Math 228B - HW2

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1 Problem No.1

1.1 Problem Description:

Consider the forward time, centered space discretization

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}=b\frac{u_{j-1}^n-2u_j^n+u_{j+1}^n}{\Delta x^2}$$

to the convection-diffusion equation

$$u_t+au_x=bu_{xx},\quad b>0$$

1. Let $\nu = a\Delta t/\Delta x$ and $\mu = b\Delta t/\Delta x^2$. Since the solution of the PDE does not grow in time, it seems reasonable to require that the numerical solution not grow in time. Use von Neumann analysis to show that the numerical solution does not grow (in 2-norm) if and only if $\nu^2 \leq 2\mu \leq 1$

2. Suppose that we use the mixed implicit-explicit scheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}=b\frac{u_{j-1}^{n+1}-2u_j^{n+1}+u_{j+1}^{n+1}}{2\Delta x^2}$$

   Use von Neumann analysis to derive a stability restriction on the time step.

1.2 Solution:

Part 1: Using von Neumann analysis, we start by assuming that the solution at a grid point $U_j^n = e^{i\zeta\Delta j}$, where ζ is the wave number and $i=\sqrt{-1}$. Thus, the solution at $U_j^{n+1}=g(\zeta)e^{i\zeta\Delta j}$, where $g(\zeta)$ is the amplification factor at the wave number ζ. Showing that the numerical solution does not grow (in 2-norm) means that we should derive that restrictions for $|g(\zeta)|^2 \leq 1.0$.

Substituting in the equation, we obtain

$$g(\zeta)e^{ij\zeta\Delta x}-e^{ij\zeta\Delta x}+\frac{\nu}{2}(e^{i(j+1)\zeta\Delta x}-e^{i(j-1)\zeta\Delta x})=\mu(e^{i(j-1)\zeta\Delta x}-2e^{ij\zeta\Delta x}+e^{i(j+1)\zeta\Delta x}) \tag{1}$$

Dividing by $e^{ij\zeta\Delta x}$, using geometric identity and rearranging, we obtain

$$g(\zeta)=1-\nu jsin(\zeta\Delta x)+2\mu(cos(\zeta\Delta x)-1)=1-4\mu sin^2(\theta)-2\nu jsin(\theta)cos(\theta)$$

where $\theta = \zeta\Delta x/2$. This complex number has modulus as

$$|g(\zeta)|^2 = (1 - 4\mu sin^2(\theta))^2 + 4\nu^2 sin^2(\theta)cos^2(\theta)$$
$$|g(\zeta)|^2 = 1 + 16\mu^2 sin^4(\theta) - 8\mu sin^2(\theta) + 4\nu^2 sin^2(\theta)cos^2(\theta)$$
$$|g(\zeta)|^2 = 1 + 4sin^2(\theta)(4\mu^2 sin^2(\theta) - 2\mu + \nu^2 cos^2(\theta))$$
$$|g(\zeta)|^2 = 1 + 4sin^2(\theta)(4\mu^2 sin^2(\theta) - 2\mu + \nu^2(1 - sin^2(\theta)))$$
$$|g(\zeta)|^2 = 1 + 4sin^2(\theta)((4\mu^2 - \nu^2)sin^2(\theta) - 2\mu + \nu^2)$$

The stability requirement suggests that $|g(\zeta)|^2 \leq 1$. Thus,

$$|g(\zeta)|^2 = 1 + 4sin^2(\theta)((4\mu^2 - \nu^2)sin^2(\theta) - 2\mu + \nu^2) \leq 1$$
$$(4\mu^2 - \nu^2)sin^2(\theta) - 2\mu + \nu^2 \leq 0$$
$$sin^2(\theta) \leq \frac{2\mu - \nu^2}{4\mu^2 - \nu^2}$$

We know that $sin^2(\theta)$ is bounded between $[0 - 1]$. So, we check the two extreme cases and from which we can obtain bounds over μ and ν. For $sin^2(\theta) = 0$, we get $\nu^2 \leq 2\mu$. For $sin^2(\theta) = 1$, we get $2\mu \leq 1$. Thus, the numerical solution does not grow (in 2-norm) iff $\nu^2 \leq s\mu \leq 1$.

