
CEGEP Linear Algebra Problems

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CEGEP LEVEL LINEAR ALGEBRA PROBLEMS

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Contents

1	Systems of Linear Equations	1
1.1	Introduction to Systems of Linear Equations	1
1.2	Gaussian and Gauss-Jordan Elimination	2
1.3	Applications of Linear Systems	4
2	Matrix Algebra	5
2.1	Introduction to Matrices and Matrix Operations	5
2.2	Matrix Inverses and Algebraic Properties	7
2.3	Elementary Matrices	8
2.4	Linear Systems and Matrices	8
3	Determinants	9
3.1	The Laplace Expansion	9
3.2	Determinants and Elementary Operations	10
3.3	Properties of Determinants and Matrix Inverses	11
3.4	Applications of the Determinant	11
4	Vector Geometry	13
4.1	Introduction to Vectors and Lines	13
4.2	Dot Product and Projections	13
4.3	Cross Product and Planes	13
4.4	Areas, Volumes and Distances	13
4.5	Geometry of Solutions of Linear Systems	13
5	Vector Spaces	15
5.1	Introduction to Vector Spaces	15
5.2	Subspaces	16
5.3	Spanning Sets	16
5.4	Linear Independence	16
5.5	Basis	17
5.6	Dimension	17
A	Answers to Exercises	19
	References	28
	Index	29

Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 [GH] State which of the following equations is a linear equation. If it is not, state why.

- | | |
|--------------------------------|--------------------------------------|
| a. $x + y + z = 10$ | f. $\sqrt{x_1^2 + x_2^2} = 25$ |
| b. $xy + yz + xz = 1$ | g. $x_1 + y + t = 1$ |
| c. $-3x + 9 = 3y - 5z + x - 7$ | h. $\frac{1}{x} + 9 = 3\cos(y) - 5z$ |
| d. $\sqrt{5}y + \pi x = -1$ | i. $\cos(15)y + \frac{x}{4} = -1$ |
| e. $(x - 1)(x + 1) = 0$ | j. $2^x + 2^y = 16$ |

1.1.2 [GH] Solve the system of linear equations using substitution, comparison and/or elimination.

- | | |
|------------------|-----------------------|
| a. $x + y = -1$ | $x - y + z = 1$ |
| $2x - 3y = 8$ | c. $2x + 6y - z = -4$ |
| b. $2x - 3y = 3$ | $4x - 5y + 2z = 0$ |
| $3x + 6y = 8$ | $x + y - z = 1$ |
| | d. $2x + y = 2$ |
| | $y + 2z = 0$ |

1.1.3 [GH] Convert the given system of linear equations into an augmented matrix.

- | |
|---------------------------------|
| $3x + 4y + 5z = 7$ |
| a. $-x + y - 3z = 1$ |
| $2x - 2y + 3z = 5$ |
| $2x + 5y - 6z = 2$ |
| b. $9x - 8z = 10$ |
| $-2x + 4y + z = -7$ |
| $x_1 + 3x_2 - 4x_3 + 5x_4 = 17$ |
| c. $-x_1 + 4x_3 + 8x_4 = 1$ |
| $2x_1 + 3x_2 + 4x_3 + 5x_4 = 6$ |
| $3x_1 - 2x_2 = 4$ |
| d. $2x_1 = 3$ |
| $-x_1 + 9x_2 = 8$ |
| $5x_1 - 7x_2 = 13$ |

1.1.4 [GH] Convert given augmented matrix into a system of linear equations. Use the variables x_1, x_2, \dots

- | | |
|--|---|
| a. $\left[\begin{array}{cc c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array} \right]$ | d. $\left[\begin{array}{cccc c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right]$ |
| b. $\left[\begin{array}{cc c} -3 & 4 & 7 \\ 0 & 1 & -2 \end{array} \right]$ | e. $\left[\begin{array}{cccc c} 1 & 0 & 1 & 0 & 7 \\ 0 & 1 & 3 & 2 & 0 & 5 \end{array} \right]$ |
| c. $\left[\begin{array}{cccc c} 1 & 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & 5 & 7 \end{array} \right]$ | |

1.1.5 [GH] Perform the given row operations on

$$\left[\begin{array}{ccc} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{array} \right].$$

- | | |
|--------------------------------|--|
| a. $-1R_1 \rightarrow R_1$ | d. $2R_2 + R_3 \rightarrow R_3$ |
| b. $R_2 \leftrightarrow R_3$ | e. $\frac{1}{2}R_2 \rightarrow R_2$ |
| c. $R_1 + R_2 \rightarrow R_2$ | f. $-\frac{5}{2}R_1 + R_3 \rightarrow R_3$ |

1.1.6 [GH] Give the row operation that transforms A into B where

$$A = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{array} \right].$$

- | | |
|--|--|
| a. $B = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{array} \right]$ | d. $B = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{array} \right]$ |
| b. $B = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{array} \right]$ | e. $B = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{array} \right]$ |
| c. $B = \left[\begin{array}{ccc} 3 & 5 & 7 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{array} \right]$ | |

1.1.7 [JH] In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy -plane. By geometrical reasoning, show that there are three possibilities:

there is a unique solution, there is no solution, and there are infinitely many solutions.

1.1.8 [JH] Is there a two-unknowns linear system whose solution set is all of \mathbb{R}^2 ?

1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [GH] State whether or not the given matrices are in reduced row echelon form.

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	h. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	m. $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$
b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	i. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	n. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
c. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	j. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	o. $\begin{bmatrix} 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
d. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$	k. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	
e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	l. $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$	
f. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$		
g. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$		

1.2.2 [GH] Use Gauss-Jordan Elimination to put the given matrix into reduced row echelon form.

a. $\begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$	h. $\begin{bmatrix} 4 & 5 & -6 \\ -12 & -15 & 18 \end{bmatrix}$	m. $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$
b. $\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}$	i. $\begin{bmatrix} -2 & -4 & -8 \\ -2 & -3 & -5 \\ 2 & 3 & 6 \end{bmatrix}$	n. $\begin{bmatrix} 2 & -1 & 1 & 5 \\ 3 & 1 & 6 & -1 \\ 3 & 0 & 5 & 0 \end{bmatrix}$
c. $\begin{bmatrix} 4 & 12 \\ -2 & -6 \end{bmatrix}$	j. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$	o. $\begin{bmatrix} 1 & 1 & -1 & 7 \\ 2 & 1 & 0 & 10 \\ 3 & 2 & -1 & 17 \end{bmatrix}$
d. $\begin{bmatrix} -5 & 7 \\ 10 & 14 \end{bmatrix}$	k. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & -3 & 0 \end{bmatrix}$	p. $\begin{bmatrix} 4 & 1 & 8 & 15 \\ 1 & 1 & 2 & 7 \\ 3 & 1 & 5 & 11 \end{bmatrix}$
e. $\begin{bmatrix} -1 & 1 & 4 \\ -2 & 1 & 1 \end{bmatrix}$	l. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 6 & 9 \end{bmatrix}$	
f. $\begin{bmatrix} 7 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$		
g. $\begin{bmatrix} 3 & -3 & 6 \\ -1 & 1 & -2 \end{bmatrix}$		

1.2.3 [JH] Use Gauss's Method to find the unique solution for each system.

a. $\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$	b. $\begin{aligned} x - z &= 0 \\ 3x + y &= 1 \\ -x + y + z &= 4 \end{aligned}$
---	---

1.2.4 [GH] Find the solution to the given linear system. If the system has infinite solutions, give two particular solutions.

