

Math 228B - HW4

Ahmed H. Mahmoud

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1 Problem No.1

1.1 Problem Description:

Write programs to solve the advection equation

$$u_t + au_x = 0,$$

on $[0, 1]$ with periodic boundary conditions using upwinding and Lax-Wendroff. For smooth solutions we expect upwinding to be first-order accurate and Lax-Wendroff to be second-order accurate, but it is not clear what accuracy to expect for nonsmooth solution.

1. Let $a = 1$ and solve the problem up to time $t = 1$. Perform a refinement study for both upwinding and Lax-Wendroff with $\Delta t = 0.9a\Delta x$ with a smooth initial condition. Compute the rate of converge in the 1-norm and 2-norm, and max-norm. Note that the exact solution at time $t = 1$ is the initial condition, and so computing the error is easy.
2. Repeat the previous problem with the discontinuous initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } |x - \frac{1}{2}| < \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

1.2 Solution:

Part 1: Under upwinding scheme, the advection equation is discretized as:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$
$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

For which we can define *Courant* number ν as $\nu = a\Delta t/\Delta x$. For Lax-Wendroff scheme, the discretization would be

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - a^2 \Delta t \frac{u_j^{n-1} - 2u_j^n + u_{j+1}^n}{2\Delta x^2} = 0$$
$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2 \Delta t^2}{2\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

For smooth initial conditions we use $\cos(2.0 * x * PI) + \sin(6.0 * x * PI)/2.0$ as initial conditions which is smooth continuous function for the periodic domain $[0,1]$. The periodic boundary conditions mean that the value of the last grid point is equal to the value of the first grid point. We can consider this as solving over a circle.

Since the problem asking for solving such that $\Delta t = 0.9a\Delta x$ and $a = 1.0$, thus $\nu = 0.9$ and the numerical scheme is stable. This applies for upwinding and Lax-Wendroff.

The error was calculated for 1-norm as $\|e\|_1 = h \sum_j |e_j|$, where e_j is the absolute difference (error) between the numerical solution and exact solution at grid point j . For 2-norm, the error is $\|e\|_2 = h \sqrt{\sum_j |e_j|^2}$. For max-norm, the error becomes $\|e\|_{max} = \max_j |e_j|$.

The exact and numerical solutions for both the smooth and discontinuous initial conditions are shown in Figure 1.

