

# The basic `larcc` module \*

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February 15, 2016

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\*This document is part of the *Linear Algebraic Representation with CoChains* (LAR-CC) framework [\[CL13\]](#). February 15, 2016

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# 1 Basic representations

A few basic representation of topology are used in LARCC. They include some common sparse matrix representations: CSR (Compressed Sparse Row), CSC (Compressed Sparse Column), COO (Coordinate Representation), and BRC (Binary Row Compressed).

## 1.1 BRC (Binary Row Compressed)

We denote as BRC (Binary Row Compressed) the standard input representation of our LARCC framework. A BRC representation is an array of arrays of integers, with no requirement of equal length for the component arrays. The BRC format is used to represent a (normally sparse) binary matrix. Each component array corresponds to a matrix row, and contains the indices of columns that store a 1 value. No storage is used for 0 values.

**BRC format example** Let  $A = (a_{i,j} \in \{0,1\})$  be a binary matrix. The notation  $\text{BRC}(A)$  is used for the corresponding data structure.

$$A = \begin{pmatrix} 0, 1, 0, 0, 0, 0, 0, 1, 0, 0 \\ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 \\ 1, 0, 0, 1, 0, 0, 0, 0, 0, 1 \\ 1, 0, 0, 0, 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 0, 0, 1, 1, 1, 0, 0 \\ 0, 0, 1, 0, 1, 0, 0, 0, 1, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, 0, 0, 0, 1, 0, 1 \\ 0, 0, 0, 1, 0, 0, 0, 0, 1, 0 \\ 0, 1, 1, 0, 1, 0, 0, 0, 0, 0 \end{pmatrix} \mapsto \text{BRC}(A) = \begin{array}{l} [[1,7], \\ [2], \\ [0,3,9], \\ [0,6], \\ [5,6,7], \\ [2,4,8], \\ [], \\ [1,7,9], \\ [3,8], \\ [1,2,4]] \end{array}$$

## 1.2 Format conversions

**From triples to `scipy.sparse`** The function `brc2Coo` transforms a BRC representation in a list of triples (*row*, *column*, 1) ordered by row.

```

⟨Brc to Coo transformation 2⟩ ≡
def brc2Coo(ListOfListOfInt):
    COOm = [[k,col,1] for k,row in enumerate(ListOfListOfInt)
              for col in row ]
    return COOm

```

Macro referenced in [49b](#).

Two coordinate compressed sparse matrices `cooFV` and `cooEV` are created below, starting from the BRC representation `FV` and `EV` of the incidence of vertices on faces and edges, respectively, for a very simple plane triangulation.

⟨ Test example of Brc to Coo transformation 3a ⟩ ≡

```
print "\n>>> brc2Coo"
V = [[0, 0], [1, 0], [2, 0], [0, 1], [1, 1], [2, 1]]
FV = [[0, 1, 3], [1, 2, 4], [1, 3, 4], [2, 4, 5]]
EV = [[0,1],[0,3],[1,2],[1,3],[1,4],[2,4],[2,5],[3,4],[4,5]]
cooFV = brc2Coo(FV)
cooEV = brc2Coo(EV)
assert cooFV == [[0,0,1],[0,1,1],[0,3,1],[1,1,1],[1,2,1],[1,4,1],[2,1,1],
[2,3,1],[2,4,1],[3,2,1],[3,4,1],[3,5,1]]
assert cooEV == [[0,0,1],[0,1,1],[1,0,1],[1,3,1],[2,1,1],[2,2,1],[3,1,1],
[3,3,1],[4,1,1],[4,4,1],[5,2,1],[5,4,1],[6,2,1],[6,5,1],[7,3,1],[7,4,1],
[8,4,1],[8,5,1]]
◇
```

Macro referenced in 50.

**Conversion to csr format** Then we give the function `triples2mat` to make the transformation from the sparse matrix, given as a list of triples *row,column,value* (non-zero elements), to the `scipy.sparse` format corresponding to the `shape` parameter, set by default to "csr", that stands for *Compressed Sparse Row*, the normal matrix format of the LARCC framework.

⟨ From list of triples to scipy.sparse 3b ⟩ ≡

```
def triples2mat(triples,shape="csr"):
    n = len(triples)
    data = arange(n)
    ij = arange(2*n).reshape(2,n)
    for k,item in enumerate(triples):
        ij[0][k],ij[1][k],data[k] = item
    return scipy.sparse.coo_matrix((data, ij)).asformat(shape)
◇
```

Macro referenced in 49b.

The conversion from triples to `csr` format is provided below.

⟨ Coo to Csr transformation 3c ⟩ ≡

```
def coo2Csr(COOm):
    CSRm = triples2mat(COOm,"csr")
    return CSRm
◇
```

Macro referenced in 49b.

Two CSR sparse matrices `csrFV` and `csrEV` are generated (by `scipy.sparse`) in the following example:

⟨ Test example of Coo to Csr transformation 3d ⟩ ≡

```

csrFV = coo2Csr(cooFV)
csrEV = coo2Csr(cooEV)
print "\ncsr(FV) =\n", repr(csrFV)
print "\ncsr(EV) =\n", repr(csrEV)
◇

```

Macro referenced in 50.

The *scipy* printout of the last two lines above is the following:

```

csr(FV) = <4x6 sparse matrix of type '<type 'numpy.int64'>'
        with 12 stored elements in Compressed Sparse Row format>
csr(EV) = <9x6 sparse matrix of type '<type 'numpy.int64'>'
        with 18 stored elements in Compressed Sparse Row format>

```

**Conversion from BRC to CSR format** The transformation from BRC to CSR format is implemented slightly differently, according to the fact that the matrix dimension is either unknown (`shape=(0,0)`) or known.

```

⟨Brc to Csr transformation 4a⟩ ≡
def csrCreate(BRCmatrix,lenV=0,shape=(0,0)):
    triples = brc2Coo(BRCmatrix)
    if shape == (0,0):
        CSRmatrix = coo2Csr(triples)
    else:
        CSRmatrix = scipy.sparse.csr_matrix(shape)
        for i,j,v in triples: CSRmatrix[i,j] = v
    return CSRmatrix
◇

```

Macro referenced in 49b.

**Example** The conversion to CSR format of the characteristic matrix *faces-vertices* **FV** is given below for our simple example made by four triangle of a manifold 2D space, graphically shown in Figure 1a. The LAR representation with CSR matrices does not make difference between manifolds and non-manifolds, conversely than most modern solid modelling representation schemes, as shown by removing from **FV** the third triangle, giving the model in Figure 1b.

```

⟨Test example of Brc to Csr transformation 4b⟩ ≡
print "\n>>> brc2Csr"
V = [[0, 0], [1, 0], [2, 0], [0, 1], [1, 1], [2, 1]]
FV = [[0, 1, 3], [1, 2, 4], [1, 3, 4], [2, 4, 5]]
EV = [[0,1],[0,3],[1,2],[1,3],[1,4],[2,4],[2,5],[3,4],[4,5]]
csrFV = csrCreate(FV)
csrEV = csrCreate(EV)
print "\ncsrCreate(FV) =\n", csrFV

```

```
VIEW(STRUCT(MKPOLS((V,FV))))
VIEW(STRUCT(MKPOLS((V,EV))))
```

◇

Macro referenced in [7](#), [50](#).



Figure 1: (a) Simplicial 2-complex; (b) its 1-skeleton.

## 2 Matrix operations

As we know, the LAR representation of topology is based on CSR representation of sparse binary (and integer) matrices. In this section we hence discuss the stack of matrix representations and operations implemented by this module. The current python prototype makes reference to the scipy implementation of sparse matrices. Later implementations in different languages will necessarily make reference to different matrix packages.

### 2.1 Basic operations

Two utility functions allow to query the number of rows and columns of a CSR matrix, independently from the low-level implementation (that in the following is provided by *scipy.sparse*).

```
< Query Matrix shape 5a > ≡
def csrGetNumberOfRows(CSRmatrix):
    Int = CSRmatrix.shape[0]
    return Int

def csrGetNumberOfColumns(CSRmatrix):
    Int = CSRmatrix.shape[1]
    return Int
◇
```

Macro referenced in [49b](#).

```
< Test examples of Query Matrix shape 5b > ≡
```

```

print "\n>>> csrGetNumberOfRows"
print "\ncsrGetNumberOfRows(csrFV) =", csrGetNumberOfRows(csrFV)
print "\ncsrGetNumberOfRows(csrEV) =", csrGetNumberOfRows(csrEV)
print "\n>>> csrGetNumberOfColumns"
print "\ncsrGetNumberOfColumns(csrFV) =", csrGetNumberOfColumns(csrFV)
print "\ncsrGetNumberOfColumns(csrEV) =", csrGetNumberOfColumns(csrEV)

```

Macro referenced in 50.

**Sparse to dense matrix transformation** The Scipy package provides the useful method `.todense()` in order to transform any sparse matrix format in the corresponding dense format. The function `csr2DenseMatrix` is given here for the sake of generality and portability.

```

⟨Sparse to dense matrix transformation 6a⟩ ≡
def csr2DenseMatrix(CSRm):
    nrows = csrGetNumberOfRows(CSRm)
    ncolumns = csrGetNumberOfColumns(CSRm)
    ScipyMat = zeros((nrows,ncolumns),int)
    C = CSRm.tocoo()
    for triple in zip(C.row,C.col,C.data):
        ScipyMat[triple[0],triple[1]] = triple[2]
    return ScipyMat

```

Macro referenced in 49b.

```

⟨Test examples of Sparse to dense matrix transformation 6b⟩ ≡
print "\n>>> csr2DenseMatrix"
print "\nFV =\n", csr2DenseMatrix(csrFV)
print "\nEV =\n", csr2DenseMatrix(csrEV)

```

Macro referenced in 7, 50.

**Matrix product and transposition** The following macro provides the IDE interface for the two main matrix operations required by LARCC, the binary product of compatible matrices and the unary transposition of matrices.

```

⟨Matrix product and transposition 6c⟩ ≡
def matrixProduct(CSRm1,CSRm2):
    CSRm = CSRm1 * CSRm2
    return CSRm

def csrTranspose(CSRm):
    CSRm = CSRm.T
    return CSRm

```

Macro referenced in 49b.

## 2.2 Characteristic matrices

We define as *characteristic matrices*  $M_k$  ( $0 \leq k \leq d$ ) the binary matrices having as rows the images of the characteristic functions of the  $k$ -cells  $\alpha_k \subset V$  of a cellular complex with vertices  $V$ . Remember that characteristic (or *indicator*) function is

$$\mathbf{1}_A: V \rightarrow \{0, 1\},$$

which for a given subset  $A$  of  $X$ , has value 1 at points of  $A$  and 0 at points of  $V - A$ .

**Example: from BRC to CSR to dense matrix** Let us compute and show in dense form the characteristic matrices of 2- and 1-cells of the simple manifold given in Figure 1. By running the file `test/py/larcc/test08.py` the reader will get the two matrices shown in Example 2

```
"test/py/larcc/test08.py" 7 ≡
    """ Characteristic matrices """
    from larlib import *

    < Test example of Brc to Csr transformation 4b >
    < Test examples of Sparse to dense matrix transformation 6b >
    ◇
```

**Example 1** (Dense Characteristic matrices). *Let us notice that the two matrices below have the same numbers of columns (indexed by vertices of the cell decomposition). This very fact allows to multiply one matrix for the other transposed, and hence to compute the matrix form of linear operators between the spaces of cells of various dimensions.*

$$\begin{array}{rcl}
 & & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \\
 FV = & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} & EV =
 \end{array}$$

**Example 2** (Operators from edges to faces and vice-versa). *As a general rule for operators between two spaces of chains of different dimensions supported by the same cellular complex, we use names made by two characters, whose first letter correspond to the target space, and whose second letter to the domain space. Hence FE must be read as the operator from edges to faces. Of course, since this use correspond to see the first letter as the space generated by rows, and the second letter as the space generated by columns. Notice that the element*



$(i, j)$  of such matrices stores the number of vertices shared between the (row-)cell  $i$  and the (column-)cell  $j$ .

$$FE = FV EV^\top = \begin{bmatrix} 2 & 2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 2 \end{bmatrix}$$

$$EF = EV FV^\top = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

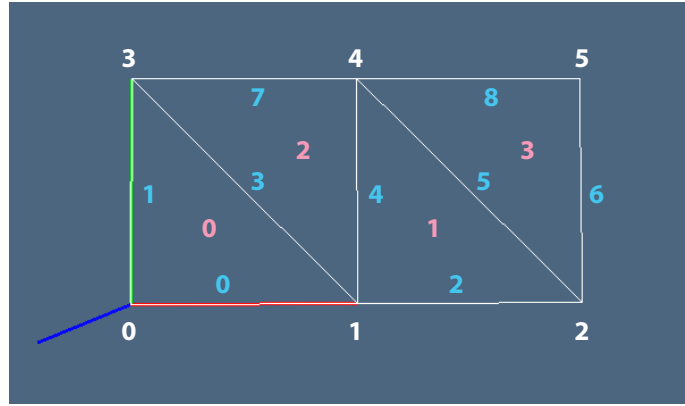


Figure 2: example caption

**Matrix elements filtering** Some filtering operations on matrix elements are needed in the implementation of various topological operators. Some of such filtering operations are given below.

⟨Matrix filtering to produce the boundary matrix 8⟩ ≡

```
def csrBoundaryFilter(CSRm, facetLengths):
    maxs = [max(CSRm[k].data) for k in range(CSRm.shape[0])]
    inputShape = CSRm.shape
    coo = CSRm.tocoo()
    for k in range(len(coo.data)):
        if coo.data[k] == maxs[coo.row[k]]: coo.data[k] = 1
        else: coo.data[k] = 0
    mtx = coo_matrix((coo.data, (coo.row, coo.col)), shape=inputShape)
    out = mtx.tocsr()
```

```
    return out
```

◇

Macro referenced in 49b.

⟨ Test example of Matrix filtering to produce the boundary matrix 9a ⟩ ≡

```
print "\n>>> csrBoundaryFilter"
csrEF = matrixProduct(csrFV, csrTranspose(csrEV)).T
facetLengths = [csrCell.getnnz() for csrCell in csrEV]
CSRm = csrBoundaryFilter(csrEF, facetLengths).T
print "\ncsrMaxFilter(csrFE) =\n", csr2DenseMatrix(CSRm)
```

◇

Macro referenced in 50.

⟨ Matrix filtering via a generic predicate 9b ⟩ ≡

```
def csrPredFilter(CSRm, pred):
    # can be done in parallel (by rows)
    coo = CSRm.tocoo()
    triples = [[row,col,val] for row,col,val
                in zip(coo.row,coo.col,coo.data) if pred(val)]
    i, j, data = TRANS(triples)
    CSRm = scipy.sparse.coo_matrix((data,(i,j)),CSRm.shape).tocsr()
    return CSRm
```

◇

Macro referenced in 49b.

⟨ Test example of Matrix filtering via a generic predicate 9c ⟩ ≡

```
print "\n>>> csrPredFilter"
CSRm = csrPredFilter(matrixProduct(csrFV, csrTranspose(csrEV)).T, GE(2)).T
print "\nccsrPredFilter(csrFE) =\n", csr2DenseMatrix(CSRm)
```

◇

Macro referenced in 50.

**Correction to boundary operator** The more deep issue with LAR (that already appeared at the beginning of LAR experiments, two years ago) is discussed and solved here.

The boundary operator (and the whole LAR approach) works well with convex cells, but MAY give errors (like  $\partial\partial \neq 0$ ) with non-convex cells. At that time, we understood the source of the problem, due to the fact that ALL the vertices of a  $(d-1)$ -cell EXTERNAL to a NON-CONVEX  $d$ -cell may belong to its boundary ... (e.g. BOTH the vertices of a 1-cell external to a non-convex 2-cell)

You might say: convexity is a geometric concept, boundary is topological, so what? Because the input is certainly geometrical, via drawings and/or machine generated EV,FV starting from drawings, so the two aspects are strongly interconnected ...

We implement here a general resolution strategy, since it is pretty easy to discover the (d-1)-facets where the problem appears: they are shared by MORE than TWO d-cells ... and this one is a topological ERROR. So, testing  $\partial\partial$  on each face, we eliminate the problem *within the boundary operator*, but not within the LAR data structure (to be marked in some way where the problem may appear ... TODO).