- a. $2x_1 + 4x_2 = 2$
 $x_1 + 2x_2 = 1$
- b. $-x_1 + 5x_2 = 3$
 $2x_1 - 10x_2 = -6$
- c. $x_1 + x_2 = 3$
 $2x_1 + x_2 = 4$
- d. $-3x_1 + 7x_2 = -7$
 $2x_1 - 8x_2 = 8$
- e. $-2x_1 + 4x_2 + 4x_3 = 6$
 $x_1 - 3x_2 + 2x_3 = 1$
- f. $-x_1 + 2x_2 + 2x_3 = 2$
 $2x_1 + 5x_2 + x_3 = 2$
- g. $-x_1 - x_2 + x_3 + x_4 = 0$
 $-2x_1 - 2x_2 + x_3 = -1$
- h. $x_1 + x_2 + 6x_3 + 9x_4 = 0$
 $x_1 + x_3 + 2x_4 = 3$
 $x_1 + 2x_2 + 2x_3 = 1$
- i. $2x_1 + x_2 + 3x_3 = 1$
 $3x_1 + 3x_2 + 5x_3 = 2$
- j. $2x_1 + 4x_2 + 6x_3 = 2$
 $1x_1 + 2x_2 + 3x_3 = 1$
 $3x_1 + 6x_2 + 9x_3 = 3$
- k. $2x_1 + 3x_2 = 1$
 $-2x_1 - 3x_2 = 1$
- l. $2x_1 + x_2 + 2x_3 = 0$
 $x_1 + x_2 + 3x_3 = 1$
 $3x_1 + 2x_2 + 5x_3 = 3$

1.2.5 [YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 + x_5 &= 3 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 + 17x_3 - 2x_4 - x_5 &= 1 \end{aligned}$$

- a. Solve the following system by Gauss-Jordan elimination.
- b. Find two particular solution to the above system.
- c. Find a solution to the above system when $x_3 = 1$.

1.2.6 [YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 &= 0 \\ 4x_1 + 17x_3 - 2x_4 &= 0 \\ 9x_1 + 6x_2 + 27x_3 - 4x_4 &= 0 \end{aligned}$$

- a. Solve the system by Gauss-Jordan elimination.
- b. Find two particular nontrivial solution to the system.
- c. Find a solution to the system when $x_1 = 1$.

1.2.7 [JH] Find the coefficients a , b , and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.2.8 [JH] True or false: a system with more unknowns than equations has at least one solution. (As always, to say ‘true’ you must prove it, while to say ‘false’ you must produce a counterexample.)

1.2.9 [JH] For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

1.2.10 [GH] State for which values of k the given system will have exactly 1 solution, infinite solutions, or no solution.

- a. $x_1 + 2x_2 = 1$
 $2x_1 + 4x_2 = k$
- b. $x_1 + 2x_2 = 1$
 $x_1 + kx_2 = 1$
- c. $x_1 + 2x_2 = 1$
 $x_1 + kx_2 = 2$
- d. $x_1 + 2x_2 = 1$
 $x_1 + 3x_2 = k$

1.2.11 [YL] Given the augmented matrix of a linear system:

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a^2 - 1 & b^2 - a^2 \end{array} \right]$$

If possible for what values of a and b the system has

- a. no solution? Justify.
- b. exactly one solution? Justify.
- c. infinitely many solutions? Justify.

1.2.12 [YL] Given the augmented matrix of a linear system

$$\left[\begin{array}{ccccc} 1 & 3 & 1 & -4 & b_1 \\ 3 & -2 & 4 & 5 & b_2 \\ 4 & 1 & 5 & 1 & b_3 \\ 7 & -1 & 9 & 6 & b_4 \end{array} \right]$$

Determine the restrictions on the b_i 's for the system to be consistent.

1.2.13 [JH] Prove that, where a, b, \dots, e are real numbers and $a \neq 0$, if

$$ax + by = c$$

has the same solution set as

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

1.2.14 [JH] Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

1.3 Applications of Linear Systems

1.3.1 Place Holder

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [JH] Find the indicated entry of the following matrix.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{bmatrix}$$

- a. $a_{2,1}$ b. $a_{1,2}$ c. $a_{2,2}$ d. $a_{3,1}$

2.1.2 [JH] Determine the size of each matrix.

a. $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{bmatrix}$ c. $\begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$

2.1.3 [GH] Simplify the given expression where

$$A = \begin{bmatrix} 1 & -1 \\ 7 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 2 \\ 5 & 9 \end{bmatrix}$$

- a. $A + B$ c. $3(A - B) + B$
b. $2A - 3B$ d. $2(A - B) - (A - 3B)$

2.1.4 [GH] The row and column matrix U and V are defined. Find the product UV , where possible.

a. $U = \begin{bmatrix} 1 & -4 \end{bmatrix}, \quad V = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ c. $U = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad V = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
b. $U = \begin{bmatrix} 6 & 2 & -1 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 3 \\ 2 \\ 9 \\ 5 \end{bmatrix}$ d. $U = \begin{bmatrix} 2 & -5 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2.1.5 [GH] State the dimensions of A and B . State the dimensions of AB and BA , if the product is defined. Then compute the product AB and BA , if possible.

a. $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 7 \\ 4 & 2 & 9 \end{bmatrix}$

c. $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$

d. $A = \begin{bmatrix} -2 & -1 \\ 9 & -5 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 6 & -4 \\ 0 & 6 & -3 \end{bmatrix}$

e. $A = \begin{bmatrix} 2 & 6 \\ 6 & 2 \\ 5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 5 & 0 \\ -4 & 4 & -4 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 4 \\ 7 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & -5 & 5 \\ -2 & 1 & 3 & -5 \end{bmatrix}$

g. $A = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $A = \begin{bmatrix} -4 & -1 & 3 \\ 2 & -3 & 5 \\ 1 & 5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 & 3 \\ -1 & 1 & -1 \\ 4 & 0 & 2 \end{bmatrix}$

2.1.6 [GH] Given a diagonal matrix D and a matrix A , compute the product DA and AD , if possible.

a. $D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

b. $D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

c. $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

d. $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

2.1.7 [GH] Given a matrix A compute A^2 and A^3 .