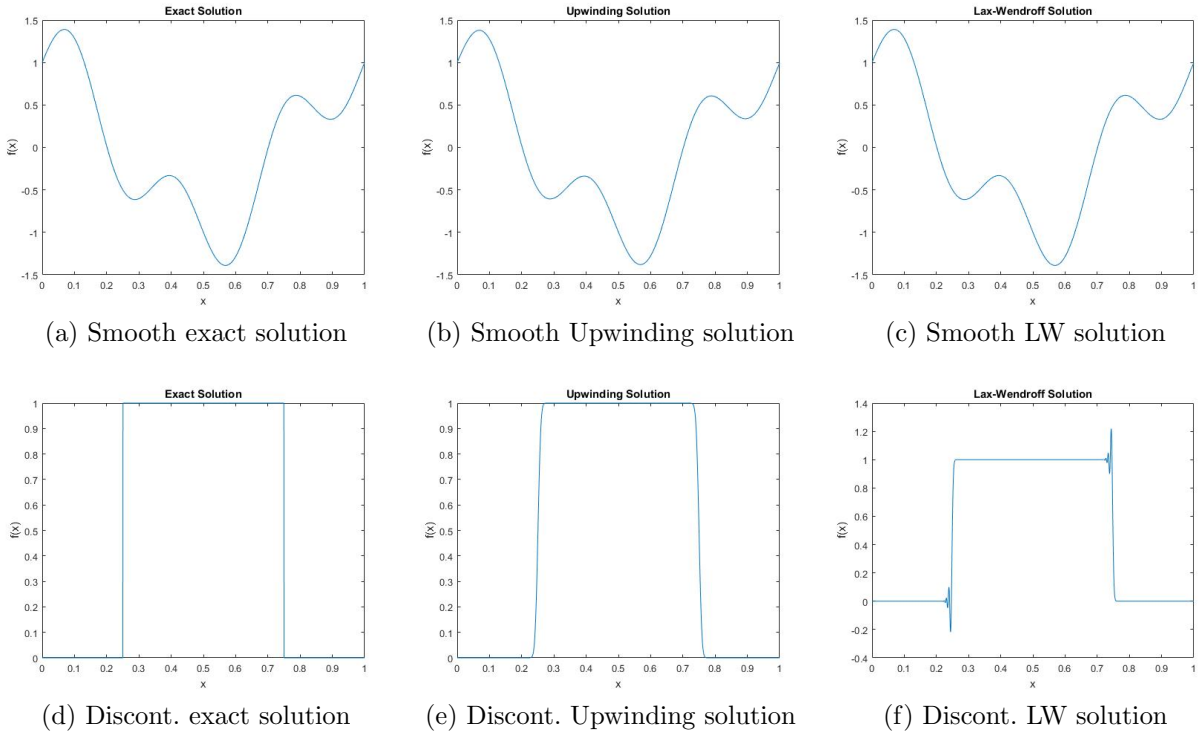


Figure 1: The exact solution (first column), upwinding solution (second column) and Lax-Wendroff solution (third column) for both smooth (first row) and discontinuous (second row) initial conditions for the 1d advection equation with $\Delta x = \frac{1}{2048}$.