My guess is that this source of errors might appear anytime the Boolean input becomes pretty complex, and that it was the greatest source of random errors.

```
"test/py/larcc/test20.py" 10 ≡
    """ Correction to boundary operator """
    from larlib import *

    V,[VV,EV,FV] = larCuboids([2,1],True)
    complex = Struct([(V,FV,EV), t(.5,.25), s(.5,.5), (V,FV,EV)])
    V,FV,EV = struct2lar(complex)
    lines = [[V[v] for v in edge] for edge in EV]
    V,FV,EV,polygons = larFromLines(lines)

    VV = AA(LIST)(range(len(V)))
    submodel = STRUCT(MKPOLS((V,EV)))
    VIEW(larModelNumbering(1,1,1)(V,[VV,EV,FV],submodel,.5))

    csrBoundaryMat = boundary(FV,EV)
    print "wrong boundary matrix =",csrBoundaryMat.todense()
    csrBoundaryMat = boundary1(FV,EV,VV) # <<<<< NOTE !!
    print "right boundary matrix =",csrBoundaryMat.todense()

    ⟨2-boundary example 11, ... ⟩

    submodel = viewBoundaryChain((V,FV,EV))([1,1,1,1])
    VIEW(larModelNumbering(1,1,1)(V,[VV,EV,FV],submodel,.5))
    submodel = viewBoundaryChain((V,FV,EV))([0,0,1,1])
    VIEW(larModelNumbering(1,1,1)(V,[VV,EV,FV],submodel,.5))
    ◇
```

The printing from example `test/py/larcc/test20.py` is given below. Notice the difference between the fourth lines from the bottom of the two matrices, generated by the function applications `boundary(FV,EV)` and `boundary1(FV,EV,VV)`, respectively.

<pre> wrong boundary matrix = [   [0 0 1 0]   [0 0 1 1]   [0 1 1 0]   [0 1 1 0]   [0 0 1 0]   [0 0 0 1]   [1 0 0 1]   [0 0 0 1]   [1 0 0 1]   [0 0 1 0]   [0 0 0 1]   [1 1 1 1]   [1 0 0 1]   [0 1 1 0]   [0 0 1 1]] </pre>	<pre> right boundary matrix = [   [0 0 1 0]   [0 0 1 1]   [0 1 1 0]   [0 1 1 0]   [0 0 1 0]   [0 0 0 1]   [1 0 0 1]   [0 0 0 1]   [1 0 0 1]   [0 0 1 0]   [0 0 0 1]   [0 0 0 0]   [1 0 0 1]   [0 1 1 0]   [0 0 1 1]] </pre>
---	---

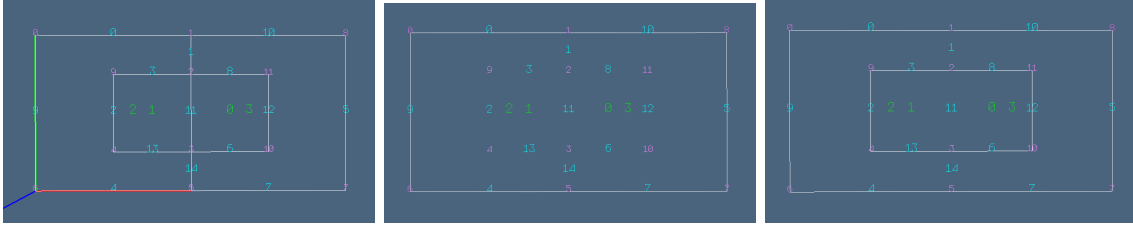


Figure 3: (a) A 2-complex with non-convex cells; (b) boundary of 2-chain  $[1,1,1,1]$ ; (c) boundary of 2-chain  $[0,0,1,1]$ .

Whereas the boundary 2-chain is

```

<2-boundary example 11> ≡
""" View boundary chain """
def viewBoundaryChain(larModel):
    V,FV,EV = larModel
    VV = AA(LIST)(range(len(V)))
    def viewBoundaryChain0 (chain):
        BE = chain2BoundaryChain(boundary1(FV,EV,VV))(chain)
        if chain == [1,1,1,1]: assert BE == [0,4,5,7,9,10]
        if chain == [0,0,1,1]: assert BE == [0,2,3,4,5,6,7,8,9,10,12,13]
        BEV = [EV[e] for e in BE]
        if chain == [1,1,1,1]: assert BEV == [(0,1),(5,6),(7,8),(5,7),(0,6),(1, 8)]
        submodel = STRUCT(MKPOLS((V,BEV)))
        return submodel
    return viewBoundaryChain0

```

◇

Macro defined by [11](#), [12a](#).  
Macro referenced in [10](#).

The cellular 2-complex generated by example The printing from example `test/py/larcc/test20.py` is displayed in Figure ?? . The corresponding LAR is the following:

```

<2-boundary example 12a> ≡
  V = [[0.0,1.0],[1.0,1.0],[1.0,0.75],[1.0,0.25],[0.5,0.25],[1.0,0.0],
        [0.0,0.0],[2.0,0.0],[2.0,1.0],[0.5,0.75],[1.5,0.25],[1.5,0.75]]

  FV = [[3,2,11,10],[9,2,3,4],[1,2,9,4,3,5,6,0],[1,8,7,5,3,10,11,2]]

  EV = [(0,1),(1,2),(4,9),(2,9),(5,6),(7,8),(3,10),(5,7),(2,11),(0,6),
        (1,8),(2,3),(10,11),(3,4),(3,5)]
  ◇

```

Macro defined by [11](#), [12a](#).  
Macro referenced in [10](#).

## 2.3 Boundary operator on 3-complex with non-convex cells

### Correction to boundary operator

```

"test/py/larcc/test21.py" 12b ≡
  """ Correction to boundary operator """
  from larlib import *

  V = [[0.0,0.0,0.0],[0.0,0.0,1.0],[0.0,1.0,0.0],[0.0,1.0,1.0],[1.0,0.0,0.0],
        [1.0,0.0,1.0],[1.0,1.0,0.0],[1.0,1.0,1.0],[2.0,0.0,0.0],[2.0,0.0,1.0],
        [2.0,1.0,0.0],[2.0,1.0,1.0],[0.5,0.25,0.25],[0.5,0.25,0.75],[0.5,0.75,
        0.25],[0.5,0.75,0.75],[1.0,0.25,0.25],[1.0,0.25,0.75],[1.0,0.75,0.25],
        [1.0,0.75,0.75],[1.5,0.25,0.25],[1.5,0.25,0.75],[1.5,0.75,0.25],[1.5,
        0.75,0.75]]

  CV = [(16,17,18,19,20,21,22,23),(4,5,6,7,8,9,10,11,16,17,18,19,20,21,
        22,23),(12,13,14,15,16,17,18,19),(0,1,2,3,4,5,6,7,12,13,14,15,16,
        17,18,19)]

  FV = [(0,2,4,6),(12,13,14,15),(4,6,8,10),(0,1,4,5),(16,17,18,19),(4,5,6,
        7,16,17,18,19),(17,19,21,23),(16,17,20,21),(0,1,2,3),(18,19,22,23),
        (14,15,18,19),(1,3,5,7),(12,13,16,17),(12,14,16,18),(13,15,17,19),
        (4,5,8,9),(2,3,6,7),(16,18,20,22),(5,7,9,11),(6,7,10,11),(8,9,10,11),
        (20,21,22,23)]

  EV = [[0,1],[2,3],[4,5],[6,7],[8,9],[10,11],[0,2],[1,3],[4,6],[5,7],
        [8,10],[9,11],[0,4],[1,5],[2,6],[3,7],[4,8],[5,9],[6,10],[7,11],[12,13],
        [14,15],[16,17],[18,19],[20,21],[22,23],[12,14],[13,15],[16,18],[17,19],
        [20,22],[21,23],[12,16],[13,17],[14,18],[15,19],[16,20],[17,21],[18,22],
        [19,23]]

```

```

csrBoundaryMat = boundary(CV,FV)
print "wrong boundary matrix =",csrBoundaryMat.todense()
csrBoundaryMat = boundary1(CV,FV,EV)
print "right boundary matrix =",csrBoundaryMat.todense()

VV = AA(LIST)(range(len(V)))
submodel = STRUCT(MKPOLS((V,EV)))
VIEW(larModelNumbering(1,1,1)(V,[VV,EV,FV,CV],submodel,.5))
◇

```

**Testing correction to boundary operator for general (non-convex) LAR** The test example exported in `test/py/larcc/test22.py` is aimed to check the working of LAR on chain complexes supported by cellular decompositions with non-convex but connected cells, possibly of topological genus  $> 1$ , i.e. with more than one hole in some face.

```

"test/py/larcc/test22.py" 13a ≡
    """ Testing correction to boundary operator for general (non-convex) LAR """
    from larlib import *

    ⟨Input and visualization of a general cellular complex 13b⟩
    ⟨Visualization of the skeletons 15a⟩
    ⟨Computation of corrected boundary operator 15b⟩
    ⟨Visualization of some boundary chains 16⟩
    ◇

```

### Input and visualization of a general cellular complex

```

⟨Input and visualization of a general cellular complex 13b⟩ ≡
    """ Input and visualization of a general cellular complex """

    mod1 = larQuote1D([0.2,-0.2,0.2])
    mod2 = larQuote1D([0.5,0.5])
    mod3 = larQuote1D([0.5])
    mod = larModelProduct([larModelProduct([mod2,mod1]),mod3])

    mx = larQuote1D([1,1])
    my = larQuote1D([1])
    m = larModelProduct([larModelProduct([mx,my]),my])

    complex = Struct([m, t(.5,.2,.25), mod])
    V,CV0 = struct2lar(complex)
    CV = copy(CV0)
    CV[1] += CV[0] + CV[4]
    CV[3] += CV[2] + CV[5]
    VIEW(SKEL_1(STRUCT(MKPOLS((V,CV)))))
    ◇

```

Macro referenced in 13a.

The LAR of the cellular decomposition shown in Figure ?? is given in Figure 4.

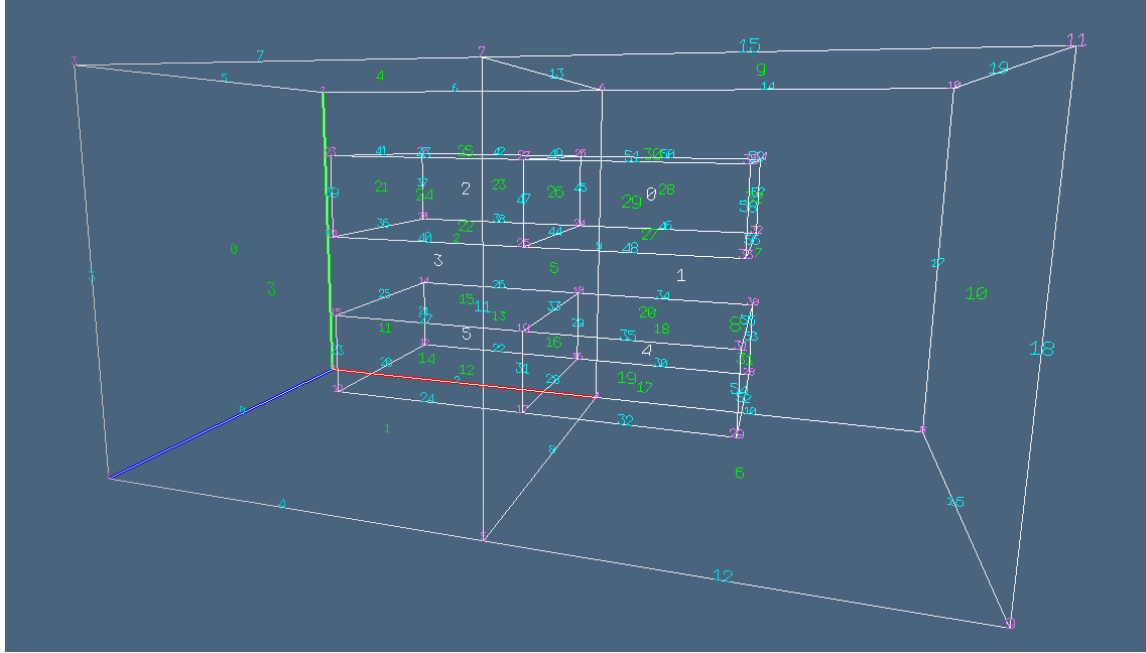


Figure 4: The cellular 3-complex defined below. Cells of different dimension are numbered in different colours.

We have 36 vertices, indexed from 0 to 35, and six 3-cells, numbered from 0 to 5. Notice that the subsets of integers  $CV[1]$  and  $CV[3]$  provide the description of the two non-convex cells.

$V = [[0.0,0.0,0.0],[0.0,0.0,1.0],[0.0,1.0,0.0],[0.0,1.0,1.0],[1.0,0.0,0.0],$   
 $[1.0,0.0,1.0],[1.0,1.0,0.0],[1.0,1.0,1.0],[2.0,0.0,0.0],[2.0,0.0,1.0],$   
 $[2.0,1.0,0.0],[2.0,1.0,1.0],[0.5,0.2,0.25],[0.5,0.2,0.75],[0.5,0.4,0.25],$   
 $[0.5,0.4,0.75],[1.0,0.2,0.25],[1.0,0.2,0.75],[1.0,0.4,0.25],[1.0,0.4,0.75],$   
 $[0.5,0.6,0.25],[0.5,0.6,0.75],[0.5,0.8,0.25],[0.5,0.8,0.75],[1.0,0.6,0.25],$   
 $[1.0,0.6,0.75],[1.0,0.8,0.25],[1.0,0.8,0.75],[1.5,0.2,0.25],[1.5,0.2,0.75],$   
 $[1.5,0.4,0.25],[1.5,0.4,0.75],[1.5,0.6,0.25],[1.5,0.6,0.75],[1.5,0.8,0.25],$   
 $[1.5,0.8,0.75]]$

$CV = [(24,25,26,27,32,33,34,35),(4,5,6,7,8,9,10,11,24,25,26,27,$   
 $32,33,34,35,16,17,18,19,28,29,30,31),(20,21,22,23,24,25,26,27),$   
 $(0,1,2,3,4,5,6,7,20,21,22,23,24,25,26,27,12,13,14,15,16,17,18,19),$   
 $(16,17,18,19,28,29,30,31),(12,13,14,15,16,17,18,19)]$

## Computation and visualization of the skeletons

⟨ Visualization of the skeletons 15a ⟩ ≡

```

""" Visualization of the skeletons """

def cubeFV(verts):
    [a,b,c,d, e,f,g,h] = verts
    return [[a,b,c,d],[e,f,g,h],[a,c,e,g],[b,d,f,h],[a,b,e,f],[c,d,g,h]]

FV = COMP([sorted,set,AA(tuple),CAT,AA(cubeFV)])(CV0)
V,EV = larFacets((V,FV),dim=2)
VV = AA(LIST)(range(len(V)))
submodel = STRUCT(MKPOLS((V,EV)))
VIEW(larModelNumbering(1,1,1)(V,[VV,EV,FV,CV],submodel,.25))
FV[5] += FV[16] + FV[26]
assert 30==len(boundaryCells(CV0,FV)) # assembly of 3 unrelated 3-complexes
◇

```

Macro referenced in [13a](#).

```

FV = [(0,1,2,3),(0,1,4,5),(0,2,4,6),(1,3,5,7),(2,3,6,7),
(4,5,6,7,16,17,18,19,24,25,26,27),
(4,5,8,9),(4,6,8,10),(5,7,9,11),(6,7,10,11),(8,9,10,11),(12,13,14,15),
(12,13,16,17),(12,14,16,18),(13,15,17,19),(14,15,18,19),(16,17,18,19),
(16,17,28,29),(16,18,28,30),(17,19,29,31),(18,19,30,31),(20,21,22,23),
(20,21,24,25),(20,22,24,26),(21,23,25,27),(22,23,26,27),(24,25,26,27),
(24,25,32,33),(24,26,32,34),(25,27,33,35),(26,27,34,35),(28,29,30,31),
(32,33,34,35)]

```

**Computation of corrected boundary operator** Of course, `totalChain = len(CV)*[1]` is the binary coordinate `[1,1,1,1,1,1]` of the cellular chain made by all the six 3-cells in `CV`. Analogously, `BF` is the list of integer indices of boundary 2-faces, corresponding to the coordinate representation given below, i.e. to the binary string of length 33, equating the size of the 2-skeleton of the cellular complex.

```

BF = [0, 1, 2, 3, 4, 6, 7, 8, 9, 10]
\texttt{[1,1,1,1,1,0,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]}

```

Conversely, `BEV` is (a superset of) the 1-skeleton cells of the boundary, according to the image in Figure ??.

