Curves, surfaces and splines with LAR [*]

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Abstract

In this module we implement above LAR most of the parametric methods for polynomial and rational curves, surfaces and splines discussed in the book [Pao03], and implemented in the PLaSM language and in the python package pyplasm.

Contents

[*]This document is part of the *Linear Algebraic Representation with CoChains* (LAR-CC) framework [CL13]. May 7, 2014

1 Introduction

2 Tensor product surfaces

The tensor product form of surfaces will be primarily used, in the remainder of this module, to support the LAR implementation of polynomial (rational) surfaces. For this purpose, we start by defining some basic operators on function tensors. In particular, a toolbox of basic tensor operations is given in Script 12.3.1. The ConstFunTensor operator produces a tensor of constant functions starting from a tensor of numbers; the recursive FlatTensor may be used to ?flatten? a tensor with any number of indices by producing a corresponding one index tensor; the InnerProd and TensorProd are used to compute the inner product and the tensor product of conforming tensors of functions, respectively.

Toolbox of tensor operations

⟨ Multidimensional transfinite Bernstein-Bezier Basis 1 ⟩ ≡
```
""" Toolbox of tensor operations """
def larBernsteinBasis (U):
   def BERNSTEIN0 (N):
      def BERNSTEIN1 (I):
         def map_fn(point):
            t = U(point)
            out = CHOOSE([N,I])*math.pow(1-t,N-I)*math.pow(t,I)
            return out
         return map_fn
      return [BERNSTEIN1(I) for I in range(0,N+1)]
   return BERNSTEIN0
◇
```
Macro referenced in 8b.

2.1 Tensor product surface patch

⟨ Tensor product surface patch 2a ⟩ ≡
```
""" Tensor product surface patch """
def larTensorProdSurface (args):
   ubasis , vbasis = args
   def TENSORPRODSURFACE0 (controlpoints_fn):
      def map_fn(point):
         u,v=point
         U=[f([u]) for f in ubasis]
         V=[f([v]) for f in vbasis]
         controlpoints=[f(point) if callable(f) else f
            for f in controlpoints_fn]
         target_dim = len(controlpoints[0][0])
```

```
            ret=[0 for x in range(target_dim)]
            for i in range(len(ubasis)):
               for j in range(len(vbasis)):
                  for M in range(len(ret)):
                     for M in range(target_dim):
                        ret[M] += U[i]*V[j] * controlpoints[i][j][M]
            return ret
         return map_fn
      return TENSORPRODSURFACE0
   ◇
```

Macro referenced in 8b.

Bilinear tensor product surface patch

⟨ Bilinear surface patch 2b ⟩ ≡

```
   """ Bilinear tensor product surface patch """
   def larBilinearSurface(controlpoints):
      basis = larBernsteinBasis(S1)(1)
      return larTensorProdSurface([basis,basis])(controlpoints)
   ◇
```

Macro referenced in 8b.

Biquadratic tensor product surface patch

⟨ Biquadratic surface patch 3a ⟩ ≡

```
   """ Biquadratic tensor product surface patch """
   def larBiquadraticSurface(controlpoints):
      basis1 = larBernsteinBasis(S1)(2)
      basis2 = larBernsteinBasis(S1)(2)
      return larTensorProdSurface([basis1,basis2])(controlpoints)
   ◇
```

Macro referenced in 8b.

Bicubic tensor product surface patch

⟨ Bicubic surface patch 3b ⟩ ≡

```
   """ Bicubic tensor product surface patch """
   def larBicubicSurface(controlpoints):
      basis1 = larBernsteinBasis(S1)(3)
      basis2 = larBernsteinBasis(S1)(3)
      return larTensorProdSurface([basis1,basis2])(controlpoints)
   ◇
```

Macro referenced in 8b.

