Math 228B - HW4

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1 Problem No.1

1.1 Problem Description:

Write programs to solve the advection equation

$$u_t + au_x = 0,$$

on $[0, 1]$ with periodic boundary conditions using upwinding and Lax-Wendroff. For smooth solutions we expect upwinding to be first-order accurate and Lax-Wendroff to be second-order accurate, but it is not clear what accuracy to expect for nonsmooth solution.

1. Let $a = 1$ and solve the problem up to time $t = 1$. Perform a refinement study for both upwinding and Lax-Wendroff with $\Delta t = 0.9a\Delta x$ with a smooth initial condition. Compute the rate of converge in the 1-norm and 2-norm, and max-norm. Note that the exact solution at time $t = 1$ is the initial condition, and so computing the error is easy.

2. Repeat the previous problem with the discontinuous initial condition

$$u(x,0) = \begin{cases} 1 & if \;\; |x - \frac{1}{2}| < \frac{1}{4} \\ 0 & otherwise \end{cases}$$

1.2 Solution:

Part 1: Under upwinding scheme, the advection equation is discretized as:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

For which we can define *Courant* number ν as $\nu = a\Delta t/\Delta x$. For Lax-Wendroff scheme, the discretization would be

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - a^2\Delta t\frac{u_n^{j-1} - 2u_j^n + u_{j+1}^n}{2\Delta x^2} = 0$$

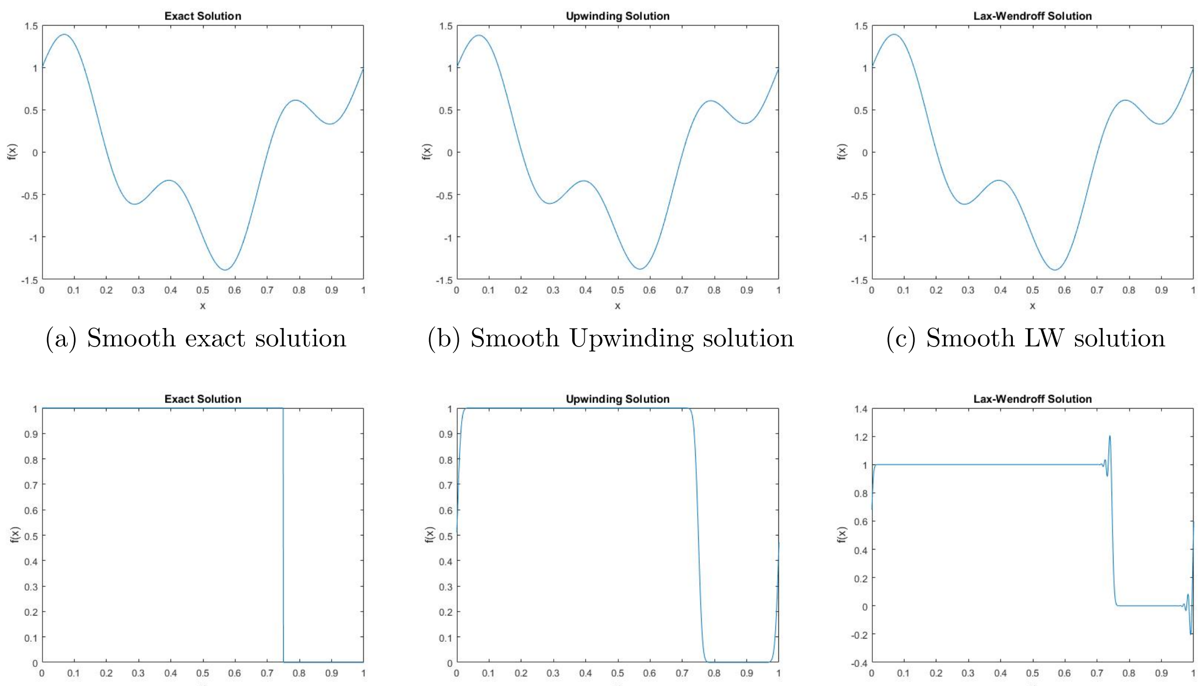
$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2\Delta t^2}{2\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

For smooth initial conditions we use $cos(2.0 * x * PI) + sin(6.0 * x * PI)/2.0$ as initial conditions which is smooth continuous function for the periodic domain [0,1]. The periodic boundary conditions mean that the value of the last grid point is equal to the value of the first gird point. We can consider this as solving over a circle.