⟨ Computation of corrected boundary operator 15b ⟩ ≡

```

""" Computation of corrected boundary operator """

totalChain = len(CV)*[1]

```



```

BF = chain2BoundaryChain(boundary1(CV,FV,EV))(totalChain)
BFV = [FV[f] for f in BF]
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,BFV))))
glass = MATERIAL([1,0,0,0.1, 0,1,0,0.1, 0,0,1,0.1, 0,0,0,0.1, 100])
VIEW(STRUCT(AA(glass)(MKPOLs((V,BFV))))))
VIEW(glass(EXPLODE(1.2,1.2,1.2)(MKPOLs((V,BFV))))))
V,BEV = larFacets((V,BFV),dim=1)
VIEW(EXPLODE(1.2,1.2,1.2)(MKPOLs((V,[e for e in BEV if len(e)==2]))))
◇

```

Macro referenced in [13a](#).

## Visualization of some boundary chains

```

⟨ Visualization of some boundary chains 16 ⟩ ≡
    """ Visualization of some boundary chains """

    BF = chain2BoundaryChain(boundary1(CV,FV,EV))([1,0,1,1,0,0])
    BFV = [FV[f] for f in BF]
    V,BEV = larFacets((V,BFV),dim=1)
    BEV = [e for e in BEV if len(e)==2]
    VIEW(glass(STRUCT(MKPOLs((V,BFV))))))
    submodel = STRUCT(MKPOLs((V,BEV)))
    VIEW(larModelNumbering(1,1,1)(V,[VV,BEV,BFV],submodel,.25))

    obj = Struct([(V,BFV,BEV)])
    W,FW,EW = struct2lar(obj)
    VIEW(EXPLODE(1.2,1.2,1.2)(MKPOLs((W,EW))))
    VIEW(SKEL_1(EXPLODE(1.2,1.2,1.2)(MKTRIANGLES((W,FW,EW))))))

    ◇

```

Macro referenced in [13a](#).

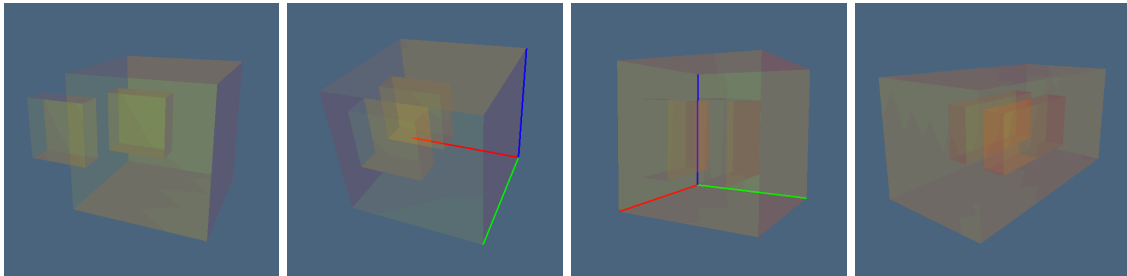


Figure 5: example caption

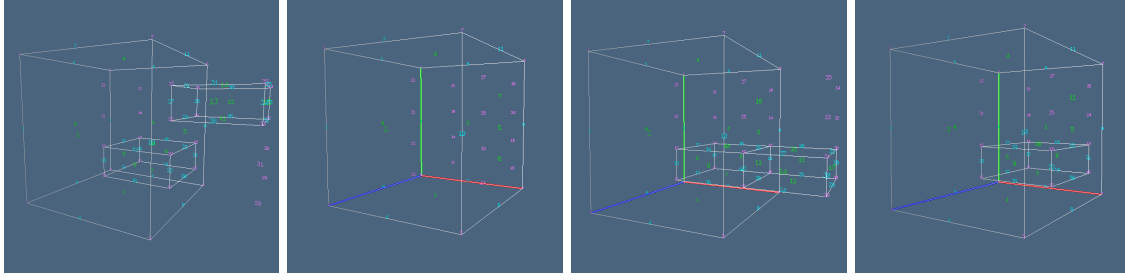


Figure 6: example caption

**Relational inversion (characteristic matrix transposition)** The operation could be executed by simple matrix transposition of the CSR (Compressed Sparse Row) representation of the sparse characteristic matrix  $M_d \equiv CV$ . A simple relational inversion using Python lists is given here. The `invertRelation` function is given here, linear in the size of the `CV` list, where the complexity of each cell is constant and small in most cases.

```
< Characteristic matrix transposition 17 > ≡
    """ Characteristic matrix transposition """
    def invertRelation(CV):
        def myMax(List):
            if List==[]: return -1
            else: return max(List)

        columnNumber = max(AA(myMax)(CV))+1
        VC = [[] for k in range(columnNumber)]
        for k,cell in enumerate(CV):
            for v in cell: VC[v] += [k]
        return VC
    ◇
```

Macro referenced in [49b](#).

## 2.4 Computation of lower-dimensional skeletons

In most cases, in particular when the cellular complex is made by convex cells, the only cells of maximal dimension must be entered to gain a complete knowledge of the whole complex. Here we show how to compute the  $(d - 1)$ -skeleton of a complex starting from its  $d$ -dimensional skeleton.

**Extraction of facets of a cell complex** The following `larFacets` function returns the LAR model `v,cellFacets` starting from the input `model` parameter. Two optional parameters define the (intrinsic) dimension of the input cells, with default value equal to three, and the eventual presence of a `emptyCellNumber` of empty cells. Their number

default to zero when the complex is closed, for example in the case it provides the  $d$ -boundary of a  $(d+1)$ -complex. If empty cells are present, their subset must be located at the end of the `cell` list.

⟨ Extraction of facets of a cell complex 18a ⟩  $\equiv$

```
def setup(model,dim):
    V, cells = model
    csr = csrCreate(cells)
    csrAdjSquareMat = larCellAdjacencies(csr)
    csrAdjSquareMat = csrPredFilter(csrAdjSquareMat, GE(dim)) # ? HOWTODO ?
    return V,cells,csr,csrAdjSquareMat

def larFacets(model, dim=3, emptyCellNumber=0):
    """ Estraction of (d-1)-cellFacets from "model" := (V,d-cells)
        Return (V, (d-1)-cellFacets)
    """
    V,cells,csr,csrAdjSquareMat = setup(model,dim)
    solidCellNumber = len(cells) - emptyCellNumber
    cellFacets = []
    # for each input cell i
    for i in range(len(cells)):
        adjCells = csrAdjSquareMat[i].tocoo()
        cell1 = csr[i].tocoo().col
        pairs = zip(adjCells.col,adjCells.data)
        for j,v in pairs:
            if (i<j) and (i<solidCellNumber):
                cell2 = csr[j].tocoo().col
                cell = list(set(cell1).intersection(cell2))
                cellFacets.append(sorted(cell))
    # sort and remove duplicates
    cellFacets = sorted(AA(list)(set(AA(tuple)(cellFacets))))
    return V,cellFacets
```

◇

Macro referenced in 49b.

⟨ Computation of cell adjacencies 18b ⟩  $\equiv$

```
def larCellAdjacencies(CSRm):
    CSRm = matrixProduct(CSRm,csrTranspose(CSRm))
    return CSRm
```

◇

Macro referenced in 49b.

**Examples** Two simple complexes are defined below by providing the pair  $V, FV$ . In both cases the EV relation is computed via the `larFacets` function.

```

⟨ Test examples of Extraction of facets of a cell complex 19a ⟩ ≡
    """ A first (simplicial) example """
    V = [[0.,0.],[3.,0.],[0.,3.],[3.,3.],[1.,2.],[2.,2.],[1.,1.],[2.,1.]]
    FV = [[0,1,3],[1,2,4],[2,4,5],[3,4,6],[4,6,7],[5,7,8], # full
          [1,3,4],[4,5,7], # empty
          [0,1,2],[6,7,8],[0,3,6],[2,5,8]] # exterior
    _,EV = larFacets((V,FV),dim=2)
    print "\nEV =",EV
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,EV))))

    """ Another (cuboidal) example """
    FV = [[0,1,6,7],[0,2,4,6],[4,5,6,7],[1,3,5,7],[2,3,4,5],[0,1,2,3]]
    _,EV = larFacets((V,FV),dim=2)
    print "\nEV =",EV
    VV = AA(LIST)(range(len(V)))
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,EV))))
    ◇

```

Macro referenced in 50.

**Visualization of cell numbers** The adjacency matrices between 2-cells and 1-cells are printed here. Finally, the complex is displayed by numbering with different colours and sizes (depending on the rank) the complex cells.

```

⟨ Test examples of Computation of cell adjacencies 19b ⟩ ≡

    print "\n>>> larCellAdjacencies"
    adj_2_cells = larCellAdjacencies(csrCreate(FV))
    print "\nadj_2_cells =\n", csr2DenseMatrix(adj_2_cells)
    adj_1_cells = larCellAdjacencies(csrCreate(EV))
    print "\nadj_1_cells =\n", csr2DenseMatrix(adj_1_cells)

    submodel = mkSignedEdges((V,EV))
    VIEW(submodel)
    VIEW(larModelNumbering(scalx=1,scaly=1,scalz=1)(V,[VV,EV,FV],submodel,2))
    ◇

```

Macro referenced in 50.

### 3 Topological operations

In this section we provide the matrix representation of operators to compute the more important and useful topological operations on cellular complexes, and/or the indexed relations they return. We start the section by giving a graphical tool used to test the developed software, concerning the graphical writing of the full set of indices of the cells of every dimension in a 3D cuboidal complex.

### 3.1 Visualization of cellular complexes

It is often necessary to have a visual picture of the generated structures and computations. This section provides some quite versatile visualisation tools of both the cells and/or their integer indices.

**Visualization of cell indices** As already outlined, the `modelIndexing` function return the *hpc* value assembling both the 1-skeletons of the cells of every dimensions, and the graphical output of their indices, located on the centroid of each cell, and displayed using colors and sizes depending on the *rank* of the cell.

⟨ Visualization of cell indices 20a ⟩ ≡

```
""" Visualization of cell indices """
from larlib import *

def modelIndexing(shape):
    V, bases = larCuboids(shape,True)
    # bases = [[cell for cell in cellComplex if len(cell)==2**k] for k in range(4)]
    color = [ORANGE,CYAN,GREEN,WHITE]
    nums = AA(range)(AA(len)(bases))
    hpcs = []
    for k in range(4):
        hpcs += [SKEL_1(STRUCT(MKPOLS((V,bases[k]))))]
        hpcs += [cellNumbering((V,bases[k]),hpcs[2*k])(nums[k],color[k],0.3+0.2*k)]
    return STRUCT(hpcs)
```

◇

Macro defined by [20ab](#).  
Macro referenced in [49b](#).

⟨ Visualization of cell indices 20b ⟩ ≡

```
""" Numbered visualization of a LAR model """
def larModelNumbering(scalx=1,scaly=1,scalz=1):
    def larModelNumbering0(V,bases,submodel,numberScaling=1):
        color = [ORANGE,CYAN,GREEN,WHITE]
        nums = AA(range)(AA(len)(bases))
        hpcs = [submodel]
        for k in range(len(bases)):
            hpcs += [cellNumbering((V,bases[k]),submodel)
                    (nums[k],color[k],(0.5+0.1*k)*numberScaling)]
        return STRUCT(hpcs)
    #return EXPLODE(scalx,scaly,scalz)(hpcs)
    return larModelNumbering0
```

◇

Macro defined by [20ab](#).  
Macro referenced in [49b](#).

**Drawing of oriented edges** The following function return the `hpc` of the drawing with arrows of the oriented 1-cells of a 2D cellular complex. Of course, each edge orientation is from second to first vertex, independently from the vertex indices. Therefore, the edge orientation can be reversed by swapping the vertex indices in the 1-cell definition.

```

⟨Drawing of oriented edges 21a⟩ ≡
    """ Drawing of oriented edges (2D) """
    def mkSignedEdges (model,scalingFactor=1):
        V,EV = model
        assert len(V[0])==2
        hpcs = []
        times = C(SCALARVECTPROD)
        frac = 0.06*scalingFactor
        for e0,e1 in EV:
            v0,v1 = V[e0], V[e1]
            vx,vy = DIFF([ v1, v0 ])
            nx,ny = [-vy, vx]
            v2 = SUM([ v0, times(0.66)([vx,vy]) ])
            v3 = SUM([ v0, times(0.6-frac)([vx,vy]), times(frac)([nx,ny]) ])
            v4 = SUM([ v0, times(0.6-frac)([vx,vy]), times(-frac)([nx,ny]) ])
            verts,cells = [v0,v1,v2,v3,v4],[[1,2],[3,4],[3,5]]
            hpcs += [MKPOL([verts,cells,None])]
        hpc = STRUCT(hpcs)
        return hpc
    ◇

```

Macro referenced in [49b](#).

**Example of oriented edge drawing** An example of drawing of oriented edges is given in `test/py/larcc/test11.py` file, and in Figure [7](#), showing both the numbering of the cells and the arrows indicating the edge orientation is illustrated in Figure [7](#), where also the oriented boundary is shown.

"test/py/larcc/test11.py" 21b ≡

```

    ⟨Example of oriented edge drawing 21c⟩
    ◇

    ⟨Example of oriented edge drawing 21c⟩ ≡
        """ Example of oriented edge drawing """
        from larlib import *

        V = [[9,0],[13,2],[15,4],[17,8],[14,9],[13,10],[11,11],[9,10],[7,9],[5,9],[3,
            8],[0,6],[2,3],[2,1],[5,0],[7,1],[4,2],[12,10],[6,3],[8,3],[3,5],[5,5],[7,6],
            [8,5],[10,5],[11,4],[10,2],[13,4],[14,6],[13,7],[11,9],[9,7],[7,7],[4,7],[2,
            6],[12,7],[12,5]]

```



Figure 7: Example of numbered polytopal complex, including edge orientations, and its oriented boundary.

```

FV = [[0,1,26],[5,6,17],[6,7,17,30],[7,30,31],[7,8,31,32],[24,30,31,35],[3,4,
28],[4,5,17,29,30,35],[4,28,29],[28,29,35,36],[8,9,32,33],[9,10,33],[11,10,
33,34],[11,20,34],[20,33,34],[20,21,32,33],[18,21,22],[21,22,32],[22,23,31,
32],[23,24,31],[11,12,20],[12,16,18,20,21],[18,22,23],[18,19,23],[19,23,24],
[15,19,24,26],[0,15,26],[24,25,26],[24,25,35,36],[2,3,28],[1,2,27,28],[12,13,
16],[13,14,16],[14,15,16,18,19],[1,25,26,27],[25,27,36],[36,27,28]]

VIEW(EXPLODE(1.2,1.2,1)(MKPOLLS((V,FV))))
VV = AA(LIST)(range(len(V)))
_,EV = larFacets((V,FV+[range(16)]),dim=2,emptyCellNumber=1)

submodel = mkSignedEdges((V,EV))
VIEW(submodel)
VIEW(larModelNumbering(scalx=1,scaly=1,scalz=1)(V,[VV,EV,FV],submodel,2))

orientedBoundary = signedCellularBoundaryCells(V,[VV,EV,FV])
cells = [EV[e] if sign==1 else REVERSE(EV[e]) for (sign,e) in zip(*orientedBoundary)]
submodel = mkSignedEdges((V,cells))
VIEW(submodel)
◇

```

Macro referenced in [21b](#), [23](#).