3 Transfinite Bézier

⟨ Multidimensional transfinite Bézier 3c ⟩ ≡

```
""" Multidimensional transfinite Bezier """
def larBezier(U):
   def BEZIER0(controldata_fn):
      N = len(controldata_fn)-1
      def map_fn(point):
         t = U(point)
         controldata = [fun(point) if callable(fun) else fun
            for fun in controldata_fn]
         out = [0.0 for i in range(len(controldata[0]))]
         for I in range(N+1):
            weight = CHOOSE([N,I])*math.pow(1-t,N-I)*math.pow(t,I)
            for K in range(len(out)):  out[K] += weight*(controldata[I][K])
         return out
      return map_fn
   return BEZIER0

def larBezierCurve(controlpoints):
   return larBezier(S1)(controlpoints)
◇
```

Macro referenced in 8b.

4 Coons patches

⟨ Transfinite Coons patches 4 ⟩ ≡

```
""" Transfinite Coons patches """
def larCoonsPatch (args):
   su0_fn , su1_fn , s0v_fn , s1v_fn = args
   def map_fn(point):
      u,v=point
      su0 = su0_fn(point) if callable(su0_fn) else su0_fn
      su1 = su1_fn(point) if callable(su1_fn) else su1_fn
      s0v = s0v_fn(point) if callable(s0v_fn) else s0v_fn
      s1v = s1v_fn(point) if callable(s1v_fn) else s1v_fn
      ret=[0.0 for i in range(len(su0))]
      for K in range(len(ret)):
         ret[K] = ((1-u)*s0v[K] + u*s1v[K]+(1-v)*su0[K] + v*su1[K] +
         (1-u)*(1-v)*s0v[K] + (1-u)*v*s0v[K] + u*(1-v)*s1v[K] + u*v*s1v[K])
      return ret
   return map_fn
◇
```

Macro referenced in 8b.

5 Bsplines

The B-splines discussed in this section are called *non-uniform* because different spline segments may correspond to different intervals in parameter space, unlike uniform B-splines. The basis polynomials, and consequently the spline shape and the other properties, are defined by a non-decreasing sequence of real numbers

$$t_0 \leq t_1 \leq \cdots \leq t_n,$$

called the *knot sequence*. Splines of this kind are also named *NUB-splines* in the remainder of this book,[1] where the name stands for Non-Uniform B-splines.

The knot sequence is used to define the basis polynomials which blend the control points. In particular, each subset of $k+2$ adjacent knot values is used to compute a basis polynomial of degree k. Notice that some subsequent knots may coincide. In this case we speak of *multiplicity* of the knots.

Note In non-uniform B-splines the number $n+1$ of *knot values* is greater than the number $m+1$ of control points $\mathbf{p}_0, \ldots, \mathbf{p}_m$. In particular, the relation

$$n = m + k + 1, \tag{1}$$

where k is the *degree* of spline segments, must hold between the number of knots and the number of control points. The quantity $h = k + 1$ is called the *order* of the spline. It will be useful when giving recursive formulas to compute the B-basis polynomials. Let us remember, e.g., that a spline of order four is made of cubic segments.

Non-uniform B-spline flexibility Such splines have a much greater flexibility than the uniform ones. The basis polynomial associated with each control point may vary depending on the subset of knots it depends on. Spline segments may be parametrized over intervals of different size, and even reduced to a single point. Therefore, the continuity at a joint may be reduced, e.g. from C^2 to C^1 to C^0 and even to none by suitably increasing the multiplicity of a knot.

5.1 Definitions

5.1.1 Geometric entities

In order to fully understand the construction of a non-uniform B-spline, it may be useful to recall the main inter-relationships among the 5 geometric entities that enter the definition.

[1]Some authors call them non-uniform non-rational B-splines. We prefer to emphasize that they are polynomial splines.

Control points are denoted as $\mathbf{p}_i$, with $0 \leq i \leq m$. A non-uniform B-spline usually approximates the control points.

Knot values are denoted as t_i, with $0 \leq i \leq n$. It must be $n = m + k + 1$, where k is the spline degree. Knot values are used to define the B-spline polynomials. They also define the join points (or joints) between adjacent spline segments. When two consecutive knots coincide, the spline segment associated with their interval reduces to a point.