Since the problem asking for solving such that $\Delta t = 0.9a\Delta x$ and $a = 1.0$, thus $\nu = 0.9$ and the numerical scheme is stable. This applies for upwinding and Lax-Wendroff.

The error was calculated for 1-norm as $||e||_1 = h\sum_j |e_j|$, where e_j is the absolute difference (error) between the numerical solution and exact solution at grid point j. For 2-norm, the error is $||e||_2 = h\sqrt{\sum_j |e_j|^2}$. For max-norm, the error becomes $||e||_{max} = max_j|e_j|$.

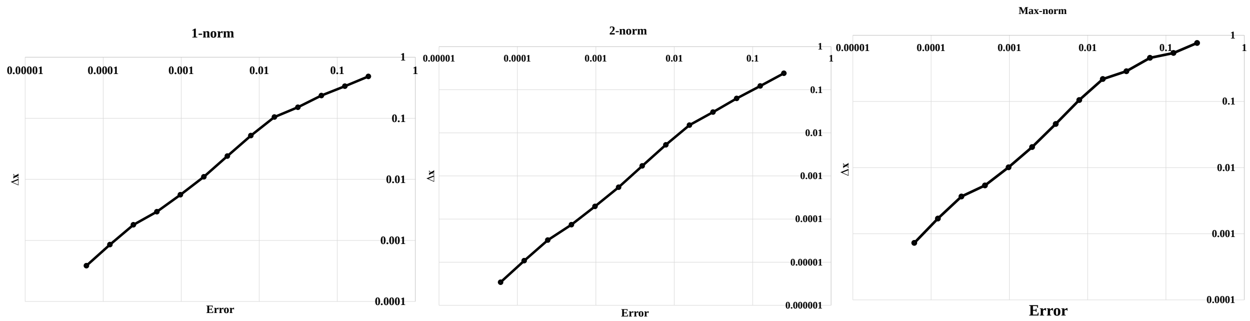
The exact and numerical solutions for both the smooth and discontinuous initial conditions are shown in Figure 1.



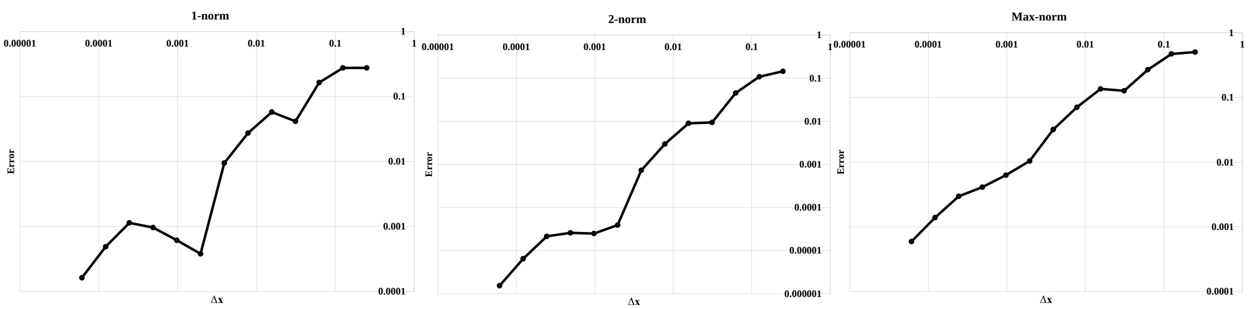
(a) Smooth exact solution (b) Smooth Upwinding solution (c) Smooth LW solution

(d) Discont. exact solution (e) Discont. Upwinding solution (f) Discont. LW solution

