General Physics (1) For Engineering

Textbook:

Physics for Scientists and Engineers, seventh edition, Jewett / Serway

Dr. Awos Alsalman

Chapter 3

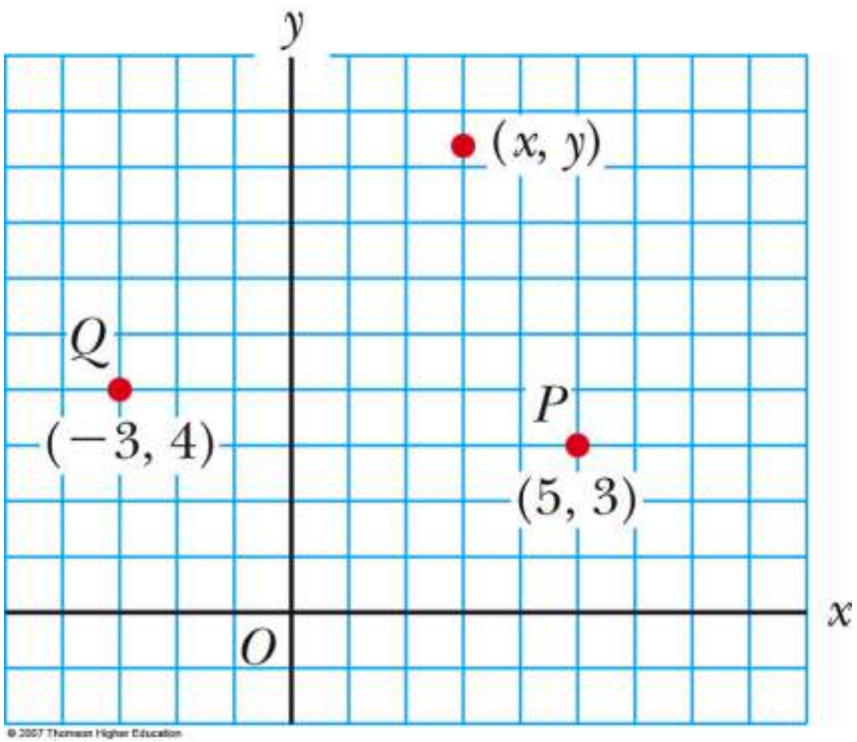
Vectors

Coordinate Systems

- Used to describe the position of a point in space
- Coordinate system consists of
  - A fixed reference point called the origin
  - Specific axes with scales and labels
  - Instructions on how to label a point relative to the origin and the axes

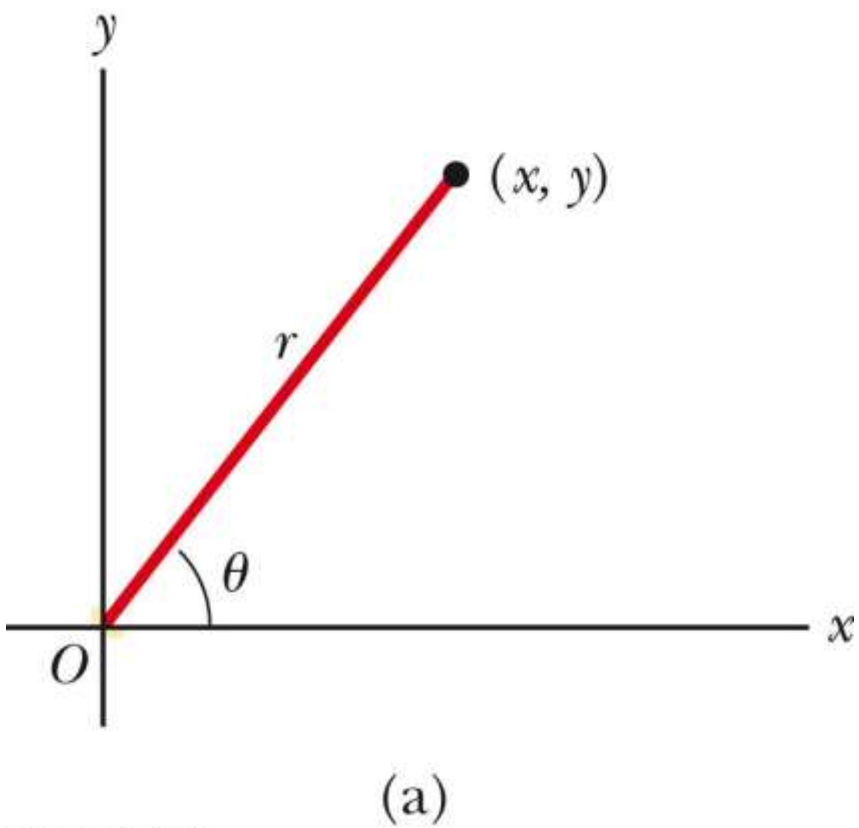
Cartesian Coordinate System

- Also called rectangular coordinate system
- *x*- and *y*- axes intersect at the origin
- Points are labeled (*x*,*y*)



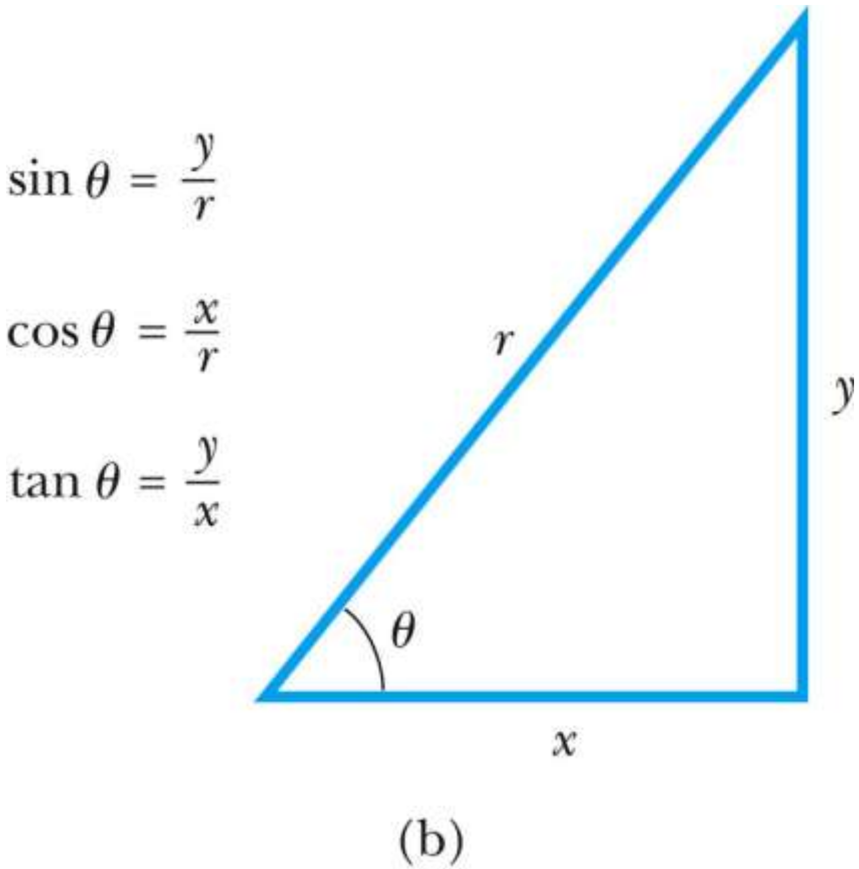
Polar Coordinate System (r, θ)

- **r is the distance from the origin to the point having Cartesian coordinates (x,y).**
- **θ is the angle between a fixed axis (+x axis) and a line drawn from the origin to the point. θ *usually measured counterclockwise***
- **Origin and reference line are noted**
- **Points are labeled (r,θ)**



Polar to Cartesian Coordinates

- Based on forming a right triangle from *r* and θ
- $x = r \cos \theta$
- $y = r \sin \theta$



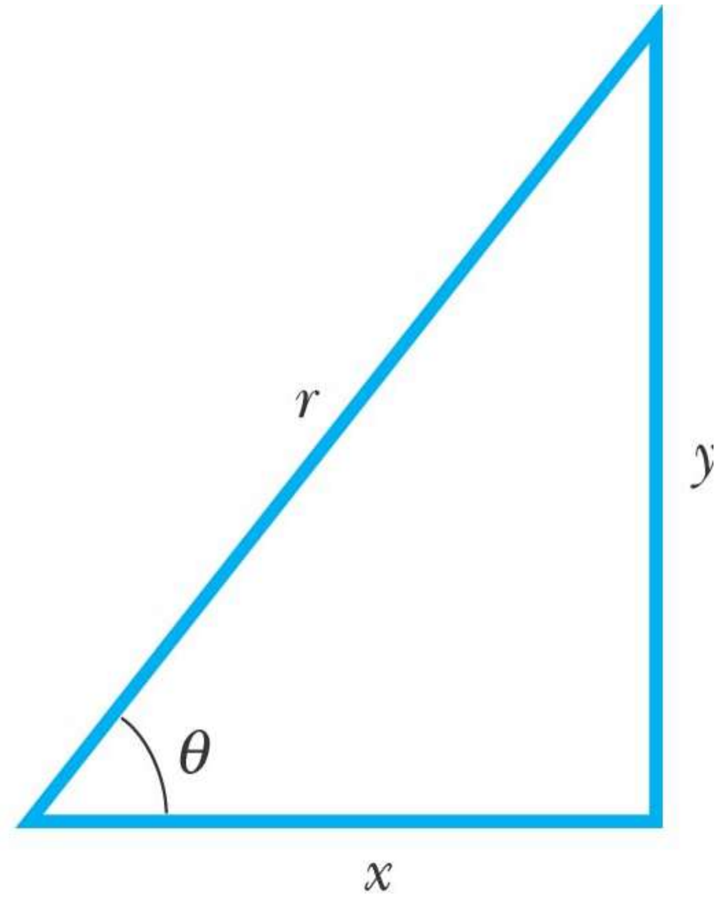
Cartesian to Polar Coordinates

- *r* is the hypotenuse and θ an angle

$$\tan\theta = \frac{y}{x}$$

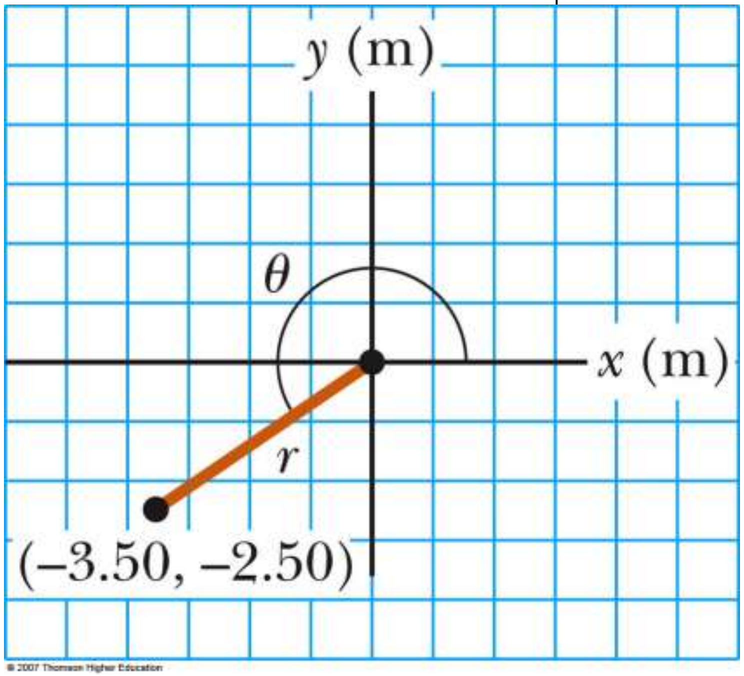
$$r = \sqrt{x^2 + y^2}$$

- θ must be counterclockwise from positive *x* axis for these equations to be valid



Example 3.1

- The Cartesian coordinates of a point in the *xy* plane are (*x*,*y*) = (-3.50, -2.50) m, as shown in the figure. Find the polar coordinates of this point.