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

d. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

e. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

a. $\begin{bmatrix} 4 & 1 & 1 \\ -2 & 0 & 0 \\ -1 & -2 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -5 \\ 9 & 5 \end{bmatrix}$

c. $\begin{bmatrix} -10 & 6 & -7 & -9 \\ -2 & 1 & 6 & -9 \\ 0 & 4 & -4 & 0 \\ -3 & -9 & 3 & -10 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 6 & 4 \\ -1 & 8 & -10 \end{bmatrix}$

e. Any skew-symmetric matrix.

f. I_n

2.1.8 [HE] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

a. $3C - 4D$

b. $A - (D + 2C)$

c. $A - E$

d. AE

e. $3BC - 4BD$

f. $CB + D$

g. GC

h. FG

i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A , B , and C are matrices above.

2.1.9 [GH] In each part a matrix A is given. Find A^T . State whether A is upper/lower triangular, diagonal, symmetric and/or skew symmetric.

a. $\begin{bmatrix} -9 & 4 & 10 \\ 6 & -3 & -7 \\ -8 & 1 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 2 & -9 \\ 5 & -4 & -10 \\ -6 & 6 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -7 & -4 & -9 \\ -9 & 6 & 3 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$

e. $\begin{bmatrix} 4 & 0 & 0 \\ -2 & -7 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

f. $\begin{bmatrix} -3 & -4 & -5 \\ 0 & -3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$

2.1.11 [GH] Find values for the scalars a and b that satisfy the given equation.

a. $a \begin{bmatrix} -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

b. $a \begin{bmatrix} 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

c. $a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

d. $a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$

2.1.12 [GH] The following statement

$$(A + B)^2 = A^2 + 2AB + B^2$$

is false. We investigate that claim here.

a. Let $A = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} -5 & -5 \\ -2 & 1 \end{bmatrix}$. Compute $A + B$

b. Find $(A + B)^2$ by using the previous part.

c. Compute $A^2 + 2AB + B^2$.

d. Are the results from the two previous parts equal?

e. Carefully expand the expression $(A + B)^2 = (A + B)(A + B)$ and show why this is not equal to $A^2 + 2AB + B^2$.

2.1.13 [YL]

a. Prove: If A and B are $n \times n$ matrices then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

b. Prove: If A and B are $n \times n$ matrices then $\text{tr}(AB) = \text{tr}(BA)$.

2.1.14 [YL] A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$.

Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

2.1.15 [YL] A square matrix A is called *idempotent* if $A^2 = A$.

Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A .

2.1.10 [GH] Find the trace of the given matrix.

2.2 Matrix Inverses and Algebraic Properties

2.2.5 [YL] Solve for X given that it satisfies

$$DXD^T = \text{tr}(BC)BC$$

2.2.1 [GH] Given the matrices A and B below. Find X where that satisfies the equation.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}$$

a. $2A + X = B$

c. $3A + 2X = -1B$

b. $A - X = 3B$

d. $A - \frac{1}{2}X = -B$

2.2.6 [YL] Given

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & \frac{1}{2} \end{bmatrix}.$$

2.2.2 [GH] Given the matrices A . Find A^{-1} , if possible.

a. $\begin{bmatrix} 1 & 5 \\ -5 & -24 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$

a. Find A^{-1} .

b. Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2.2.3 [GH] Given the matrices A . Find A^{-1} , if possible.

a. $\begin{bmatrix} 25 & -10 & -4 \\ -18 & 7 & 3 \\ -6 & 2 & 1 \end{bmatrix}$

i. $\begin{bmatrix} 2 & 3 & 4 \\ -3 & 6 & 9 \\ -1 & 9 & 13 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -7 \\ 20 & 7 & -48 \end{bmatrix}$

j. $\begin{bmatrix} 5 & -1 & 0 \\ 7 & 7 & 1 \\ -2 & -8 & -1 \end{bmatrix}$

c. $\begin{bmatrix} -4 & 1 & 5 \\ -5 & 1 & 9 \\ -10 & 2 & 19 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -19 & -9 & 0 & 4 \\ 33 & 4 & 1 & -7 \\ 4 & 2 & 0 & -1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -5 & 0 \\ -2 & 15 & 4 \\ 4 & -19 & 1 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 27 & 1 & 0 & 4 \\ 18 & 0 & 1 & 4 \\ 4 & 0 & 0 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 25 & -8 & 0 \\ -78 & 25 & 0 \\ 48 & -15 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -29 & -110 \\ 0 & -3 & -5 & -19 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & 8 \\ -2 & -2 & -3 \end{bmatrix}$

n. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.2.7 [YL] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.2.8 [YL] Prove: If AB and BA are both invertible then A and B are both invertible.

2.2.9 [YL] Prove: If B and C are $n \times n$ matrices such that $A = B^T C + C^T B$ is invertible then A^{-1} is symmetric.

2.2.4 [YL] Solve for A given that it satisfies

$$(I - A^T)^{-1} = (\text{tr}(B)B^2)^T$$

where

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.3 Elementary Matrices

2.3.1 [YL] Write the given matrix as a product of elementary matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

2.3.2 [YL] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.3.3 [YL] Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3 E_2 E_1 A = B$.

2.4 Linear Systems and Matrices

2.4.1 [YL] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1} .

b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

2.4.2 [GH] Given the matrices A and b below. Find x that satisfies the equation $Ax = b$ by using the inverse of A

a. $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix},$

$$b = \begin{bmatrix} 21 \\ 13 \end{bmatrix}$$

b. $A = \begin{bmatrix} 1 & -4 \\ 4 & -15 \end{bmatrix},$

$$b = \begin{bmatrix} 21 \\ 77 \end{bmatrix}$$

c. $A = \begin{bmatrix} 1 & 2 & 12 \\ 0 & 1 & 6 \\ -3 & 0 & 1 \end{bmatrix},$

$$b = \begin{bmatrix} -17 \\ -5 \\ 20 \end{bmatrix}$$

d. $A = \begin{bmatrix} 1 & 0 & -3 \\ 8 & -2 & -13 \\ 12 & -3 & -20 \end{bmatrix},$

$$b = \begin{bmatrix} -34 \\ -159 \\ -243 \end{bmatrix}$$

Chapter 3

Determinants

3.1 The Laplace Expansion

Is it the same for lower triangular matrices?