Spline degree is defined as the degree of the B-basis functions which are combined with the control points. The degree is denoted as k. It is connected to the spline order $h = k+1$. The most used non-uniform B-splines are either cubic or quadratic. The image of a linear non-uniform B-spline is a polygonal line. The image of a non-uniform B-spline of degree 0 coincides with the sequence of control points.

B-basis polynomials are denoted as $B_{i,h}(t)$. They are univariate polynomials in the t indeterminate, computed by using the recursive formulas of Cox and de Boor. The i index is associated with the first one of values in the knot subsequence $(t_i, t_{i+1}, \ldots, t_{i+h})$ used to compute $B_{i,h}(t)$. The second index is called *order* of the polynomial.

Spline segments are defined as polynomial vector functions of a single parameter. Such functions are denoted as $\mathbf{Q}_i(t)$, with $k \leq i \leq m$. A $\mathbf{Q}_i(t)$ spline segment is obtained by a combination of the i-th control point and the k previous points with the basis polynomials of order h associated to the same indices. It is easy to see that the number of spline segments is $m - K + 1$.

5.2 Computation of a B-spline mapping

The B-spline mapping, i.e. the vector-valued polynomial to be mapped over a 1D domain discretisation by the `larMap` operator, is computed by making reference to the `pyplasm` implementation given by the `BSPLINE` contained in the `fenvs.py` library in the `pyplasm` package.

`BSPLINE` is a third-order function, that must be ordinately applied to `degree`, `knots`, and `controlpoints`.

5.3 Domain computation

⟨Domain decomposition for 1D bspline maps 6a⟩ ≡

```
""" Domain decomposition for 1D bspline maps """
def larDom(knots,tics=32):
   domain = knots[-1]-knots[0]
   return larIntervals([tics*domain])([domain])
◇
```

Macro referenced in 8b.

5.4 Examples

Two examples of B-spline curves using lar-cc

"test/py/splines/test08.py" 6b ≡
```
""" Two examples of B-spline curves using lar-cc """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

controls = [[0,0],[-1,2],[1,4],[2,3],[1,1],[1,2],[2.5,1],[2.5,3],[4,4],[5,0]];
knots = [0,0,0,0,1,2,3,4,5,6,7,7,7,7]
bspline = BSPLINE(3)(knots)(controls)
obj = larMap(bspline)(larDom(knots))
VIEW(STRUCT( MKPOLS(obj) + [POLYLINE(controls)] ))

controls = [[0,1],[1,1],[2,0],[3,0],[4,0],[5,-1],[6,-1]]
knots = [0,0,0,1,2,3,4,5,5,5]
bspline = BSPLINE(2)(knots)(controls)
obj = larMap(bspline)(larDom(knots))
VIEW(STRUCT( MKPOLS(obj) + [POLYLINE(controls)] ))
◇
```

Bezier curve as a B-spline curve

"test/py/splines/test09.py" 7a ≡
```
""" Bezier curve as a B-spline curve """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

controls = [[0,1],[0,0],[1,1],[1,0]]
bezier = larBezierCurve(controls)
dom = larIntervals([32])([1])
obj = larMap(bezier)(dom)
VIEW(STRUCT( MKPOLS(obj) + [POLYLINE(controls)] ))

knots = [0,0,0,0,1,1,1,1]
bspline = BSPLINE(3)(knots)(controls)
dom = larIntervals([100])([knots[-1]-knots[0]])
obj = larMap(bspline)(dom)
VIEW(STRUCT( MKPOLS(obj) + [POLYLINE(controls)] ))
◇
```