Solution: From Equation 3.4,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50\text{ m})^2 + (-2.50\text{ m})^2} = 4.30\text{ m}$$

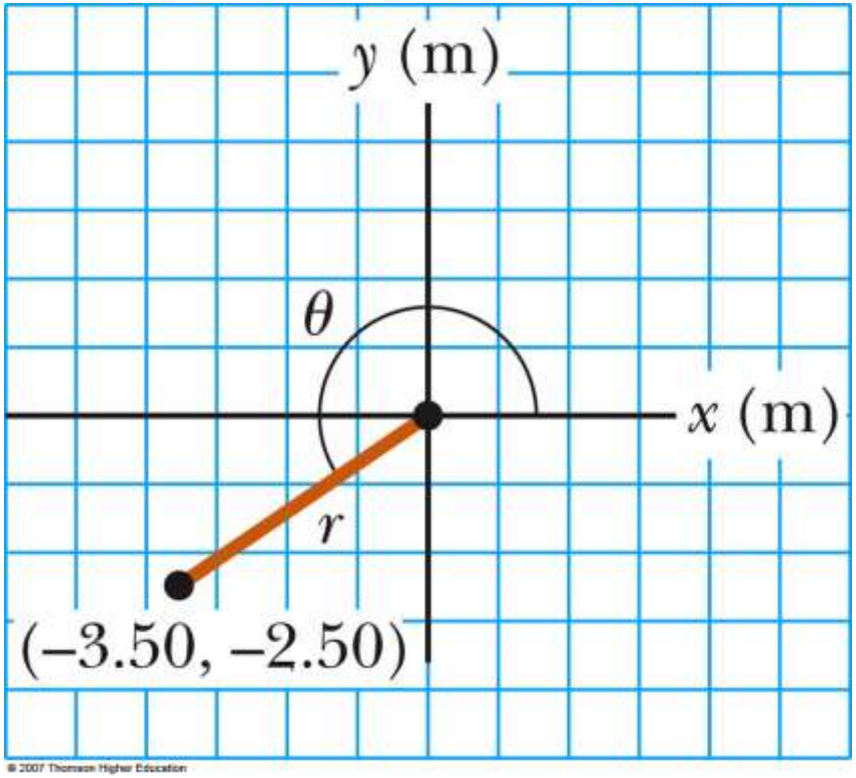
and from Equation 3.3,

$$\tan\theta = \frac{y}{x} = \frac{-2.50\text{ m}}{-3.50\text{ m}} = 0.714$$

$$\theta = 216^\circ \quad \textit{not} \quad 35.5^\circ \quad \text{(signs give quadrant)}$$

Example 3.1, cont.

- Change the point in the x-y plane
- Note its Cartesian coordinates
- Note its polar coordinates

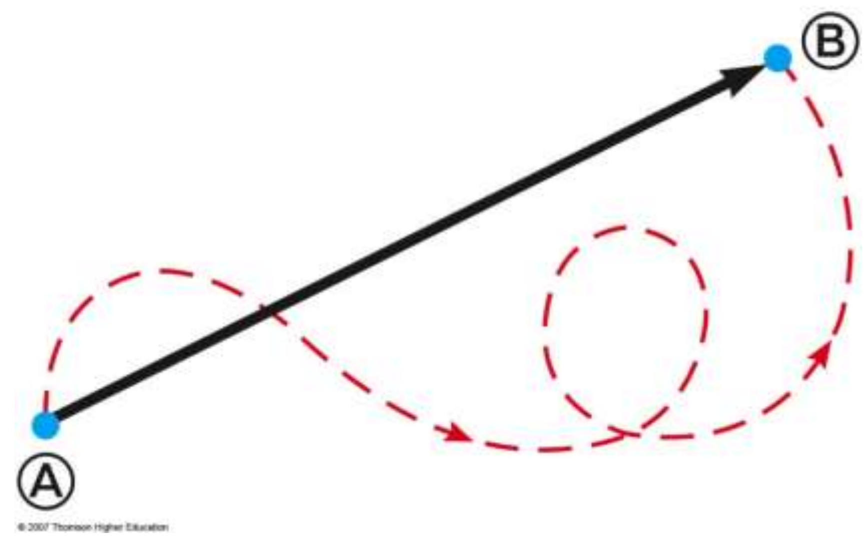


Vectors and Scalars

- A ***scalar quantity*** is completely specified by a single value with an appropriate unit and has no direction.
- A ***vector quantity*** is completely described by a number and appropriate units plus a direction.

Vector Example

- A particle travels from A to B along the path shown by the dotted red line
  - This is the ***distance*** traveled and is a scalar
- The ***displacement*** is the solid line from A to B
  - The displacement is independent of the path taken between the two points
  - Displacement is a vector

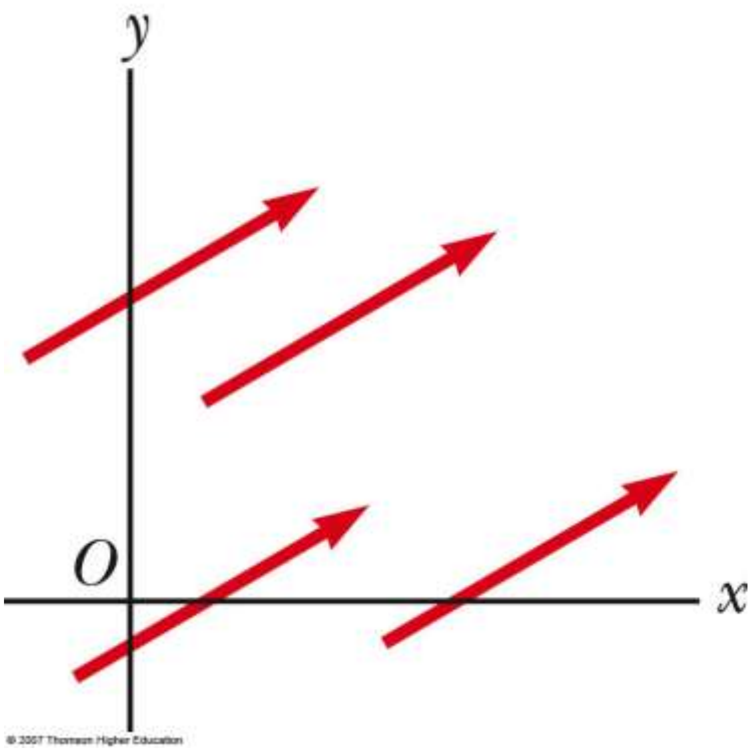


Vector Notation

- Text uses bold with arrow to denote a vector: $\vec{\mathbf{A}}$
- Also used for printing is simple bold print: **A**
- When dealing with just the magnitude of a vector in print, an italic letter will be used: *A* or $|\vec{\mathbf{A}}|$
  - The magnitude of the vector has physical units
  - The magnitude of a vector is always a positive number
- When handwritten, use an arrow: $\vec{A}$

Equality of Two Vectors

- Two vectors are ***equal*** if they have the same magnitude and the same direction
- $\vec{\mathbf{A}} = \vec{\mathbf{B}}$ if $A = B$ and they point along parallel lines
- All of the vectors shown are equal



Adding Vectors

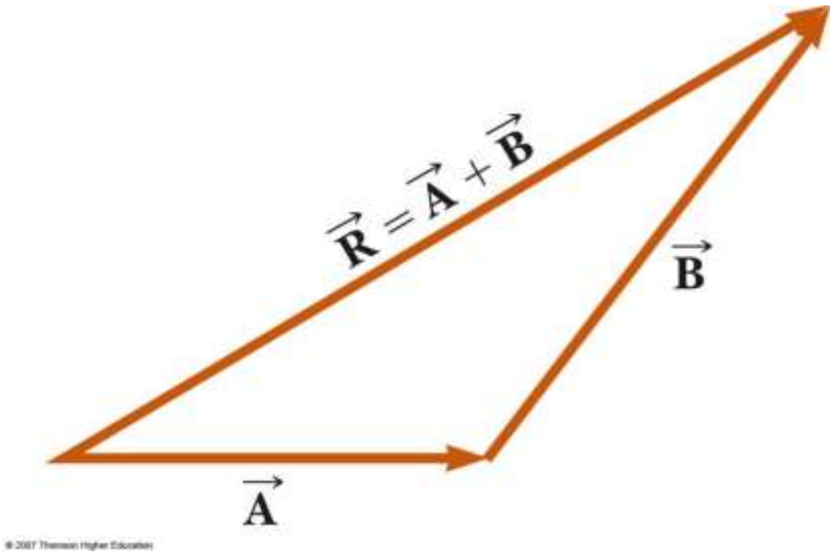
- When adding vectors, their directions must be taken into account
- Units must be the same
- Graphical Methods
  - Use scale drawings
- Algebraic Methods
  - More convenient

Adding Vectors Graphically

- Choose a scale
- Draw the first vector, $\vec{\mathbf{A}}$, with the appropriate length and in the direction specified, with respect to a coordinate system
- Draw the next vector with the appropriate length and in the direction specified, with respect to a coordinate system whose origin is the end of vector $\vec{\mathbf{A}}$