3.1.1 [GH] Compute the determinant of the following matrices.

a. $\begin{bmatrix} 10 & 7 \\ 8 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 6 & -1 \\ -7 & 8 \end{bmatrix}$

c. $\begin{bmatrix} -1 & -7 \\ -5 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -10 & -1 \\ -4 & 7 \end{bmatrix}$

3.1.2 [GH] For the following matrices, construct the submatrices used to compute the minors $M_{1,1}$, $M_{1,2}$ and $M_{1,3}$. Compute the cofactors $C_{1,1}$, $C_{1,2}$, and $C_{1,3}$.

a. $\begin{bmatrix} 7 & -3 & 10 \\ 3 & 7 & 6 \\ 1 & 6 & 10 \end{bmatrix}$

b. $\begin{bmatrix} -2 & -9 & 6 \\ -10 & -6 & 8 \\ 0 & -3 & -2 \end{bmatrix}$

c. $\begin{bmatrix} -5 & -3 & 3 \\ -3 & 3 & 10 \\ -9 & 3 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -6 & -4 & 6 \\ -8 & 0 & 0 \\ -10 & 8 & -1 \end{bmatrix}$

3.1.3 [GH] Find the determinant of the given matrix using cofactor expansion.

a. $\begin{bmatrix} 3 & 2 & 3 \\ -6 & 1 & -10 \\ -8 & -9 & -9 \end{bmatrix}$

b. $\begin{bmatrix} 8 & -9 & -2 \\ -9 & 9 & -7 \\ 5 & -1 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -1 & 0 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

f. $\begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix}$

3.1.4 [JH] Verify that the determinant of an upper-triangular 3×3 matrix is the product of the main diagonal.

$$\det \left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix} \right) = aei$$

3.1.5 [YL] Solve for λ .

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1 - \lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

3.2 Determinants and Elementary Operations

3.2.1 [GH] A matrix M and $\det(M)$ are given. Matrices A , B and C are obtained by performing operations on M . Determine the determinants of A , B and C and indicate the operations used to obtain A , B and C .

a. $M = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}$,
 $\det(M) = -41$,

$$A = \begin{bmatrix} 18 & 14 & 16 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 96 & 73 & 83 \end{bmatrix},$$

$$C = \begin{bmatrix} 9 & 1 & 6 \\ 7 & 3 & 3 \\ 8 & 7 & 3 \end{bmatrix}.$$

b. $M = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}$,
 $\det(M) = 45$,

$$A = \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & -1 \\ 3 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ 8 & 16 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & 4 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}.$$

c. $M = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$,
 $\det(M) = -16$,

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 5 & 1 & 5 \\ 4 & 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} -5 & -1 & -5 \\ -4 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 15 & 3 & 15 \\ 12 & 0 & 6 \\ 0 & 0 & 12 \end{bmatrix}.$$

d. $M = \begin{bmatrix} 5 & 4 & 0 \\ 7 & 9 & 3 \\ 1 & 3 & 9 \end{bmatrix}$,
 $\det(M) = 120$,

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 7 & 9 & 3 \\ 5 & 4 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 5 & 4 & 0 \\ 14 & 18 & 6 \\ 3 & 9 & 27 \end{bmatrix},$$

$$C = \begin{bmatrix} -5 & -4 & 0 \\ -7 & -9 & -3 \\ -1 & -3 & -9 \end{bmatrix}.$$

3.2.2 [GH] Find the determinant of the given matrix by using elementary operations to bring the matrix under triangular form.

a. $\begin{bmatrix} -4 & 3 & -4 \\ -4 & -5 & 3 \\ 3 & -4 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -2 & 1 \\ 5 & 5 & 4 \\ 4 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} -5 & 0 & -4 \\ 2 & 4 & -1 \\ -5 & 0 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

f. $\begin{bmatrix} -5 & 1 & 0 & 0 \\ -3 & -5 & 2 & 5 \\ -2 & 4 & -3 & 4 \\ 5 & 4 & -3 & 3 \end{bmatrix}$

g. $\begin{bmatrix} 2 & -1 & 4 & 4 \\ 3 & -3 & 3 & 2 \\ 0 & 4 & -5 & 1 \\ -2 & -5 & -2 & -5 \end{bmatrix}$

3.2.3 [YL] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.2.4 Vandermonde's determinant [JH] Prove:

$$\det \left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \right) = (b-a)(c-a)(c-b)$$

3.3 Properties of Determinants and 3.4 Applications of the Determinant Matrix Inverses

3.4.1 [YL] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned}x_1 - 2x_2 + 3x_3 &= 4 \\5x_2 - 6x_3 &= 7 \\8x_3 &= 9\end{aligned}$$

3.3.1 [JH] Find the adjoint of the following matrices.

a. $\begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 1 & 8 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$

3.3.2 [JH] Which real numbers x make this matrix singular?

$$\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}$$

3.3.3 [JH] Prove: If S and T are $n \times n$ matrix then $\det(TS) = \det(ST)$.

3.3.4 [JH]

- Suppose that $\det(A) = 3$ and that $\det(B) = 2$. Find $\det(A^2 B^T B^{-2} A^T)$.
- If $\det(A) = 0$ then show that $\det(6A^3 + 5A^2 + 2A) = 0$.

3.3.5 [JH]

- Give a non-identity matrix with the property that $A^T = A^{-1}$.
- Prove: If $A^T = A^{-1}$ then $\det(A) = \pm 1$.
- Does the converse to the above hold?

3.3.6 [JH] Two matrices H and G are said to be *similar* if there is a nonsingular matrix P such that $H = P^{-1}GP$. Show that similar matrices have the same determinant.

3.3.7 [JH] Show that this gives the equation of a line in \mathbb{R}^2 thru (x_2, y_2) and (x_3, y_3) .

$$\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

3.3.8 [YL] Let A and B be $n \times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in \mathbb{R}^n* : If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$.

4.3 Cross Product and Planes

4.3.1 Place Holder

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4.4 Areas, Volumes and Distances

4.4.1 Place Holder

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4.5 Geometry of Solutions of Linear Systems

4.5.1 Place Holder

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [JH] Name the zero vector for each of these vector spaces.

- The space of degree three polynomials under the natural operations.
- The space of 2×3 matrices.
- The space $\{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$.
- The space of real-valued functions of one natural number variable.

5.1.2 [JH] Find the additive inverse, in the vector space, of the vector.

- In \mathcal{P}_3 , the vector $-3 - 2x + x^2$.
- In the space $\mathcal{M}_{2 \times 2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

- In $\{ae^x + be^{-x} \mid a, b \in \mathbb{R}\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [JH] For each, list three elements and then show it is a vector space.

- The set of linear polynomials $\mathcal{P}_1 = \{a_0 + a_1x \mid a_0, a_1 \in \mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations.
- The set of linear polynomials $\{a_0 + a_1x \mid a_0 - 2a_1 = 0\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [JH] For each, list three elements and then show it is a vector space.

- The set of 2×2 matrices with real entries under the usual matrix operations.
- The set of 2×2 matrices with real entries where the 2, 1 entry is zero, under the usual matrix operations.

5.1.5 [JH] For each, list three elements and then show it is a vector space.

- The set of three-component row vectors with their

usual operations.