**Extracting the boundary of whichever chain** The boundary of whichever chain, here defined as the list of indices of its cells, then transformed to its coordinate representation (column vector in the given basis), is explicitly computed by matrix product times the matrix of the boundary operator in the given basis, transformed back in its BRC representation, and displayed as LAR model.

```

"test/py/larcc/test19.py" 23 ≡
    """ Example of oriented edge drawing """

    (Example of oriented edge drawing 21c)

    C2 = csr_matrix((len(FV),1))
    for i in [1,2, 12,13,14,15, 22,23, 29,30,31]: C2[i,0] = 1
    BD = boundary(FV,EV)
    C1 = BD * C2
    C_1 = [i for i in range(len(EV)) if ABS(C1[i,0]) == 1 ]
    C_2 = [i for i in range(len(FV)) if C2[i,0] == 1 ]

    VIEW(EXPLODE(1.2,1.2,1)(MKPOLLS((V,[EV[k] for k in C_1] + [FV[k] for k in C_2]))))
    ◇

```

**From 2D chains to boundary chains**



⟨ From 2D chains to boundary chains 24 ⟩ ≡

```

""" From 2D chains to boundary chains """
def chain2BoundaryChain(csrBoundaryMat):
    print ">>> chain2BoundaryChain"
    nedges,nfaces = csrBoundaryMat.shape
    def chain2BoundaryChain0(chain):
        row = np.array(range(len(chain)))
        col = np.array([0 for k in range(len(chain))])
        data = np.array(chain)
        csrFaceVect = scipy.sparse.coo_matrix((data, (row, col)), shape=(nfaces,1)).tocsr()
        csrEdgeVect = csrBoundaryMat*csrFaceVect
        boundaryChain = [h for h,val in
            zip(csrEdgeVect.tocoo().row, csrEdgeVect.tocoo().data) if val%2 != 0]
        return boundaryChain
    print "<<< chain2BoundaryChain"
    return chain2BoundaryChain0

```

◇

Macro referenced in [49b](#).

### 3.2 Incidence and adjacency operators

Let us start by computing the more interesting subset of the binary relationships between the 4 decompositive and/or boundary entities of 3D cellular models. Therefore, in this case we denote with C, F, E, and V, the 3-cells and their faces, edges and vertices, respectively. The input is the full-fledged LAR representation provided by

$$CV := CSR(M_3) \tag{1}$$

$$FV := CSR(M_2) \tag{2}$$

$$EV := CSR(M_1) \tag{3}$$

$$VV := CSR(M_0) \tag{4}$$

Of course,  $CSR(M_0)$  coincides with the identity matrix of dimension  $|V|$  and can be excluded by further considerations. Some binary incidence and adjacency relations we are going to compute are:

$$CF := CV \times FV^t = CSR(M_3) \times CSR(M_2)^t \tag{5}$$

$$CE := CV \times EV^t = CSR(M_3) \times CSR(M_1)^t \tag{6}$$

$$FE := FV \times EV^t = CSR(M_2) \times CSR(M_1)^t \tag{7}$$

The other possible operators follow from a similar computational pattern.

**The programming pattern for incidence computation** A high-level function `larIncidence` useful to compute the LAR representation of the incidence matrix (operator) and the incidence relations is given in the script below.

```

⟨Some incidence operators 25a⟩ ≡
    """ Some incidence operators """
    def larIncidence(cells, facets):
        csrCellFacet = csrCellFaceIncidence(cells, facets)
        cooCellFacet = csrCellFacet.tocoo()
        larCellFacet = [[] for cell in range(len(cells))]
        for i, j, val in zip(cooCellFacet.row, cooCellFacet.col, cooCellFacet.data):
            if val == 1: larCellFacet[i] += [j]
        return larCellFacet

    ⟨Cell-Face incidence operator 25b⟩
    ⟨Cell-Edge incidence operator 25c⟩
    ⟨Face-Edge incidence operator 26a⟩
    ◇

```

Macro referenced in 49b.

**Cell-Face incidence** The `csrCellFaceIncidence` and `larCellFace` functions are given below, and exported to the `larcc` module.

```

⟨Cell-Face incidence operator 25b⟩ ≡
    """ Cell-Face incidence operator """
    def csrCellFaceIncidence(CV, FV):
        return boundary(FV, CV)

    def larCellFace(CV, FV):
        return larIncidence(CV, FV)
    ◇

```

Macro referenced in 25a.

**Cell-Edge incidence** Analogously, the `csrCellEdgeIncidence` and `larCellEdge` functions are given in the following script.

```

⟨Cell-Edge incidence operator 25c⟩ ≡
    """ Cell-Edge incidence operator """
    def csrCellEdgeIncidence(CV, EV):
        return boundary(EV, CV)

    def larCellEdge(CV, EV):
        return larIncidence(CV, EV)
    ◇

```

Macro referenced in 25a.

**Face-Edge incidence** Finally, the `csrCellEdgeIncidence` and `larCellFace` functions are provided below.

```

⟨Face-Edge incidence operator 26a⟩ ≡
    """ Face-Edge incidence operator """
    def csrFaceEdgeIncidence(FV,EV):
        return boundary(EV,FV)

    def larFaceEdge(FV,EV):
        return larIncidence(FV,EV)
    ◇

```

Macro referenced in [25a](#).

**Example** The example below concerns a 3D cuboidal grid, by computing a full LAR stack of bases `CV`, `FV`, `EV`, `VV`, showing its fully numbered 3D model, and finally by computing some more useful binary relationships (`CF`, `CE`, `FE`), needed for example to compute the signed matrices of boundary operators.

```

"test/py/larcc/test10.py" 26b ≡
    """ A mesh model and various incidence operators """

    from larlib import *

    shape = [2,2,2]
    V,(VV,EV,FV,CV) = larCuboids(shape,True)
    VIEW(modelIndexing(shape))

    CF = larCellFace(CV,FV)
    CE = larCellFace(CV,EV)
    FE = larCellFace(FV,EV)
    ◇

```

### 3.2.1 Incidence chain

Let denote with `CF`, `FE`, `EV` the three consecutive incidence relations between  $k$ -cells and  $(k-1)$ -cells ( $3 \leq k \leq 0$ ) in a 3-complex. In the general multidimensional case, let us call  $\mathbf{CF}_d$  the generic *binary* incidence operator, between  $d$ -cells and  $(d-1)$ -facets, as:

$$\mathbf{CF}_d = M_{d-1} M_d^t,$$

with

$$\mathbf{CF}_d := \{a_{ij}\}, \quad a_{ij} = \begin{cases} 1 & \text{if } M_{d-1}(i)M_d(j) = |f_j| \\ 0 & \text{otherwise} \end{cases}$$

**Incidence chain computation** The function `incidenceChain`, given below, returns the full stack of BRC incidence matrices of a LAR representation for a cellular complex, starting from its list of bases, i.e. from  $[VV, EV, FV, CV, \dots]$ . Notice that the function returns the inverse sequence  $[EV, FE, CF, \dots]$ , i.e.,  $CF_k$  ( $1 \leq k \leq d$ ).

```

⟨Incidence chain computation 27a⟩ ≡
    """ Incidence chain computation """
    def incidenceChain(bases):
        #print "\n len(bases) = ",len(bases),"\n"
        pairsOfBases = zip(bases[1:],bases[:-1])
        relations = [larIncidence(cells, facets)
                     for cells, facets in pairsOfBases]
        return REVERSE(relations)
    ◇

```

Macro referenced in [49b](#).

```

"test/py/larcc/test13.py" 27b ≡
    """ Example of incidence chain computation """
    from larlib import *

    shape = (1,1,2)
    print "\n\nFor a better example provide a greater shape!"
    V, bases = larCuboids(shape, True)

    VV, EV, FV, CV = bases
    incidence = incidenceChain([VV, EV, FV, CV])
    relations = ["CF", "FE", "EV"]
    for k in range(3):
        print "\n\n incidence", relations[k], "=\n", incidence[k],
    print "\n\n"

    submodel = SKEL_1(STRUCT(MKPOLS((V, EV))))
    VIEW(larModelNumbering(scalx=1, scalz=1, scaly=1)(V, [VV, EV, FV, CV], submodel, 1))
    ◇

```

**Example of incidence chain computation** When running the `test/py/larcc/test13.py` file one obtains the following printout. Notice that it provides the links between  $d$ -cell numerations and the numerations of their faces. See [Figure 8](#) for this purpose.

```

⟨Incidence chain for a 3D cuboidal complex 27c⟩ ≡
    incidence CF = [[0,2,4,6,8,9],[1,3,5,7,9,10]]

    incidence FE = [[0,2,8,9],[1,3,9,10],[4,6,11,12],[5,7,12,13],[0,4,14,15],
    [1,5,15,16],[2,6,17,18],[3,7,18,19],[8,11,14,17],[9,12,15,18],[10,13,16,19]]

```

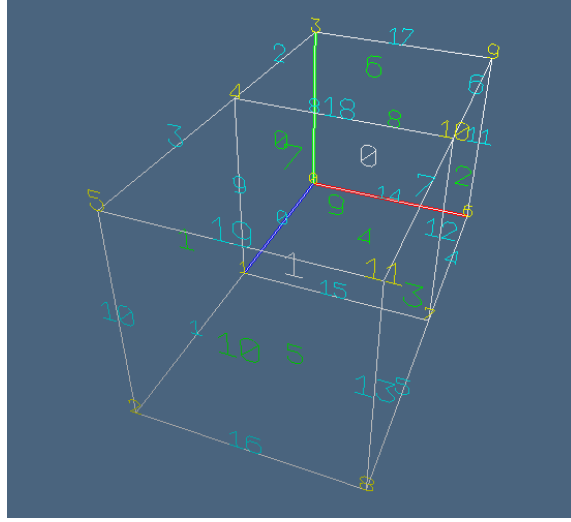


Figure 8: The stack of incidence relations gives the common links between cell numerations.

```
incidence EV = [[0,1],[1,2],[3,4],[4,5],[6,7],[7,8],[9,10],[10,11],[0,3],
[1,4],[2,5],[6,9],[7,10],[8,11],[0,6],[1,7],[2,8],[3,9],[4,10],[5,11]]
◇
```

Macro never referenced.

### 3.3 Boundary and coboundary operators

When computing the matrices of boundary and coboundary operators it may be useful to distinguish between simplicial complexes and general polytopal complexes, including cuboidal ones. In the first cases all skeletons, and hence the other topological operators, may be computed using only combinatorial methods. In the second case some reference to their geometric embedding must be done, at least to compute the *oriented* boundary and coboundary. Therefore we separate the two cases in the following sections.

**Computation of topological relations** The function `crossRelation` is used here to compute a topological relation starting from two characteristic matrices  $XV$  and  $YV$ , that associate the sets of topological objects  $X$  and  $Y$  with their vertices, respectively. The technique using sparse binary matrices stored in *CSR* (Compressed Sparse Row) format is used.

```
< Computation of topological relation 28 > ≡
""" Computation of topological relation """
def crossRelation(lenV,XV,YV,terminal=False):
    if len(YV) == len(CAT(YV)) == lenV:
```

```

    return XV
else:
    csrXV = csrCreate(XV, lenV)
    csrYV = csrCreate(YV, lenV)
    csrXY = matrixProduct(csrXV, csrYV.T)
    XY = [None for k in range(len(XV))]
    for k, face in enumerate(XV):
        data = csrXY[k].data
        col = csrXY[k].indices
        XY[k] = [col[h] for h, val in enumerate(data) if val == min(len(XV[k]), len(YV[h]))]
        # NOTE: val depends on the relation under consideration ...
    return XY

```

◇

Macro referenced in 49b.

### 3.3.1 Non-oriented operators

The `boundary` function below takes as parameters the BRC representations of  $d$ -cells and  $(d - 1)$ -facets, and returns the CSR matrix of the boundary operator. Let us notice that such operator uses a mod 2 algebra, since it takes elements within the field  $\mathbb{Z}_2 = \{0, 1\}$ .

⟨ Test examples of From cells and facets to boundary operator 29a ⟩ ≡

```

V = [[0.0, 0.0, 0.0], [1.0, 0.0, 0.0], [0.0, 1.0, 0.0], [1.0, 1.0, 0.0],
      [0.0, 0.0, 1.0], [1.0, 0.0, 1.0], [0.0, 1.0, 1.0], [1.0, 1.0, 1.0]]
CV = [[0, 1, 2, 4], [1, 2, 4, 5], [2, 4, 5, 6], [1, 2, 3, 5], [2, 3, 5, 6], [3, 5, 6, 7]]
FV = [[0, 1, 2], [0, 1, 4], [0, 2, 4], [1, 2, 3], [1, 2, 4], [1, 2, 5], [1, 3, 5], [1, 4, 5], [2, 3, 5],
      [2, 3, 6], [2, 4, 5], [2, 4, 6], [2, 5, 6], [3, 5, 6], [3, 5, 7], [3, 6, 7], [4, 5, 6], [5, 6, 7]]
EV = [[0, 1], [0, 2], [0, 4], [1, 2], [1, 3], [1, 4], [1, 5], [2, 3], [2, 4], [2, 5],
      [2, 6], [3, 5], [3, 6], [3, 7], [4, 5], [4, 6], [5, 6], [5, 7], [6, 7]]
VV = AA(LIST)(range(len(V)))

print "\ncoboundary_2 =\n", csr2DenseMatrix(coboundary(CV, FV))
print "\ncoboundary_1 =\n", csr2DenseMatrix(coboundary(FV, EV))
print "\ncoboundary_0 =\n", csr2DenseMatrix(coboundary(EV, VV))

```

◇

Macro referenced in 50.

In the script below it is necessary to guarantee that both `csrFV` and `csrCV` are created with the same number of column. The initial steps have this purpose.

⟨ From cells and facets to boundary operator 29b ⟩ ≡

```

from boundary import boundary
from boundary import boundary1

def coboundary(cells, facets):

```

```

Boundary = boundary(cells,facets)
return csrTranspose(Boundary)

def coboundary1(cells,facets):
    Boundary = boundary(cells,facets)
    return csrTranspose(Boundary)

```

◇

Macro referenced in 49b.

```

⟨ Test examples of From cells and facets to boundary cells 30 ⟩ ≡
    boundaryCells_2 = boundaryCells(CV,FV)
    boundaryCells_1 = boundaryCells([FV[k] for k in boundaryCells_2],EV)

    print "\nboundaryCells_2 =\n", boundaryCells_2
    print "\nboundaryCells_1 =\n", boundaryCells_1

    boundaryModel = (V,[FV[k] for k in boundaryCells_2])
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs(boundaryModel)))

```

◇

Macro referenced in 50.

### 3.3.2 Oriented operators

Two  $d$ -cells are said *coherently oriented* when their common  $(d-1)$ -facet has opposite orientations with respect to the two cells. When the boundary of an orientable solid partitionates its affine hull in two subsets corresponding to the *interior* and the *exterior* of the solid, then the boundary cells can be coherently oriented. This task is performed by the function `signedBoundaryCells` and `signedCellularBoundaryCells` in the following scripts. The sparse matricial structures returned by the functions `signedSimplicialBoundary` and `signedCellularBoundary` take values in the Abelian group  $\{-1, 0, 1\}$ . We call them *signed* matrices, and call *signed* operators the corresponding boundary and coboundary.

**Signed boundary matrix for simplicial complexes** The computation of the *signed* boundary matrix for simplicial complexes starts with enumerating the non-zero elements of the mod two (unoriented) boundary matrix. In particular, the `pairs` variable contains all the pairs of incident  $((d-1)$ -cell,  $d$ -cell), corresponding to each 1 elements in the binary boundary matrix. Of course, their number equates the product of the number of  $d$ -cells, times the number of  $(d-1)$ -facets on the boundary of each  $d$ -cell.

For the case of a 3-simplicial complex `CV`, we have  $4|CV|$  `pairs` elements. The actual goal of the function `signedSimplicialBoundary`, in the macro below, is to compute a sign for each of them.

The **pairs** values must be interpreted as  $(i, j)$  values in the incidence matrix **FC** (*facets-cells*), and hence as pairs of indices  $f$  and  $c$  into the characteristic matrices  $\mathbf{FV} = \text{CSR}(M_{d-1})$  and  $\mathbf{CV} = \text{CSR}(M_d)$ , respectively.