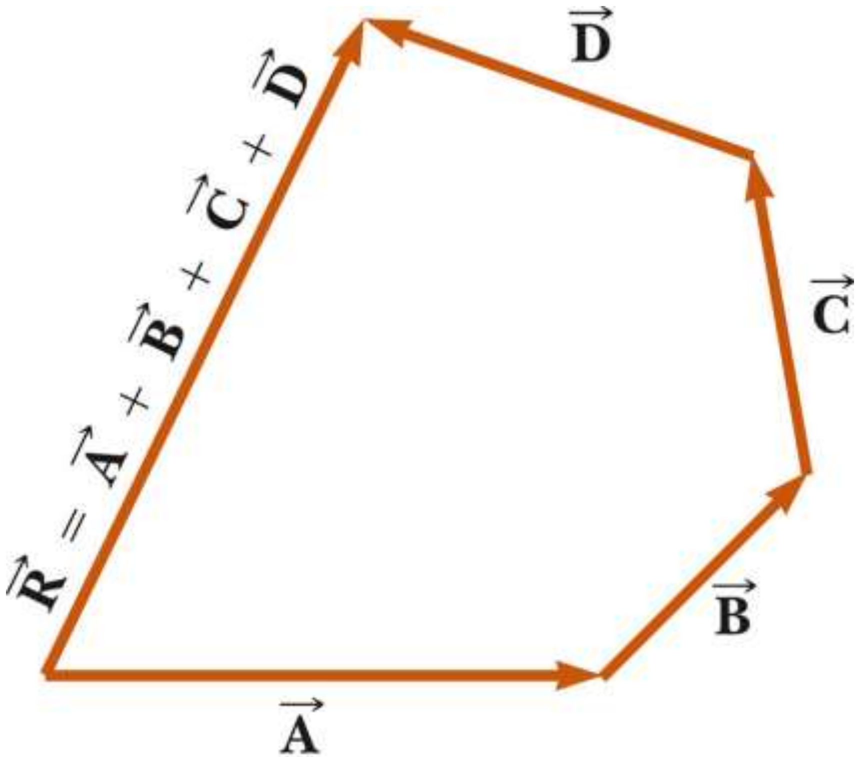
Adding Vectors Graphically, cont.

- Continue drawing the vectors "tip-to-tail"
- The resultant is drawn from the origin of $\vec{\mathbf{A}}$ to the end of the last vector
- Measure the length of $\vec{\mathbf{R}}$ and its angle
  - Use the scale factor to convert length to actual magnitude



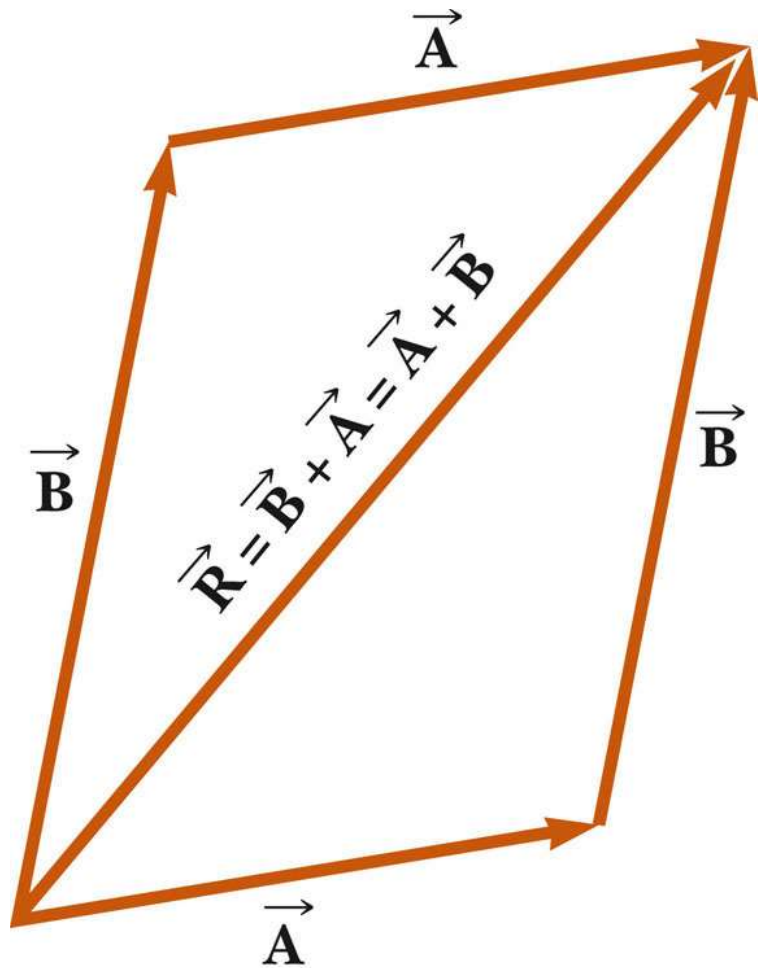
Adding Vectors Graphically, final

- When you have many vectors, just keep repeating the process until all are included
- The resultant is still drawn from the tail of the first vector to the tip of the last vector



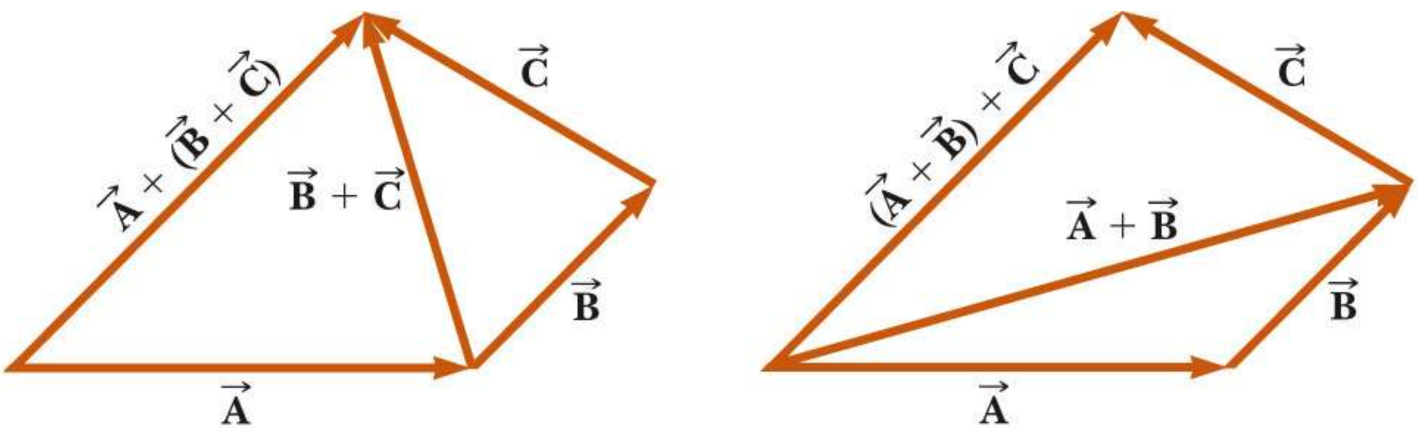
Adding Vectors, Rules

- When two vectors are added, the sum is independent of the order of the addition.
  - This is the ***Commutative Law of Addition***
- $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$



Adding Vectors, Rules cont.

- When adding three or more vectors, their sum is independent of the way in which the individual vectors are grouped
  - This is called the ***Associative Property of Addition***
- $\vec{\mathbf{A}} + \left(\vec{\mathbf{B}} + \vec{\mathbf{C}}\right) = \left(\vec{\mathbf{A}} + \vec{\mathbf{B}}\right) + \vec{\mathbf{C}}$



Adding Vectors, Rules final

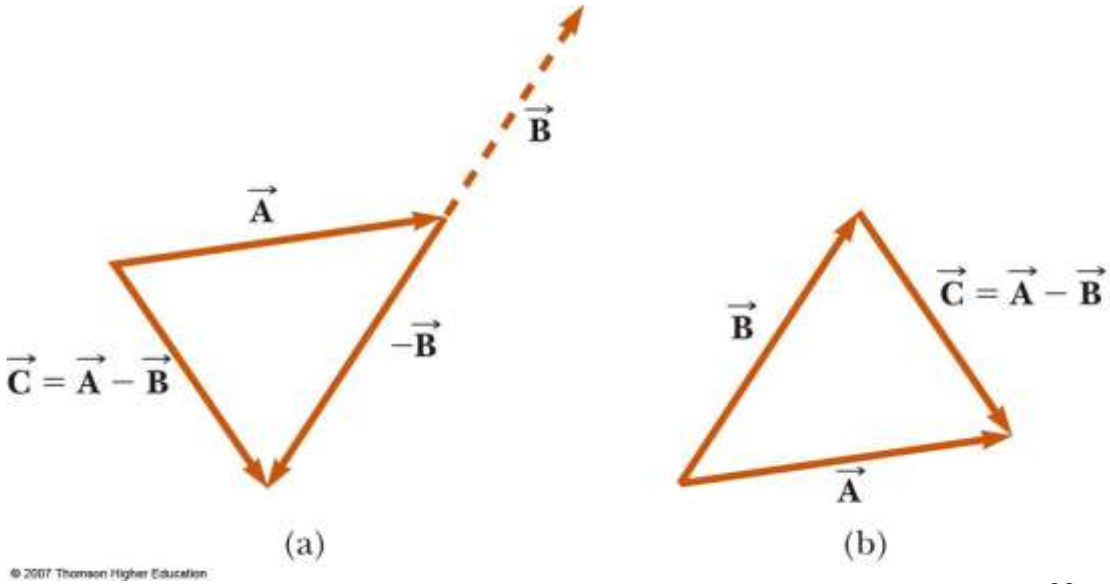
- When adding vectors, all of the vectors must have the same units
- All of the vectors must be of the same type of quantity
  - For example, you cannot add a displacement to a velocity

Negative of a Vector

- The negative of a vector is defined as the vector that, when added to the original vector, gives a resultant of zero
  - Represented as $-\vec{\mathbf{A}}$
  - $\vec{\mathbf{A}} + \left(-\vec{\mathbf{A}}\right) = 0$
- The negative of the vector will have the same magnitude, but point in the opposite direction

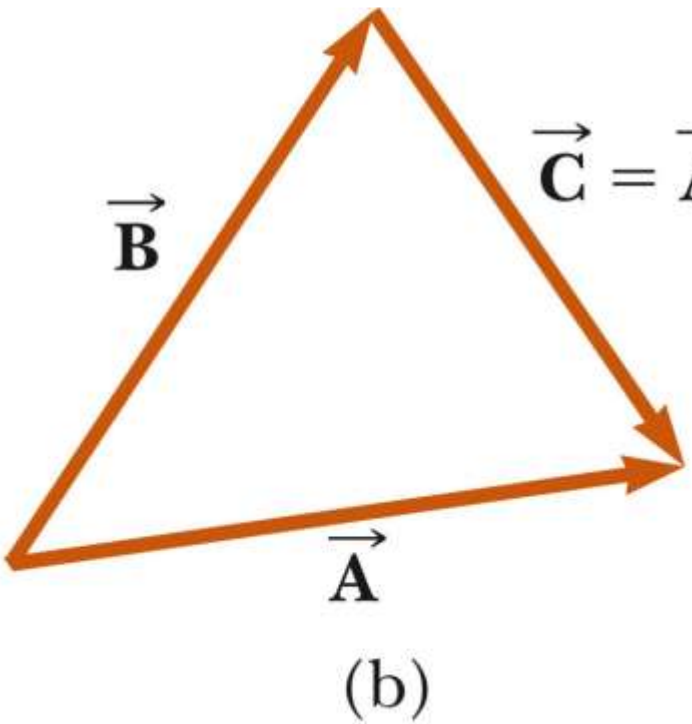
Subtracting Vectors

- Special case of vector addition
- If $\vec{\mathbf{A}} - \vec{\mathbf{B}}$, then use $\vec{\mathbf{A}} + \left(-\vec{\mathbf{B}}\right)$
- Continue with standard vector addition procedure