- The set

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$$

under the operations inherited from \mathbb{R}^4 .

5.1.6 [JH] Show that the following are not vector spaces.

- Under the operations inherited from \mathbb{R}^3 , this set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$$

- Under the operations inherited from \mathbb{R}^3 , this set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

- Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

- Under the usual polynomial operations,

$$\{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+\}$$

where \mathbb{R}^+ is the set of reals greater than zero

- Under the inherited operations,

$$\{(x, y) \in \mathbb{R}^2 \mid x + 3y = 4, 2x - y = 3 \text{ and } 6x + 4y = 10\}$$

5.1.7 [JH] Is the set of rational numbers a vector space over \mathbb{R} under the usual addition and scalar multiplication operations?

5.1.8 [JH] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \quad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [JH] Prove or disprove that \mathbb{R}^3 is a vector space under these operations.

$$\begin{aligned} \text{a. } \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix} \\ \text{b. } \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

5.1.10 [JH] For each, decide if it is a vector space; the intended operations are the natural ones.

- a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

- b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

- c. $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y + w = 1\}$
 d. The set of functions $\{f: \mathbb{R} \rightarrow \mathbb{R} \mid df/dx + 2f = 0\}$
 e. The set of functions $\{f: \mathbb{R} \rightarrow \mathbb{R} \mid df/dx + 2f = 1\}$

5.1.11 [YL] Let $V = \{A \mid A \in \mathcal{M}_{2 \times 2} \text{ and } \det(A) \neq 0\}$ with the following operations:

$$A + B = AB \text{ and } kA = kA$$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.

- a. Does V satisfy closure under vector addition? Justify.
 b. Does V contain a zero vector? If so find it. Justify.
 c. Does V contains an additive inverse for all of its vectors? Justify.
 d. Does V satisfy closure under scalar multiplication? Justify.

5.1.12 [JH] Show that the set \mathbb{R}^+ of positive reals is a vector space when we interpret ' $x + y$ ' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret ' $r \cdot x$ ' as the r -th power of x .

5.1.13 [JH] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.14 [JH]

Is $\{(x, y) \mid x, y \in \mathbb{R}\}$ a vector space under these operations?

- a. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$
 b. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, 0)$

5.1.15 [JH]

Prove the following:

- a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of \vec{v} , then \vec{v} is an additive inverse of \vec{w} . So a vector is an additive inverse of any additive inverse of itself.
 b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.16 [JH]

The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.17 [JH]

Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.18 [JH]

In a vector space every element has an additive inverse. Is the additive inverse unique (*Can some elements have two or more*)?

5.1.19 [JH]

Assume that $\vec{v} \in V$ is not $\vec{0}$.

- a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.
 b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.
 c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [JH]

- a. Prove that every point, line, or plane thru the origin in \mathbb{R}^3 is a subspace of \mathbb{R}^3 under the inherited operations.
 b. What if it doesn't contain the origin?

5.2.2 [JH] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.3 Spanning Sets

5.3.1 [YL] Given the following two subspace of \mathbb{R}^3 : $W_1 = \{x \mid A_1 x = 0\}$ and $W_2 = \{x \mid A_2 x = 0\}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.4 Linear Independence

5.4.1 [YL] Let $\vec{u} = (1, \lambda, -\lambda)$, $\vec{v} = (-2\lambda \ -2 \ 2\lambda)$ and $\vec{w} = (\lambda - 2, -5\lambda - 2, -2)$.

- For what value(s) of λ will $\{\vec{u}, \vec{v}\}$ be linearly dependent.
- For what value(s) of λ will $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent.

5.5 Basis

5.5.1 [YL] Given

$$W = \{p(x) = a_0 + a_2x^2 + a_3x^3 \mid p(-1) = 0\}$$

a subspace of \mathcal{P}_3 .

- Find a basis B for W .
- Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B .

5.6 Dimension

5.6.1 [YL] Given

$$W = \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mid p(1) = 0 \text{ and } p(-1) = 0\}$$

a subspace of \mathcal{P}_3 . Determine the dimension of W .

Appendix A

Answers to Exercises

Note that either a hint, a final answer or a complete solution is provided.

1.1.1

- a. Yes
- b. No
- c. Yes
- d. Yes
- e. No
- f. No
- g. Yes
- h. No
- i. Yes
- j. No

1.1.2

- a. $x = 1, y = -2$
- b. $x = 2, y = \frac{1}{3}$
- c. $x = -1, y = 0$, and $z = 2$.
- d. $x = 1, y = 0$, and $z = 0$.

1.1.3

- a. $\left[\begin{array}{ccc|c} 3 & 4 & 5 & 7 \\ -1 & 1 & -3 & 1 \\ 2 & -2 & 3 & 5 \end{array} \right]$
- b. $\left[\begin{array}{ccc|c} 2 & 5 & -6 & 2 \\ 9 & 0 & -8 & 10 \\ -2 & 4 & 1 & -7 \end{array} \right]$
- c. $\left[\begin{array}{cccc|c} 1 & 3 & -4 & 5 & 17 \\ -1 & 0 & 4 & 8 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{array} \right]$
- d. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 0 & 3 \\ -1 & 9 & 8 \\ 5 & -7 & 13 \end{array} \right]$

1.1.4

- a. $\begin{array}{l} x_1 + 2x_2 = 3 \\ -x_1 + 3x_2 = 9 \end{array}$
- b. $\begin{array}{l} -3x_1 + 4x_2 = 7 \\ x_2 = -2 \end{array}$

c. $\begin{array}{l} x_1 + x_2 - x_3 - x_4 = 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 = 7 \end{array}$

d. $\begin{array}{l} x_1 = 2 \\ x_2 = -1 \\ x_3 = 5 \\ x_4 = 3 \end{array}$

e. $\begin{array}{l} x_1 + x_3 + 7x_5 = 2 \\ x_2 + 3x_3 + 2x_4 = 5 \end{array}$

1.1.5

a. $\begin{bmatrix} -2 & 1 & -7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 7 \\ 5 & 0 & 3 \\ 0 & 4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -1 & 7 \\ 2 & 3 & 5 \\ 5 & 0 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 8 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 2 & -1 \\ 5 & 0 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 0 & 5/2 & -29/2 \end{bmatrix}$

1.1.6

- a. $2R_2 \rightarrow R_2$
- b. $R_1 + R_2 \rightarrow R_2$
- c. $2R_3 + R_1 \rightarrow R_1$
- d. $R_1 \leftrightarrow R_2$
- e. $-R_2 + R_3 \leftrightarrow R_3$

1.1.7 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line. Thus the lines can share one point (giving a unique solution),

share no points (giving no solutions), or share at least two points (which makes them the same line).