For each incidence pair **f,c**, the list **vertLists** contains the two lists of vertices associated to **f** and to **c**, called respectively the **face** and the **coface**. For each **face**, **coface** pair (i.e. for each unit element in the unordered boundary matrix), the **missingVertIndices** list will contain the index of the **coface** vertex not contained in the incident **face**.

Finally, the  $\pm 1$  (signed) incidence coefficients are computed and stored in the **faceSigns**, and then located in their actual positions within the **csrSignedBoundaryMat**. The sign of the incidence coefficient associated to the pair (facet,cell), also called (face,coface) in the implementation below, is computed as the sign of  $(-1)^k$ , where  $k$  is the position index of the removed vertex in the facet  $\langle v_0, \dots, v_{k-1}, v_{k+1}, \dots, v_d \rangle$  of the  $\langle v_0, \dots, v_d \rangle$  cell.

$\langle \text{Signed boundary matrix for simplicial models 31} \rangle \equiv$

```
def signedSimplicialBoundary (CV,FV):
    # compute the set of pairs of indices to [boundary face,incident coface]
    coo = boundary(CV,FV).tocoo()
    pairs = [[coo.row[k],coo.col[k]] for k,val in enumerate(coo.data) if val != 0]

    # compute the [face, coface] pair as vertex lists
    vertLists = [[FV[f], CV[c]] for f,c in pairs]

    # compute the local (interior to the coface) indices of missing vertices
    def missingVert(face,coface): return list(set(coface).difference(face))[0]
    missingVertIndices = [c.index(missingVert(f,c)) for f,c in vertLists]

    # signed incidence coefficients
    def checkPermutation(vertLists,missingVertIndices):
        sameOrientation = []
        for (face,coface),index in zip(vertLists,missingVertIndices):
            cell = tuple([vert for k,vert in enumerate(coface) if k!=index])
            if len(cell)==2 and cell==face: sameOrientation += [1]
            else: sameOrientation += [-1] # TODO: generalize for any "cell"
        return sameOrientation

    sameOrientation = checkPermutation(vertLists,missingVertIndices)
    faceSigns = AA(C(Power)(-1))(missingVertIndices)
    faceSigns = AA(Prod)(TRANS([faceSigns,sameOrientation]))

    # signed boundary matrix
    csrSignedBoundaryMat = csr_matrix( (faceSigns, TRANS(pairs)) )
    return csrSignedBoundaryMat
```

◇

Macro referenced in [49b](#).



**Computation of signed boundary simplices** The matrix of the signed boundary operator, with elements in  $\{-1, 0, 1\}$ , is computed in compressed sparse row (CSR) format, and stored in `csrSignedBoundaryMat`. In order to be able to return a list of `signedBoundaryCells` having a coherent orientation, we need to compute the coface of each boundary facet, i.e. the single  $d$ -cell having the facet on its boundary, and provide a coherent orientation to such chain of  $d$ -cells. The goal is obtained computing the sign of the determinant of the coface matrices, i.e. of square matrices having as rows the vertices of a coface, in normalised homogeneous coordinates.

The chain of boundary facets `boundaryCells`, obtained by multiplying the signed matrix of the boundary operator by the coordinate representation of the total  $d$ -chain, is coherently oriented by multiplication times the determinants of the `cofaceMats`.

The `cofaceMats` list is filled with the matrices having per row the position vectors of vertices of a coface, in normalized homogeneous coordinates. The list of signed face indices `orientedBoundaryCells` is returned by the function.

$\langle$  Orientation of general convex cells 32  $\rangle \equiv$

```
def swap(mylist): return [mylist[1]]+[mylist[0]]+mylist[2:]

def boundaryCellsCocells(cells,facets):
    csrSignedBoundaryMat = signedSimplicialBoundary(V,cells,facets)
    csrTotalChain = totalChain(cells)
    csrBoundaryChain = matrixProduct(csrSignedBoundaryMat, csrTotalChain)
    cooCells = csrBoundaryChain.tocoo()
    boundaryCells = []
    for k,v in enumerate(cooCells.data):
        if abs(v) == 1:
            boundaryCells += [int(cooCells.row[k] * cooCells.data[k])]
    boundaryCocells = []
    for k,v in enumerate(boundaryCells):
        boundaryCocells += list(csrSignedBoundaryMat[abs(v)].tocoo().col)
    return boundaryCells,boundaryCocells

def signedBoundaryCells(verts,cells,facets):
    boundaryCells,boundaryCocells = boundaryCellsCocells(cells,facets)
    boundaryCofaceMats = [[verts[v]+[1] for v in cells[c]] for c in boundaryCocells]
    boundaryCofaceSigns = AA(SIGN)(AA(np.linalg.det)(boundaryCofaceMats))
    orientedBoundaryCells = list(array(boundaryCells)*array(boundaryCofaceSigns))

    return orientedBoundaryCells
◇
```

Macro referenced in [49b](#).

## Signed boundary matrix for polytopal complexes

⟨Signed boundary matrix for polytopal complexes 33a⟩ ≡

```

""" Signed boundary matrix for polytopal complexes """
def signedCellularBoundary(V,bases):
    coo = boundary(bases[-1],bases[-2]).tocoo()
    pairs = [[coo.row[k],coo.col[k]] for k,val in enumerate(coo.data) if val != 0]
    signs = []
    dim = len(bases)-1
    chain = incidenceChain(bases)

    for pair in pairs:          # for each facet/coface pair
        flag = REVERSE(pair) # [c,f]
        #print "flag 1 =",flag
        for k in range(dim-1):
            cell = flag[-1]
            flag += [chain[k+1][cell][1]]

        verts = [CCOMB([V[v] for v in bases[dim-k][flag[k]])] for k in range(dim+1)]
        flagMat = [verts[v]+[1] for v in range(dim+1)]
        flagSign = SIGN(np.linalg.det(flagMat))
        signs += [flagSign]

    csrSignedBoundaryMat = csr_matrix( (signs, TRANS(pairs)) )
    # numpy.set_printoptions(threshold=numpy.nan)
    # print csrSignedBoundaryMat.todense()
    return csrSignedBoundaryMat

```

◇

Macro referenced in 49b.

## Oriented boundary cells for polytopal complexes

⟨Signed boundary cells for polytopal complexes 33b⟩ ≡

```

""" Signed boundary cells for polytopal complexes """
from scipy.sparse import *

def signedCellularBoundaryCells(verts,bases):
    CV = bases[-1]
    boundaryMat = signedCellularBoundary(verts,bases)
    chainCoords = csc_matrix((len(CV), 1))
    for cell in range(len(CV)): chainCoords[cell,0] = 1
    boundaryCells = list((boundaryMat * chainCoords).tocoo().row)
    orientations = list((boundaryMat * chainCoords).tocoo().data)
    return orientations,boundaryCells

```

◇

Macro referenced in 49b.

### 3.3.3 Examples

**Boundary of a 2D cuboidal grid** The `larCuboids` function, when applied to a `shape` parameter and to the optional parameter `full=True`, returns both the integer vertices `V` of the generated complex, and the list of `bases` of cells of dimension  $k$  ( $0 \leq k \leq d$ ), where  $d = \text{len}(\text{shape}) - 1$ .

```
"test/py/larcc/test14.py" 34a ≡
    """ Boundary of a 2D cuboidal grid """
    from larlib import *

    V,bases = larCuboids([6,6],True)
    [VV,EV,FV] = bases
    submodel = mkSignedEdges((V,EV))
    VIEW(submodel)
    VIEW(larModelNumbering(scalx=1,scaly=1,scalz=1)(V,bases,submodel,1))

    orientedBoundary = signedCellularBoundaryCells(V,bases)
    FV = [EV[e] if sign==1 else REVERSE(EV[e]) for (sign,e) in zip(*orientedBoundary)]
    submodel = mkSignedEdges((V,FV))
    VIEW(submodel)
    VIEW(larModelNumbering(scalx=1,scaly=1,scalz=1)(V,bases,submodel,1))
    ◇
```

**Oriented cuboidal and simplicial cells** In the example `test/py/larcc/test15.py` we generate a simplicial and a cuboidal decomposition of the space parallelepiped with `shape = [5,5,3]`. In both cases the boundary matrix is computed by using the general polytopal approach provided by the `signedCellularBoundaryCells` function, showing in both cases the oriented boundary of the two complexes (Just notice that in the cuboidal version `pyplasm` makes a wrong rendering, to be fixed).

```
"test/py/larcc/test15.py" 34b ≡
    """ Oriented cuboidal and simplicial cells (same algorithm) """
    from larlib import *

    # cuboidal grid
    V,bases = larCuboids([5,5,3],True)
    [VV,EV,FV,CV] = bases
    orientedBoundary = signedCellularBoundaryCells(V,AA(AA(REVERSE))([VV,EV,FV,CV]))
    cells = [FV[f] if sign==1 else REVERSE(FV[f]) for (sign,f) in zip(*orientedBoundary)]
    VIEW(EXPLODE(1.25,1.25,1.25)(MKPOLs((V,cells))))

    # simplicial grid
    V,CV = larSimplexGrid1([5,5,3])
    FV = larSimplexFacets(CV)
```

```

EV = larSimplexFacets(FV)
VV = AA(LIST)(range(len(V)))
bases = [VV,EV,FV,CV]
orientedBoundary = signedCellularBoundaryCells(V,bases)
cells = [FV[f] if sign==1 else REVERSE(FV[f]) for (sign,f) in zip(*orientedBoundary)]
VIEW(EXPLODE(1.25,1.25,1.25)(MKPOLs((V,cells))))
◇

"test/py/larcc/test18.py" 35a ≡
""" Oriented cuboidal cells """
""" Oriented cuboidal cells """
from larlib import *

def orientedBoundaryCells(V,(VV,EV,FV,CV)):
    boundaryMat = signedCellularBoundary(V,[VV,EV,FV,CV])
    chainCoords = csc_matrix((len(CV), 1))
    for cell in range(len(CV)): chainCoords[cell,0] = 1
    boundaryCells = list((boundaryMat * chainCoords).tocoo().row)
    orientations = list((boundaryMat * chainCoords).tocoo().data)
    return zip(orientations,boundaryCells)

def normalVector(V,facet):
    v0,v1,v2 = facet[:3]
    return VECTPROD([ DIFF([V[v1],V[v0]]), DIFF([V[v2],V[v0]]) ])

# cuboidal grid
V,bases = larCuboids([5,5,3],True)
[VV,EV,FV,CV] = bases
BCpairs = orientedBoundaryCells(V,[VV,EV,FV,CV])
orientedBoundary = [FV[face] if sign>0 else swap(FV[face]) for (sign,face) in BCpairs]
normals = [ normalVector(V,facet) for facet in orientedBoundary ]
facetCentroids = [CCOMB([V[v] for v in facet]) for facet in orientedBoundary]
appliedNormals = [[centroid,SUM([centroid,normal])] for (centroid,normal) in zip(facetCentroids,normals)]
normalVectors = AA(POLYLINE)(appliedNormals)

orientedQuads = [[sign,FV[face]] if sign>0 else [sign,swap(FV[face])] for (sign,face) in BCpairs]
FVtriangles = CAT([[v0,v1,v2],[v2,v1,v3]] if sign==1 else [[v0,v1,v2],[v0,v2,v3]]
                    for (sign,[v0,v1,v2,v3]) in orientedQuads])

VIEW(EXPLODE(1.2,1.2,1.2)(MKPOLs((V,FVtriangles))+normalVectors))
◇

```

### 3.3.4 Boundary orientation of a random (2D) cubical complex

```

"test/py/larcc/test17.py" 35b ≡
""" Boundary orientation of a random 2D cubical complex """
from larlib import *

```

```

from random import random

# test model generation
shape = 20,20
V,FV = larCuboids(shape)
cellSpan = prod(shape)
fraction = 0.5
remove = [int(random()*cellSpan) for k in range(int(cellSpan*fraction)) ]
FV = [FV[k] for k in range(cellSpan) if not (k in remove)]
_,EV = larCuboidsFacets((V,FV))
VV = AA(LIST)(range(len(V)))
orientedBoundary = signedCellularBoundaryCells(V,[VV,EV,FV])
cells = [EV[e] if sign==1 else REVERSE(EV[e]) for (sign,e) in zip(*orientedBoundary)]

# test model visualization
VIEW(STRUCT(MKPOLS((V,FV))))
VIEW(STRUCT(MKPOLS((V,EV))))
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,cells))))
VIEW(STRUCT(MKPOLS((V,cells))))
VIEW(mkSignedEdges((V,cells),2))
◇

```

### 3.3.5 Boundary orientation of a random (2D) triangulation

Here we provide a 2D example of computation of the oriented boundary of a quite convoluted random cellular complex. The steps performed by the scripts in the following paragraphs are listed below:

1. vertices are generated as random point in the unit circle
2. the Delaunay triangulation of the whole set of points is built.
3. spike-like triangles elimination
4. the 90% of triangles is randomly discarded
5. the input LAR is provided by the remaining triangles
6. the 1-cells are computed, and if  $n_i < n_j$  – oriented as  $v_i \rightarrow v_j$
7. the 2-cells are "coherently oriented" via the sign of their 3x3 determinant using normalised homogeneous coordinates of vertices: ccw if  $\det > 0$
8. the signed boundary matrix  $[\partial_2]$  is built (with elements in  $\{-1, 0, 1\}$  )
9. the signed boundary 1-chain (the red one) is computed by  $[\partial_2][\mathbf{1}_2]$ , where  $[\mathbf{1}_2]$  is the coordinate representation of the total 2-chain

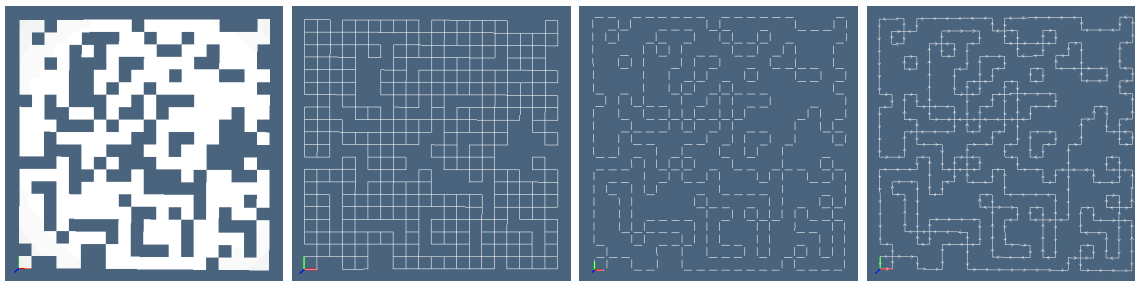


Figure 9: The orientation of the boundary of a random cuboidal 2-complex; (a) 2-cells; (b) 1-cells; (c) exploded boundary 1-chain; (d) oriented boundary 1-chain.

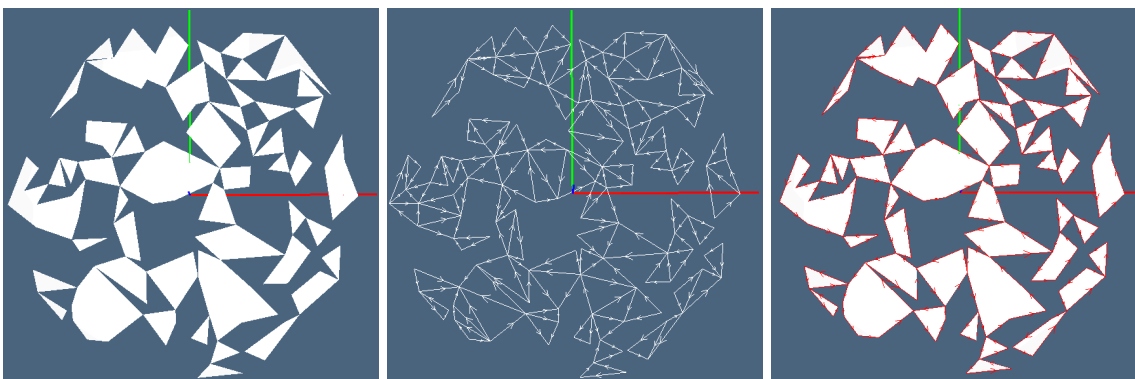


Figure 10: The orientation of the boundary of a random simplicial 2-complex; (a) 2-cells; (b) 1-cells; (c) oriented boundary 1-chain (red).

