Math 228B - HW5

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1 Problem No.1

1.1 Problem Description:

In one spatial dimension the linearized equations of acoustics (sound waves) are

$$p_t + Ku_x = 0$$
$$\rho u_t + p_x = 0$$

where u is the velocity and p is the pressure, ρ is the density, and K is the bulk modulus.

1. Show that this system is hyperbolic and find the wave speeds.

2. Write a program to solve this system using Lax-Wendroff in original variables on (0,1) using a cell centered grid $x_j = (j - \frac{1}{2})\Delta x$ for $j = 1...N$. Write the code to use ghost cells, so that different boundary conditions can be changed by simply changing the values in the ghost cells. Ghost cells are cells outside the domain whore values can be set at the beginning of a time step so that code for updating cells adjacent to the boundary is identical to the code for interior cells. Set the ghost cells at the left by

$$p_0^n = p_1^n$$
$$u_0^n = -u_1^n$$

and set the ghost cells on the right by

$$p_{N+1}^n = \frac{1}{2}\left(p_N^n + u_N^n\sqrt{K\rho}\right)$$
$$u_{N+1}^n = \frac{1}{2}\left(\frac{p_N^n}{\sqrt{K\rho}} + u_N^n\right)$$

Run simulations with different initial conditions. Explain what happens at the left and right boundaries.

3. Give a physical interpretation and a mathematical explanation of these boundary conditions

1.2 Solution:

Part 1: To prove that the system in hyperbolic, we start by writing the system of equation in matrix. Thus,

$$q_t + Aq_x = 0$$

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For the above system, we need to prove that A is diagonalizable and its eigenvalues are real and distinct (strictly hyperbolic). Since the transpose of A is a diagonal matrix, then A is diagonalizable. The eigenvalues of A can be obtain by solving the following

$$Det(A - \lambda I) = 0$$

$$\lambda^2 = \frac{K}{\rho}\lambda_{1,2} = \pm\sqrt{\frac{K}{\rho}}$$

K is the bulk modulus which measure how incompressible a substance is which always positive (ration of infinitesimal pressure increase to resulting relative decrease in volume). ρ is density which always positive. Thus, the λ is real and distinct which makes the eigenvalues of A real and distinct. Thus, the system is (strictly) hyperbolic.

2 Problem No.2

2.1 Problem Description:

Write a program to solve the linear advection equation,

$$u_t + au_x = 0$$

on the unit interval using a finite volume method of the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}})$$

Use the numerical flux function

$$F_{j-\frac{1}{2}} = F^{up}_{j-\frac{1}{2}} + \frac{|a|}{2}\Big(1 - |\frac{a\Delta t}{\Delta x}|\Big)\delta_{j-\frac{1}{2}}$$

where $F^{up}_{j-\frac{1}{2}}$ is the upwinding flux,

$$F^{up}_{j-\frac{1}{2}} = \begin{cases} au_{j-1} & if \quad a > 0 \\ au_j & if \quad a < 0 \end{cases}$$

and $\delta_{j-\frac{1}{2}}$ i the limited difference. Let $\Delta u_{j-\frac{1}{2}} = u_j - u_{j-1}$ denote the jump in u across the edge at $x_{j-\frac{1}{2}}$. The limited difference is

$$\delta_{j-\frac{1}{2}} = \phi(\theta_{j-\frac{1}{2}})\Delta u_{j-\frac{1}{2}}$$

where

$$\theta_{j-\frac{1}{2}} = \frac{\Delta u_{J_{up}-\frac{1}{2}}}{\Delta u_{j-\frac{1}{2}}}$$

$$J_{up} = \begin{cases} j-1 & if \quad a > 0 \\ j+1 & if \quad a < 0 \end{cases}$$

Note that you will need two ghost cells on each end of the domain. Write your program so that you may choose from the different limiter functions listed below.
Upwinding $\phi(\theta) = 0$
Lax-Wendroff $\phi(\theta) = 1$
Beam-Warming $\phi(\theta) = \theta$
minmod $\phi(\theta) = minmod(1, \theta)$
superbee $\phi(\theta) = max(0, min(1, 2\theta), min(2, \theta))$
MC $\phi(\theta) = max(0, min((1+\theta)/2, 2, 2\theta))$
van Leer $\phi(\theta) = (\theta + |\theta|)/(1 + |\theta|)$
The first three are linear methods that we have already studied, and the last four are high resolution methods.

Solve the advection equation with $a = 1$ with periodic boundary conditions for the different initial conditions listed below until time $t = 5$ at Courant number 0.9.

1. Wave packet: $u(x,0) = cos(16\pi x)exp(-50(x-0.5)^2)$
2. Smooth, low frequency: $u(x,0) = sin(2\pi x)sin(4\pi x)$
3. Step function:

$$u(x,0)\begin{cases} 1 & if|x-\frac{1}{2}| < \frac{1}{4} \\ 0 & otherwise \end{cases}$$

Compare the results with the exact solution, and comment on the solutions generated by the different methods. How do the different high-resolution methods perform in the different test? What high-resolution method would you choose to use in practice?

2.2 Solution: