Boolean chains *

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Abstract

A novel algorithm for computation of Boolean operations between cellular complexes is given in this module. It is based on bucketing of possibly interacting geometry using a box-extension of kd-trees, normally used for point proximity queries. Such kd-tree representation of containment boxes of cells, allow us to compute a number of independent buckets of data to be used for local intersection, followed by elimination of duplicated data. Actually we reduce the intersection of boundaries in 3D to the independent intersections of the buckets of (transformed) faces with the 2D subspace $z = 0$, in order to reconstruct each splitted facet of boolean arguments, suitably transformed ther together with the bucket of indent facets. A final tagging of cells as either belonging or not to each operand follows, allowing for fast extraction of Boolean results between any pair of chains (subsets of cells). This Boolean algorithm can be considered of a *Map-Reduce* kind, and hence suitable of a distributed implementation over big datasets. The actual engineered implementation will follow the present prototype, using some distributed NoSQL database, like MongoDB or Riak.

Contents

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*This document is part of the *Linear Algebraic Representation with CoChains* (LAR-CC) framework [CL13]. March 13, 2015

1 Introduction

2 Preview of the algorithm

The whole Boolean algorithm is composed by four stages in sequence, denoted in the following as *Unification*, *Bucketing*, *Intersection*, and *Reconstruction*. The algorithm described here is both multidimensional and variadic. Multidimensional means that the arguments are solid in Euclidean space of dimension d, with d small integer. The *arity* of a function or operation is the number of arguments or operands the function or operation accepts. In computer science, a function accepting a variable number of arguments is called *variadic*.

2.1 Unification

In this first step the boundaries of the n Boolean arguments are computed and merged together as a set of chains defined in the discrete set V made by the union of their vertices, and possibly by a discrete set of points generated by intersection of cells of complementary dimension, i.e. whose dimensions add up to the dimension of the ambient space. Actually, only the (*oriented*) boundaries V, FV_i $(1 \leq i \leq n)$ of the varius arguments are retained here, and used by the following steps of the algorithm.

2.2 Bucketing

The bounding boxes of facets FV_i are computed, and their *box-kd-tree* is worked-out, so providing a group of buckets of close cells, that can be elaborated independently, and possibly in parallel, to compute the intersections of the boundary cells.

2.3 Intersection

For each facet f of one of Boolean arguments, the subset $F(f)$ of incident or intersecting facets of boundaries of the other arguments were computed in the previous *bucketing* step. So, each F is transformed by the affine map that sends f into the $z = 0$ subspace, and there is intersected with this subspace, generating a subset $E(f)$ of coplanar edges. This one is projected in 2D, and the *regularized* cellular 2-complex $G(f)$ induced by it is computed, and mapped back to the original space position and orientation of f (providing a partition of it induced by the other boundaries).

2.4 Reconstruction

Like for in the reconstruction of 2D solid cells using the angular ordering of edges around the vertices, the coincident edges are identified in 3D , and used to sort the incident faces sing vhe falues of solid angles given with one reference face. The 3D space partition induced by $\cup_f G(f)$ is finally reconstructed, possibly in parallel, by traversing the adjacent sets of facets on the boundary of each solid cell.

3 Implementation

3.1 Box-kd-tree

Split the boxes between the (below,above) subsets

⟨Split the boxes between the below,above subsets 2⟩ ≡

```
""" Split the boxes between the below,above subsets """
def splitOnThreshold(boxes,subset,coord):
    theBoxes = [boxes[k] for k in subset]
    threshold = centroid(theBoxes,coord)
    ncoords = len(boxes[0])/2
    a = coord%ncoords
    b = a+ncoords
    below,above = [],[]
    for k in subset:
        if boxes[k][a] <= threshold: below += [k]
    for k in subset:
        if boxes[k][b] >= threshold: above += [k]
    return below,above
```
◇

Macro referenced in 10.

Test if bucket OK or append to splitting stack

⟨Test if bucket OK or append to splitting stack 3a⟩ ≡

```
""" Test if bucket OK or append to splitting stack """
def splitting(bucket,below,above, finalBuckets,splittingStack):
    if (len(below)<4 and len(above)<4) or len(set(bucket).difference(below))<7 \
        or len(set(bucket).difference(above))<7:
        finalBuckets.append(below)
        finalBuckets.append(above)
    else:
        splittingStack.append(below)
        splittingStack.append(above)
```
◇

Macro referenced in

Remove subsets from bucket list

⟨ Remove subsets from bucket list 3b ⟩ ≡

```
""" Remove subsets from bucket list """
def removeSubsets(buckets):
    n = len(buckets)
    A = zeros((n,n))
    for i,bucket in enumerate(buckets):
        for j,bucket1 in enumerate(buckets):
            if set(bucket).issubset(set(bucket1)):
                A[i,j] = 1
    B = AA(sum)(A.tolist())
    out = [bucket for i,bucket in enumerate(buckets) if B[i]==1]
    return out

def geomPartitionate(boxes,buckets):
    geomInters = [set() for h in range(len(boxes))]
    for bucket in buckets:
        for k in bucket:
            geomInters[k] = geomInters[k].union(bucket)
    for h,inters in enumerate(geomInters):
        geomInters[h] = geomInters[h].difference([h])
    return AA(list)(geomInters)
```
◇

Macro referenced in

Iterate the splitting until `splittingStack` is empty

⟨ Iterate the splitting until splittingStack is empty 4 ⟩ ≡
```
""" Iterate the splitting until \texttt{splittingStack} is empty """
def boxTest(boxes,h,k):
    B1,B2,B3,B4,B5,B6,_ = boxes[k]
```

```
        b1,b2,b3,b4,b5,b6,_ = boxes[h]
        return not (b4<B1 or B4<b1 or b5<B2 or B5<b2 or b6<B3 or B6<b3)

    def boxBuckets(boxes):
        bucket = range(len(boxes))
        splittingStack = [bucket]
        finalBuckets = []
        while splittingStack != []:
            bucket = splittingStack.pop()
            below,above = splitOnThreshold(boxes,bucket,1)
            below1,above1 = splitOnThreshold(boxes,above,2)
            below2,above2 = splitOnThreshold(boxes,below,2)

            below11,above11 = splitOnThreshold(boxes,above1,3)
            below21,above21 = splitOnThreshold(boxes,below1,3)
            below12,above12 = splitOnThreshold(boxes,above2,3)
            below22,above22 = splitOnThreshold(boxes,below2,3)

            splitting(above1,below11,above11, finalBuckets,splittingStack)
            splitting(below1,below21,above21, finalBuckets,splittingStack)
            splitting(above2,below12,above12, finalBuckets,splittingStack)
            splitting(below2,below22,above22, finalBuckets,splittingStack)

            finalBuckets = list(set(AA(tuple)(finalBuckets)))
        parts = geomPartitionate(boxes,finalBuckets)
        parts = [[h for h in part if boxTest(boxes,h,k)] for k,part in enumerate(parts)]
        return AA(sorted)(parts)
    ◇
```
Macro referenced in 10.

aaaaaa

⟨ aaaaaa 5a ⟩ ≡
```
    """ aaaaa """
```

◇

Macro never referenced.

3.2   Merging the boundaries

3.3   Elementary splitting

In this section we implement the splitting of $(d-1)$-faces, stored in FV, induced by the buckets of $(d-1)$-faces, stored in parts, and one-to-one associated to them. Of course, (a) both such arrays have the same number of elements, and (b) whereas FV contains the

indices of incident vertices for each face, `parts` contains the indices of adjacent faces for each face, with the further constraint that $i \notin \mathtt{parts}(i)$.

Computation of topological relations The function `crossRelation` is used here to compute a topological relation starting from two characteristic matrices `XV` and `YV`, that associate the sets of topological objects X and Y with their vertices, respectively. The technique using sparse binary matrices stored in `CSR` (Compressed Sparse Row) format is used.

$\langle$ Computation of topological relation 5b $\rangle \equiv$

```
""" Computation of topological relation """
def crossRelation(XV,YV):
    csrXV = csrCreate(XV)
    csrYV = csrCreate(YV)
    csrXY = matrixProduct(csrXV, csrYV.T)
    XY = [None for k in range(len(XV))]
    for k,face in enumerate(XV):
        data = csrXY[k].data
        col = csrXY[k].indices
        XY[k] = [col[h] for h,val in enumerate(data) if val==2]
        # NOTE: val depends on the relation under consideration ...
    return XY
```
$\diamond$

Macro referenced in 10.

Submanifold mapping computation The 4×4 (affine) scipy matrix `transform` of type `mat` is computed by the function `submanifoldMapping`, using as input the array `pivotFace` that contains the vertices of the so-called *pivot* face, i.e. of the face to be mapped to the coordinate subspace $z = 0$ (in 3D).

$\langle$ Submanifold mapping computation 6a $\rangle \equiv$

```
""" Submanifold mapping computation """
def submanifoldMapping(pivotFace):
    tx,ty,tz = pivotFace[0]
    transl = mat([[1,0,0,-tx],[0,1,0,-ty],[0,0,1,-tz],[0,0,0,1]])
    facet = [ VECTDIFF([v,pivotFace[0]]) for v in pivotFace ]
    m = faceTransformations(facet)
    mapping = mat([[m[0,0],m[0,1],m[0,2],0],[m[1,0],m[1,1],m[1,2],0],[m[2,0],
                   m[2,1],m[2,2],0],[0,0,0,1]])
    transform = mapping * transl
    return transform
```
$\diamond$

Macro referenced in 10.

Set of line segments partitioning a facet   The more important function of this section is the higher level `intersection` function, that accepts as input the `LAR` model `(V,FV,EV)` to be partitioned, and the pair `(k,bundledFaces)`, where `k` is the index of the pivot face (to be transformed to the $z = 0$ subspace) and where `bundledFaces` is an array of indices of faces that are guarantee to share points with face k. Such shared points may be either boundary edges of k or a segment that is internal both to face k and to some face in `bundledFaces`.

⟨Set of line segments partitioning a facet 6b⟩ ≡

```
""" Set of line segments partitioning a facet """
def intersection(V,FV,EV):
    def intersection0(k,bundledFaces):
        FE = crossRelation(FV,EV)
        pivotFace = [V[v] for v in FV[k]]
        transform = submanifoldMapping(pivotFace)  # submanifold transformation
        transformedCells,edges,faces = [],[],[]
        for face in bundledFaces:
            edge = set(FE[k]).intersection(FE[face])  # common edge index
            if edge == set():
                candidateEdges = FE[face]
                facet = []
                for e in candidateEdges:
                    cell = [V[v]+[1.0] for v in EV[e]]  # verts of incident face
                    transformedCell = (transform * (mat(cell).T)).T.tolist()
                    # vertices in local frame
                    facet += [[point[:-1] for point in transformedCell]]
                faces += [facet]
            else:  # boundary edges of face k
                e, = edge
                vs = [V[v]+[1.0] for v in EV[e]]
                ws = (transform * (mat(vs).T)).T.tolist()
                edges += [[p[:-1] for p in ws]]
        return edges,faces,transform
    return intersection0
```

◇

Macro referenced in 10.

Computation of face transformations   The faces in every `parts`(i) must be affinely transformed into the subspace $x_d = 0$, in order to compute the intersection of its elements with this subspace, that are submanifolds of dimension $d - 2$.

⟨Computation of face transformations 7a⟩ ≡

```
""" Computation of affine face transformations """
def COVECTOR(points):
```

```
            pointdim = len(points[0])
            plane = Planef.bestFittingPlane(pointdim,
                            [item for sublist in points for item in sublist])
            return [plane.get(I) for I in range(0,pointdim+1)]

        def faceTransformations(facet):
            covector = COVECTOR(facet)
            translVector = facet[0]
            # translation
            newFacet = [ VECTDIFF([v,translVector]) for v in facet ]
            # linear transformation: boundaryFacet -> standard (d-1)-simplex
            d = len(facet[0])
            transformMat = mat( newFacet[1:d] + [covector[1:]] ).T.I
            # transformation in the subspace x_d = 0
            out = (transformMat * (mat(newFacet).T)).T.tolist()
            return transformMat
    ◇
```

Macro referenced in 10.

Space partitioning via submanifold mapping   the function `spacePartition`, given in the below script, takes as input a *non-valid* (with the meaning used in solid modeling field — see [Req80]) `LAR` model of dimension $d-1$, i.e. a triple (`V,FV,EV`), and an array `parts` indexed on faces, and containing the subset of faces with greatest probability of intersecting each indexing face, respectively. The `spacePartition` function returns the *valid* `LAR` boundary model (`W,FW,EW`) of the space partition induced by `FV`.

⟨Space partitioning via submanifold mapping 7b⟩ ≡

```
    """ Space partitioning via submanifold mapping """
    def spacePartition(V,FV,EV, parts):
        transfFaces = []
        for k,bundledFaces in enumerate(parts):
            edges,faces,transform = intersection(V,FV,EV)(k,bundledFaces)
            for face in faces:
                line = []
                for edge in face:
                    (x1,y1,z1),(x2,y2,z2) = edge
                    if not verySmall(z2-z1):
                        x = (x2-x1)/(z2-z1) + x1
                        y = (y2-y1)/(z2-z1) + y1
                        p = [x,y,0]
                        line += [eval(vcode(p))]
                if line!=[]: edges += [line]
            print "k,edges =",k,edges
            v,fv,ev = larFromLines([[point[:-1] for point in edge] for edge in edges])
            if len(fv)>1: fv = fv[:-1]
```

8

```
                  lar = [w+[0.0] for w in v],fv,ev
                  transfFaces += [Struct([ larApply(transform.I)(lar) ])]
            W,FW,EW = struct2lar(Struct(transfFaces))
            return W,FW,EW
    ◇
```

Macro referenced in 10.

3.4 Circular ordering of faces around edges

Directional and orthogonal projection operators In order to sort circularly the faces incident on each edge, we need of course to compute the relation EF, and for each face f incident on $e = (v_1, v_2)$, to project a vector $w_f = (v_1, v_f)$ on the subspace ortogonal to e. This may be done by mapping w_f with the tensor $I - e \otimes e$. Finally, the angles between vectors a, b in this orthogonal space to e may be computed by using the `atan2` function, that combines both the *sin* and the *cos* of the angle:

$$angle = atan2(norm(cross(a, b)), dot(a, b)).$$

Let us just remember that, by definition, $(e \otimes e)v = (e \cdot v)e$, where e, v are vectors.

⟨ Directional and orthogonal projection operators 8 ⟩ ≡

```
    """ Directional and orthogonal projection operators """
    def dirProject (e):
        def dirProject0 (v):
            return SCALARVECTPROD([ INNERPROD([ UNITVECT(e), v ]), UNITVECT(e) ])
        return dirProject0

    def orthoProject (e):
        def orthoProject0 (v):
            return VECTDIFF([ v, dirProject(UNITVECT(e))(v) ])
        return orthoProject0
    ◇
```

Macro referenced in 10.

Slope of edges

⟨ Slope of edges 9 ⟩ ≡

```
    """ Circular ordering of faces around edges """
    from bool1 import invertRelation

    def faceSlopeOrdering(model):
        V,FV,EV = model

        print "\n V,FV,EV =",V,FV,EV
```

```python
FE = crossRelation(FV,EV)

print "\n FE =",FE

EF,EF_angle = invertRelation(FE),[]

print "\n EF,EF_angle =",EF,EF_angle

for e,ef in enumerate(EF):

    print "\n e,ef =",e,ef

    v1,v2 = EV[e]

    print "\n v1,v2 =",v1,v2

    E = array(UNITVECT(VECTDIFF([V[v2],V[v1]])))

    print "\n E =",E

    refFace = EF[e][0]

    print "\n refFace =",refFace

    ef_angle = []

    print "\n ef_angle =",ef_angle

    for face in ef:

        print "\n face =",face

        fverts = [ VECTDIFF([ V[k],V[v1] ]) for k in FV[face] if k!=v1 ]

        print "\n fverts =",fverts

        fvects = array([ E.dot( vf ) for vf in fverts ])

        print "\n fvects =",fvects

        index_min = fvects.argmin()

        print "\n index_min =",index_min

        W = fverts[index_min]
```

```
                    print "\n W =",W

                    if not verySmall(E.dot(W)): W = UNITVECT(orthoProject(list(E))(W))

                    print "\n W =",W

                    if (face == refFace): refVect = array(UNITVECT(W))

                    print "\n refVect =",refVect

                    angle = math.atan2(VECTNORM(VECTPROD([refVect,W])),refVect.dot(W))

                    print "\n angle =",angle

                    ef_angle += [180*angle/PI]

                    print "\n ef_angle =",ef_angle

                pairs = sorted(zip(ef_angle,ef))

                print "\n pairs =",pairs

                EF_angle += [[pair[1] for pair in pairs]]

                print "\n EF_angle =",EF_angle

            return EF_angle
    ◇
```

Macro referenced in 10.

3.5 Boolean chains

4 Esporting the Library

```
"lib/py/bool2.py" 10 ≡
     """ Module for Boolean computations between geometric objects """
     from pyplasm import *
     """ import modules from larcc/lib """
     import sys
     sys.path.insert(0, 'lib/py/')
     from inters import *
     DEBUG = True

     ⟨Coding utilities 17⟩
     ⟨Split the boxes between the below,above subsets 2⟩
```

5   Test examples

5.1   Random triangles

Generation of random triangles and their boxes

```
"test/py/bool2/test01.py" 11 ≡
    """ Generation of random triangles and their boxes """
    import sys
    sys.path.insert(0, 'lib/py/')
    from bool2 import *
    glass = MATERIAL([1,0,0,0.1,  0,1,0,0.1,  0,0,1,0.1, 0,0,0,0.1, 100])

    randomTriaArray = randomTriangles(10,0.99)
    VIEW(STRUCT(AA(MKPOL)([[verts, [[1,2,3]], None] for verts in randomTriaArray])))

    boxes = containmentBoxes(randomTriaArray)
    hexas = AA(box2exa)(boxes)
    cyan = COLOR(CYAN)(STRUCT(AA(MKPOL)([[verts, [[1,2,3]], None] for verts in randomTriaArray])))
    yellow = STRUCT(AA(glass)(AA(MKPOL)([hex for hex,qualifier in hexas])))
    VIEW(STRUCT([cyan,yellow]))
    ◇
```

Generation of random quadrilaterals and their boxes

```
"test/py/bool2/test02.py" 12a ≡
    """ Generation of random quadrilaterals and their boxes """
    import sys
    sys.path.insert(0, 'lib/py/')
    from bool2 import *
    glass = MATERIAL([1,0,0,0.1,  0,1,0,0.1,  0,0,1,0.1, 0,0,0,0.1, 100])

    randomQuadArray = randomQuads(10,1)
```

12

```
VIEW(STRUCT(AA(MKPOL)([[verts, [[1,2,3,4]], None] for verts in randomQuadArray])))

boxes = containmentBoxes(randomQuadArray)
hexas = AA(box2exa)(boxes)
cyan = COLOR(CYAN)(STRUCT(AA(MKPOL)([[verts, [[1,2,3,4]], None] for verts in randomQuadArray])
yellow = STRUCT(AA(glass)(AA(MKPOL)([hex for hex,qualifier in hexas])))
VIEW(STRUCT([cyan,yellow]))
◇
```
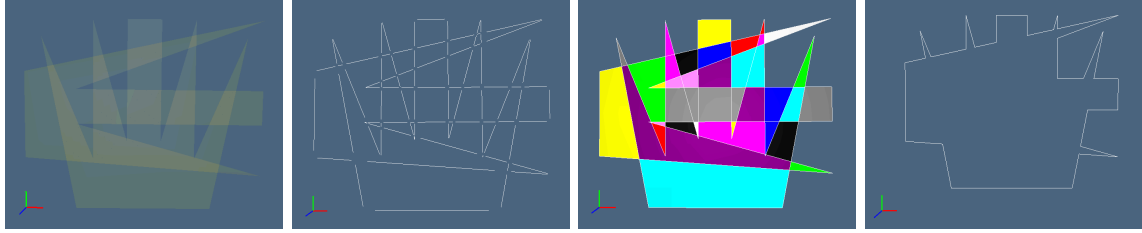

Figure 1: LAR complex from two polygons. (a) the input polygons; (b) the intersection of boundary lines; (c) the extracted *regularized* 2-complex; (d) the boundary LAR.

```
"test/py/bool2/test03.py" 12b ≡
    """ Boolean complex generated by boundaries of two complexes """
    import sys
    sys.path.insert(0, 'lib/py/')
    from inters import *
    glass = MATERIAL([1,0,0,0.1,  0,1,0,0.1,  0,0,1,0.1, 0,0,0,0.1, 100])

    V1 = [[3,0],[11,0],[13,10],[10,11],[8,11],[6,11],[4,11],[1,10],[4,3],[6,4],
            [8,4],[10,3]]
    FV1 = [[0,1,8,9,10,11],[1,2,11],[3,10,11],[4,5,9,10],[6,8,9],[0,7,8]]
    EV1 = [[0,1],[0,7],[0,8],[1,2],[1,11],[2,11],[3,10],[3,11],[4,5],[4,10],[5,
            9],[6,8],[6,9],[7,8],[8,9],[9,10],[10,11]]
    BE1 = boundaryCells(FV1,EV1)
    lines1 = [[V1[v] for v in EV1[edge]] for edge in BE1]

    V2 = [[0,3],[14,2],[14,5],[14,7],[14,11],[0,8],[3,7],[3,5]]
    FV2 = [[0,5,6,7],[0,1,7],[4,5,6],[2,3,6,7]]
    EV2 = [[0,1],[0,5],[0,7],[1,7],[2,3],[2,7],[3,6],[4,5],[4,6],[5,6],[6,7]]
    BE2 = boundaryCells(FV2,EV2)
    lines2 = [[V2[v] for v in EV2[edge]] for edge in BE2]

    VIEW(STRUCT([ glass(STRUCT(MKPOLS((V1,FV1)))), glass(STRUCT(MKPOLS((V2,FV2)))) ]))
    lines = lines1 + lines2
    VIEW(STRUCT(AA(POLYLINE)(lines)))
```

13

```
global precision
PRECISION += 2
V,FV,EV = larFromLines(lines)
VIEW(EXPLODE(1.2,1.2,1)(MKPOLS((V,EV))))

VV = AA(LIST)(range(len(V)))
submodel = STRUCT(MKPOLS((V,EV)))
VIEW(larModelNumbering(1,1,1)(V,[VV,EV,FV[:-1]],submodel,1))

polylines = [[V[v] for v in face+[face[0]]] for face in FV[:-1]]
colors = [CYAN, MAGENTA, WHITE, RED, YELLOW, GREEN, GRAY, ORANGE, BLACK, BLUE, PURPLE, BROWN]
sets = [COLOR(colors[k%12])(FAN(pol)) for k,pol in enumerate(polylines)]
VIEW(STRUCT([ T(3)(0.02)(STRUCT(AA(POLYLINE)(lines))), STRUCT(sets)]))

VIEW(EXPLODE(1.2,1.2,1)((AA(POLYLINE)(polylines))))
polylines = [ [V[v] for v in FV[-1]+[FV[-1][0]]] ]
VIEW(EXPLODE(1.2,1.2,1)((AA(POLYLINE)(polylines))))
◇
```

5.2   Testing the box-kd-trees

Visualizing with different colors the buckets of box-kd-tree

"test/py/bool2/test04.py" 13 ≡

```
""" Visualizing with different colors the buckets of box-kd-tree """
from pyplasm import *
""" import modules from larcc/lib """
import sys
sys.path.insert(0, 'lib/py/')
from bool2 import *

randomQuadArray = randomQuads(30,0.8)
VIEW(STRUCT(AA(MKPOL)([[verts, [[1,2,3,4]], None] for verts in randomQuadArray])))

boxes = containmentBoxes(randomQuadArray)
hexas = AA(box2exa)(boxes)
glass = MATERIAL([1,0,0,0.1,  0,1,0,0.1,  0,0,1,0.1, 0,0,0,0.1, 100])
yellow = STRUCT(AA(glass)(AA(MKPOL)([hex for hex,data in hexas])))
VIEW(STRUCT([#cyan,
    yellow]))

parts = boxBuckets(boxes)
for k,part in enumerate(parts):
    bunch = [glass(STRUCT( [MKPOL(hexas[h][0]) for h in part]))]
```

```
        bunch += [COLOR(RED)(MKPOL(hexas[k][0]))]
        VIEW(STRUCT(bunch))
    ◇
```

5.3   Intersection of geometry subsets

Two unit cubes

⟨ Two unit cubes 14a ⟩ ≡
```
    """ Two unit cubes """
    import sys
    sys.path.insert(0, 'lib/py/')
    from bool2 import *
    glass = MATERIAL([1,0,0,0.1,  0,1,0,0.1,  0,0,1,0.1, 0,0,0,0.1, 100])

    V,[VV,EV,FV,CV] = larCuboids([1,1,1],True)
    cube1 = Struct([(V,FV,EV)],"cube1")
    #twoCubes = Struct([cube1,t(-1,.5,1),cube1])      # other test example
    twoCubes = Struct([cube1,t(.5,.5,.5),cube1])
    #twoCubes = Struct([cube1,t(.5,.5,0),cube1])     # other test example
    #twoCubes = Struct([cube1,t(.5,0,0),cube1])        # other test example
    V,FV,EV = struct2lar(twoCubes)
    VIEW(EXPLODE(1.2,1.2,1.2)(MKPOLS((V,FV))))

    quadArray = [[V[v] for v in face] for face in FV]
    boxes = containmentBoxes(quadArray)
    hexas = AA(box2exa)(boxes)
    parts = boxBuckets(boxes)
    ◇
```
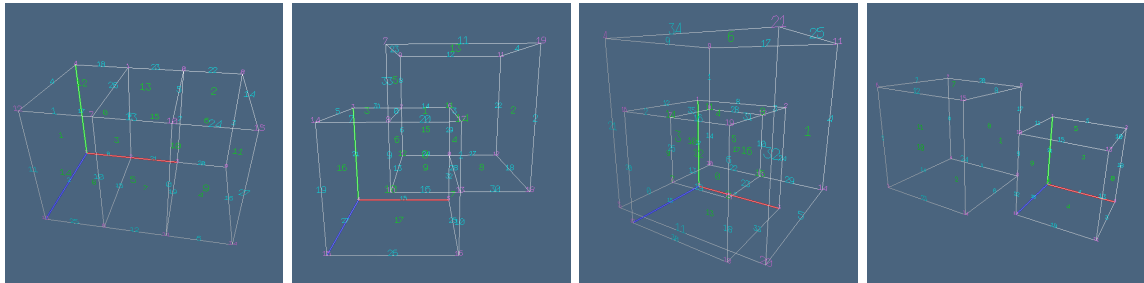
Macro referenced in 14b, 15.

Figure 2: LAR complex of the space decomposition generated by two cubes in special positions. (a) translation on one coordinate; (b) translation on two coordinates; (c) translation on three coordinates; (d) non-manifold position along an edge.

Face (and incident faces) transformation

`"test/py/bool2/test05.py"` 14b ≡
```
""" non-valid -> valid solid representation of a space partition """

⟨Two unit cubes 14a⟩

W,FW,EW = spacePartition(V,FV,EV, parts)

from architectural import *
polylines = lar2polylines((W,FW))
VIEW(EXPLODE(1.2,1.2,1.2)(AA(POLYLINE)(polylines)))

WW = AA(LIST)(range(len(W)))
submodel = STRUCT(MKPOLS((W,EW)))
VIEW(larModelNumbering(1,1,1)(W,[WW,EW,FW],submodel,0.5))
◇
```

3-cell reconstruction from LAR space partition

`"test/py/bool2/test06.py"` 15 ≡
```
""" 3-cell reconstruction from LAR space partition """
⟨Two unit cubes 14a⟩
W,FW,EW = spacePartition(V,FV,EV, parts)
WW = AA(LIST)(range(len(W)))
submodel = STRUCT(MKPOLS((W,EW)))
VIEW(larModelNumbering(1,1,1)(W,[WW,EW,FW],submodel,0.6))

faceSlopeOrdering((W,FW,EW))
◇
```

2D polygon triangulation   Here a 2D polygon is imported from an SVG file made of boundary lines, and the `V,FV,EV` LAR model is generated. Then the unique polygonal face in `FV` is embedded in 3D ($z = 0$), and triangulated using the tassellation algorithm extracted from pyOpenGL and pyGLContext, stored in the `lib/py/support.py` file. The generated triangles are finally coherently oriented, by testing the z-component of their normal vector.

`"test/py/bool2/test07.py"` 16a ≡
```
""" 2D polygon triangulation """
import sys
sys.path.insert(0, 'lib/py/')
from inters import *
from support import PolygonTessellator,vertex
```

```
filename = "test/py/bool2/interior.svg"
lines = svg2lines(filename)
V,FV,EV = larFromLines(lines)
VIEW(EXPLODE(1.2,1.2,1)(MKPOLS((V,FV[:-1]+EV)) + AA(MK)(V)))

pivotFace = [V[v] for v in FV[0]+[FV[0][0]]]
pol = PolygonTessellator()
vertices = [ vertex.Vertex( (x,y,0) ) for (x,y) in pivotFace  ]
verts = pol.tessellate(vertices)
ps = [list(v.point) for v in verts]
trias = [[ps[k],ps[k+1],ps[k+2],ps[k]] for k in range(0,len(ps),3)]
VIEW(STRUCT(AA(POLYLINE)(trias)))

def orientTriangle(pointTriple):
    v1 = array(pointTriple[1])-pointTriple[0]
    v2 = array(pointTriple[2])-pointTriple[0]
    if cross(v1,v2)[2] < 0: return REVERSE(pointTriple)
    else: return pointTriple

triangles = DISTR([AA(orientTriangle)(trias),[[0,1,2]]])
VIEW(STRUCT(CAT(AA(MKPOLS)(triangles))))
◇
```

3D boundary triangulation of the space partition   The function `boundaryTriangulation` given below is used to guarantee that there is a unique (simple) facet incident to an edge and contained in one LAR facet. More clearly, the Boolean decompositions generated by LAR allow for non convex cells, and in particular for nonconvex boundary facets of *d*-cells. This fact may induce errors in the computation of circularly sorted faces around edges. Conversely, by decomposing the faces into triangles, such ordering problems cannot appear.