## Top-down implementation

```
"test/py/larcc/test16.py" 38a ≡
    """ Boundary orientation of a random 2D triangulation """
    from larlib import *
    from random import random
    from scipy import spatial,linalg

    ⟨ Vertices V generated as random point in the unit circle 38b ⟩
    ⟨ Delaunay triangulation of the whole set V of points 38c ⟩
    ⟨ Fraction of triangles randomly discarded 39a ⟩
    ⟨ Coherently orient the input LAR model (V,FV) 39b ⟩
    ⟨ Compute the 1-cell and 0-cell bases EV and VV 39c ⟩
    ⟨ Signed 2-boundary matrix and signed boundary 1-chain 39d ⟩
    ⟨ Display the boundary 1-chain 40a ⟩
    ◇
```

## Vertices V generated as random point in the unit circle

```
⟨ Vertices V generated as random point in the unit circle 38b ⟩ ≡
    """ Vertices V generated as random point in the unit circle """
    verts = []
    npoints = 50
    for k in range(npoints):
        t = 2*pi*random()
        u = random()+random()
        if u > 1: r = 2-u
        else: r = u
        verts += [[r*cos(t), r*sin(t)]]
    VIEW(STRUCT(AA(MK)(verts)))
    ◇
```

Macro referenced in 38a.

## Delaunay triangulation of the whole set V of points

```
⟨ Delaunay triangulation of the whole set V of points 38c ⟩ ≡
    """ Delaunay triangulation of the whole set V of points """
    triangles = spatial.Delaunay(verts)
    def area(cell): return linalg.det([verts[v]+[1] for v in cell])/2
    cells = [ cell for cell in triangles.vertices.tolist() if area(cell)>PI/(3*npoints)]
    V, FV = AA(list)(verts), cells
    ◇
```

Macro referenced in 38a.

### Fraction of triangles randomly discarded

```
⟨ Fraction of triangles randomly discarded 39a ⟩ ≡  
    """ Fraction of triangles randomly discarded """  
    fraction = 0.25  
    cellSpan = len(FV)  
    remove = [int(random()*cellSpan) for k in range(int(cellSpan*fraction)) ]  
    FV = [FV[k] for k in range(cellSpan) if not k in remove]  
    ◇
```

Macro referenced in 38a.

### Coherent orientation of input LAR model (V,FV)

```
⟨ Coherently orient the input LAR model (V,FV) 39b ⟩ ≡  
    """ Coherently orient the input LAR model (V,FV) """  
    def positiveOrientation(model):  
        V,simplices = model  
        out = []  
        for simplex in simplices:  
            theMat = [V[v]+[1] for v in simplex]  
            if sign(linalg.det(theMat)) > 0: out += [simplex]  
            else: out += [REVERSE(simplex)]  
        return V,out  
  
    V,FV = positiveOrientation((V,FV))  
    ◇
```

Macro referenced in 38a.

### Compute the 1-cell and 0-cell bases EV and VV

```
⟨ Compute the 1-cell and 0-cell bases EV and VV 39c ⟩ ≡  
    """ Compute the 1-cell and 0-cell bases EV and VV """  
    EV = larSimplexFacets(FV)  
    VV = AA(LIST)(range(len(V)))  
    VIEW(mkSignedEdges((V,EV)))  
    ◇
```

Macro referenced in 38a.

### Signed boundary matrix $[\partial_2]$ and signed boundary 1-chain

```
⟨ Signed 2-boundary matrix and signed boundary 1-chain 39d ⟩ ≡  
    """ Signed 2-boundary matrix and signed boundary 1-chain """  
    orientedBoundary = signedCellularBoundaryCells(V,[VV,EV,FV])  
    cells = [EV[e] if sign==1 else REVERSE(EV[e]) for (sign,e) in zip(*orientedBoundary)]  
    ◇
```

Macro referenced in 38a.



## Display the boundary 1-chain

```

⟨Display the boundary 1-chain 40a⟩ ≡
    """ Display the boundary 1-chain """
    VIEW(STRUCT(MKPOLS((V,FV))))
    VIEW(STRUCT(
        MKPOL((V,FV)) +
        [COLOR(RED)(mkSignedEdges((V,cells))))])
    )
    ◇

```

Macro referenced in [38a](#).

## 3.4 Orienting polytopal cells

An orientation can be allocated to a general convex (polytopal) cell by computing the biggest simplex in its interior, and attributing to the cell the orientation of the contained simplex. It is in fact easy to see that the orientation can be propagated via adjacent coherently oriented simplexes, until to cover the whole cell.

The variables in the following script have the meaning specified below: `input` : "cell" indices of a convex and solid polytopes and "V" vertices; `output` : biggest "simplex" indices spanning the polytope; `m` : number of cell vertices; `d` : dimension (number of coordinates) of cell vertices; `d+1` : number of simplex vertices; `vcell` : cell vertices; `vsimplex` : simplex vertices; `Id` : identity matrix; `basis` : orthonormal spanning set of vectors  $e_k$ ; `vector` : position vector of a simplex vertex in translated coordinates; `unUsedIndices` : cell indices not moved to simplex.

```

⟨Oriented boundary cells for simplicial models 40b⟩ ≡
def pivotSimplices(V,CV,d=3):
    simplices = []
    for cell in CV:
        vcell = np.array([V[v] for v in cell])
        m, simplex = len(cell), []
        # translate the cell: for each k, vcell[k] -= vcell[0], and simplex[0] := cell[0]
        for k in range(m-1,-1,-1): vcell[k] -= vcell[0]
        # simplex = [0], basis = [], tensor = Id(d+1)
        simplex += [cell[0]]
        basis = []
        tensor = np.array(IDNT(d))
        # look for most distant cell vertex
        dists = [SUM([SQR(x) for x in v])**0.5 for v in vcell]
        maxDistIndex = max(enumerate(dists),key=lambda x: x[1])[0]
        vector = np.array([vcell[maxDistIndex]])
        # normalize vector
        den=(vector**2).sum(axis=-1) **0.5
        basis = [vector/den]
        simplex += [cell[maxDistIndex]]

```

```

unUsedIndices = [h for h in cell if h not in simplex]

# for k in {2,d+1}:
for k in range(2,d+1):
    # update the orthonormal tensor
    e = basis[-1]
    tensor = tensor - np.dot(e.T, e)
    # compute the index h of a best vector
    # look for most distant cell vertex
    dists = [SUM([SQR(x) for x in np.dot(tensor,v)])**0.5
    if h in unUsedIndices else 0.0
    for (h,v) in zip(cell,vcell)]
    # insert the best vector index h in output simplex
    maxDistIndex = max(enumerate(dists),key=lambda x: x[1])[0]
    vector = np.array([vcell[maxDistIndex]])
    # normalize vector
    den=(vector**2).sum(axis=-1) **0.5
    basis += [vector/den]
    simplex += [cell[maxDistIndex]]
    unUsedIndices = [h for h in cell if h not in simplex]
    simplices += [simplex]
return simplices

def simplexOrientations(V,simplices):
    vcells = [[V[v]+[1.0] for v in simplex] for simplex in simplices]
    return [SIGN(np.linalg.det(vcell)) for vcell in vcells]

```

◇

Macro referenced in [49b](#).

**Affine transformations of points** Some primitive maps of points to points are given in the following, including translation, rotation and scaling of array of points via direct transformation of their coordinates. Second-order functions are used in order to employ their curried version to transform geometric assemblies.

⟨ Affine transformations of  $d$ -points 41 ⟩  $\equiv$

```

def larTranslate (tvect):
    def larTranslate0 (points):
        return [VECTSUM([p,tvect]) for p in points]
    return larTranslate0

def larRotate (angle):          # 2-dimensional !! TODO: n-dim
    def larRotate0 (points):
        a = angle
        return [[x*COS(a)-y*SIN(a), x*SIN(a)+y*COS(a)] for x,y in points]
    return larRotate0

```

```

def larScale (svect):
    def larScale0 (points):
        print "\n points =",points
        print "\n svect =",svect
        return [AA(PROD)(TRANS([p,svect])) for p in points]
    return larScale0

```

◇

Macro referenced in [49b](#).

## 4 Piecewise-linear mapping of topological spaces

A very simple but foundational software subsystem is developed in this section, by giving a general mechanism to produce curved maps of topological spaces, via the simplicial decomposition of a chart, i.e. of a planar embedding of the fundamental polygon of a  $d$ -dimensional manifold, and the definition of coordinate functions to be applied to its vertices (0-cells of the decomposition) to generate an embedding of the manifold.

### 4.1 Domain decomposition

A simplicial map is a map between simplicial complexes with the property that the images of the vertices of a simplex always span a simplex. Simplicial maps are thus determined by their effects on vertices, and provide a piecewise-linear approximation of their underlying polyhedra.

Since double simmeries are always present in the curved primitives generated in the module, an alternative cellular decomposition with cuboidal cells is provided. The default choice is "cuboid".

**Standard and scaled decomposition of unit domain** The `larDomain` of given `shape` is decomposed by `larSimplexGrid1` as an hypercube of dimension  $d \equiv \text{len}(\text{shape})$ , where the `shape` tuple provides the number or row, columns, pages, etc. of the decomposition.

```

⟨ Generate a simplicial decomposition of the  $[0,1]^d$  domain 42 ⟩ ≡
    """ cellular decomposition of the unit d-cube """
    def larDomain(shape, cell='cuboid'):
        if cell=='simplex': V,CV = larSimplexGrid1(shape)
        elif cell=='cuboid': V,CV = larCuboids(shape)
        V = larScale( [1./d for d in shape])(V)
        return [V,CV]

```

◇

Macro referenced in [49b](#).

A scaled simplicial decomposition is provided by the second-order `larIntervals` function, with `len(shape)` and `len(size)` parameters, where the  $d$ -dimensional vector `len(size)` is assumed as the scaling vector to be applied to the point  $\mathbf{1} \in \mathbb{E}^d$ .

⟨ Scaled simplicial decomposition of the  $[0,1]^d$  domain 43a ⟩  $\equiv$

```
def larIntervals(shape, cell='cuboid'):
    def larIntervals0(size):
        V,CV = larDomain(shape,cell)
        V = larScale( size)(V)
        return [V,CV]
    return larIntervals0
```

◇

Macro referenced in 49b.

## 4.2 Mapping domain vertices

The second-order `textttlarMap` function is the LAR implementation of the PLaSM primitive `MAP`. It is applied to the array `coordFuncs` of coordinate functions and to the simplicially decomposed `domain`, returning an embedded and/or curved `domain` instance.

⟨ Primitive mapping function 43b ⟩  $\equiv$

```
def larMap(coordFuncs):
    if isinstance(coordFuncs, list): coordFuncs = CONS(coordFuncs)
    def larMap0(domain,dim=2):
        V,CV = domain
        V = AA(coordFuncs)(V) # plasm CONstruction
        return [V,CV]
        # checkModel([V,CV]) TODO
    return larMap0
```

◇

Macro referenced in 49b.

## 4.3 Identify close or coincident points

The function `checkModel`, applied to a `model` parameter, i.e. to a (vertices, cells) pair, returns the model after identification of vertices with coincident or very close position vectors. The `checkModel` function works as follows: first a dictionary `vertDict` is created, with key a suitably approximated position converted into a string by the `vcode` converter (given in the Appendix), and with value the list of vertex indices with the same (approximated) position. Then, an `invertedindex` array is created, associating each original vertex index with the new index produced by enumerating the (distinct) keys of the dictionary. Finally, a new list `CV` of cells is created, by substituting the new vertex indices for the old ones.

⟨ Create a dictionary with key the point location 43c ⟩  $\equiv$

```

from collections import defaultdict
def checkModel(model,dim=2):
    V,CV = model; n = len(V)
    vertDict = defaultdict(list)
    for k,v in enumerate(V): vertDict[vcode(v)].append(k)
    points,verts = TRANS(vertDict.items())
    invertedindex = [None]*n
    V = []
    for k,value in enumerate(verts):
        V.append(eval(points[k]))
        for i in value:
            invertedindex[i]=k
    CV = [[invertedindex[v] for v in cell] for cell in CV]
    # filter out degenerate cells
    CV = [list(set(cell)) for cell in CV if len(set(cell))>=dim+1]
    return [V, CV]

```

◇

Macro referenced in 49b.

## 2-cell with high topological genus

```

"test/py/larcc/test23.py" 44a ≡
    """ 2-cell with high topological genus """
    from larlib import *

    ⟨Output of 2-cell with high topological genus 0. 44b⟩
    ⟨Output of 2-cell with high topological genus 1. 45a⟩
    ⟨Output of 2-cell with high topological genus 2. 45b⟩

```

◇

```

⟨Output of 2-cell with high topological genus 0. 44b⟩ ≡
    """ 2-cell with high topological genus 0. """
    side = QUOTE(5*[1,-1])
    holes = PROD([side,side])
    V,FV,_ = UKPOL(holes)
    FV = [[v-1 for v in f] for f in FV]
    EV = CAT([[[v,f[(k+1)%4]] for k,v in enumerate(f+[f[0]][:-1])] for f in FV])
    VIEW(STRUCT(MKPOLS((V,EV))))

    (W,_ ) = larBox((-1,-1),(10,10))
    (_,_,EW,FW)) = larCuboids((1,1),True)
    complex = Struct([(W,FW,EW),(V,FV,EV)])
    V,FV,EV = struct2lar(complex)

```

◇

Macro referenced in 44a, 46a.

```

< Output of 2-cell with high topological genus 1. 45a > ≡
    """ 2-cell with high topological genus 1. """
    FV = [sorted(CAT(FV))]
    # EV = sorted(AA(sorted)(EV)) # p2t bug if uncommented: CHECK
    VIEW(STRUCT(MKPOLS((V,EV)))

    VV = AA(LIST)(range(len(V)))
    submodel = STRUCT(MKPOLS((V,EV)))
    VIEW(larModelNumbering(1,1,1)(V,[VV,EV,FV],submodel,1.5))
    ◇

```

Macro referenced in [44a](#).

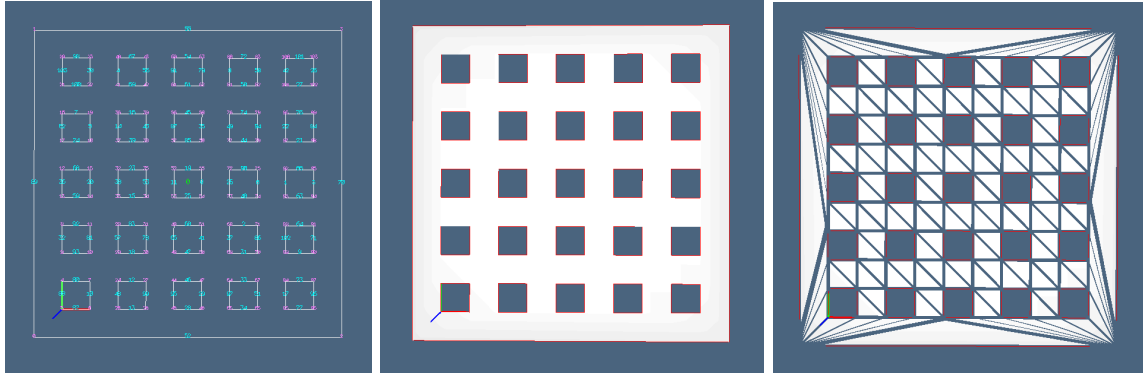


Figure 11: LAR of a 2-complex: (a) numbering of complex 0-, 1- and 2-cells (just one, numbered 0); (b) solid display of the complex, with boundary 1-cells (all 1-cells) colored in red; (c) triangulation of the interior, made for display purpose.

```

< Output of 2-cell with high topological genus 2. 45b > ≡
    """ 2-cell with high topological genus 2. """

    # LAR data structures
    csrBoundaryMat = boundary(FV,EV)
    boundaryChain = chain2BoundaryChain(csrBoundaryMat)([1])
    triangleSet = larTriangulation( (V,EV) )

    # PyPLaSM data structures
    hpcChain = AA(JOIN)(AA(AA(MK))(CAT(triangleSet)))
    hpcChainBoundary = AA(COLOR(RED))(MKPOLs((V,[EV[e] for e in boundaryChain])))

    VIEW(STRUCT( hpcChain + hpcChainBoundary ))
    VIEW(EXPLODE(1.2,1.2,1.2)( hpcChain + hpcChainBoundary ))
    ◇

```

Macro referenced in [44a](#).

