0 & 0 \\ C & A & B & 0 & \dots & \dots & 0 & 0 \\ 0 & C & A & B & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & C & A & B \\ 0 & 0 & \dots & \dots & 0 & 0 & C & A \end{vmatrix} \begin{vmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \dots \\ \dots \\ \dots \\ u_{nx-3}^{n+1} \\ u_{nx-2}^{n+1} \end{vmatrix} = \begin{vmatrix} D_1^n - Cu_0^{n+1} \\ D_2^n \\ D_3^n \\ \dots \\ \dots \\ \dots \\ D_{nx-3}^n \\ D_{nx-2}^n - Bu_{nx-1}^{n+1} \end{vmatrix}$$

The system is then solved using LU-Decomposition.

2.3 Results:

Using the above formula, we can obtain the solution for different grid resolution as shown in Figure 1. For each grid resolution, we calculated the solution at different time steps (x-axis) and plotted them against the space (y-axis). For the iso-contour, red color means the function value is close to 1 while blue is for function values closer to 0. We notice that at different grid resolution, the effect of the boundary wash out as we advance in time. Additionally, the rate of washing out (convergence) is accelerated by increasing the grid resolution.

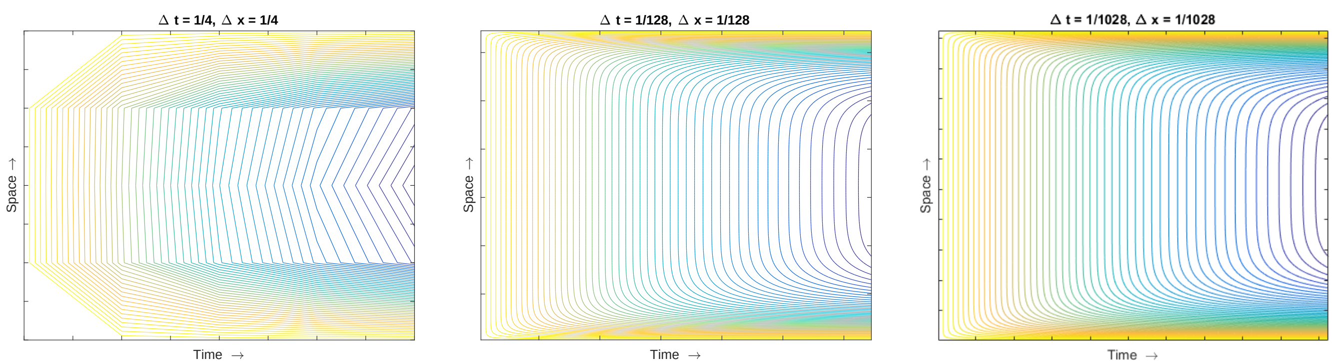


Figure 1: Iso-contour of the the solution for the given PDE using Crank-Nicolson scheme for different Δt and Δx values. The horizontal axis represents the time (advances from right to left) while the vertical axis represents the space.

Refinement Study: The goal of this study is to confirm that Crank-Nicolson scheme is second-order accurate in space and time. Ideally, we would calculate the solution for different grid resolution and graph the error as the normalized difference from the analytical solution. Since the analytical solution is not provided, and since we are interested only in the order of the error, we can calculate it based on the relative error between each two successive grid resolution.