"test/py/bool2/test08.py" 16b ≡
```
""" 3D boundary triangulation of the space partition """
import sys
sys.path.insert(0, 'test/py/bool2/')
from test06 import *
from support import PolygonTessellator,vertex

def orientTriangle(pointTriple):
    v1 = array(pointTriple[1])-pointTriple[0]
    v2 = array(pointTriple[2])-pointTriple[0]
    if cross(v1,v2)[2] < 0: return REVERSE(pointTriple)
    else: return pointTriple

def boundaryTriangulation(W,FW):
```

```
        triangles = []
        for face in FW:
            pivotFace = [W[v] for v in face+(face[0],)]
            transform = submanifoldMapping(pivotFace)
            mappedVerts = (transform * (mat([p+[1.0] for p in pivotFace]).T)).T.tolist()
            facet = [point[:-2] for point in mappedVerts]
            pol = PolygonTessellator()
            vertices = [ vertex.Vertex( (x,y,0) ) for (x,y) in facet  ]
            verts = pol.tessellate(vertices)
            ps = [list(v.point) for v in verts]
            trias = [[ps[k],ps[k+1],ps[k+2],ps[k]] for k in range(0,len(ps),3)]
            mappedVerts = (transform.I * (mat([p+[1.0] for p in ps]).T)).T.tolist()
            points = [p[:-1] for p in mappedVerts]
            trias = [[points[k],points[k+1],points[k+2],points[k]] for k in range(0,len(points),3)
                if scipy.linalg.norm(cross(array(points[k+1])-points[k],
                                           array(points[k+2])-points[k])) != 0 ]
            triangles += DISTR([AA(orientTriangle)(trias),[[0,1,2]]])
        return triangles

triangles = boundaryTriangulation(W,FW)
VIEW(EXPLODE(1.2,1.2,1.2)(CAT(AA(MKPOLS)(triangles))))
◇
```
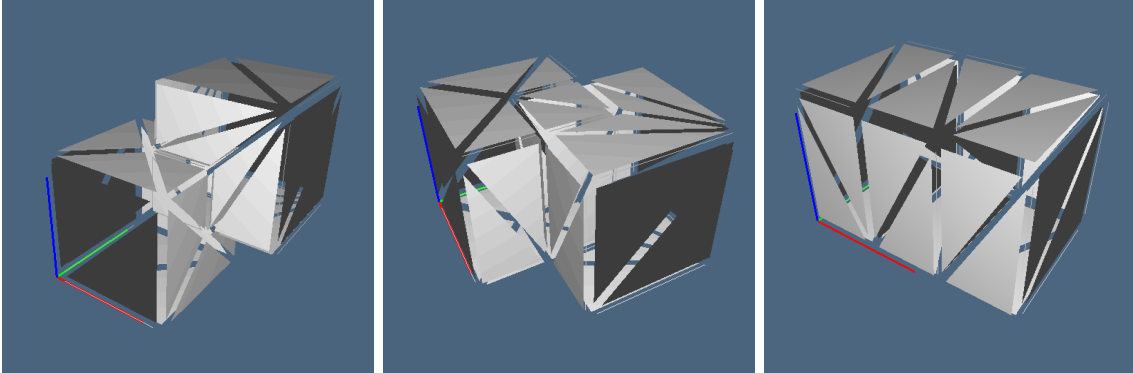

Figure 3: The triangulated boundaries of the space partition induced by two cubes (one is variously translated).

A   Code utilities

Coding utilities   Some utility fuctions used by the module are collected in this appendix. Their macro names can be seen in the below script.

18

⟨ Coding utilities 17 ⟩ ≡
     """ Coding utilities """
     ⟨ Generation of a random 3D point 19b ⟩
     ⟨ Generation of random 3D triangles 18 ⟩
     ⟨ Generation of random 3D quadrilaterals 19a ⟩
     ⟨ Generation of a single random triangle 19c ⟩
     ⟨ Containment boxes 20a ⟩
     ⟨ Transformation of a 3D box into an hexahedron 20b ⟩
     ⟨ Computation of the 1D centroid of a list of 3D boxes 21 ⟩
        ◇
Macro referenced in 10.

Generation of random triangles The function `randomTriangles` returns the array
`randomTriaArray` with a given number of triangles generated within the unit 3D interval.
The `scaling` parameter is used to scale every such triangle, generated by three randow
points, that could be possibly located to far from each other, even at the distance of the
diagonal of the unit cube.

The arrays `xs`, `ys` and `zs`, that contain the x, y, z coordinates of triangle points, are
used to compute the minimal translation `v` needed to transport the entire set of data within
the positive octant of the 3D space.

⟨ Generation of random 3D triangles 18 ⟩ ≡
     """ Generation of random triangles """
     def randomTriangles(numberOfTriangles=400,scaling=0.3):
         randomTriaArray = [rtriangle(scaling) for k in range(numberOfTriangles)]
         [xs,ys,zs] = TRANS(CAT(randomTriaArray))
         xmin, ymin, zmin = min(xs), min(ys), min(zs)
         v = array([-xmin,-ymin, -zmin])
         randomTriaArray = [[list(v1+v), list(v2+v), list(v3+v)] for v1,v2,v3 in randomTriaArray]
         return randomTriaArray
        ◇
Macro referenced in 17.

Generation of random 3D quadrilaterals

⟨ Generation of random 3D quadrilaterals 19a ⟩ ≡
     """ Generation of random 3D quadrilaterals """
     def randomQuads(numberOfQuads=400,scaling=0.3):
         randomTriaArray = [rtriangle(scaling) for k in range(numberOfQuads)]
         [xs,ys,zs] = TRANS(CAT(randomTriaArray))
         xmin, ymin, zmin = min(xs), min(ys), min(zs)
         v = array([-xmin,-ymin, -zmin])
         randomQuadArray = [AA(list)([v1+v, v2+v, v3+v, v+v2-v1+v3]) for v1,v2,v3 in randomTriaAr
         return randomQuadArray
        ◇
Macro referenced in 17.