**Explicit LAR of 2-cell with high topological genus** The explicit definition of the LAR of the 2-complex shown in Figura 12 is given below. The reader should notice that the LAR contains the 2-faces description **EV** of a single 2-cell—of cardinality  $O(V)$ , and that the bulk of this representation is the array of edge indices **EV**, given here in lexicographic order.

```
"test/py/larcc/test24.py" 46a ≡
    """ Explicit LAR of 2-cell with high topological genus """
    from larlib import *
    <Output of 2-cell with high topological genus 0. 44b>

    FV = [CAT(FV)] + sorted(AA(list)(FV))[1:]
    # EV = sorted(AA(sorted)(EV)) # p2t bug if uncommented: CHECK
    model = V,FV,EV

    VV = AA(LIST)(range(len(V)))
    submodel = STRUCT(MKPOLS((V,EV)))
    VIEW(larModelNumbering(1,1,1)(V,[VV,EV,FV],submodel,1.5))

    # LAR data structures
    chain = [1]+[0]*(len(FV)-1)
    outModel,triangleSet = larComplexChain(model)(chain)

    viewLarComplexChain(model)([0]+[1]*(len(FV)-1))
    viewLarComplexChain(model)([1]+[0]*(len(FV)-1))
    viewLarComplexChain(model)([1]+[0]*5 + [1]*(len(FV)-6))
    viewLarComplexChain(model)([1]*(len(FV)))
    ◇

"test/py/larcc/test25.py" 46b ≡
    """ 2-cell with high topological genus """
    from larlib import *

    V = [[-1.0,-1.0],[-1.0,10.0],[10.0,-1.0],[10.0,10.0],[0.0,1.0],[0.0,0.0],[1.0,0.0],
    [1.0,1.0],[0.0,3.0],[0.0,2.0],[1.0,2.0],[1.0,3.0],[0.0,5.0],[0.0,4.0],[1.0,4.0],
    [1.0,5.0],[0.0,7.0],[0.0,6.0],[1.0,6.0],[1.0,7.0],[0.0,9.0],[0.0,8.0],[1.0,8.0],
    [1.0,9.0],[2.0,1.0],[2.0,0.0],[3.0,0.0],[3.0,1.0],[2.0,3.0],[2.0,2.0],[3.0,2.0],
    [3.0,3.0],[2.0,5.0],[2.0,4.0],[3.0,4.0],[3.0,5.0],[2.0,7.0],[2.0,6.0],[3.0,6.0],
    [3.0,7.0],[2.0,9.0],[2.0,8.0],[3.0,8.0],[3.0,9.0],[4.0,1.0],[4.0,0.0],[5.0,0.0],
    [5.0,1.0],[4.0,3.0],[4.0,2.0],[5.0,2.0],[5.0,3.0],[4.0,5.0],[4.0,4.0],[5.0,4.0],
    [5.0,5.0],[4.0,7.0],[4.0,6.0],[5.0,6.0],[5.0,7.0],[4.0,9.0],[4.0,8.0],[5.0,8.0],
    [5.0,9.0],[6.0,1.0],[6.0,0.0],[7.0,0.0],[7.0,1.0],[6.0,3.0],[6.0,2.0],[7.0,2.0],
    [7.0,3.0],[6.0,5.0],[6.0,4.0],[7.0,4.0],[7.0,5.0],[6.0,7.0],[6.0,6.0],[7.0,6.0],
    [7.0,7.0],[6.0,9.0],[6.0,8.0],[7.0,8.0],[7.0,9.0],[8.0,1.0],[8.0,0.0],[9.0,0.0],
    [9.0,1.0],[8.0,3.0],[8.0,2.0],[9.0,2.0],[9.0,3.0],[8.0,5.0],[8.0,4.0],[9.0,4.0],
```

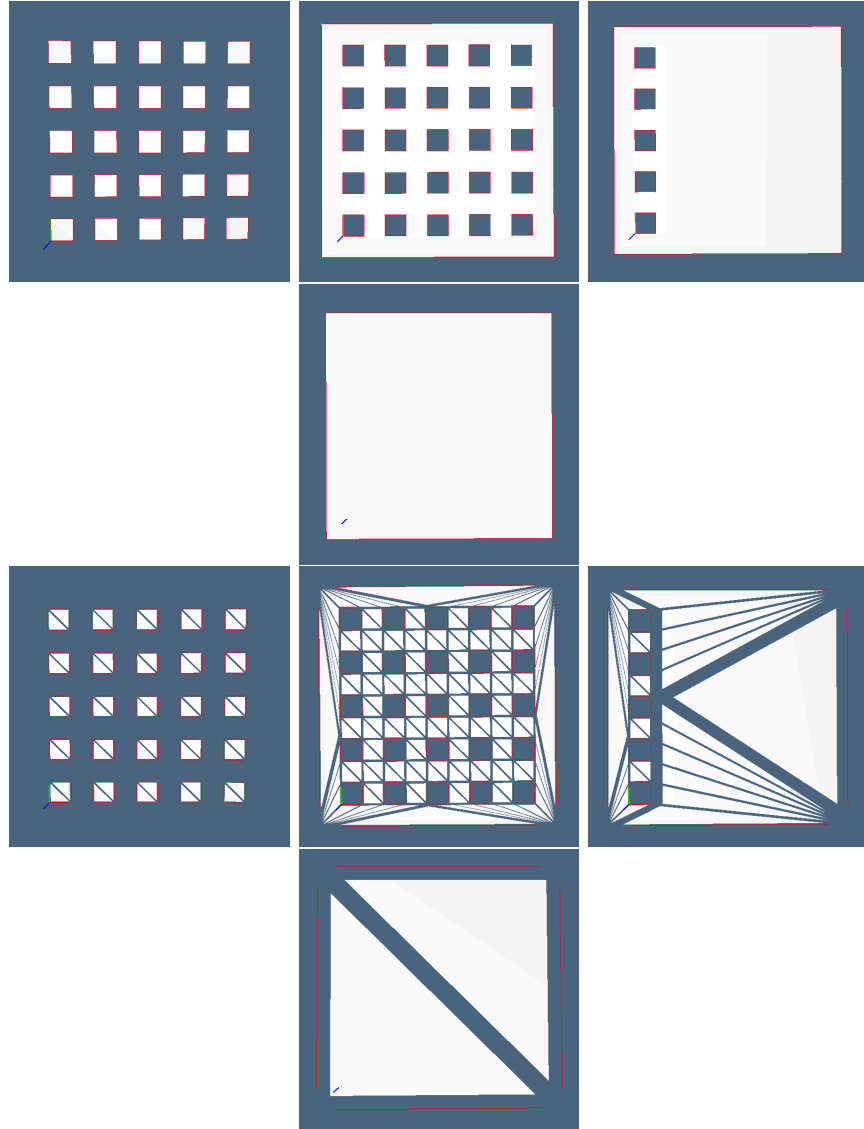


Figure 12: LAR of a 2-complex. Upper row: interior (white) and boundary (red) of different 2-chains; Lower row: triangulations of the chains. Warning: all extracted from LAR of a complex with the same boundary operator ... :o)



$$\begin{aligned}
 \text{FV} = & [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, \\
 & 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, \\
 & 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, \\
 & 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103], [4, \\
 & 5, 6, 7], [8, 9, 10, 11], [12, 13, 14, 15], [16, 17, 18, 19], [20, 21, 22, 23], [24, 25, 26, 27], [28, \\
 & 29, 30, 31], [32, 33, 34, 35], [36, 37, 38, 39], [40, 41, 42, 43], [44, 45, 46, 47], [48, 49, 50, 51], \\
 & [52, 53, 54, 55], [56, 57, 58, 59], [60, 61, 62, 63], [64, 65, 66, 67], [68, 69, 70, 71], [72, 73, 74, \\
 & 75], [76, 77, 78, 79], [80, 81, 82, 83], [84, 85, 86, 87], [88, 89, 90, 91], [92, 93, 94, 95], [96, 97, \\
 & 98, 99], [100, 101, 102, 103]
 \end{aligned}$$

```
chain1 = [0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]
chain2 = [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]
chain3 = [1,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]
chain4 = [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]
```

## 5 Exporting the library

 $\langle \text{The MIT Licence 48} \rangle \equiv$ 

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 """

◇

Macro referenced in [49b](#).

## 5.2 Importing of modules or packages

⟨Importing of modules or packages [49a](#)⟩ ≡

```
from larlib import *

import collections
import numpy as np
from scipy import zeros, arange, mat, amin, amax, array
from scipy.sparse import vstack, hstack, csr_matrix, coo_matrix, lil_matrix, triu
```

◇

Macro referenced in [49b](#).

## 5.3 Writing the library file

"larlib/larlib/larcc.py" [49b](#) ≡

```
# -*- coding: utf-8 -*-
""" Basic LARCC library """
from larlib import *
```

⟨The MIT Licence [48](#)⟩

⟨Importing of modules or packages [49a](#)⟩

⟨Affine transformations of  $d$ -points [41](#)⟩

⟨ From list of triples to scipy.sparse 3b ⟩  
 ⟨ Brc to Co0 transformation 2 ⟩  
 ⟨ Co0 to Csr transformation 3c ⟩  
 ⟨ Brc to Csr transformation 4a ⟩  
 ⟨ Query Matrix shape 5a ⟩  
 ⟨ Sparse to dense matrix transformation 6a ⟩  
 ⟨ Matrix product and transposition 6c ⟩  
 ⟨ Matrix filtering to produce the boundary matrix 8 ⟩  
 ⟨ Matrix filtering via a generic predicate 9b ⟩  
 ⟨ Characteristic matrix transposition 17 ⟩  
 ⟨ From cells and facets to boundary operator 29b ⟩  
 ⟨ Computation of topological relation 28 ⟩  
 ⟨ Signed boundary matrix for simplicial models 31 ⟩  
 ⟨ Orientation of general convex cells 32 ⟩  
 ⟨ Computation of cell adjacencies 18b ⟩  
 ⟨ Extraction of facets of a cell complex 18a ⟩  
 ⟨ Some incidence operators 25a ⟩  
 ⟨ Visualization of cell indices 20a, ... ⟩  
 ⟨ Numbered visualization of a LAR model ? ⟩  
 ⟨ Drawing of oriented edges 21a ⟩  
 ⟨ Incidence chain computation 27a ⟩  
 ⟨ Signed boundary matrix for polytopal complexes 33a ⟩  
 ⟨ Signed boundary cells for polytopal complexes 33b ⟩  
 ⟨ Oriented boundary cells for simplicial models 40b ⟩  
 ⟨ Generate a simplicial decomposition of the  $[0, 1]^d$  domain 42 ⟩  
 ⟨ Scaled simplicial decomposition of the  $[0, 1]^d$  domain 43a ⟩  
 ⟨ Create a dictionary with key the point location 43c ⟩  
 ⟨ Primitive mapping function 43b ⟩  
 ⟨ From 2D chains to boundary chains 24 ⟩

```

if __name__ == "__main__":
    ⟨ Test examples 50 ⟩

```

## 6 Unit tests

⟨ Test examples 50 ⟩ ≡

⟨ Test example of Brc to Co0 transformation 3a ⟩  
 ⟨ Test example of Co0 to Csr transformation 3d ⟩  
 ⟨ Test example of Brc to Csr transformation 4b ⟩  
 ⟨ Test examples of Query Matrix shape 5b ⟩  
 ⟨ Test examples of Sparse to dense matrix transformation 6b ⟩  
 ⟨ Test example of Matrix filtering to produce the boundary matrix 9a ⟩  
 ⟨ Test example of Matrix filtering via a generic predicate 9c ⟩

{ Test examples of From cells and facets to boundary operator 29a }  
 { Test examples of From cells and facets to boundary cells 30 }  
 { Test examples of Computation of cell adjacencies 19b }  
 { Test examples of Extraction of facets of a cell complex 19a }  
 ◇

Macro referenced in 49b.

## Comparing oriented and unoriented boundary

"test/py/larcc/test09.py" 51a ≡

```

""" comparing oriented boundary and unoriented boundary extraction on a simple example """
import sys; sys.path.insert(0, 'lib/py/')
from largrid import *
from larcc import *

V,CV = larSimplexGrid1([1,1,1])
FV = larSimplexFacets(CV)

orientedBoundary = signedBoundaryCells(V,CV,FV)
orientedBoundaryFV = [FV[-k] if k<0 else swap(FV[k]) for k in orientedBoundary]
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,orientedBoundaryFV))))

BF = boundaryCells(CV,FV)
boundaryCellsFV = [FV[k] for k in BF]
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,boundaryCellsFV))))
◇

```

"test/py/larcc/test12.py" 51b ≡

```

""" comparing edge orientation and oriented boundary extraction """
import sys; sys.path.insert(0, 'lib/py/')
from largrid import *
from larcc import *

V,FV = larSimplexGrid1([5,5])
EV = larSimplexFacets(FV)
VIEW(mkSignedEdges((V,EV)))

orientedBoundary = signedBoundaryCells(V,FV,EV)
orientedBoundaryEV = [EV[-k] if k<0 else swap(EV[k]) for k in orientedBoundary]
VIEW(mkSignedEdges((V,orientedBoundaryEV)))
◇

```

## A Appendix: Tutorials

### A.1 Model generation, skeleton and boundary extraction

```
"test/py/larcc/test01.py" 52a ≡
    """ Model generation, skeleton and boundary extraction """
    from larlib import *
    ⟨input of 2D topology and geometry data 52b⟩
    ⟨characteristic matrices 52c⟩
    ⟨incidence matrix 52d⟩
    ⟨boundary and coboundary operators 52e⟩
    ⟨product of cell complexes 53a⟩
    ⟨2-skeleton extraction 53b⟩
    ⟨1-skeleton extraction 53c⟩
    ⟨0-coboundary computation 54a⟩
    ⟨1-coboundary computation 54b⟩
    ⟨2-coboundary computation 54c⟩
    ⟨boundary chain visualisation 54d⟩
    ◇

    ⟨input of 2D topology and geometry data 52b⟩ ≡

    # input of geometry and topology
    V2 = [[4,10],[8,10],[14,10],[8,7],[14,7],[4,4],[8,4],[14,4]]
    EV = [[0,1],[1,2],[3,4],[5,6],[6,7],[0,5],[1,3],[2,4],[3,6],[4,7]]
    FV = [[0,1,3,5,6],[1,2,3,4],[3,4,6,7]]
    ◇
```

Macro referenced in 52a.

```
⟨characteristic matrices 52c⟩ ≡
    # characteristic matrices
    csrFV = csrCreate(FV)
    csrEV = csrCreate(EV)
    print "\nFV =\n", csr2DenseMatrix(csrFV)
    print "\nEV =\n", csr2DenseMatrix(csrEV)
    ◇
```

Macro referenced in 52a.

```
⟨incidence matrix 52d⟩ ≡
    # product
    csrEF = matrixProduct(csrEV, csrTranspose(csrFV))
    print "\nEF =\n", csr2DenseMatrix(csrEF)
    ◇
```

Macro referenced in 52a.

```
⟨boundary and coboundary operators 52e⟩ ≡
```

```

# boundary and coboundary operators
facetLengths = [csrCell.getnnz() for csrCell in csrEV]
boundary = csrBoundaryFilter(csrEF,facetLengths)
coboundary_1 = csrTranspose(boundary)
print "\ncoboundary_1 =\n", csr2DenseMatrix(coboundary_1)
◇

```

Macro referenced in 52a.

⟨product of cell complexes 53a⟩ ≡

```

# product operator
mod_2D = (V2,FV)
V1,topol_0 = [[0.],[1.],[2.]], [[0],[1],[2]]
topol_1 = [[0,1],[1,2]]
mod_0D = (V1,topol_0)
mod_1D = (V1,topol_1)
V3,CV = larModelProduct([mod_2D,mod_1D])
mod_3D = (V3,CV)
VIEW(EXPLODE(1.2,1.2,1.2)(MKPOLs(mod_3D)))
print "\nk_3 =", len(CV), "\n"
◇

```

Macro referenced in 52a.