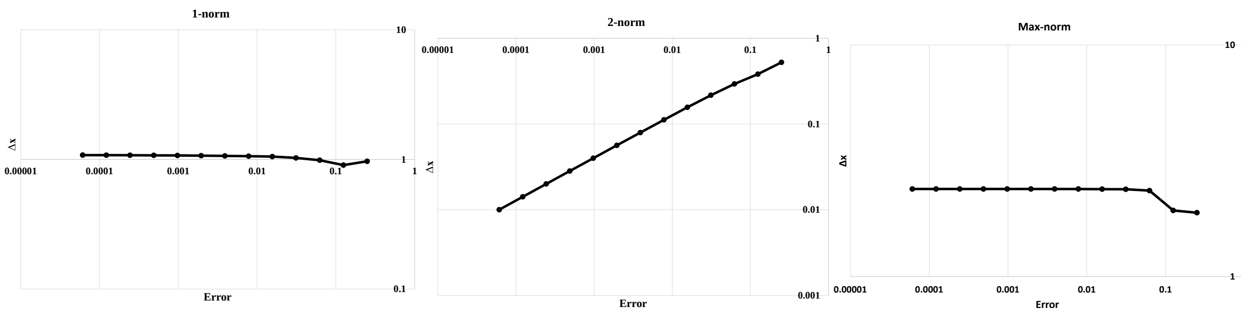
Figure 1: The exact solution (first column), upwinding solution (second column) and Lax-Wendroff solution (third column) for both smooth (first row) and discontinuous (second row) initial conditions for the 1d advection equation.



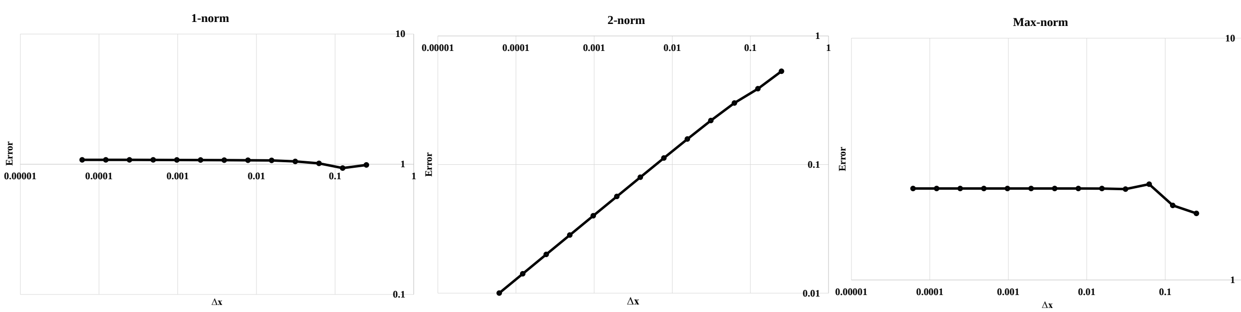
(a) Smooth Upwinding, 1-norm (b) Smooth Upwinding, 2-norm (c) Smooth Upwinding, max-norm



(d) Smooth LW, 1-norm (e) Smooth LW, 2-norm (f) Smooth LW, max-norm



(g) Discont. Upwinding, 1-norm (h) Discont. Upwinding, 2-norm (i) Discont. Upwinding, max-norm



(j) Discont. LW, 1-norm (k) Discont. LW, 2-norm (l) Discont. LW, max-norm

Figure 2: The rate convergence in 1-norm (first column),2-norm (second column) and max-norm (third column) for smooth (first two rows) and discontinuous (last two rows) initial condition using upwinding and Lax-Wendroff scheme to solve the 1d advection equation.

2 Problem No.2

2.1 Problem Description:

For solving the heat equation we frequently use Crank-Nicolson, which is trapezoidal rule time integration with a second-order space discretization. The analogous scheme for the linear advection equation is

$$u_j^{n+1} - u_j^n + \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = 0,$$

where $\nu = a\Delta t/\Delta x$

1. Use von Neumann analysis to show that this scheme is unconditionally stable and that $||u^n||_2 = ||u^0||_2$. This scheme is said to be non-dissipative - i.e. there is no amplitude error. This seems reasonable because this is a property of the PDE.

2. Solve the advection equation on the periodic domain $[0, 1]$ with the initial condition from problem 1 (part 2). Show the solution and comment on your results.

3. Compute the relative phase error as $arg(g(\theta)/(-\nu\theta))$, where g is the amplification factor and $\theta = \zeta\Delta t$, and plot it for $\theta \in [0, \pi]$. How does the relative phase error and lack of amplitude error relate to the numerical solutions you observed in part 2.

2.2 Solution:

Part 1:

Appendix