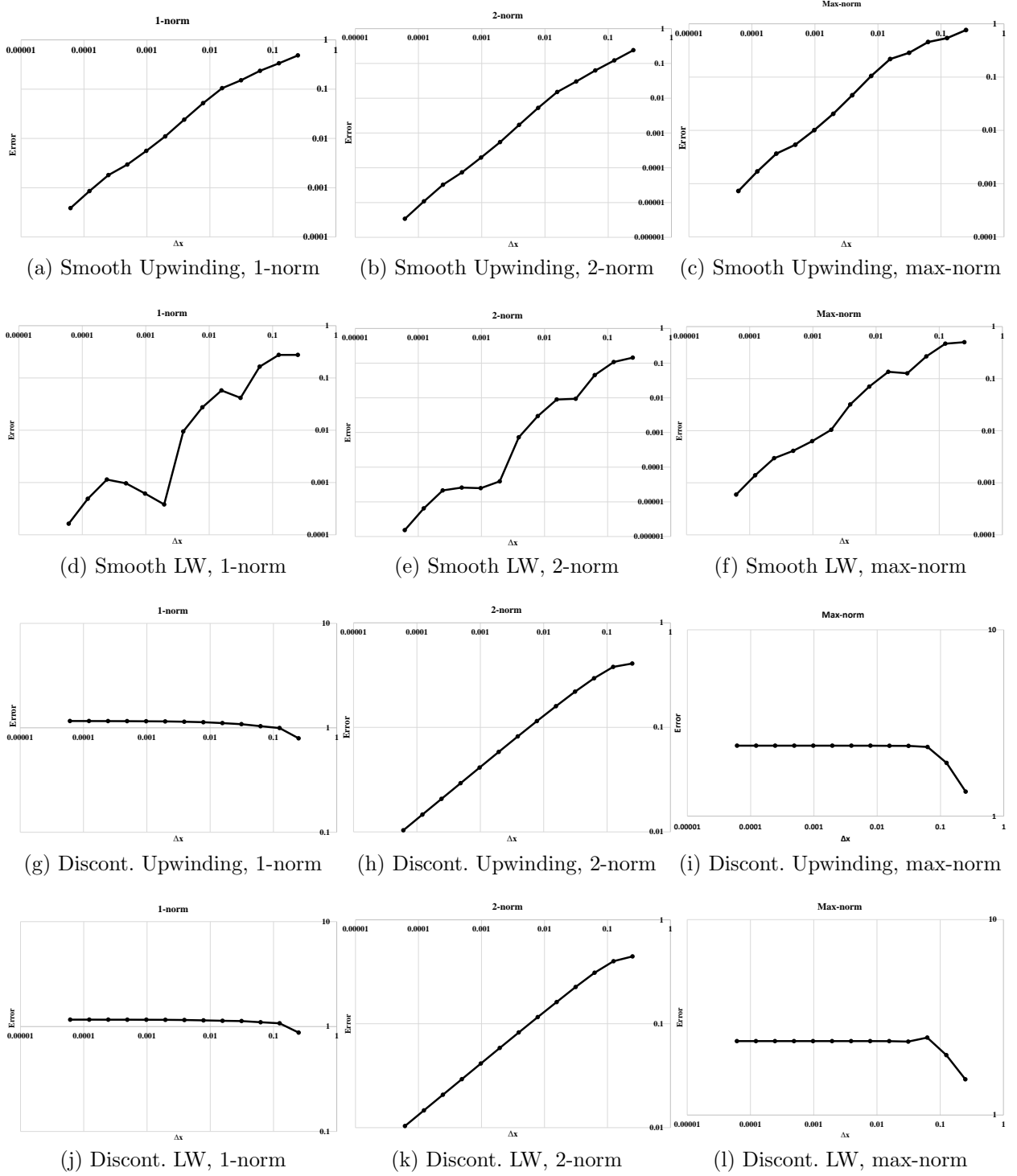


Figure 2: The rate convergence in 1-norm (first column), 2-norm (second column) and max-norm (third column) for smooth (first two rows) and discontinuous (last two rows) initial condition using upwinding and Lax-Wendroff scheme to solve the 1d advection equation.

2 Problem No.2

2.1 Problem Description:

For solving the heat equation we frequently use Crank-Nicolson, which is trapezoidal rule time integration with a second-order space discretization. The analogous scheme for the linear advection equation

is

$$u_j^{n+1} - u_j^n + \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = 0,$$

where $\nu = a\Delta t/\Delta x$

1. Use von Neumann analysis to show that this scheme is unconditionally stable and that $\|u^n\|_2 = \|u^0\|_2$. This scheme is said to be non-dissipative - i.e. there is no amplitude error. This seems reasonable because this is a property of the PDE.
2. Solve the advection equation on the periodic domain $[0, 1]$ with the initial condition from problem 1 (part 2). Show the solution and comment on your results.
3. Compute the relative phase error as $\arg(g(\theta))/(-\nu\theta)$, where g is the amplification factor and $\theta = \zeta\Delta x$, and plot it for $\theta \in [0, \pi]$. How does the relative phase error and lack of amplitude error relate to the numerical solutions you observed in part 2.

2.2 Solution:

Part 1: To show that the scheme is unconditionally stable using von Neumann analysis, we start by assuming that the solution is of the form

$$u_j^n = e^{i\zeta x_j}$$

$$u_j^{n+1} = g(\zeta)e^{i\zeta x_j}$$

where $g(\zeta)$ is the amplification factor and ζ is the wave number. In order to show the scheme is stable, $|g(\zeta)| \leq 1$. We start by plugging in the solution above in advection equation, the result will be

$$g(\zeta)e^{i\zeta x_j} - e^{i\zeta x_j} + \frac{\nu}{4}(e^{i\zeta(x_j+\Delta x)} - e^{i\zeta(x_j-\Delta x)}) + \frac{\nu}{4}(g(\zeta)e^{i\zeta(x_j+\Delta x)} - g(\zeta)e^{i\zeta(x_j-\Delta x)}) = 0$$

Diving by $e^{i\zeta x_j}$ and arranging, the equation becomes

$$g(\zeta) - 1 + \frac{\nu}{4}(e^{i\zeta\Delta x} - e^{-i\zeta\Delta x}) + \frac{\nu}{4}(g(\zeta)e^{i\zeta\Delta x} - g(\zeta)e^{-i\zeta\Delta x}) = 0$$

$$g(\zeta) = \frac{1 - \frac{\nu}{4}(e^{i\zeta\Delta x} - e^{-i\zeta\Delta x})}{1 + \frac{\nu}{4}(e^{i\zeta\Delta x} - e^{-i\zeta\Delta x})}$$

$$g(\zeta) = \frac{1 - i\frac{\nu}{2}\sin(\zeta\Delta x)}{1 + i\frac{\nu}{2}\sin(\zeta\Delta x)}$$