Methodology: Here we derive how to extract the order of accuracy without the analytical solution. We are going to derive the order of n order forward finite different equation $f'(x_o) = f(x_o + ih_1)/h_1 + (O(h_1))^n$ where h_1 is the grid spacing. We can write the same equation for the same x_o for another grid resolution h_2 and subtract. The result would be

$$\frac{f(x_o + ih_1)}{h_1} - \frac{f(x_o + ih_2)}{h_2} = (O(h_2))^n - (O(h_1))^n$$

The LHS represents the difference between two solution computed for two different grid resolution. Thus, we can substitute it symbolically by $C_1 - C_2 = (O(h_2))^n - (O(h_1))^n$; where C_1 and C_2 represent the computed solution. Additionally, we can represent the RHS by a single term of order n since we are looking for the order not the exact value. This simplifies the equation to be $C_1 - C_2 = (Error)^n$, where $Error$ is a function of grid spacing. By taking the log of both sides, the equations becomes $log(C_1 - C_2) = n * log(Error)$. Thus, by taking the difference of two successive solutions at a point in time, and draw this difference against the grid spacing on log-log scale, the results is a straight line whose slope is the order of accuracy. So, we computed the solution at time =1 for certain refinement level and then compute the normalized sum. We did the same thing for another refinement level. The difference between the two normalized sums substitutes $C_1 - C_2$ in the above equation. Note that this is the error over both space and time.

We use the same method with different space and time steps. We starts by 4 space and time steps and increase both by factor of 2 at each solution. We plot the total difference between each two successive solution at $t = 1$ against the grid resolution on a log-log scale (Figure 2). We notice that the first point does not lie on the same line. This point is computed as the error between grid of size 4×4 and 8×8 which is too coarse and the results generally could be considered unreliable. The slope of the resulting line is -2.1. Thus, we can conclude that the method is second order accurate in space and time.

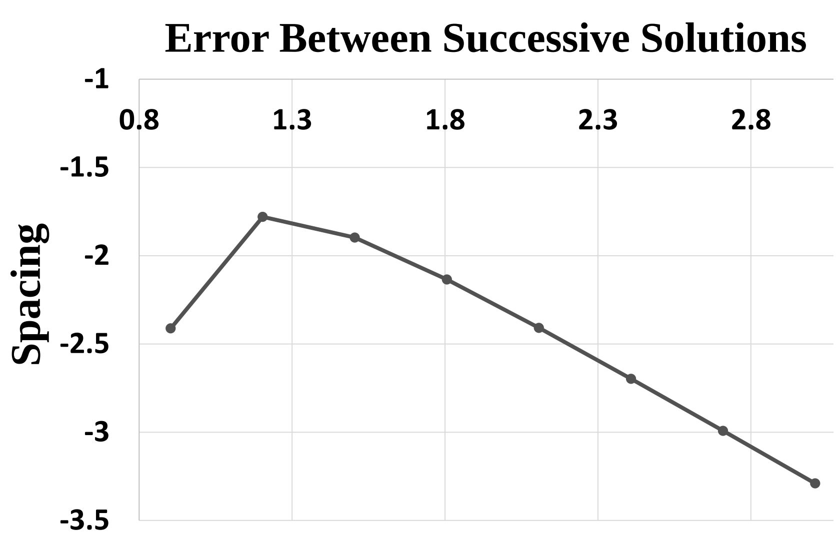


Figure 2: For the refinement study, we draw the error between two successive solution at $t = 1$ for different grid resolution and the slope of output line represents the order of accuracy in space and time.

3 Problem No.3

3.1 Problem Description:

$$
\begin{aligned}
&u_t = u_{xx}, \quad 0 < x < 1 \\
&u(0,t) = 1, \quad u(1,t) = 0 \\
&u(x,0) = \begin{cases} 1 & if \;\; x < 0.5 \\ 0 & if \;\; x \geq 0.5 \end{cases}
\end{aligned}
$$

1. Use Crank-Nicolson with grid spacing $\Delta x = 0.02$ and time step 0.1 to solve the problem up to time $t = 1$. Comment on your results. What is wrong with the solution?
2. Give a mathematical argument to explain the unphysical behavior you observed in the numerical solution.
3. Repeat the simulation using BDF2, and discuss why the unphysical behavior is not present in the numerical solution for any time step.