1.1.8 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

1.2.1

- a. Yes
- b. No
- c. No
- d. Yes
- e. Yes
- f. Yes
- g. No
- h. Yes
- i. No
- j. Yes
- k. Yes
- l. Yes
- m. No
- n. Yes
- o. Yes

1.2.2

- a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- c. $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$
- d. $\begin{bmatrix} 1 & -7/5 \\ 0 & 0 \end{bmatrix}$
- e. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \end{bmatrix}$
- f. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$
- g. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
- h. $\begin{bmatrix} 1 & \frac{5}{4} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$
- i. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- j. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- k. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- l. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- m. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- n. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$
- o. $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- p. $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.2.3

- a. $x = 2, y = 3$
- b. $x = -1, y = 4$, and $z = -1$.

1.2.4

- a. $x_1 = 1 - 2t; x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 1, x_2 = 0$ and $x_1 = -1, x_2 = 1$.
- b. $x_1 = -3 + 5t; x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 3, x_2 = 0$ and $x_1 = -8, x_2 = -1$.
- c. $x_1 = 1; x_2 = 2$.
- d. $x_1 = 0; x_2 = -1$.
- e. $x_1 = -11 + 10t; x_2 = -4 + 4t; x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -11, x_2 = -4, x_3 = 0$ and $x_1 = -1, x_2 = 0$ and $x_3 = 1$.
- f. $x_1 = -\frac{2}{3} + \frac{8}{9}t; x_2 = \frac{2}{3} - \frac{5}{9}t; x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -\frac{2}{3}, x_2 = \frac{2}{3}, x_3 = 0$ and $x_1 = \frac{4}{9}, x_2 = -\frac{1}{9}, x_3 = 1$.
- g. $x_1 = 1 - s - t; x_2 = s; x_3 = 1 - 2t; x_4 = t$ where $s, t \in \mathbb{R}$. Possible solutions: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ and $x_1 = -2, x_2 = 1, x_3 = -3, x_4 = 2$.
- h. $x_1 = 3 - s - 2t; x_2 = -3 - 5s - 7t; x_3 = s; x_4 = t$ where $s, t \in \mathbb{R}$. Possible solutions: $x_1 = 3, x_2 = -3, x_3 = 0, x_4 = 0$ and $x_1 = 0, x_2 = -5, x_3 = -1, x_4 = 1$.
- i. $x_1 = \frac{1}{3} - \frac{4}{3}t; x_2 = \frac{1}{3} - \frac{1}{3}t; x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = \frac{1}{3}, x_2 = \frac{1}{3}, x_3 = 0$ and $x_1 = -1, x_2 = 0, x_3 = 1$.
- j. $x_1 = 1 - 2s - 3t; x_2 = s; x_3 = t$ where $s, t \in \mathbb{R}$. Possible solutions: $x_1 = 1, x_2 = 0, x_3 = 0$ and $x_1 = 8, x_2 = 1, x_3 = -3$.
- k. No solution; the system is inconsistent.
- l. No solution; the system is inconsistent.

1.2.5

- a. $(x_1, x_2, x_3, x_4, x_5) = (60s - 55t + 30, -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3}, -14s + 13t - 7, s, t)$ where $s, t \in \mathbb{R}$.

- b. If $s = t = 0$ then $(x_1, x_2, x_3, x_4, x_5) = (30, -\frac{38}{3}, -7, 0, 0)$.
 If $s = 0$ and $t = 1$ then $(x_1, x_2, x_3, x_4, x_5) = (-25, \frac{35}{3}, 6, 0, 1)$.
 c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1, x_2, x_3, x_4, x_5) = (-\frac{30}{7}, \frac{316}{21}, 1, \frac{4}{7}, 0)$.

1.2.6

- a. $(x_1, x_2, x_3, x_4) = (60t, -\frac{79}{3}t, -14t, t)$ where $t \in \mathbb{R}$.
 b. If $t = 1$ then $(x_1, x_2, x_3, x_4) = (60, -\frac{79}{3}, -14, 1)$.
 If $t = 3$ then $(x_1, x_2, x_3, x_4) = (180, -79, 42, 3)$.
 c. If $t = \frac{1}{60}$ then $(x_1, x_2, x_3, x_4) = (1, -\frac{79}{180}, -\frac{14}{60}, \frac{1}{60})$.

1.2.7 Because $f(1) = 2$, $f(-1) = 6$, and $f(2) = 3$ we get a linear system.

$$\begin{aligned} 1a + 1b + c &= 2 \\ 1a - 1b + c &= 6 \\ 4a + 2b + c &= 3 \end{aligned}$$

After performing Gaussian elimination we obtain

$$\begin{aligned} a + b + c &= 2 \\ -2b &= 4 \\ -3c &= -9 \end{aligned}$$

which shows that the solution is $f(x) = 1x^2 - 2x + 3$.

1.2.8 The following system with more unknowns than equations

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

has no solution.

1.2.9 After performing Gaussian elimination the system becomes

$$\begin{aligned} x - y &= 1 \\ 0 &= -3 + k \end{aligned}$$

This system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

1.2.10

- a. Never exactly 1 solution; infinite solutions if $k = 2$; no solution if $k \neq 2$.
 b. Exactly 1 solution if $k \neq 2$; infinite solutions if $k = 2$; never no solution.
 c. Exactly 1 solution if $k \neq 2$; no solution if $k = 2$; never infinite solutions.
 d. Exactly 1 solution for all k .

1.2.11

- a. Possible if $a = \pm 1$ and $a \neq \pm b$.
 b. Not possible.
 c. Possible if $a \neq \pm 1$ or $a = \pm b$.

1.2.12 Consistent if $b_3 - b_2 - b_1 = 0$ and $b_4 - 2b_2 - b_1 = 0$.

1.2.13 If $a \neq 0$ then the solution set of the first equation is $\{(x, y) \mid x = (c - by)/a\}$. Taking $y = 0$ gives the solution $(c/a, 0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a) + d \cdot 0 = e$, so $c = e$. Then taking $y = 1$ in $x = (c - by)/a$ gives that $a((c - b)/a) + d \cdot 1 = e$, which gives that $b = d$. Hence they are the same equation.

When $a = 0$ the equations can be different and still have the same solution set: e.g., $0x + 3y = 6$ and $0x + 6y = 12$.

1.2.14 We take three cases: that $a \neq 0$, that $a = 0$ and $c \neq 0$, and that both $a = 0$ and $c = 0$.

For the first, we assume that $a \neq 0$. Then Gaussian elimination

$$\begin{aligned} ax + by &= j \\ -(cb/a) + d)y &= -(cj/a) + k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a) + d \neq 0$; remember that $a \neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a) + d = (ad - bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad - bc \neq 0$.