B-spline curve: effect of double or triple control points

"test/py/splines/test10.py" 7b ≡

```
""" B-spline curve: effect of double or triple control points """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

controls1 = [[0,0],[2.5,5],[6,1],[9,3]]
controls2 = [[0,0],[2.5,5],[2.5,5],[6,1],[9,3]]
controls3 = [[0,0],[2.5,5],[2.5,5],[2.5,5],[6,1],[9,3]]
knots = [0,0,0,0,1,1,1,1]
bspline1 = larMap( BSPLINE(3)(knots)(controls1) )(larDom(knots))
knots = [0,0,0,0,1,2,2,2,2]
bspline2 = larMap( BSPLINE(3)(knots)(controls2) )(larDom(knots))
knots = [0,0,0,0,1,2,3,3,3,3]
bspline3 = larMap( BSPLINE(3)(knots)(controls3) )(larDom(knots))

VIEW(STRUCT( CAT(AA(MKPOLS)([bspline1,bspline2,bspline3])) +
   [POLYLINE(controls1)]) )
◇
```

Periodic B-spline curve

"test/py/splines/test11.py" 8a ≡

```
""" Periodic B-spline curve """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

controls = [[0,1],[0,0],[1,0],[1,1],[0,1]]
knots = [0,0,0,1,2,3,3,3]           # non-periodic B-spline
bspline = BSPLINE(2)(knots)(controls)
obj = larMap(bspline)(larDom(knots))
VIEW(STRUCT( MKPOLS(obj) + [POLYLINE(controls)] ))

knots = [0,1,2,3,4,5,6,7]           # periodic B-spline
bspline = BSPLINE(2)(knots)(controls)
obj = larMap(bspline)(larDom(knots))
VIEW(STRUCT( MKPOLS(obj) + [POLYLINE(controls)] ))
◇
```

6 Computational framework

6.1 Exporting the library

"lib/py/splines.py" 8b ≡

```
""" Mapping functions and primitive objects """
⟨ Initial import of modules 12 ⟩
⟨ Tensor product surface patch 2a ⟩
⟨ Bilinear surface patch 2b ⟩
⟨ Biquadratic surface patch 3a ⟩
⟨ Bicubic surface patch 3b ⟩
⟨ Multidimensional transfinite Bernstein-Bezier Basis 1 ⟩
⟨ Multidimensional transfinite Bézier 3c ⟩
⟨ Transfinite Coons patches 4 ⟩
⟨ Domain decomposition for 1D bspline maps 6a ⟩
◇
```

7 Examples

Examples of larBernsteinBasis generation

⟨ Examples of larBernsteinBasis 9a ⟩ ≡

```
larBernsteinBasis(S1)(3)
""" [<function __main__.map_fn>,
   <function __main__.map_fn>,
   <function __main__.map_fn>,
   <function __main__.map_fn>] """
larBernsteinBasis(S1)(3)[0]
""" <function __main__.map_fn> """
larBernsteinBasis(S1)(3)[0]([0.0])
""" 1.0 """
◇
```

Macro never referenced.

Graph of Bernstein-Bezier basis

"test/py/splines/test04.py" 9b ≡

```
""" Graph of Bernstein-Bezier basis """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

def larBezierBasisGraph(degree):
   basis = larBernsteinBasis(S1)(degree)
   dom = larDomain([32])
   graphs = CONS(AA(larMap)(DISTL([S1, basis])))(dom)
   return graphs

graphs = larBezierBasisGraph(4)
VIEW(STRUCT( CAT(AA(MKPOLS)( graphs )) ))
◇
```

Some examples of curves

"test/py/splines/test01.py" 9c ≡

```
""" Example of Bezier curve """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

controlpoints = [[-0,0],[1,0],[1,1],[2,1],[3,1]]
dom = larDomain([32])
obj = larMap(larBezierCurve(controlpoints))(dom)
VIEW(STRUCT(MKPOLS(obj)))

obj = larMap(larBezier(S1)(controlpoints))(dom)
VIEW(STRUCT(MKPOLS(obj)))
◇
```