3.2 Solution:

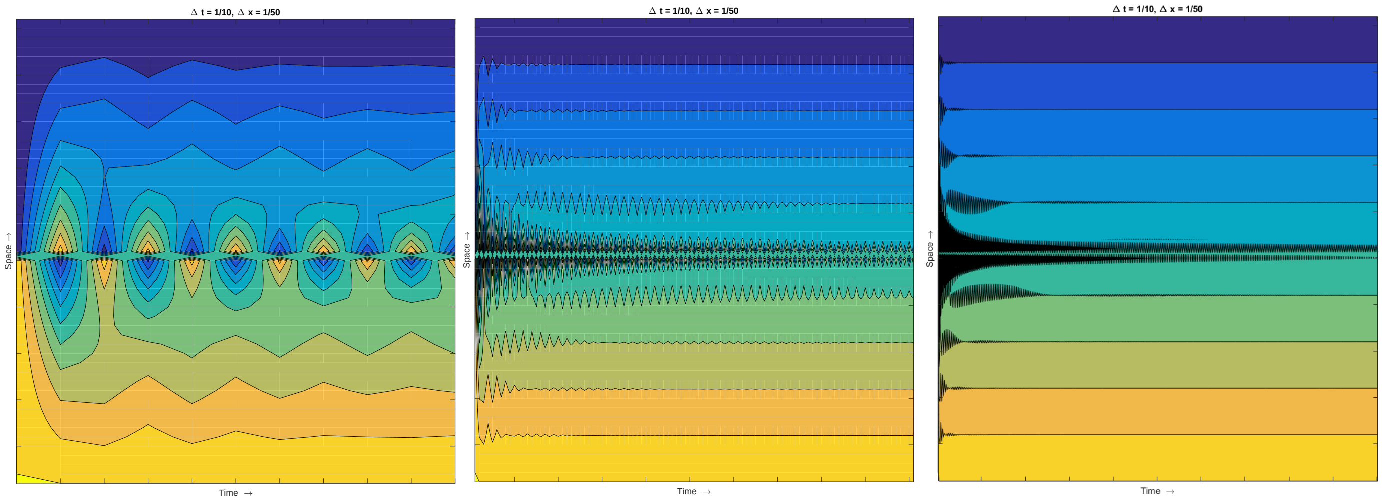
Crank-Nicolson: Following the same procedure described in Section 2.2, we can derive a similar set of linear equations for Crank-Nicolson. The difference lies withing the values of the coefficients $A, B, C,$ and D as follows:

$$
\begin{aligned}
&A = 1 + \Delta t \frac{1}{(\Delta x)^2}, \\
&B = C = -\frac{\Delta t}{2}\frac{1}{(\Delta x)^2}, \\
&D_i^n = \Big(1 - \Delta t \frac{1}{(\Delta x)^2}\Big) u_i^n + \frac{\Delta t}{2}\frac{1}{(\Delta x)^2}(u_{i+1}^n + u_{i-1}^n)
\end{aligned}
$$

Thus, the system can be solved using the same LU-Decomposition technique.

Using $\Delta x = 0.02$ and $\Delta t = 0.1$, the solution is shown in Figure 3(a). It is clear that the solution exhibits some nonphysical behavior. It is expected that the initial condition and its discontinuity to wash out at the end. But at $x = 0.4$ the solution's discontinuity still persists. Additionally, the amplitude of the exact solution (from the lecture notes) is $|u| = e^{-D\zeta^2 t}$, where $D = 1$. This indicates that if the initial conditions are discontinuities i.e., high spatial frequency (quick change in the function within small distance), it will instantly smoothed out which is not the case here.

We notice that the unphysical behavior of the solution diminishes if we increase the total time. When the total time is increased up to 10 and 50 (Figure 3(b) and (c)), the high frequency components starts to decay. Since Crank-Nicolson scheme is stable, then it is possible that the nonphysical behavior is due to the rate of decay of the high frequency components specially that we have large discontinuity in the initial conditions. This shall be investigated next.



