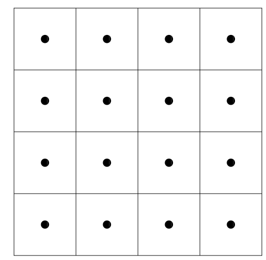
Math 228B
Homework 3
Due Friday, March 3rd

1. Consider

$$u_t = 0.1\,\Delta u \text{ on } \Omega = (0,1)\times(0,1)$$
$$\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega$$
$$u(x,y,0) = \exp\left(-10((x-0.3)^2 + (y-0.4)^2)\right)$$



Write a program to solve this PDE using the Peaceman-Rachford ADI scheme on a cell-centered grid. Use a direct solver for the tridiagonal systems. In a cell-centered discretization the solution is stored at the grid points $(x_i, y_j) = (\Delta x(i-0.5), \Delta x(j-0.5))$ for $i,j = 1\ldots N$ and $dx = 1/N$. This discretization is natural for handling Neumann boundary conditions, and it is often used to discretize conservation laws. At the grid points adjacent to the boundary, the one-dimensional discrete Laplacian for homogeneous Neumann boundary conditions is

$$u_{xx}(x_1) \approx \frac{-u_1 + u_2}{\Delta x^2}.$$

(a) Perform a refinement study to show that your numerical solution is second-order accurate in space and time (refine time and space simultaneously using $\Delta t = \Delta x$) at time $t = 1$.

(b) Time you code for different grid sizes. Show how the computational time scales with the grid size. Compare your timing results with those from the previous homework assignment for Crank-Nicolson. If you had errors in your codes from HW2, fix them.

Note, many of you are using script languages (MATLAB and python), and loops in these languages can be slow. You can program a single time step of ADI (in 2D) without any loops. For example, you can solve $LU = F$, where U and F are stored as arrays rather than vectors, and L is the 1D operator to invert. Avoiding loops in not necessary for this assignment. Write your code for correctness and clarity first, and efficiency later if you have time.

(c) Show that the spatial integral of the solution to the PDE does not change in time. That is

$$\frac{d}{dt}\int_\Omega u\,dV = 0.$$

(d) Show that the solution to the discrete equations satisfies the discrete conservation property

$$\sum_{i,j} u_{i,j}^n = \sum_{i,j} u_{i,j}^0$$

for all n. Demonstrate this property with your code.

2. The FitzHugh-Nagumo equations

$$\begin{aligned}\frac{\partial v}{\partial t} &= D\Delta v + (a-v)(v-1)v - w + I \\ \frac{\partial w}{\partial t} &= \epsilon(v - \gamma w).\end{aligned}$$

are used in electrophysiology to model the cross membrane electrical potential (voltage) in cardiac tissue and in neurons. Assuming that the spatial coupling is local and passive results the term which looks like the diffusion of voltage. The state variables are the voltage v and the recovery variable w.

(a) Write a program to solve the FitzHugh-Nagumo equations on the unit square with homogeneous Neumann boundary conditions for v (meaning electrically insulated). Use a fractional step method to handle the diffusion and reactions separately. Use an ADI method for the diffusion solve. Describe what ODE solver you used for the reactions and what fractional stepping your chose.

(b) Use the following parameters $a = 0.1$, $\gamma = 2$, $\epsilon = 0.005$, $I = 0$, $D = 5 \cdot 10^{-5}$, and initial conditions

$$\begin{aligned}v(x, y, 0) &= \exp\left(-100(x^2 + y^2)\right) \\ w(x, y, 0) &= 0.0.\end{aligned}$$

Note that $v = 0$, $w = 0$ is a stable steady state of the system. Call this the rest state. For these initial conditions the voltage has been raised above rest in the bottom corner of the domain. Generate a numerical solution up to time $t = 300$. Visualize the voltage and describe the solution. Pick space and time steps to resolve the spatiotemporal dynamics of the solution you see. Discuss what grid size and time step you used and why.

(c) Use the same parameters from part (b), but use the initial conditions

$$\begin{aligned}v(x, y, 0) &= 1 - 2x \\ w(x, y, 0) &= 0.05y,\end{aligned}$$

and run the simulation until time $t = 600$. Show the voltage at several points in time (pseudocolor plot, or contour plot, or surface plot $z = V(x, y, t)$) and describe the solution.