Subtracting Vectors, Method 2

- Another way to look at subtraction is to find the vector that, added to the second vector gives you the first vector
- $\vec{\mathbf{A}} + \left(-\vec{\mathbf{B}}\right) = \vec{\mathbf{C}}$
  - As shown, the resultant vector points from the tip of the second to the tip of the first



Multiplying or Dividing a Vector by a Scalar

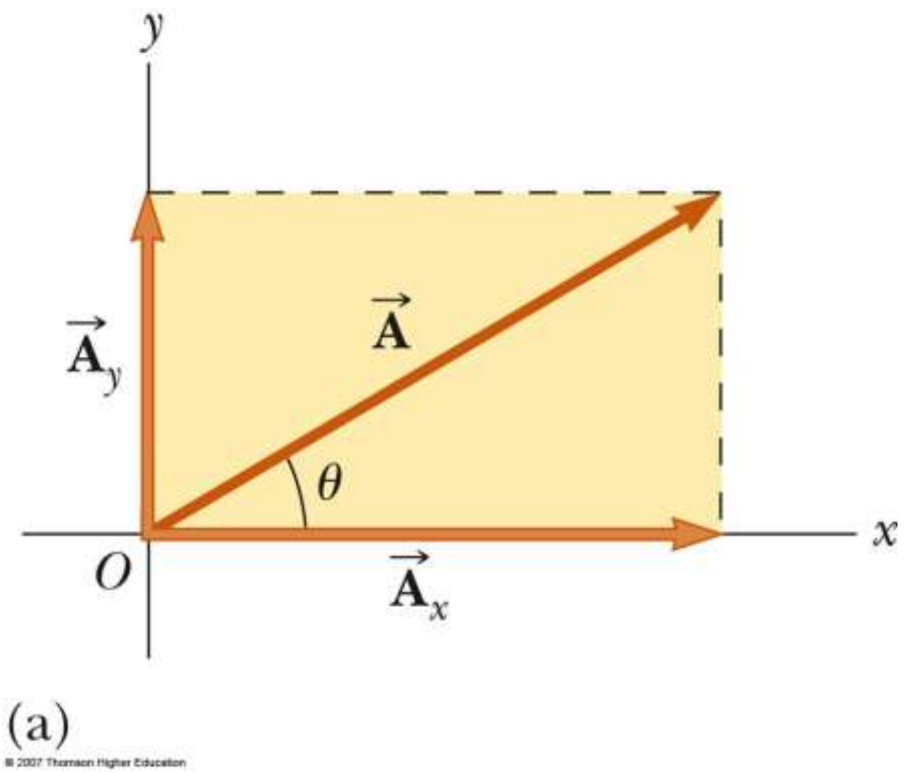
- The result of the multiplication or division of a vector by a scalar is a vector
- The magnitude of the vector is multiplied or divided by the scalar
- If the scalar is positive, the direction of the result is the same as of the original vector
- If the scalar is negative, the direction of the result is opposite that of the original vector

Component Method of Adding Vectors

- Graphical addition is not recommended when
  - High accuracy is required
  - If you have a three-dimensional problem
- Component method is an alternative method
  - It uses projections of vectors along coordinate axes

Components of a Vector, Introduction

- A **component** is a projection of a vector along an axis
  - Any vector can be completely described by its components
- It is useful to use **rectangular components**
  - These are the projections of the vector along the x- and y-axes

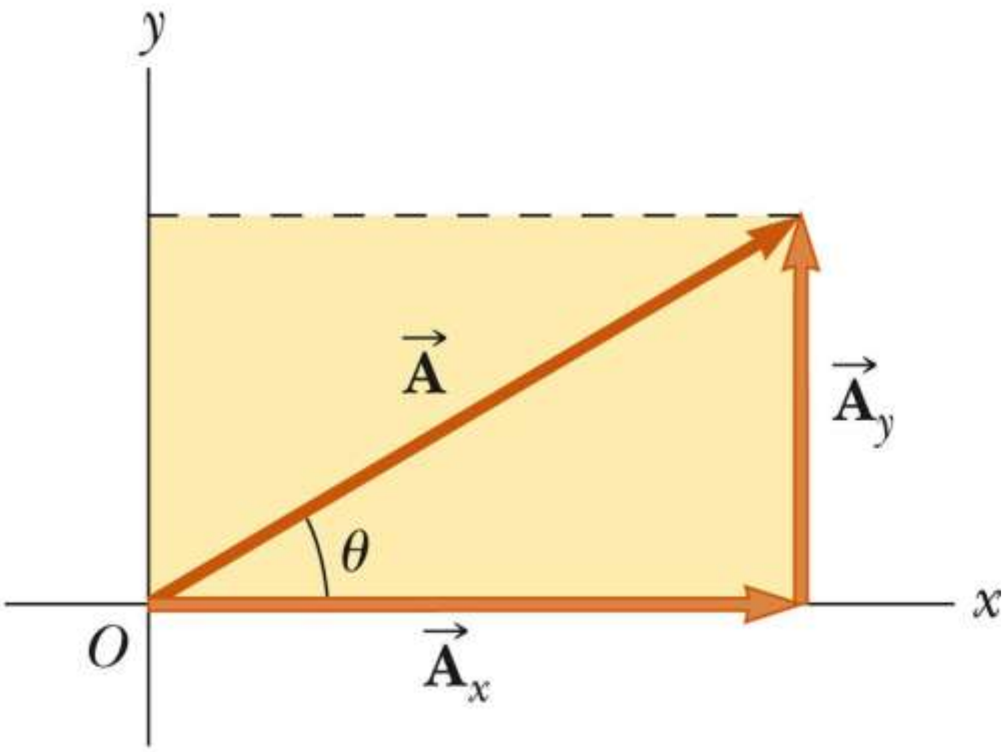


Vector Component Terminology

- $\vec{\mathbf{A}}_x$ *and* $\vec{\mathbf{A}}_y$ are the ***component vectors*** of $\vec{\mathbf{A}}$
  - They are vectors and follow all the rules for vectors
- A_x and A_y are scalars, and will be referred to as the ***components*** of $\vec{\mathbf{A}}$

Components of a Vector

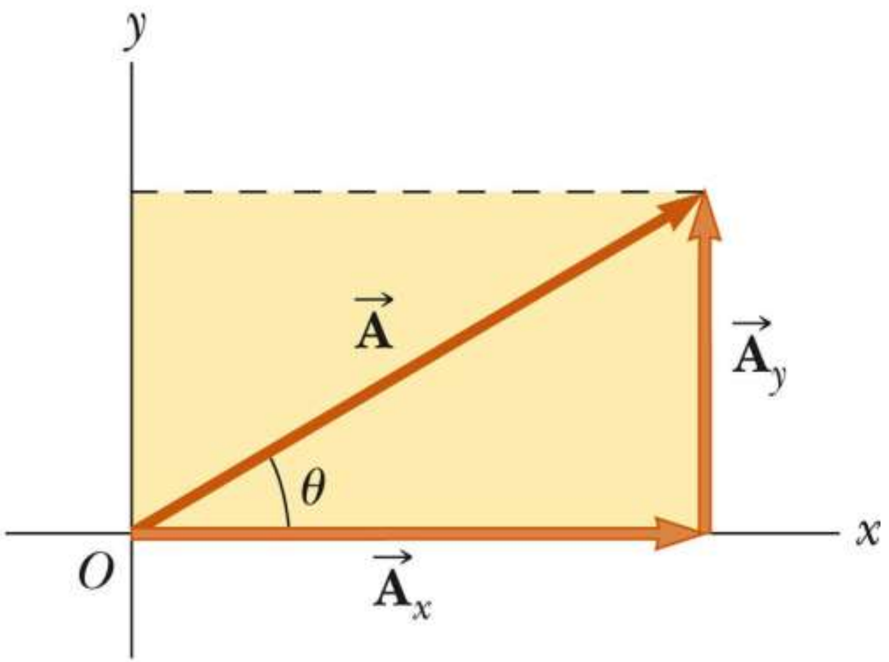
- Assume you are given a vector $\vec{\mathbf{A}}$
- It can be expressed in terms of two other vectors, $\vec{\mathbf{A}}_x$ and $\vec{\mathbf{A}}_y$
- These three vectors form a right triangle
- $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$



(b)

Components of a Vector, 2

- The *y*-component is moved to the end of the *x*-component
- This is due to the fact that any vector can be moved parallel to itself without being affected
  - This completes the triangle



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Components of a Vector, 3

- The x-component of a vector is the projection along the x-axis

$$A_x = A\cos\theta$$

- The y-component of a vector is the projection along the y-axis

$$A_y = A\sin\theta$$

- This assumes the angle θ is measured with respect to the x-axis
  - If not, do not use these equations, use the sides of the triangle directly

Components of a Vector, 4

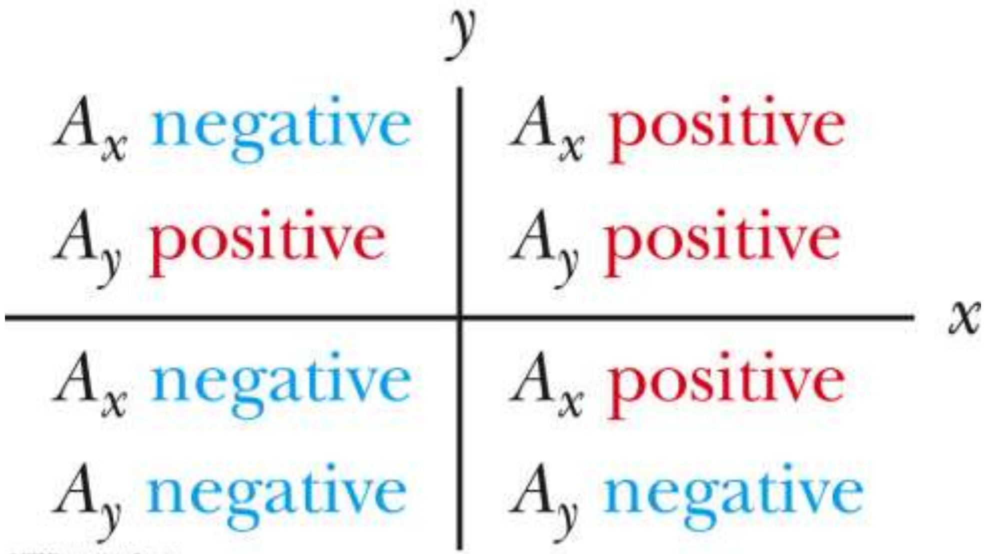
- The components are the legs of the right triangle whose hypotenuse is the length of *A*