In the second case, if $a = 0$ but $c \neq 0$, then we swap

$$\begin{aligned} cx + dy &= k \\ by &= j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b \neq 0$ (we use the case assumption that $c \neq 0$ to get a unique x in back substitution). But where $a = 0$ and $c \neq 0$ the condition " $b \neq 0$ " is equivalent to the condition " $ad - bc \neq 0$ ". That finishes the second case.

Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x + by &= j \\ 0x + dy &= k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x, y) will do), but it never has a unique solution. Note that $a = 0$ and $c = 0$ gives that $ad - bc = 0$.

1.3.1 Lorem ipsum dolor sit amet, consectetur adipiscing elit. Aliquam tincidunt cursus volutpat. Quisque non congue sem. Vivamus nec nibh sed est dapibus auctor eu sed nulla. Praesent ornare eleifend nibh a finibus. Proin rutrum neque nec massa tincidunt, non malesuada dolor interdum. Nam a massa sit amet diam efficitur pharetra. Nulla interdum efficitur sem, sit amet commodo orci mattis non. Duis tortor ex, maximus a sapien id, molestie maximus risus.

2.1.1

- a. 2
- b. 3
- c. -1
- d. Not defined.

2.1.2

- a. 2×3
- b. 3×2
- c. 2×2

2.1.3

- a. $\begin{bmatrix} -2 & -1 \\ 12 & 13 \end{bmatrix}$
- b. $\begin{bmatrix} 11 & -8 \\ -1 & -19 \end{bmatrix}$
- c. $\begin{bmatrix} 9 & -7 \\ 11 & -6 \end{bmatrix}$
- d. $\begin{bmatrix} -2 & 1 \\ 12 & 13 \end{bmatrix}$

2.1.4

- a. -22
- b. -2
- c. 23
- d. Not possible.
- e. Not possible.

2.1.5

- a. $AB = \begin{bmatrix} 8 & 3 \\ 10 & -9 \end{bmatrix}, BA = \begin{bmatrix} -3 & 24 \\ 4 & 2 \end{bmatrix}$
- b. $AB = \begin{bmatrix} -1 & -2 & 12 \\ 10 & 4 & 32 \end{bmatrix}, BA$ is not defined
- c. $AB = \begin{bmatrix} 3 & 8 \\ -5 & -8 \\ -8 & -32 \end{bmatrix}, BA$ is not defined
- d. $AB = \begin{bmatrix} 10 & -18 & 11 \\ -45 & 24 & -21 \\ -15 & 12 & -9 \end{bmatrix}, BA = \begin{bmatrix} 52 & -21 \\ 45 & -27 \end{bmatrix}$
- e. $AB = \begin{bmatrix} -32 & 34 & -24 \\ -32 & 38 & -8 \\ -16 & 21 & 4 \end{bmatrix}, BA = \begin{bmatrix} 22 & -14 \\ -4 & -12 \end{bmatrix}$
- f. $AB = \begin{bmatrix} -7 & 3 & 7 & -15 \\ -5 & -1 & -17 & 5 \end{bmatrix}, BA$ is not defined
- g. $AB = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ -2 & 0 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 & 4 \\ -3 & 6 & 1 \\ -1 & 2 & 1 \end{bmatrix}$
- h. $AB = \begin{bmatrix} 21 & -17 & -5 \\ 19 & 5 & 19 \\ 5 & 9 & 4 \end{bmatrix}, BA = \begin{bmatrix} 19 & 5 & 23 \\ 5 & -7 & -1 \\ -14 & 6 & 18 \end{bmatrix}$

2.1.6

- a. $DA = \begin{bmatrix} 2 & 2 & 2 \\ -6 & -6 & -6 \\ -15 & -15 & -15 \end{bmatrix}, AD = \begin{bmatrix} 2 & -3 & 5 \\ 4 & -6 & 10 \\ -6 & 9 & -15 \end{bmatrix}$
- b. $DA = \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix}, AD = \begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix}$
- c. $DA = \begin{bmatrix} d_1a & d_1b \\ d_2c & d_2d \end{bmatrix}, AD = \begin{bmatrix} d_1a & d_2b \\ d_1c & d_2d \end{bmatrix}$
- d. $DA = \begin{bmatrix} d_1a & d_1b & d_1c \\ d_2d & d_2e & d_2f \\ d_3g & d_3h & d_3i \end{bmatrix}, AD = \begin{bmatrix} d_1a & d_2b & d_3c \\ d_1d & d_2e & d_3f \\ d_1g & d_2h & d_3i \end{bmatrix}$

2.1.7

- a. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- b. $A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}$
- c. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}, A^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 125 \end{bmatrix}$
- d. $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- e. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2.1.8

- a. $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$
- b. $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$
- c. Not possible, since dimension of A and E are not the same.
- d. $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$
- e. $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$
- f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D .
- g. $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$
- h. $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.1.9

- a. $\begin{bmatrix} -9 & 6 & -8 \\ 4 & -3 & 1 \\ 10 & -7 & -1 \end{bmatrix}$
- b. $\begin{bmatrix} 4 & 5 & -6 \\ 2 & -4 & 6 \\ -9 & -10 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -9 \\ -7 & 6 \\ -4 & 3 \\ -9 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$, symmetric

e. $\begin{bmatrix} 4 & -2 & 4 \\ 0 & -7 & -2 \\ 0 & 0 & 5 \end{bmatrix}$, A is lower triangular and A^T is upper triangular.

f. $\begin{bmatrix} -3 & 0 & 0 \\ -4 & -3 & 0 \\ -5 & 5 & -3 \end{bmatrix}$, A is upper triangular and A^T is lower triangular.

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, diagonal.

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$, symmetric.

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$, skew-symmetric.

2.1.10

- a. -9
- b. 6
- c. -23
- d. Not defined; the matrix must be square.
- e. 0
- f. n

2.1.11

- a. $a = -1, b = 1/2$
- b. $a = 5/2 + 3/2t, b = t$ where $t \in \mathbb{R}$
- c. $a = 5, b = 0$
- d. No solution.

2.1.12

- a. $\begin{bmatrix} 0 & -2 \\ -5 & -1 \end{bmatrix}$
- b. $\begin{bmatrix} 10 & 2 \\ 5 & 11 \end{bmatrix}$
- c. $\begin{bmatrix} -11 & -15 \\ 37 & 32 \end{bmatrix}$
- d. No.
- e. $(A+B) = AA+AB+BA+BB = A^2+AB+BA+B^2$

2.1.13

- a. Hint: Apply the definition of the trace to arbitrary matrices A and B .
- b. Hint: Analyse the ij product of the elements of the main diagonal.