Part 2: Following the same approach, we can reuse equation (1) while doing the necessary changes over the time step on the right hand side as follows

$$g(\zeta)e^{ij\zeta\Delta x} - e^{ij\zeta\Delta x} + \frac{\nu}{2}(e^{i(j+1)\zeta\Delta x} - e^{i(j-1)\zeta\Delta x}) =$$
$$\mu(g(\zeta)e^{i(j-1)\zeta\Delta x} - 2g(\zeta)e^{ij\zeta\Delta x} + g(\zeta)e^{i(j+1)\zeta\Delta x})$$

After dividing by $e^{ij\zeta\Delta x}$, using geometric identity and rearranging, we obtain

$$g(\zeta)(1 - 2\mu(cos(\zeta\Delta x) - 1)) = 1 - i\nu sin(\zeta\Delta x)g(\zeta) = \frac{1 - i2\nu sin(\theta)cos(\theta)}{1 + 4\mu sin^2(\theta)}$$

where $\theta = \zeta\Delta x/2$. The complex number has modulus as

$$|g(\zeta)|^2 = \frac{1 + 4\nu^2 sin^2(\theta)cos^2(\theta)}{1 + 8\mu sin^2(\theta) + 16\mu^2 sin^4(\theta)}$$

Applying the stability requirements, such that $|g(\zeta)|^2 \leq 1$, we obtain

$$\nu^2 cos^2(\theta) \leq 2\mu + 4\mu^2 sin^2(\theta) = 2\mu + 4\mu^2(1 - cos^2(\theta))$$
$$cos^2(\theta) \leq \frac{2\mu + 4\mu^2}{\nu^2 + 4\mu^2}$$

Since $0 \leq cos^2(\theta) \leq 1$, we can check these two bounds to get the stability restrictions. For $cos^2(\theta) = 0$, we obtain restriction such that $-1 \leq 2\mu$. For $cos^2(\theta) = 1$, we get $\nu^2 \leq 2\mu$. The second restriction implies that the first one is preserved. Thus, the final restriction we get is $\nu^2 \leq 2\mu$. Further, we can substitute by the values of ν and μ to get the time step restriction. Thus, the time step restriction is $\Delta t \leq 2b/a^2$.

2 Problem No.2

2.1 Problem Description:

Write programs to solve

$$u_t = \Delta u \;\; on \;\; \Omega = (0,1) \times (0,1)$$
$$u = 0 \;\; on \;\; \partial\Omega$$
$$u(x,y,0) = exp(-100((x-0.3)^2 + (y-0.4)^2))$$

to time t=1 using forward Euler and Crank-Nicolson. For Crank-Nicolson use a fixed time step of $\Delta t = 0.01$, and for forward Euler use a time step just below the stability limit. For Crank-Nicolson use Gaussian elimination to solve the linear system that arises, but make sure to account the banded structure of the matrix.

1. Time your codes for different grid and compare the time to solve using forward Euler and Crank-Nicolson.
2. In theory, how should the time scale as the grid is refined for each algorithm? How did the time scale with the gird size in practice?
3. For this problem we could use and FFT-based Poisson solver which will perform the direct solve in $\mathcal{O}(Nlog(N))$, where N is the total number of grid points. We could also use multigrid and perform the solve in $\mathcal{O}(N)$ time. How should the time scale as the grid is refined for Crank-Nicolson if we used an $\mathcal{O}(N)$ solver?

2.2 Solution:

Part 1: We record the timing for different grid resolution for both forward Euler and Crank-Nicolson schemes and the timings are show in Figure **??**.

Part 2: With Crank-Nicolson, the system of equations is banded matrix which can be solved using LU-decomposition [?] which has complexity of $\mathcal{O}(N^2)$ for N unknowns. For this problem, the number of unknowns at each time step is $N_x - 2 \times N_y - 2$, where N_x and N_y is the number of grid points in x and y direction respectively. Thus, it is expected for the banded matrix solver (and thus for Crank-Nicolson scheme) to be of order $\mathcal{O}(N^2)$. For the explicit forward Euler scheme, the number of operations is of order of the number of unknowns. Thus, the complexity is of order $\mathcal{O}(N^2)$ for forward Euler.

There is hidden overhead with forward Euler due to its stability criterion as we have to perform temporal refinement as we do spatial refinement. In contrast to Crank-Nicolson which is unconditionally stable. For forward Euler, the method is only stable only when $\Delta t/\Delta x^2 \leq \frac{1}{4}$. Since we take time steps just below the stability limit, the complexity of forward Euler at each time step increases to be $\mathcal{O}(N^4)$. Thus if we double the space steps (N_x and N_y) we should exhibit 16 times increase in time. For Crank-Nicolson, doubling the space steps would give the same increase in time.