$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{A_y}{A_x}$$

  - May still have to find θ with respect to the positive x-axis

Components of a Vector, final

- The components can be positive or negative and will have the same units as the original vector
- The signs of the components will depend on the angle



Unit Vectors

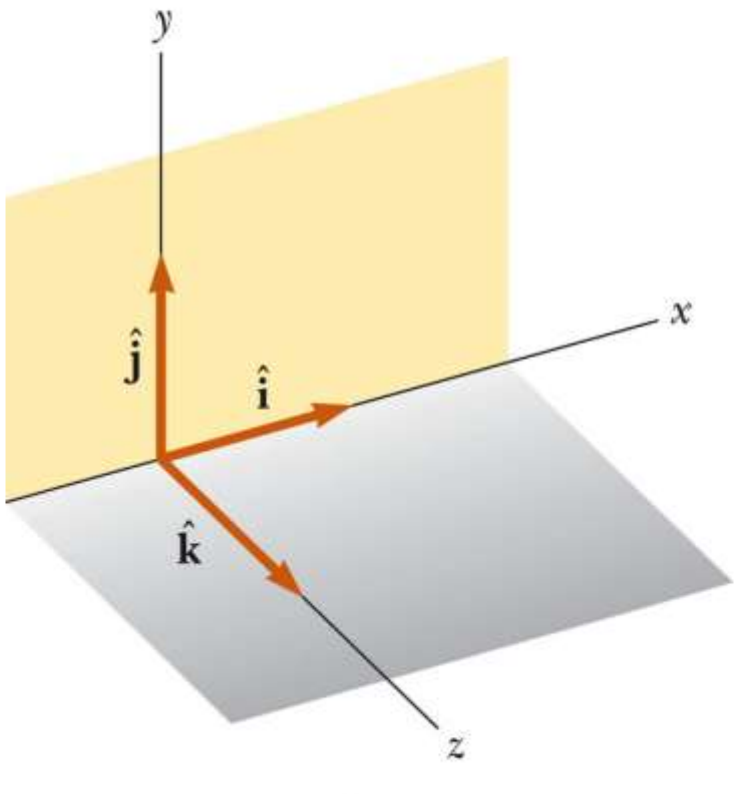
- A ***unit vector*** is a dimensionless vector with a magnitude of exactly 1.
- Unit vectors are used to specify a direction and have no other physical significance

Unit Vectors, cont.

- The symbols

  $$\hat{\mathrm{i}}, \hat{\mathrm{j}}, \text{ and } \hat{\mathrm{k}}$$

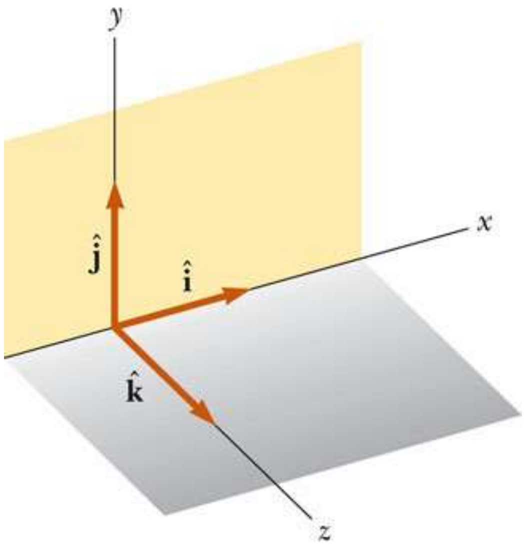
  represent unit vectors
- They form a set of mutually perpendicular vectors in a right-handed coordinate system
- Remember, $\left|\hat{\mathbf{i}}\right| = \left|\hat{\mathbf{j}}\right| = \left|\hat{\mathbf{k}}\right| = 1$



(a)

Viewing a Vector and Its Projections

- Rotate the axes for various views
- Study the projection of a vector on various planes
  - x, y
  - x, z
  - y, z



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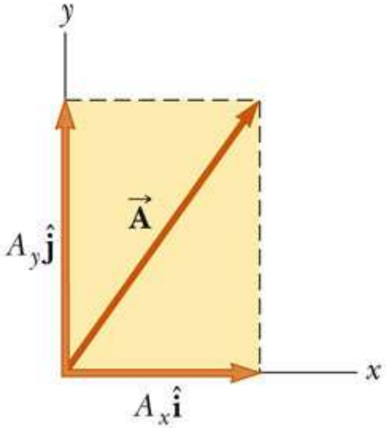
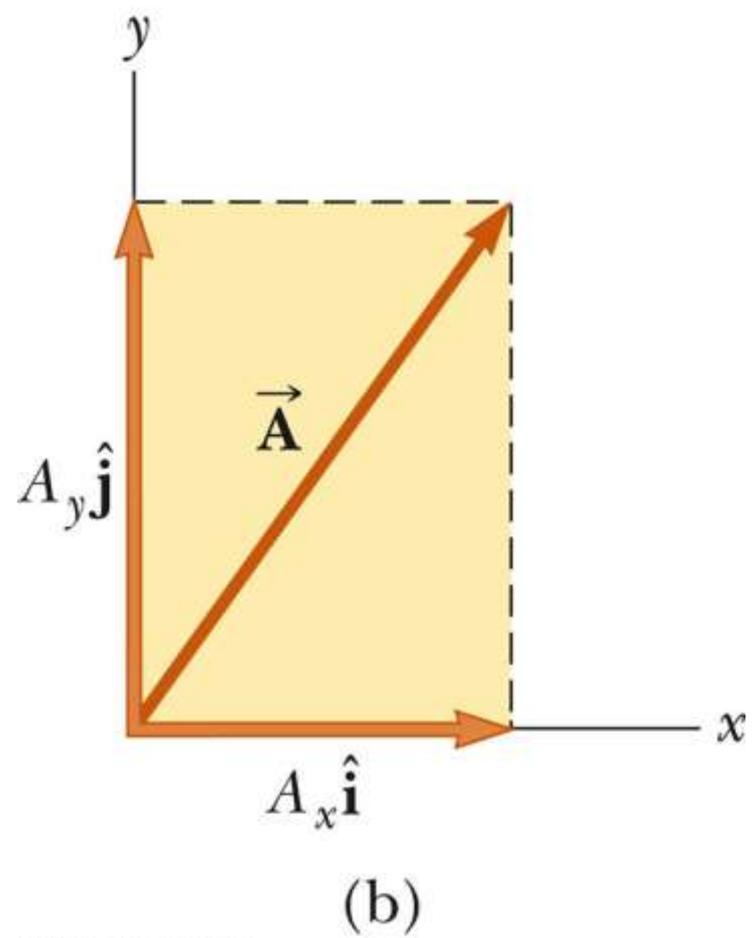


(b)

Unit Vectors in Vector Notation

- $\mathbf{A_x}$ is the same as $A_x\,\hat{\mathbf{i}}$ and $\mathbf{A_y}$ is the same as $A_y\hat{\mathbf{j}}$ etc.
- The complete vector can be expressed as

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$



Adding Vectors Using Unit Vectors

- Using $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$
- Then $\vec{\mathbf{R}} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}\right)$

$$\vec{\mathbf{R}} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}}$$

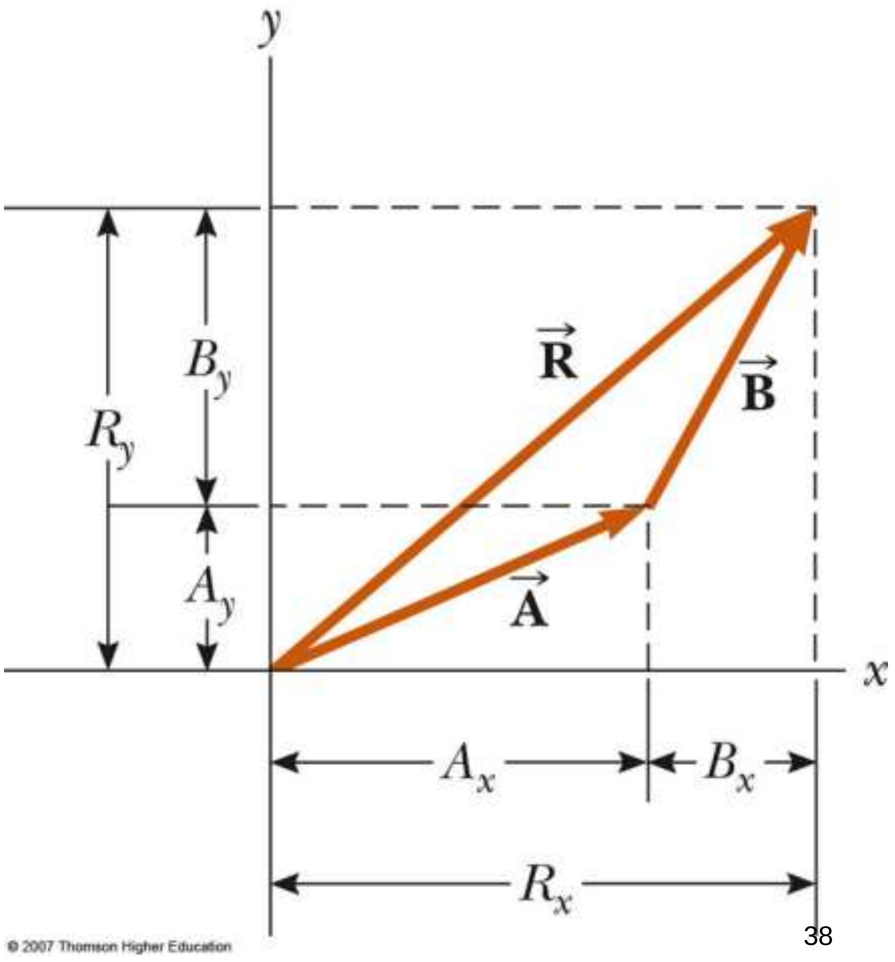
$$\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$$

- and so $R_x = A_x + B_x$ and $R_y = A_y + B_y$

$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \tan^{-1}\frac{R_y}{R_x}$$

Adding Vectors with Unit Vectors

- Note the relationships among the components of the resultant and the components of the original vectors
- $R_x = A_x + B_x$
- $R_y = A_y + B_y$