(a) time = 1 (b) time = 10 (c) time = 50

Figure 3: Solution of Problem 3 using Crank-Nicolson scheme with $\Delta x = 0.02$ and $\Delta t = 0.1$ up to time t=1 (a), t=10 (b) and t=50 (c).

Mathematical argument for the nonphysical behavior: We would like here to investigate the *amplification factor* of diffusion equation above under Crank-Nicolson scheme. The amplification factor relates the solution of two successive time steps in the Fourier space such that $\hat{u}^{n+1} = g(\zeta)\hat{u}^n$, where $g(\zeta)$ is the amplification factor for the method at wave number ζ[2]. Thus, the amplification factor gives a good indication to what happens to the different frequencies components as we march in time. Notice that for Crank-Nicolson, $|g(\zeta)| \leq 1$ and thus the scheme is A-stable. But if $|g(\zeta)| \approx 1$, then the solution from one time step to another does not change much and the high frequencies persists which looks like the case here.

Following the derivation of the amplification factor for the diffusion equation under the Crank-Nicolson scheme in [2] (equations 9.27 and 9.28),

$$g(\zeta) = \frac{1 + \dfrac{\Delta t}{\Delta x^2}(cos(\Delta x \zeta) - 1)}{1 - \dfrac{\Delta t}{\Delta x^2}(cos(\Delta x \zeta) - 1)}$$

The high frequencies occur at $cos(\Delta x \zeta) \approx \pi$, and the amplification factor becomes

$$g(\zeta) = \frac{1 - 2\dfrac{\Delta t}{\Delta x^2}}{1 + 2\dfrac{\Delta t}{\Delta x^2}}$$

For $\Delta x = 0.02$ and $\Delta t = 0.1$, $|g(\zeta)| = 0.995$ which is too close to 1.0 which explains the nonphysical behavior in the solution. This also gives an intuition for how to solve such behavior, either by allowing enough time for the solution to converge and for the high frequncies to wash out as shown in Figure 3(c) or choose different time and space steps such that $|g(\zeta)| \leq 0.5$.

BDF2: Here we try to solve the same problem using BDF2 scheme using the same boundary and initial conditions. The discretized PDE under BDF2 is

$$\frac{3u_i^{n+1} - 4u_i^n + u_i^{n-1}}{2\Delta t} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$

After rearrangement, it becomes

$$\Big(3 + \frac{4\Delta t}{(\Delta x)^2}\Big)u_i^{n+1} - \frac{2\Delta t}{(\Delta x)^2}u_{i+1}^{n+1} - \frac{2\Delta t}{(\Delta x)^2}u_{i-1}^{n+1} = 4u_i^n - u_i^{n-1}$$

or

$$Au_i^{n+1} + Bu_{i+1}^{n+1} + Cu_{i-1}^{n+1} = D_i^n$$

where $A = (3 + 4\Delta t/(\Delta x)^2),\ B = C = -2\Delta t/(\Delta x)^2$ and $D_i^n = 4u_i^n - u_i^{n-1}$

Thus, to solve at certain time step, information from two previous time steps should be available. To solve this issue, we use BDF1 to solve the PDE for the first time step $n = 1$. Then we use BDF2 system to solve subsequent time steps.

For BDF1, the discretized PDE is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$

Which can be written in the form

$$Au_i^{n+1} + Bu_{i+1}^{n+1} + Cu_{i-1}^{n+1} = D_i^n$$

where $A = (1 + 2\Delta t/(\Delta x)^2),\ B = C = -\Delta t/(\Delta x)^2$ and $D_i^n = u_i^n$.

The system of equation from BDF1 and BDF2 can be solved using the same LU-Decomposition solver. The result is shown in Figure 4, where the nonphysical behavior no longer exists.

BDF2 Amplification Factor: To explain why the nonphysical behavior no longer exists under BDF2 scheme, we derive the amplification factor for the specified boundary and initial conditions. It is expected that the amplification factor to have a value less than 0.5 since the high frequency components vanish rapidly in the solution (Figure 4). Starting with the form the descretized form of the PDE

$$(3 + 4\alpha)u_i^{n+1} - 2\alpha u_{i+1}^{n+1} - 2\alpha u_{i-1}^{n+1} = 4u_i^n - u_i^{n-1}$$

where $\alpha = \Delta t/(\Delta x)^2$. The solution is of the form $u_i^n = e^{ij\Delta x\zeta}$, where $j = \sqrt{-1}$. We expect that $u_{i+1}^n = g(\zeta)e^{ij\Delta x\zeta}$, where $g(\zeta)$ is the amplification factor at the wave number ζ. Plugging in these expressions in the descretized form of the PDE gives

$$(3 + 4\alpha)g(\zeta)e^{ij\Delta x\zeta} - 2\alpha g(\zeta)e^{(i+1)j\Delta x\zeta} - 2\alpha g(\zeta)e^{(i-1)j\Delta x\zeta} = 4e^{ij\Delta x\zeta} - (g(\zeta))^{-1}e^{ij\Delta x\zeta}$$

dividing by $e^{ij\Delta x\zeta}$

$$(3 + 4\alpha)g(\zeta) - 2\alpha g(\zeta)e^{j\Delta x\zeta} - 2\alpha g(\zeta)e^{-j\Delta x\zeta} = 4 - (g(\zeta))^{-1}$$

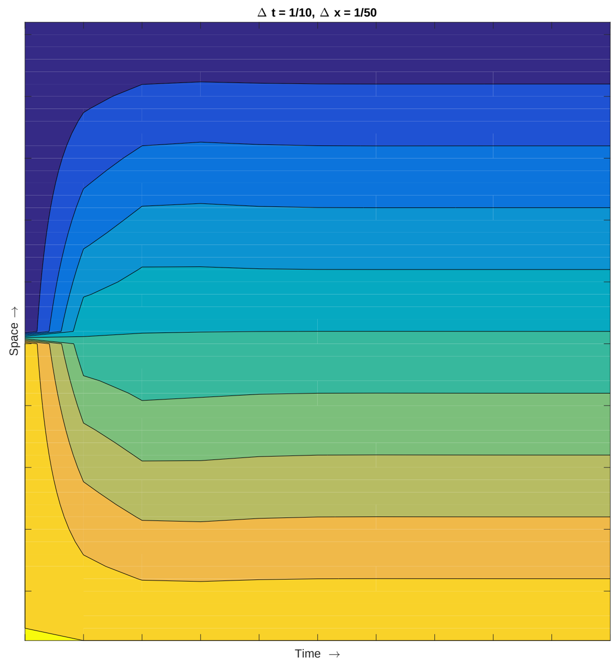


Figure 4: Solution of Problem 3 using BDF2 scheme with $\Delta x = 0.02$ and $\Delta t = 0.1$ up to time t=1, and using BDF1 to solve the first time step.

hence

$$(g(\zeta))^2(3 + 4\alpha - 2\alpha(e^{j\Delta x\zeta} + e^{-j\Delta x\zeta})) - 4g(\zeta) + 1 = 0$$

$$(g(\zeta))^2(3 + 4\alpha - 4\alpha cos(\Delta x\zeta)) - 4g(\zeta) + 1 = 0$$

Solving the above equation for $g(\zeta)$ gives

$$g(\zeta) = \frac{4 \pm \sqrt{16 - 12 - 16\alpha + 16\alpha cos(\Delta x\zeta)}}{6 + 8 + \alpha - 8\alpha cos(\Delta x\zeta)}$$

As mentioned previously, the high frequency components occurs at $cos(\Delta x\zeta) \approx \pi$. This gives

$$g(\zeta) = \frac{4 \pm \sqrt{4 - 32\alpha}}{6 + 16\alpha}$$

Substituting with the given initial and boundary conditions values gives $\alpha = 250$. Thus the amplification factor becomes

$$g(\zeta) = \frac{4 \pm 89.42j}{4006}$$

Thus $|g(\zeta)| = 0.02234$, which explains why the high frequency components vanished rapidly in the solution. Additionally, since $g(\zeta) \leq 1$, the scheme is stable.