The slope of the curve in Figure **??** is 3.4 for both schemes which deviates by a factor of 15% from the what is expected. For Crank-Nicolson, this could be justified since we don't go through the LU-decomposition of the banded matrix at every solution step. Instead, half the work done once (finding the pivot), and the other half of the work is solved per time step (backward and forward substitution). For forward Euler, it is not clear why such deviation occurs.

Forward Euler			Crank-Nicolson		
Δt	Δx	time	Δt	Δx	time
0.000976	0.0625	2.0E-06	0.01	0.0625	1.00E-06
0.000244	0.03125	2.0E-07	0.01	0.03125	1.00E-05
6.1E-05	0.01563	8.0E-05	0.01	0.01563	0.000108
1.53E-05	0.00782	0.001047	0.01	0.00782	0.000831
3.81E-06	0.00391	0.015268	0.01	0.00391	0.011299

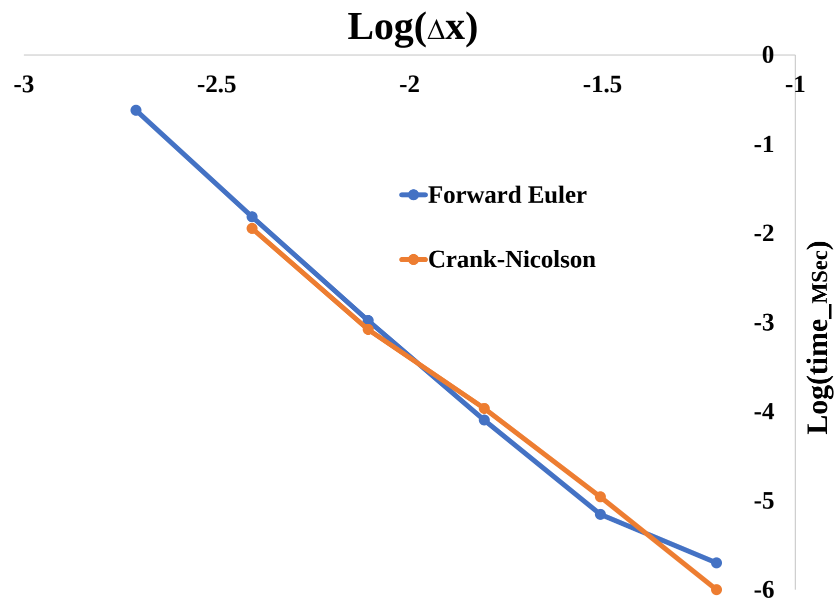

Figure 1: Timing of the refinement study on forward Euler and Crank-Nicolson schemes applied on problem 2. The tables show the actual time in milliseconds and the figure shows the curve drawn on a log-log scale to indicate the order of the complexity.

Part 4: Using a $\mathcal{O}(N)$ solver, the time scale will be four for Crank-Nicolson since no temporal refinement is needed given the same reasoning we give for a $\mathcal{O}(N^2)$ solver.

Appendix

Construction of the penta-diagonal system raised in Crank-Nicolson scheme when used to solve the 2D heat equation. The system will be derived for the heat equation in Section **??** with the same initial and boundary conditions. We start by applying the trapezoidal rule on the following equation

$$u_t = \Delta u = u_{xx} + u_{yy} \;\; on \;\; \Omega = (0,1) \times (0,1)$$

We obtain

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \frac{1}{2}\left(\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{(\Delta y)^2} + \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2}\right)$$

For simplicity, we assume $\Delta x = \Delta y$. After rearranging the above equation we get

$$(1+2r)u_{i,j}^{n+1} - \frac{r}{2}(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}) = (1-2r)u_{i,j}^{n} + \frac{r}{2}(u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n})$$

where $r = \Delta t/\Delta x^2$. Additionally, we can write the equation in more compact way as

$$\alpha u_{i,j}^{n+1} + \beta(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}) = D_{i,j}^{n}$$

where $D_{i,j}^{n} = \gamma u_{i,j}^{n} - \beta(u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n})$, $\alpha = 1+2r$, $\beta = -r/2$ and $\gamma = 1-2r$.