Three-Dimensional Extension

- Using $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$
- Then $\vec{\mathbf{R}} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right)$

$$\vec{\mathbf{R}} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} + \left(A_z + B_z\right)\hat{\mathbf{k}}$$

$$\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$$

- and so $R_x = A_x + B_x$, $R_y = A_y + B_y$, and $R_z = A_x + B_z$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \qquad \theta = \cos^{-1}\frac{R_x}{R}, etc.$$

Example 3.3

Find the sum of two vectors $\vec{A}$ and $\vec{B}$ lying in the xy plane and given by

$$\vec{A} = \left(2\hat{i} + 2\hat{j}\right)\ \mathrm{m}$$

$$\vec{B} = \left(2\hat{i} - 4\hat{j}\right)\ \mathrm{m}$$

Example 3.3, Solution:

$$\vec{A} = \left(2\hat{i} + 2\hat{j}\right)\ \text{m} \qquad \vec{B} = \left(2\hat{i} - 4\hat{j}\right)\ \text{m}$$

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

$$\vec{R} = (2 + 2)\hat{i} + (2 - 4)\hat{j} + (0 + 0)\hat{k} = 4\hat{i} - 2\hat{j}$$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{(4)^2 + (-2)^2 + (0)^2} = 4.5\ \text{m}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2}{4} = -0.5 \quad \Rightarrow \theta = -27^\circ$$

Example 3.4

Aparticle undergoes three consecutive displacements:

$$\Delta\vec{r}_1 = \left(15\hat{i} + 30\hat{j} + 12\hat{k}\right)\ \text{cm} \quad , \quad \Delta\vec{r}_2 = \left(23\hat{i} - 14\hat{j} - 5\hat{k}\right)\ \text{cm, and}$$

$$\Delta\vec{r}_3 = \left(-13\hat{i} + 15\hat{j}\right)\ \text{cm}$$

Find the components of the resultant displacement and its magnitude.

Example 3.4, Solution:

$$\Delta\vec{r}_1 = \left(15\hat{i} + 30\hat{j} + 12\hat{k}\right)\ \text{cm} \quad , \quad \Delta\vec{r}_2 = \left(23\hat{i} - 14\hat{j} - 5\hat{k}\right)\ \text{cm, and}$$

$$\Delta\vec{r}_3 = \left(-13\hat{i} + 15\hat{j}\right)\ \text{cm}$$

$$\Delta\vec{r} = \Delta\vec{r}_1 + \Delta\vec{r}_2 + \Delta\vec{r}_3 = \left(15 + 23 - 13\right)\hat{i} + \left(30 - 14 + 15\right)\hat{j} + \left(12 - 5 + 0\right)\hat{k}$$

$$\Delta\vec{r} = (25\hat{i} + 31\hat{j} + 7\hat{k})\ \text{cm}$$

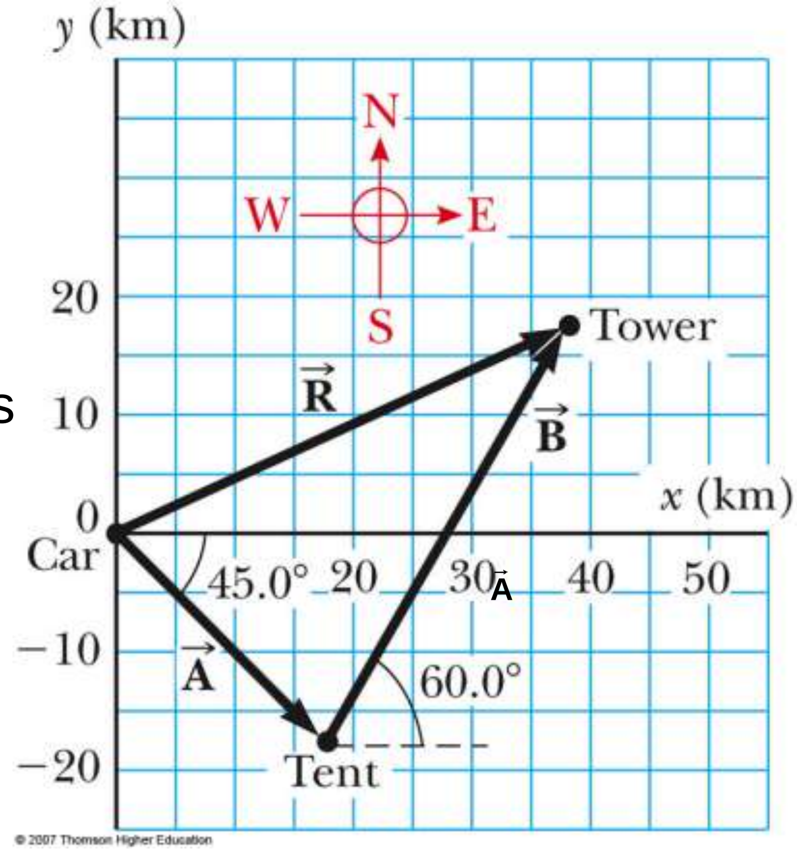
$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{(25)^2 + (31)^2 + (7)^2} = 40\ \text{cm}$$

Example 3.5 – Taking a Hike

- A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

Example 3.5

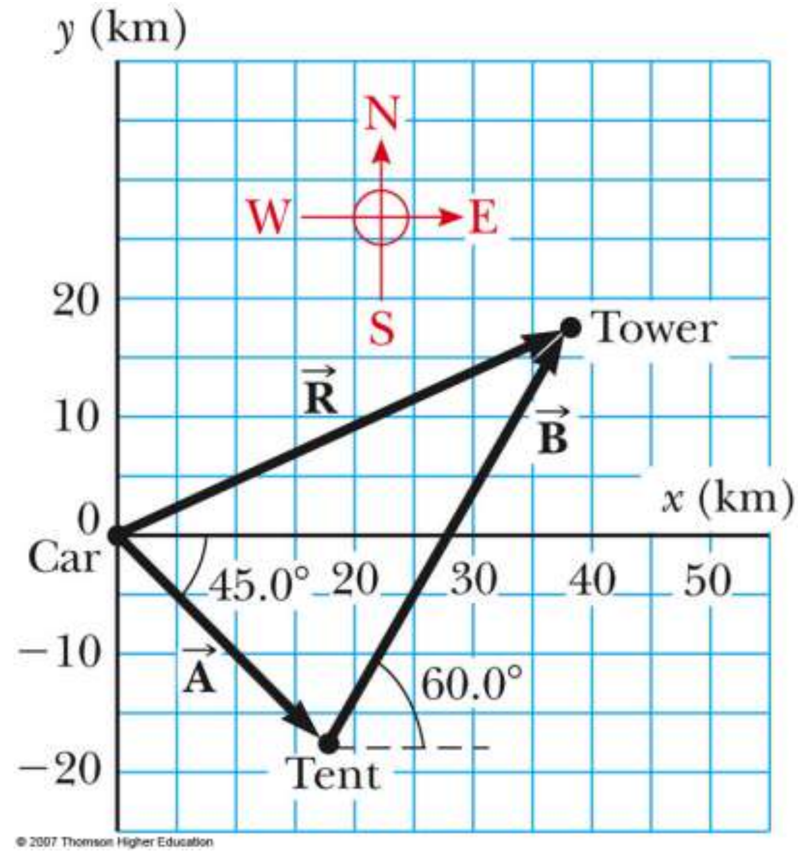
- (A) Determine the components of the hiker's displacement for each day.



Solution: We *conceptualize* the problem by drawing a sketch as in the figure above. If we denote the displacement vectors on the first and second days by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ respectively, and use the car as the origin of coordinates, we obtain the vectors shown in the figure. Drawing the resultant $\vec{\mathbf{R}}$, we can now *categorize* this problem as an addition of two vectors.

Example 3.5

- We will analyze this problem by using our new knowledge of vector components. Displacement $\vec{\mathbf{A}}$ has a magnitude of 25.0 km and is directed 45.0° below the positive *x* axis.



From Equations 3.8 and 3.9, its components are:

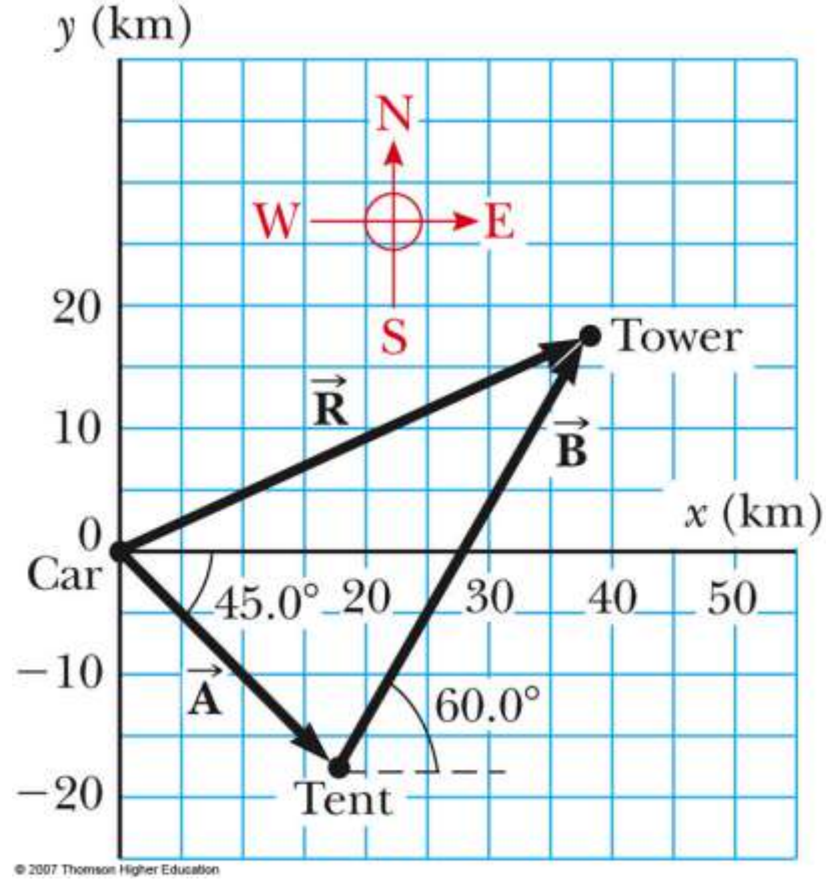
$$A_x = A\cos(-45.0°) = (25.0\text{ km})(0.707) = 17.7\text{ km}$$

$$A_y = A\sin(-45.0°) = (25.0\text{ km})(-0.707) = -17.7\text{ km}$$

The negative value of A_y indicates that the hiker walks in the negative y direction on the first day. The signs of A_x and A_y also are evident from the figure above.

Example 3.5

- The second displacement $\vec{\mathbf{B}}$ has a magnitude of 40.0 km and is 60.0° north of east.