Taking the absolute value of the above

$$|g(\zeta)|^2 = \frac{1 + \frac{\nu^2}{4}\sin^2(\zeta\Delta x)}{1 + \frac{\nu^2}{4}\sin^2(\zeta\Delta x)} = 1$$

Since the $|g(\zeta)| \leq 1$, then the scheme is unconditionally stable. \square

To prove that $\|u^n\|_2 = \|u^0\|_2$, we start by arranging the advection equation so follows

$$u_j^{n+1} + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n - \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n)$$

Taking the sum squared of both sides

$$\begin{aligned} \sum_j \left(u_j^{n+1} + \frac{\nu}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) \right)^2 &= \sum_j \left(u_j^n - \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n) \right)^2 \\ \sum_j \left((u_j^{n+1})^2 + \frac{\nu}{2}u_j^{n+1}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + \frac{\nu^2}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1})^2 \right) &= \\ \sum_j \left((u_j^n)^2 - \frac{\nu}{2}u_j^n(u_{j+1}^n - u_{j-1}^n) + \frac{\nu^2}{4}(u_{j+1}^n - u_{j-1}^n)^2 \right) & \\ \sum_j (u_j^{n+1})^2 = \sum_j \left((u_j^n)^2 - \frac{\nu}{2}(u_j^n(u_{j+1}^n - u_{j-1}^n) + u_j^{n+1}(u_{j+1}^{n+1} - u_{j-1}^{n+1})) + \right. & \\ \left. \left(\frac{\nu}{4} \right)^2 ((u_{j+1}^n - u_{j-1}^n)^2 - (u_{j+1}^{n+1} - u_{j-1}^{n+1})^2) \right) & \end{aligned}$$

Since it's proven that $u_j^{n+1} = |g(\zeta)|u_j^n = u_j^n$, then

$$\begin{aligned} \sum_j (u_j^{n+1})^2 &= \sum_j \left((u_j^n)^2 - \frac{\nu}{2}(u_j^n(u_{j+1}^n - u_{j-1}^n) + u_j^n(u_{j+1}^n - u_{j-1}^n)) + \right. \\ &\quad \left. \left(\frac{\nu}{4} \right)^2 ((u_{j+1}^n - u_{j-1}^n)^2 - (u_{j+1}^n - u_{j-1}^n)^2) \right) \\ \sum_j (u_j^{n+1})^2 &= \sum_j ((u_j^n)^2 - \nu u_j^n(u_{j+1}^n - u_{j-1}^n)) \\ \sum_j (u_j^{n+1})^2 &= \sum_j ((u_j^n)^2) + \nu \left(\sum_j u_j^n u_{j-1}^n - \sum_j u_j^n u_{j+1}^n \right) \end{aligned}$$

The product at grid point j and the next point is the same as the product of j and the previous point since it is a domain with periodic boundary conditions and we can imagine this as moving over a circle.

$$\begin{aligned} \sum_j (u_j^{n+1})^2 &= \sum_j ((u_j^n)^2) \\ \|u^{n+1}\|_2 &= \|u^n\|_2 \end{aligned}$$

Thus, $\|u^n\|_2 = \|u^{n-1}\|_2 = \|u^{n-2}\|_2 \dots = \|u^0\|_2$ \square

Part 2: We can rewrite the equation as

$$Au_j^{n+1} + Bu_{j+1}^{n+1} + Cu_{j-1}^{n+1} = D_j^n$$

where

$$A = 1, \quad B = \frac{\nu}{4}, \quad C = \frac{-\nu}{4}, \quad D_i^n = u_i^n - \frac{\nu}{4}(u_{j+1}^n - u_{j-1}^n)$$