Transfinite cubic surface

"test/py/splines/test02.py" 10a ≡

```
""" Example of transfinite surface """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

dom = larDomain([20],'simplex')
C0 = larBezier(S1)([[0,0,0],[10,0,0]])
C1 = larBezier(S1)([[0,2,0],[8,3,0],[9,2,0]])
C2 = larBezier(S1)([[0,4,1],[7,5,-1],[8,5,1],[12,4,0]])
C3 = larBezier(S1)([[0,6,0],[9,6,3],[10,6,-1]])
dom2D = larExtrude1(dom,20*[1./20])
obj = larMap(larBezier(S2)([C0,C1,C2,C3]))(dom2D)
VIEW(STRUCT(MKPOLS(obj)))
◇
```

Coons patch interpolating 4 boundary curves

"test/py/splines/test03.py" 10b ≡
```
""" Example of transfinite Coons surface """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *
Su0 = larBezier(S1)([[0,0,0],[10,0,0]])
Su1 = larBezier(S1)([[0,10,0],[2.5,10,3],[5,10,-3],[7.5,10,3],[10,10,0]])
Sv0 = larBezier(S2)([[0,0,0],[0,0,3],[0,10,3],[0,10,0]])
Sv1 = larBezier(S2)([[10,0,0],[10,5,3],[10,10,0]])
dom = larDomain([20])
dom2D = larExtrude1(dom, 20*[1./20])
out = larMap(larCoonsPatch([Su0,Su1,Sv0,Sv1]))(dom2D)
VIEW(STRUCT(MKPOLS(out)))
◇
```

Bilinear tensor product patch

"test/py/splines/test05.py" 10c ≡
```
""" Example of bilinear tensor product surface patch """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

controlpoints = [
   [[0,0,0],[2,-4,2]],
   [[0,3,1],[4,0,0]]]
dom = larDomain([20])
dom2D = larExtrude1(dom, 20*[1./20])
mapping = larBilinearSurface(controlpoints)
patch = larMap(mapping)(dom2D)
VIEW(STRUCT(MKPOLS(patch)))
◇
```

Biquadratic tensor product patch

"test/py/splines/test06.py" 11a ≡
```
""" Example of bilinear tensor product surface patch """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *
```

```
controlpoints=[
   [[0,0,0],[2,0,1],[3,1,1]],
   [[1,3,-1],[2,2,0],[3,2,0]],
   [[-2,4,0],[2,5,1],[1,3,2]]]
dom = larDomain([20])
dom2D = larExtrude1(dom, 20*[1./20])
mapping = larBiquadraticSurface(controlpoints)
patch = larMap(mapping)(dom2D)
VIEW(STRUCT(MKPOLS(patch)))
◇
```

Bicubic tensor product patch

"test/py/splines/test07.py" 11b ≡

```
""" Example of bilinear tensor product surface patch """
import sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from splines import *

controlpoints=[
   [[ 0,0,0],[0 ,3  ,4],[0,6,3],[0,10,0]],
   [[ 3,0,2],[2 ,2.5,5],[3,6,5],[4,8,2]],
   [[ 6,0,2],[8 ,3 , 5],[7,6,4.5],[6,10,2.5]],
   [[10,0,0],[11,3  ,4],[11,6,3],[10,9,0]]]
dom = larDomain([20])
dom2D = larExtrude1(dom, 20*[1./20])
mapping = larBicubicSurface(controlpoints)
patch = larMap(mapping)(dom2D)
VIEW(STRUCT(MKPOLS(patch)))
◇
```

A Utility functions

Initial import of modules

⟨Initial import of modules 12⟩ ≡

```
from pyplasm import *
from scipy import *
import os,sys
""" import modules from larcc/lib """
sys.path.insert(0, 'lib/py/')
from lar2psm import *
from simplexn import *
from larcc import *
from largrid import *
```

```
    from mapper import *
    ◇
```

Macro referenced in 8b.

References

[CL13] CVD-Lab, *Linear algebraic representation*, Tech. Report 13-00, Roma Tre University, October 2013.

[Pao03] A. Paoluzzi, *Geometric programming for computer aided design*, John Wiley & Sons, Chichester, UK, 2003.