⟨2-skeleton extraction 53b⟩ ≡

```

# 2-skeleton of the 3D product complex
mod_2D_1 = (V2,EV)
mod_3D_h2 = larModelProduct([mod_2D,mod_0D])
mod_3D_v2 = larModelProduct([mod_2D_1,mod_1D])
_,FV_h = mod_3D_h2
_,FV_v = mod_3D_v2
FV3 = FV_h + FV_v
SK2 = (V3,FV3)
VIEW(EXPLODE(1.2,1.2,1.2)(MKPOLs(SK2)))
print "\nk_2 =", len(FV3), "\n"
◇

```

Macro referenced in 52a.

⟨1-skeleton extraction 53c⟩ ≡

```

# 1-skeleton of the 3D product complex
mod_2D_0 = (V2,AA(LIST)(range(len(V2))))
mod_3D_h1 = larModelProduct([mod_2D_1,mod_0D])
mod_3D_v1 = larModelProduct([mod_2D_0,mod_1D])
_,EV_h = mod_3D_h1
_,EV_v = mod_3D_v1
EV3 = EV_h + EV_v
SK1 = (V3,EV3)

```

```
VIEW(EXPLODE(1.2,1.2,1.2)(MKPOLSK1)))
print "\nk_1 =", len(EV3), "\n"
◇
```

Macro referenced in 52a.

```
<0-coboundary computation 54a> ≡
# boundary and coboundary operators
np.set_printoptions(threshold=sys.maxint)
csrFV3 = csrCreate(FV3)
csrEV3 = csrCreate(EV3)
csrVE3 = csrTranspose(csrEV3)
facetLengths = [csrCell.getnnz() for csrCell in csrEV3]
boundary = csrBoundaryFilter(csrVE3,facetLengths)
coboundary_0 = csrTranspose(boundary)
print "\ncoboundary_0 =\n", csr2DenseMatrix(coboundary_0)
◇
```

Macro referenced in 52a.

```
<1-coboundary computation 54b> ≡
csrEF3 = matrixProduct(csrEV3, csrTranspose(csrFV3))
facetLengths = [csrCell.getnnz() for csrCell in csrFV3]
boundary = csrBoundaryFilter(csrEF3,facetLengths)
coboundary_1 = csrTranspose(boundary)
print "\ncoboundary_1.T =\n", csr2DenseMatrix(coboundary_1.T)
◇
```

Macro referenced in 52a.

```
<2-coboundary computation 54c> ≡
csrCV = csrCreate(CV)
csrFC3 = matrixProduct(csrFV3, csrTranspose(csrCV))
facetLengths = [csrCell.getnnz() for csrCell in csrCV]
boundary = csrBoundaryFilter(csrFC3,facetLengths)
coboundary_2 = csrTranspose(boundary)
print "\ncoboundary_2 =\n", csr2DenseMatrix(coboundary_2)
◇
```

Macro referenced in 52a.

```
<boundary chain visualisation 54d> ≡
# boundary chain visualisation
boundaryCells_2 = boundaryCells(CV,FV3)
boundary = (V3,[FV3[k] for k in boundaryCells_2])
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLSK1)))
◇
```

Macro referenced in 52a.

## A.2 Boundary of 3D simplicial grid

```
"test/py/larcc/test02.py" 55a ≡
    """ Boundary of 3D simplicial grid """
    from larlib import *

    ⟨boundary of 3D simplicial grid 55b⟩
    ◇

⟨boundary of 3D simplicial grid 55b⟩ ≡

    V,CV = larSimplexGrid1([10,10,3])
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,CV))))
    SK2 = (V,larSimplexFacets(CV))
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs(SK2)))
    _,FV = SK2
    SK1 = (V,larSimplexFacets(FV))
    _,EV = SK1
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs(SK1)))

    boundaryCells_2 = boundaryCells(CV,FV)
    boundary = (V,[FV[k] for k in boundaryCells_2])
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs(boundary)))
    print "\nboundaryCells_2 =\n", boundaryCells_2

    boundaryCells_2 = signedBoundaryCells(V,CV,FV)
    boundaryFV = [FV[-k] if k<0 else swap(FV[k]) for k in boundaryCells_2]

    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,boundaryFV))))
    print "\nboundaryCells_2 =\n", boundaryFV
    ◇

Macro referenced in 55a.
```

## A.3 Oriented boundary of a random simplicial complex

```
"test/py/larcc/test03.py" 55c ≡
    """ Oriented boundary of a random simplicial complex """
    from larlib import *

    ⟨Importing external modules 55d⟩
    ⟨Generating and viewing a random 3D simplicial complex 56a⟩
    ⟨Computing and viewing its non-oriented boundary 56b⟩
    ⟨Computing and viewing its oriented boundary 56c⟩
    ◇

⟨Importing external modules 55d⟩ ≡
```



```

""" Importing external modules """
from scipy.spatial import Delaunay
import numpy as np

```

Macro referenced in 55c.

```

⟨ Generating and viewing a random 3D simplicial complex 56a ⟩ ≡
    verts = np.random.rand(10000, 3) # 1000 points in 3-d
    verts = [AA(lambda x: 2*x)(VECTDIFF([vert,[0.5,0.5,0.5]])) for vert in verts]
    verts = [vert for vert in verts if VECTNORM(vert) < 1.0]
    tetra = Delaunay(verts)
    cells = [cell for cell in tetra.vertices.tolist()
              if ((verts[cell[0]][2]<0) and (verts[cell[1]][2]<0)
                  and (verts[cell[2]][2]<0) and (verts[cell[3]][2]<0) ) ]
    V, CV = verts, cells
    VIEW(MKPOL([V,AA(AA(lambda k:k+1))(CV),[]]))

```

Macro referenced in 55c.

```

⟨ Computing and viewing its non-oriented boundary 56b ⟩ ≡
    FV = larSimplexFacets(CV)
    VIEW(MKPOL([V,AA(AA(lambda k:k+1))(FV),[]]))
    boundaryCells_2 = boundaryCells(CV,FV)
    print "\nboundaryCells_2 =\n", boundaryCells_2
    bndry = (V,[FV[k] for k in boundaryCells_2])
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOL(bndry)))

```

Macro referenced in 55c.

```

⟨ Computing and viewing its oriented boundary 56c ⟩ ≡
    boundaryCells_2 = signedBoundaryCells(V,CV,FV)
    print "\nboundaryCells_2 =\n", boundaryCells_2
    boundaryFV = [FV[-k] if k<0 else swap(FV[k]) for k in boundaryCells_2]
    boundaryModel = (V,boundaryFV)
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOL(boundaryModel)))

```

Macro referenced in 55c.

## A.4 Oriented boundary of a simplicial grid

```

"test/py/larcc/test04.py" 56d ≡
    ⟨ Generate and view a 3D simplicial grid 57a ⟩
    ⟨ Computing and viewing the 2-skeleton of simplicial grid 57b ⟩
    ⟨ Computing and viewing the oriented boundary of simplicial grid 57c ⟩

```

```

⟨Generate and view a 3D simplicial grid 57a⟩ ≡
    """ Generate and view a 3D simplicial grid """
    from larlib import *

    V,CV = larSimplexGrid1([4,4,4])
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,CV))))
    ◇

```

Macro referenced in [56d](#).

```

⟨Computing and viewing the 2-skeleton of simplicial grid 57b⟩ ≡
    FV = larSimplexFacets(CV)
    EV = larSimplexFacets(FV)
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,FV))))
    ◇

```

Macro referenced in [56d](#).

```

⟨Computing and viewing the oriented boundary of simplicial grid 57c⟩ ≡
    csrSignedBoundaryMat = signedSimplicialBoundary (V,CV,FV)
    boundaryCells_2 = signedBoundaryCells(V,CV,FV)
    boundaryFV = [FV[-k] if k<0 else swap(FV[k]) for k in boundaryCells_2]
    boundary = (V,boundaryFV)
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs(boundary)))
    ◇

```

Macro referenced in [56d](#).

## A.5 Skeletons and oriented boundary of a simplicial complex

```

"test/py/larcc/test05.py" 57d ≡
    """ Skeletons and oriented boundary of a simplicial complex """
    from larlib import *

    ⟨Skeletons computation and visualisation ?⟩
    ⟨Oriented boundary matrix visualization 58a⟩
    ⟨Computation of oriented boundary cells 58b⟩
    ◇

```

```

⟨Skeletons computation and visualisation 57e⟩ ≡
    """ Skeletons computation and visualisation """
    V,FV = larSimplexGrid1([3,3])
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,FV))))
    EV = larSimplexFacets(FV)
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,EV))))
    VV = larSimplexFacets(EV)
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs((V,VV))))
    ◇

```

Macro never referenced.

```
< Oriented boundary matrix visualization 58a > ≡  
    """ Oriented boundary matrix visualization """  
    np.set_printoptions(threshold='nan')  
    csrSignedBoundaryMat = signedSimplicialBoundary (V,FV,EV)  
    Z = csr2DenseMatrix(csrSignedBoundaryMat)  
    print "\ncsrSignedBoundaryMat =\n", Z  
    import matplotlib.pyplot  
    from pylab import *  
    matshow(Z)  
    show()  
    ◇
```

Macro referenced in [57d](#).

```
< Computation of oriented boundary cells 58b > ≡  
    """ Computation of oriented boundary cells """  
    boundaryCells_1 = signedBoundaryCells(V,FV,EV)  
    print "\nboundaryCells_1 =\n", boundaryCells_1  
    boundaryEV = [EV[-k] if k<0 else swap(EV[k]) for k in boundaryCells_1]  
    bndry = (V,boundaryEV)  
    VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLs(bndry)))  
    ◇
```

Macro referenced in [57d](#).

## A.6 Boundary of random 2D simplicial complex

```
"test/py/larcc/test06.py" 58c ≡  
    """ Boundary of random 2D simplicial complex """  
    from larlib import *  
    from scipy.spatial import Delaunay  
  
    < Test for quasi-equilateral triangles 58d >  
    < Generation and selection of random triangles 59a >  
    < Boundary computation and visualisation 59b >  
    ◇
```

```
< Test for quasi-equilateral triangles 58d > ≡  
    """ Test for quasi-equilateral triangles """  
    def quasiEquilateral(tria):  
        a = VECTNORM(VECTDIFF(tria[0:2]))  
        b = VECTNORM(VECTDIFF(tria[1:3]))  
        c = VECTNORM(VECTDIFF([tria[0],tria[2]]))  
        m = max(a,b,c)
```

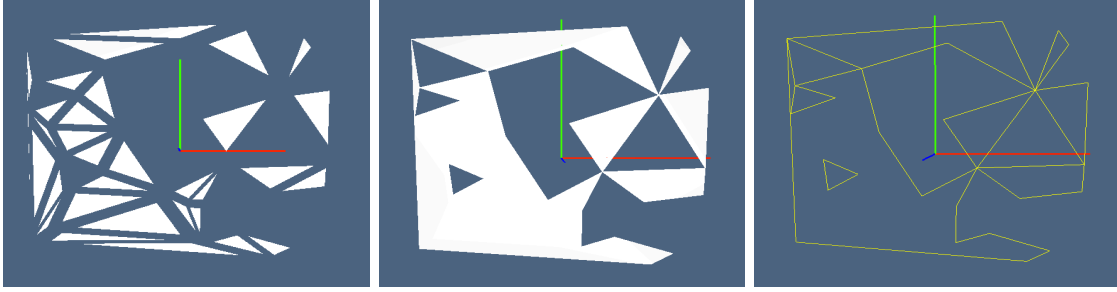


Figure 13: example caption

```

    if m/a < 1.7 and m/b < 1.7 and m/c < 1.7: return True
    else: return False

```

◇

Macro referenced in 58c.

⟨ Generation and selection of random triangles 59a ⟩ ≡

```

""" Generation and selection of random triangles """
verts = np.random.rand(50,2)
verts = (verts - [0.5,0.5]) * 2
triangles = Delaunay(verts)
cells = [ cell for cell in triangles.vertices.tolist()
          if (not quasiEquilateral([verts[k] for k in cell])) ]
V, FV = AA(list)(verts), cells
EV = larSimplexFacets(FV)
pols2D = MKPOLs((V,FV))
VIEW(EXPLODE(1.5,1.5,1.5)(pols2D))

```

◇

Macro referenced in 58c.

⟨ Boundary computation and visualisation 59b ⟩ ≡

```

""" Boundary computation and visualisation """
orientedBoundary = signedBoundaryCells(V,FV,EV)
submodel = mkSignedEdges((V,orientedBoundary))
VIEW(submodel)

```

◇

Macro referenced in 58c.

⟨ Decompose a permutation into cycles 59c ⟩ ≡

```

""" Decompose a permutation into cycles """
def permutationOrbits(List):
    d = dict((i,int(x)) for i,x in enumerate(List))
    out = []

```

```

while d:
    x = list(d)[0]
    orbit = []
    while x in d:
        orbit += [x],
        x = d.pop(x)
    out += [CAT(orbit)+orbit[0]]
return out

if __name__ == "__main__":
    print [2, 3, 4, 5, 6, 7, 0, 1]
    print permutationOrbits([2, 3, 4, 5, 6, 7, 0, 1])
    print [3,9,8,4,10,7,2,11,6,0,1,5]
    print permutationOrbits([3,9,8,4,10,7,2,11,6,0,1,5])

```

◇  
Macro never referenced.

## A.7 Assemblies of simplices and hypercubes

```

"test/py/larcc/test07.py" 60a ≡
    """ Assemblies of simplices and hypercubes """
    from larlib import *

    ⟨ Definition of 1-dimensional LAR models 60b ⟩
    ⟨ Assembly generation of squares and triangles 61a ⟩
    ⟨ Assembly generation of cubes and tetrahedra 61b ⟩
    ◇

```

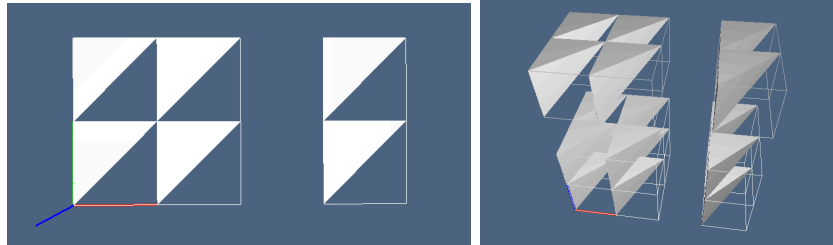


Figure 14: (a) Assemblies of squares and triangles; (b) assembly of cubes and tetrahedra.

```

⟨ Definition of 1-dimensional LAR models 60b ⟩ ≡
    """ Definition of 1-dimensional LAR models """
    geom_0,topol_0 = [[0.],[1.],[2.],[3.],[4.]], [[0,1],[1,2],[3,4]]
    geom_1,topol_1 = [[0.],[1.],[2.]], [[0,1],[1,2]]
    mod_0 = (geom_0,topol_0)
    mod_1 = (geom_1,topol_1)
    ◇

```

Macro referenced in 60a.

```

⟨ Assembly generation of squares and triangles 61a ⟩ ≡
    """ Assembly generation of squares and triangles """
    squares = larModelProduct([mod_0,mod_1])
    V,FV = squares
    simplices = pivotSimplices(V,FV,d=2)
    VIEW(STRUCT([ MKPOL([V,AA(AA(C(SUM)(1)))(simplices),[]]),
                    SKEL_1(STRUCT(MKPOLS((V,FV)))) ]))
    ◇

```

Macro referenced in 60a.

```

⟨ Assembly generation of cubes and tetrahedra 61b ⟩ ≡
    """ Assembly generation of cubes and tetrahedra """
    cubes = larModelProduct([squares,mod_0])
    V,CV = cubes
    simplices = pivotSimplices(V,CV,d=3)
    VIEW(STRUCT([ MKPOL([V,AA(AA(C(SUM)(1)))(simplices),[]]),
                    SKEL_1(STRUCT(MKPOLS((V,CV)))) ]))
    ◇

```

Macro referenced in 60a.

## References

- [CL13] CVD-Lab, *Linear algebraic representation*, Tech. Report 13-00, Roma Tre University, October 2013.