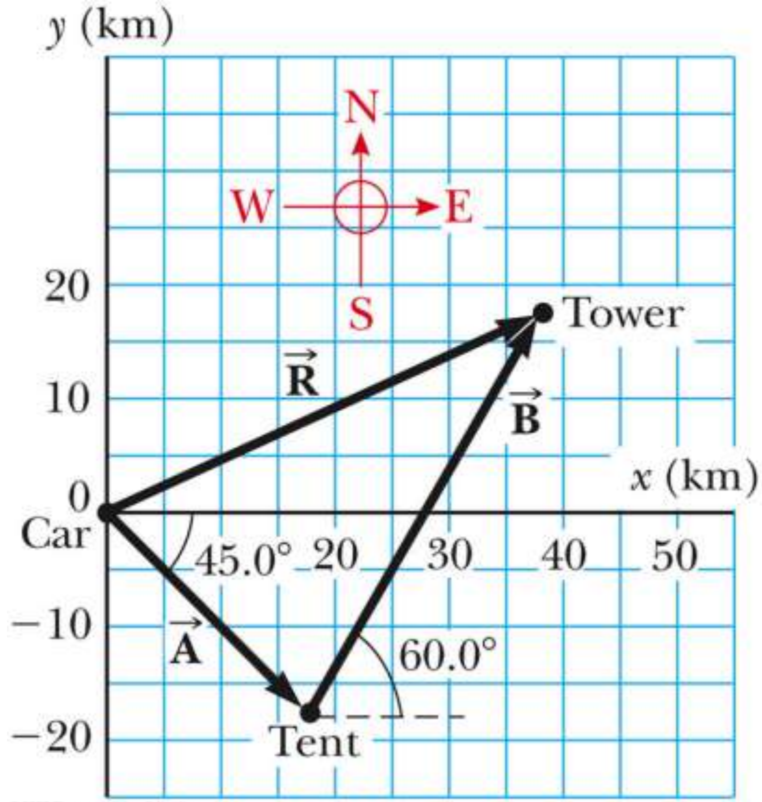
Its components are:

$$B_x = B\cos 60.0° = (40.0\text{ km})(0.500) = 20.0\text{ km}$$

$$B_y = B\sin 60.0° = (40.0\text{ km})(0.866) = 34.6\text{ km}$$

Example 3.5

- (B) Determine the components of the hiker's resultant displacement $\vec{\mathbf{R}}$ for the trip. Find an expression for $\vec{\mathbf{R}}$ in terms of unit vectors.



Solution: The resultant displacement for the trip $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ has components given by Equation 3.15:

$$R_x = A_x + B_x = 17.7 \text{ km} + 20.0 \text{ km} = 37.7 \text{ km}$$

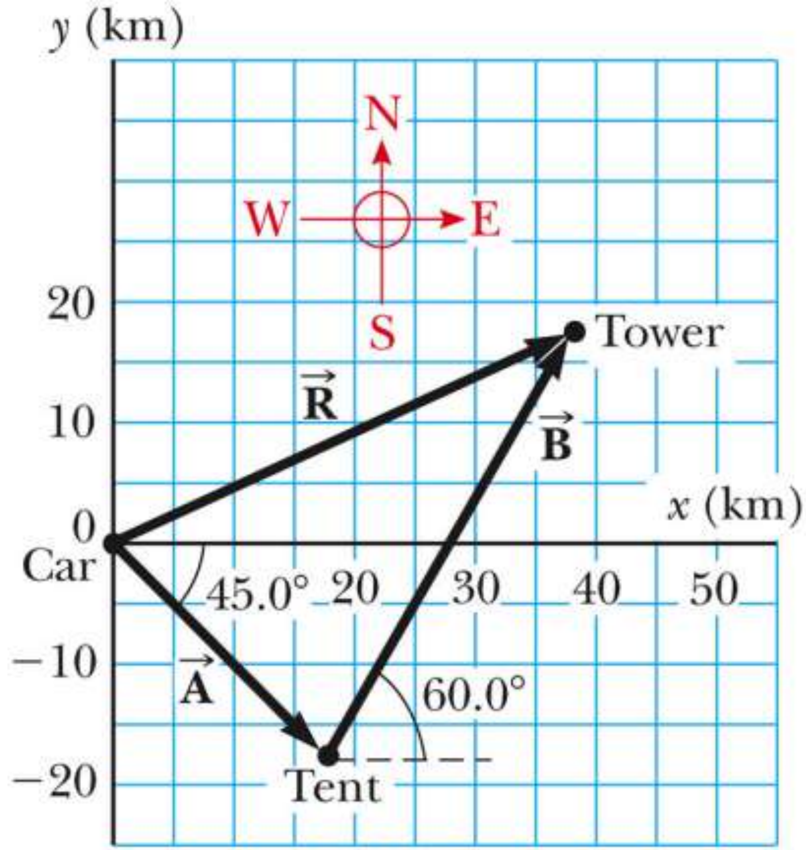
$$R_y = A_y + B_y = -17.7 \text{ km} + 34.6 \text{ km} = 16.9 \text{ km}$$

In unit-vector form, we can write the total displacement as

$$\vec{\mathrm{R}} = (37.7\hat{\mathbf{i}} + 16.9\hat{\mathbf{j}}) \text{ km}$$

Example 3.5

- Using Equations 3.16 and 3.17, we find that the resultant vector has a magnitude of 41.3 km and is directed 24.1° north of east.



Let us *finalize*. The units of $\vec{\mathbf{R}}$ are km, which is reasonable for a displacement. Looking at the graphical representation in the figure above, we estimate that the final position of the hiker is at about (38 km, 17 km) which is consistent with the components of $\vec{\mathbf{R}}$ in our final result. Also, both components of $\vec{\mathbf{R}}$ are positive, putting the final position in the first quadrant of the coordinate system, which is also consistent with the figure.

Exercise:

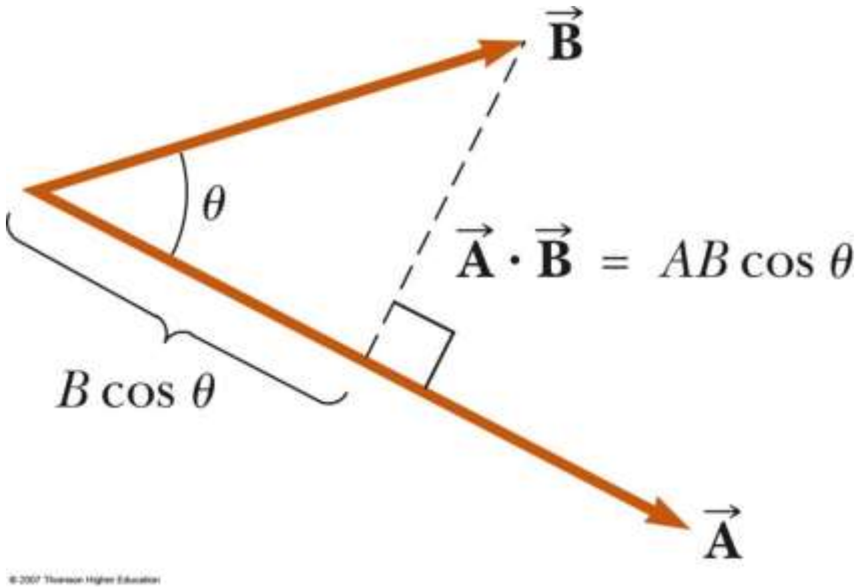
What is the magnitude of the sum of the following vectors?

$$\vec{\mathbf{A}} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}} \qquad \vec{\mathbf{B}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

$$\vec{\mathbf{C}} = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

Scalar Product of Two Vectors
Page: 167

- The scalar product of two vectors is written as $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$
  - It is also called the dot product
- $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} \equiv A\,B\cos\theta$
  - θ is the angle *between A* and *B*



Scalar Product, cont

- The scalar product is commutative
  - $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$
- The scalar product obeys the distributive law of multiplication
  - $\vec{\mathbf{A}} \cdot \left(\vec{\mathbf{B}} + \vec{\mathbf{C}} \right) = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \vec{\mathbf{C}}$

Dot Products of Unit Vectors

- $$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

  $$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$$

- Using component form with vectors:

  $$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}$$

  $$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}}$$

  $$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z$$

Scalar Product:

Very important note: **The result of the scalar product is a scalar quantity (means it is a number)**

Example 7.2

The vectors $\vec{A}$ and $\vec{B}$ are given by

$$\vec{A} = 2\hat{i} + 3\hat{j} \qquad \vec{B} = -\hat{i} + 2\hat{j}$$

(A) Determine the scalar product $\vec{A} \bullet \vec{B}$

(B) Find the angle θ between $\vec{A}$ and $\vec{B}$

Example 7.2, Solution:

$$\vec{A} = 2\hat{i} + 3\hat{j} \qquad \vec{B} = -\hat{i} + 2\hat{j}$$

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z = 2(-1) + 3(2) + 0 = 4$$

(B)

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{13} \quad , \quad B = \sqrt{B_x^2 + B_y^2} = \sqrt{5}$$

$$\cos\theta = \frac{\vec{A} \bullet \vec{B}}{AB} = \frac{4}{\sqrt{13}\sqrt{5}} \quad \Rightarrow \theta = \cos^{-1}\frac{4}{\sqrt{13}\sqrt{5}} = 60.3^\circ$$

What is the angle ϕ between $\vec{a} = 3.0\vec{i} - 4.0\vec{j}$ and $\vec{b} = -2.0\vec{i} + 3.0\vec{k}$.