The matrix formulation for the above equation for the periodic domain is

$$\begin{vmatrix} A & B & 0 & 0 & \dots & \dots & 0 & C \\ C & A & B & 0 & \dots & \dots & 0 & 0 \\ 0 & C & A & B & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & C & A & B \\ B & 0 & \dots & \dots & 0 & 0 & C & A \end{vmatrix} \begin{vmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ \dots \\ \dots \\ \dots \\ u_{nx-2}^{n+1} \\ u_{nx-1}^{n+1} \end{vmatrix} = \begin{vmatrix} D_0^n \\ D_1^n \\ D_2^n \\ \dots \\ \dots \\ \dots \\ D_{nx-2}^n \\ D_{nx-1}^n \end{vmatrix}$$

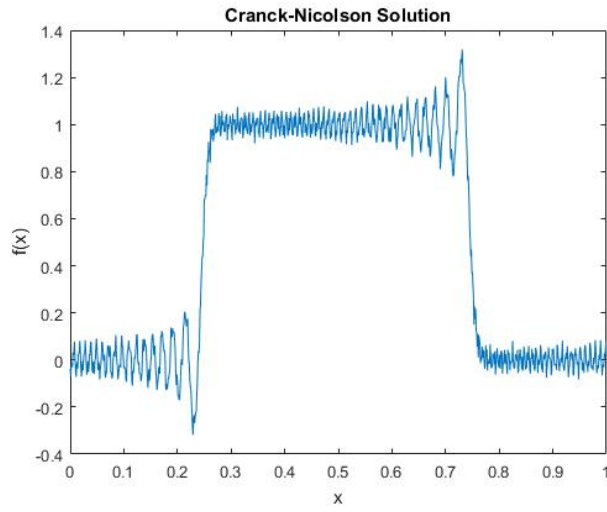
Part 3: We start by deriving the expression $arg(g(\theta))$, where $\theta = \zeta \Delta x$ form the amplification factor such that

$$\begin{aligned} g(\zeta) &= \frac{1 - i\frac{\nu}{2}\sin(\zeta \Delta x)}{1 + i\frac{\nu}{2}\sin(\zeta \Delta x)} \\ g(\zeta) &= \frac{1 - \frac{\nu^2}{4}\sin^2(\zeta \Delta x)}{1 + \frac{\nu^2}{4}\sin^2(\zeta \Delta x)} - i \frac{\nu \sin(\zeta \Delta x)}{1 + \frac{\nu^2}{4}\sin^2(\zeta \Delta x)} \\ g(\zeta) &= \frac{1 - \frac{\nu^2}{4}\sin^2(\theta)}{1 + \frac{\nu^2}{4}\sin^2(\theta)} - i \frac{\nu \sin(\theta)}{1 + \frac{\nu^2}{4}\sin^2(\theta)} \end{aligned}$$

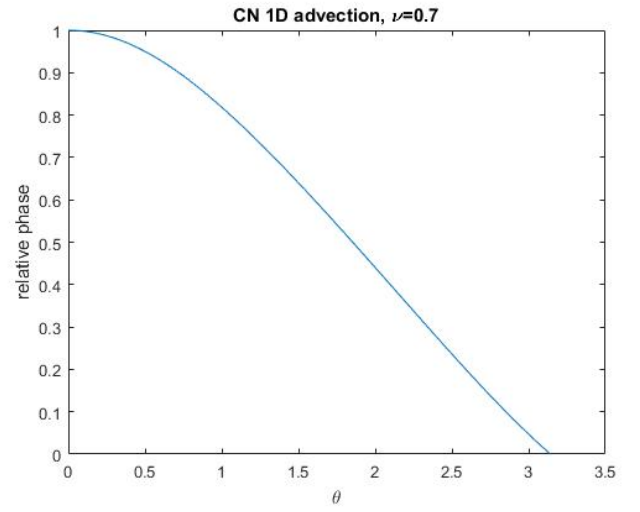
This gives the phase as

$$arg(g(\theta)) = \tan^{-1} \left(-\frac{\nu \sin(\theta)}{1 - \frac{\nu^2}{4}\sin^2(\theta)} \right)$$

Figure 3 (b) shows the relative phase error. The plot shows



(a) Solution



(b) Relative Phase

Figure 3: Solution for of 1D advection equation using CN method; (a) shows the solution with discontinuous initial conditions and (b) shows the the relative phase error.