19

Generation of a random 3D point   A single random point, codified in floating point format, and with a fixed (quite small) number of digits, is returned by the `rpoint()` function, with no input parameters.

⟨ Generation of a random 3D point 19b ⟩ ≡

```
""" Generation of a random 3D point """
def rpoint():
    return eval( vcode([ random.random(), random.random(), random.random() ]) )
```
◇

Macro referenced in 17.

Generation of a single random triangle   A single random triangle, scaled about its centroid by the `scaling` parameter, is returned by the `rtriangle()` function, as a tuple ot two random points in the unit square.

⟨ Generation of a single random triangle 19c ⟩ ≡

```
""" Generation of a single random triangle """
def rtriangle(scaling):
    v1,v2,v3 = array(rpoint()), array(rpoint()), array(rpoint())
    c = (v1+v2+v3)/3
    pos = rpoint()
    v1 = (v1-c)*scaling + pos
    v2 = (v2-c)*scaling + pos
    v3 = (v3-c)*scaling + pos
    return tuple(eval(vcode(v1))), tuple(eval(vcode(v2))), tuple(eval(vcode(v3)))
```
◇

Macro referenced in 17.

Containment boxes   Given as input a list `randomTriaArray` of pairs of 2D points, the function `containmentBoxes` returns, in the same order, the list of *containment boxes* of the input lines. A *containment box* of a geometric object of dimension d is defined as the minimal d-cuboid, equioriented with the reference frame, that contains the object. For a 2D line it is given by the tuple $(x1, y1, x2, y2)$, where $(x1, y1)$ is the point of minimal coordinates, and $(x2, y2)$ is the point of maximal coordinates.

⟨ Containment boxes 20a ⟩ ≡

```
""" Containment boxes """
def containmentBoxes(randomPointArray,qualifier=0):
    if len(randomPointArray[0])==2:
        boxes = [eval(vcode([min(x1,x2), min(y1,y2), min(z1,z2),
                             max(x1,x2), max(y1,y2), max(z1,z2)]))+[qualifier]
                for ((x1,y1,z1),(x2,y2,z2)) in randomPointArray]
    elif len(randomPointArray[0])==3:
        boxes = [eval(vcode([min(x1,x2,x3), min(y1,y2,y3), min(z1,z2,z3),
```

20

```
                              max(x1,x2,x3), max(y1,y2,y3), max(z1,z2,z3)]))+[qualifier]
                    for ((x1,y1,z1),(x2,y2,z2),(x3,y3,z3)) in randomPointArray]
         elif len(randomPointArray[0])==4:
             boxes = [eval(vcode([min(x1,x2,x3,x4), min(y1,y2,y3,y4), min(z1,z2,z3,z4),
                              max(x1,x2,x3,x4), max(y1,y2,y3,y4), max(z1,z2,z3,z4)]))+[qualifie
                    for ((x1,y1,z1),(x2,y2,z2),(x3,y3,z3),(x4,y4,z4)) in randomPointArray]
         return boxes
    ◇
```

Macro referenced in 17.

Transformation of a 3D box into an hexahedron   The transformation of a 2D box
into a closed rectangular polyline, given as an ordered sequwncw of 2D points, is produced
by the function `box2exa`

⟨ Transformation of a 3D box into an hexahedron 20b ⟩ ≡

```
    """ Transformation of a 3D box into an hexahedron """
    def box2exa(box):
        x1,y1,z1,x2,y2,z2,type = box
        verts = [[x1,y1,z1], [x1,y1,z2], [x1,y2,z1], [x1,y2,z2], [x2,y1,z1], [x2,y1,z2], [x2,y2,z1]
        cell = [range(1,len(verts)+1)]
        return [verts,cell,None],type


    def lar2boxes(model,qualifier=0):
        V,CV = model
        boxes = []
        for k,cell in enumerate(CV):
            verts = [V[v] for v in cell]
            x1,y1,z1 = [min(coord) for coord in TRANS(verts)]
            x2,y2,z2 = [max(coord) for coord in TRANS(verts)]
            boxes += [eval(vcode([min(x1,x2),min(y1,y2),min(z1,z2),max(x1,x2),max(y1,y2),max(z1,z2
        return boxes
    ◇
```

Macro referenced in 17.

Computation of the 1D centroid of a list of 3D boxes   The 1D `centroid` of a list
of 3D boxes is computed by the function given below. The direction of computation (either
x, y or z) is chosen depending on the value of the `coord` parameter.

⟨ Computation of the 1D centroid of a list of 3D boxes 21 ⟩ ≡

```
    """ Computation of the 1D centroid of a list of 3D boxes """
    def centroid(boxes,coord):
        delta,n = 0,len(boxes)
        ncoords = len(boxes[0])/2
        a = coord%ncoords
        b = a+ncoords
```

21

```
        for box in boxes:
            delta += (box[a] + box[b])/2
        return delta/n
    ◇
```

Macro referenced in 17.

References

[CL13]   CVD-Lab, *Linear algebraic representation*, Tech. Report 13-00, Roma Tre University, October 2013.

[Req80] Aristides G. Requicha, *Representations for rigid solids: Theory, methods, and systems*, ACM Comput. Surv. **12** (1980), no. 4, 437–464.