- What is the **scalar** product of the following vectors? $\overrightarrow{\mathbf{A}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}, \overrightarrow{\mathbf{B}} = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$

The Vector Product
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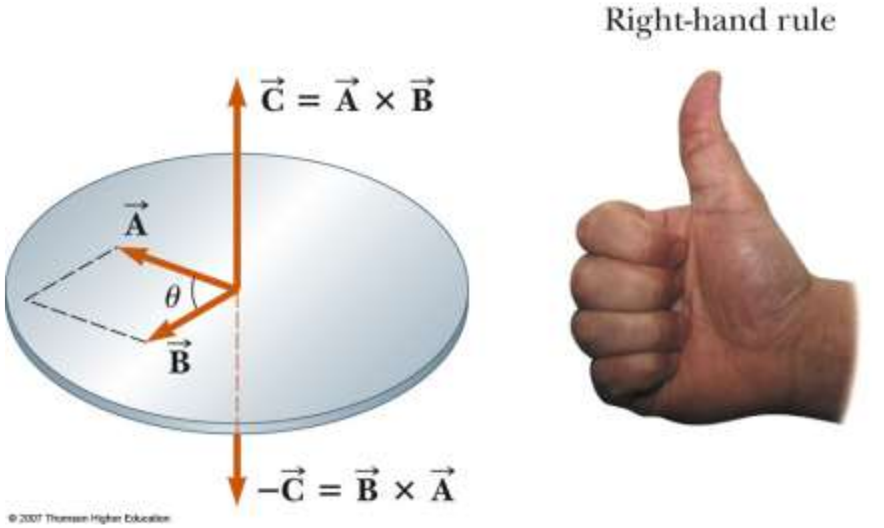
- There are instances where the product of two vectors is another vector
  - Earlier we saw where the product of two vectors was a scalar
    - This was called the dot product
- The vector product of two vectors is also called the cross product

The Vector Product Defined

- Given two vectors, $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$
- The vector (cross) product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined as a *third vector*, $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$
  - **C** is read as "**A** cross **B**"
- The magnitude of vector C is *AB* sin θ
  - θ is the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$

More About the Vector Product

- The quantity $AB \sin \theta$ is equal to the area of the parallelogram formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$
- The direction of $\vec{\mathbf{C}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$
- The best way to determine this direction is to use the right-hand rule



Properties of the Vector Product

- The vector product is *not* commutative. The order in which the vectors are multiplied is important
  - To account for order, remember

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

- If $\vec{\mathbf{A}}$ is parallel to $\vec{\mathbf{B}}$ ($\theta = 0^o$ or 180^o), then

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = 0$$

  - Therefore $\vec{\mathbf{A}} \times \vec{\mathbf{A}} = 0$

More Properties of the Vector Product

- If $\vec{\mathbf{A}}$ is perpendicular to $\vec{\mathbf{B}}$, then $\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB$
- The vector product obeys the distributive law
  - $\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$

Final Properties of the Vector Product

- The derivative of the cross product with respect to some variable such as *t* is

$$\frac{d}{dt}\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) = \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt}$$

  where it is important to preserve the multiplicative order of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$

Vector Products of Unit Vectors

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -\hat{\mathbf{j}} \times \hat{\mathbf{i}} = \hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -\hat{\mathbf{k}} \times \hat{\mathbf{j}} = \hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -\hat{\mathbf{i}} \times \hat{\mathbf{k}} = \hat{\mathbf{j}}$$

Vector Products of Unit Vectors, cont

- Signs are interchangeable in cross products
  - $\vec{\mathbf{A}} \times \left(-\vec{\mathbf{B}}\right) = -\vec{\mathbf{A}} \times \vec{\mathbf{B}}$
  - and $\hat{\mathbf{i}} \times \left(-\hat{\mathbf{j}}\right) = -\hat{\mathbf{i}} \times \hat{\mathbf{j}}$

Using Determinants

- The cross product can be expressed as

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{\mathbf{k}}$$

- Expanding the determinants gives

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \left(A_y B_z - A_z B_y\right) \hat{\mathbf{i}} - \left(A_x B_z - A_z B_x\right) \hat{\mathbf{j}} + \left(A_x B_y - A_y B_x\right) \hat{\mathbf{k}}$$

Vector Product:

Always remember this important note:

The result of the vector product is a vector quantity but its magnitude is a scalar quantity (means it is a number)

Vector Product Example

- Given $\vec{\mathbf{A}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}};\ \vec{\mathbf{B}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$
- Find $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$
- Result

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \times (-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \\ &= 2\hat{\mathbf{i}} \times (-\hat{\mathbf{i}}) + 2\hat{\mathbf{i}} \times 2\hat{\mathbf{j}} + 3\hat{\mathbf{j}} \times (-\hat{\mathbf{i}}) + 3\hat{\mathbf{j}} \times 2\hat{\mathbf{j}} \\ &= 0 + 4\hat{\mathbf{k}} + 3\hat{\mathbf{k}} + 0 = 7\hat{\mathbf{k}}\end{aligned}$$

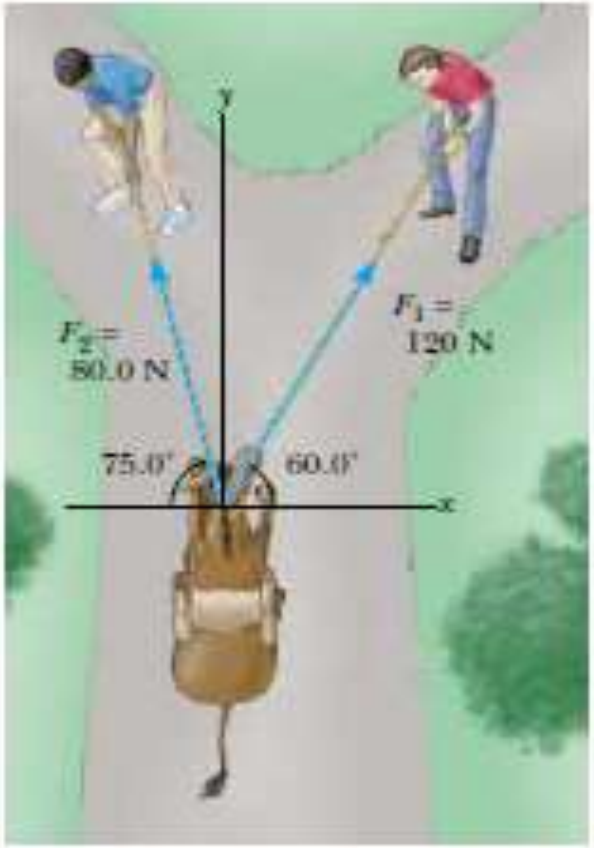
Exercises:

- If $\vec{\mathbf{a}} = 3\vec{\mathbf{i}} - 4\vec{\mathbf{j}}$ and $\vec{\mathbf{b}} = -2\vec{\mathbf{i}} + 3\vec{\mathbf{k}}$, what is $\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$?

- What is the **vector** product of the following vectors? $\overrightarrow{\mathbf{A}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}, \overrightarrow{\mathbf{B}} = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$
- What is the angle between the following vectors: $\vec{a} = 3\vec{i} - 4\vec{j}$ and $\vec{b} = -\vec{i} + 3\vec{k}$ then find the vector product $\vec{C} = \vec{a} \times \vec{b}$.

Example

- The helicopter view in next Fig. shows two people pulling
- on a stubborn mule.
- Find (a) the single force that is equivalent to the two forces shown, and
- (b) the force that a third person would have to exert on the mule to make the resultant force equal to zero. The forces are measured in units of Newtons (abbreviated N).



(a) $\vec{F} = \vec{F}_1 + \vec{F}_2$

$$\vec{F} = 120\cos(60.0°)\,\hat{i} + 120\sin(60.0°)\,\hat{j} - 80.0\cos(75.0°)\,\hat{i} + 80.0\sin(75.0°)\,\hat{j}$$

$$\vec{F} = 60.0\hat{i} + 104\hat{j} - 20.7\hat{i} + 77.3\hat{j} = \left(39.3\hat{i} + 181\hat{j}\right)\text{ N}$$

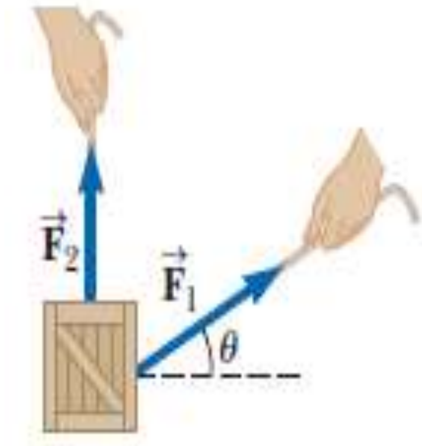
$$\left|\vec{F}\right| = \sqrt{39.3^2 + 181^2} = \boxed{185\text{ N}}$$

$$\theta = \tan^{-1}\left(\frac{181}{39.3}\right) = \boxed{77.8°}$$

(b) $\vec{F}_3 = -\vec{F} = \boxed{\left(-39.3\hat{i} - 181\hat{j}\right)\text{ N}}$

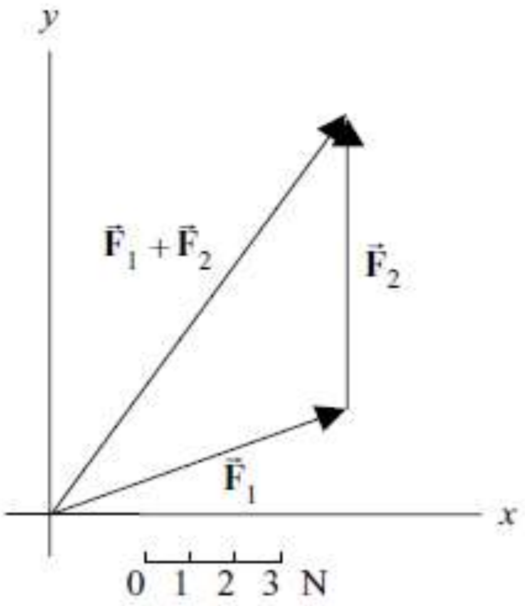
Example

- A force $\mathbf{F_1}$ of magnitude 6.00 units
- acts on an object at the origin in a
- direction θ = 30.0° above the positive
- *x axis (next Fig.). A second* force $\mathbf{F_2}$
- of magnitude 5.00 units acts on the object in the direction of the positive *y axis. Find graphically and analytically*
- the magnitude and direction
- of the resultant force **F**



Solution

Find the resultant $\vec{F}_1 + \vec{F}_2$ graphically by placing the tail of $\vec{F}_2$ at the head of $\vec{F}_1$. The resultant force vector $\vec{F}_1 + \vec{F}_2$ is of magnitude $\boxed{9.5\text{ N}}$ and at an angle of $\boxed{57^\circ \text{ above the } x \text{ axis}}$.



Consider the three displacement vectors $\vec{A} = (3\hat{i} - 3\hat{j})$ m, $\vec{B} = (\hat{i} - 4\hat{j})$ m, and $\vec{C} = (-2\hat{i} + 5\hat{j})$ m. Use the component method to determine (a) the magnitude and direction of the vector $\vec{D} = \vec{A} + \vec{B} + \vec{C}$ and (b) the magnitude and direction of $\vec{E} = -\vec{A} - \vec{B} + \vec{C}$.

Solution

(a) $\vec{D} = \vec{A} + \vec{B} + \vec{C} = 2\hat{i} - 2\hat{j}$

$$\left|\vec{D}\right| = \sqrt{2^2 + 2^2} = \boxed{2.83 \text{ m at } \theta = 315^\circ}$$

(b) $\vec{E} = -\vec{A} - \vec{B} + \vec{C} = -6\hat{i} + 12\hat{j}$

$$\left|\vec{E}\right| = \sqrt{6^2 + 12^2} = \boxed{13.4 \text{ m at } \theta = 117^\circ}$$

Example

A person going for a walk follows the path shown in Figure P3.47. The total trip consists of four straight-line paths. At the end of the walk, what is the person's resultant displacement measured from the starting point?