Appendix

We derive here the function $R(z)$ for the advection equation under the classical fourth-order Runge-Kutta method (mentioned in Section 1.2).

Starting with the definition of fourth-order Runge-Kutta scheme:

$$U^{n+1} = U^n + \frac{\Delta t}{6}\Big[f(Y_1, t_n) + 2f\Big(Y_2, t_n + \frac{\Delta t}{2}\Big) + 2f\Big(Y_3, t_n + \frac{\Delta t}{2}\Big) + f(Y_4, t_n + \Delta t)\Big]$$

such that

$$\begin{aligned}
Y_1 &= U^n \\
Y_2 &= U^n + \frac{1}{2}\,\Delta t f(Y_1, t_n) \\
Y_3 &= U^n + \frac{1}{2}\,\Delta t f\Big(Y_2, t_n + \frac{\Delta t}{2}\Big) \\
Y_4 &= U^n + \Delta t f\Big(Y_3, t_n + \frac{\Delta t}{2}\Big)
\end{aligned}$$

where Δt is the time step. We first expand each value of Y term, such that

$$\begin{aligned}
Y_1 &= U^n \\
Y_2 &= U^n + \frac{1}{2}\,\Delta t f(U^n, t_n) \\
Y_3 &= U^n + \frac{1}{2}\,\Delta t f\Big(U^n + \frac{1}{2}\,\Delta t f(U^n, t_n), t_n + \frac{\Delta t}{2}\Big) \\
Y_4 &= U^n + \Delta t f\Big(U^n + \frac{1}{2}\,\Delta t f\Big(U^n + \frac{1}{2}\,\Delta t f(U^n, t_n), t_n + \frac{\Delta t}{2}\Big), t_n + \frac{\Delta t}{2}\Big)
\end{aligned}$$

Since $f(U, t)$ is a function of U, we can drop t. Additionally, each of these functions can be expressed in terms of eigenvalue λ. By substituting in fourth-order Runge-Kutta equation, we obtain

$$U^{n+1} = U^n + \frac{\Delta t}{6}[\lambda U^n + 2\lambda U^n + \lambda^2 U^n \Delta t + 2\lambda U^n +$$
$$\Delta t \lambda^2 U^n + \tfrac{1}{2}\,\Delta t^2 \lambda^3 U^n + U^n \lambda + \Delta t \lambda^2 U^2 + \tfrac{1}{2}\,\Delta t^2 \lambda^3 U^n + \tfrac{1}{4}\,\lambda^4 \Delta t^3 U^n]$$

$$U^{n+1} = U^n[1 + \lambda \Delta t + \tfrac{1}{2}\,\lambda^2 \Delta t^2 + \tfrac{1}{6}\,\Delta t^3 \lambda^3 + \tfrac{1}{24}\,\Delta t^4 \lambda^4]$$

But we have $z = \lambda \Delta t$. Thus,

$$U^{n+1} = U^n[1 + z + \tfrac{1}{2}\,z^2 + \tfrac{1}{6}\,z^3 + \tfrac{1}{24}\,z^4]$$

Thus,

$$R(z) = 1 + z + \tfrac{1}{2}\,z^2 + \tfrac{1}{6}\,z^3 + \tfrac{1}{24}\,z^4$$

References

[1] Klaus A Hoffmann and Steve T Chiang. Computational fluid dynamics Volume I. *Engineering Education System, Wichita, Kan, USA*, 2000.

[2] Randall J LeVeque. *Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems*. SIAM, 2007.