2.1.14 Disprove: Show that it is impossible to obtain a nonzero matrix.

2.1.15 Hint: Apply the definition of an idempotent matrix.

2.2.1

a. $X = \begin{bmatrix} -5 & 9 \\ -1 & -14 \end{bmatrix}$

b. $X = \begin{bmatrix} 0 & -22 \\ -7 & 17 \end{bmatrix}$

c. $X = \begin{bmatrix} -5 & -2 \\ -9/2 & -19/2 \end{bmatrix}$

d. $X = \begin{bmatrix} 8 & 12 \\ 10 & 2 \end{bmatrix}$

2.2.2

a. $\begin{bmatrix} -24 & -5 \\ 5 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/7 \end{bmatrix}$

c. $\begin{bmatrix} -4/7 & 5/7 \\ 3/7 & -2/7 \end{bmatrix}$

d. The inverse does not exist.

2.2.3

a. $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 6 & 10 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 52 & -48 & 7 \\ 8 & -7 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -9 & 4 \\ 5 & -26 & 11 \\ 0 & -2 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 91 & 5 & -20 \\ 18 & 1 & -4 \\ -22 & -1 & 5 \end{bmatrix}$

e. $\begin{bmatrix} 25 & 8 & 0 \\ 78 & 25 & 0 \\ -30 & -9 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 5 & -3 & -8 \\ -4 & 2 & 5 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- i. The inverse does not exist.
- j. The inverse does not exist.

$$\mathbf{k.} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & -4 \\ -35 & -10 & 1 & -47 \\ -2 & -2 & 0 & -9 \end{bmatrix}$$

$$\mathbf{l.} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -11 & 1 & 0 & -4 \\ -2 & 0 & 1 & -4 \\ -4 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{m.} \begin{bmatrix} 1 & 28 & -2 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 254 & -19 & 110 \\ 0 & -67 & 5 & -29 \end{bmatrix}$$

$$\mathbf{n.} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{o.} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1/4 \end{bmatrix}$$

$$\mathbf{2.2.4} \quad A = \begin{bmatrix} -\frac{3}{4} & 3 \\ 1 & -\frac{3}{4} \end{bmatrix}$$

$$\mathbf{2.2.5} \quad A = \begin{bmatrix} 0 & -1 \\ -11 & -\frac{17}{2} \end{bmatrix}$$

2.2.6

$$\mathbf{a.} \quad A = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$$

$$\mathbf{b.} \quad X = \begin{bmatrix} -\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & \frac{3}{2} & -4 & 2 \end{bmatrix}$$

2.2.7 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.2.8 Hint: Use the definition of the inverse of a matrix.

2.2.9 Hint: Apply the definition of symmetric matrices.

2.3.1

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.2

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

$$\mathbf{2.3.3} \quad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.4.1

$$\mathbf{a.} \quad A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\mathbf{b.} \quad x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$$

2.4.2

$$\mathbf{a.} \quad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{b.} \quad x = \begin{bmatrix} -7 \\ -7 \end{bmatrix}$$

$$\mathbf{c.} \quad x = \begin{bmatrix} -7 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{d.} \quad x = \begin{bmatrix} -7 \\ -7 \\ 9 \end{bmatrix}$$

3.1.1

a. 34

b. 41

c. -44

d. -74

3.1.2

$$\mathbf{a.} \quad M_{1,1} = \begin{bmatrix} 7 & 6 \\ 6 & 10 \end{bmatrix}, M_{1,2} = \begin{bmatrix} 3 & 6 \\ 1 & 10 \end{bmatrix}, M_{1,3} = \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix}.$$

$$C_{1,1} = 43, C_{1,2} = -24, C_{1,3} = 11.$$

$$\mathbf{b.} \quad M_{1,1} = \begin{bmatrix} -6 & 8 \\ -3 & -2 \end{bmatrix}, M_{1,2} = \begin{bmatrix} -10 & 8 \\ 0 & -2 \end{bmatrix}, M_{1,3} = \begin{bmatrix} 10 & -6 \\ 0 & -3 \end{bmatrix}.$$

$$C_{1,1} = 36, C_{1,2} = -20, C_{1,3} = -30.$$

$$\mathbf{c.} \quad M_{1,1} = \begin{bmatrix} 3 & 10 \\ 3 & 9 \end{bmatrix}, M_{1,2} = \begin{bmatrix} -3 & 10 \\ -9 & 9 \end{bmatrix}, M_{1,3} = \begin{bmatrix} -3 & 3 \\ -9 & 3 \end{bmatrix}.$$

$$C_{1,1} = -3, C_{1,2} = -63, C_{1,3} = 18.$$

$$\mathbf{d.} \quad M_{1,1} = \begin{bmatrix} 0 & 0 \\ 8 & -1 \end{bmatrix}, M_{1,2} = \begin{bmatrix} -8 & 0 \\ -10 & -1 \end{bmatrix}, M_{1,3} = \begin{bmatrix} -8 & 0 \\ -10 & 8 \end{bmatrix}.$$

$$C_{1,1} = 0, C_{1,2} = -8, C_{1,3} = -64.$$

3.1.3

a. -59

b. 250

- c. 3
- d. 0
- e. 0
- f. 2

3.1.4 Evaluate the determinant using a cofactor expansion. The same is true for lower triangular matrices.

3.1.5 $\lambda = \frac{3 \pm \sqrt{33}}{4}$

3.2.1

- a. $\det(A) = 90$; $2R_1 \rightarrow R_1$.
 $\det(B) = 45$; $10R_1 + R_3 \rightarrow R_3$.
 $\det(C) = 45$; $C = A^T$.
- b. $\det(A) = 41$; $R_2 \leftrightarrow R_3$.
 $\det(B) = 164$; $-4R_3 \rightarrow R_3$.
 $\det(C) = -41$; $R_2 + R_1 \rightarrow R_1$.
- c. $\det(A) = -16$; $R_1 \leftrightarrow R_2$ then $R_1 \leftrightarrow R_3$.
 $\det(B) = -16$; $-R_1 \rightarrow R_1$ and $-R_2 \rightarrow R_2$.
 $\det(C) = -432$; $C = 3M$.
- d. $\det(A) = -120$; $R_1 \leftrightarrow R_2$ then $R_1 \leftrightarrow R_3$ then $R_2 \leftrightarrow R_3$.
 $\det(B) = 720$; $2R_2 \rightarrow R_2$ and $3R_3 \rightarrow R_3$.
 $\det(C) = -120$; $C = -M$.

3.2.2

- a. 15
- b. -52
- c. 0
- d. 1
- e. -113
- f. 179

3.2.3 $\det(A) = -\frac{5}{12}$

3.2.4 Hint: Use elementary operations to bring the matrix under triangular form.

3.3.1

- a. $\begin{bmatrix} 0 & -1 & 2 \\ 3 & -2 & -8 \\ 0 & 1 & 1 \end{bmatrix}$
- b. $\begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$
- c. $\begin{bmatrix} 0 & -1 \\ -5 & 1 \end{bmatrix}$
- d. $\begin{bmatrix} -24 & -12 & 12 \\ 12 & 6 & -6 \\ -8 & -4 & 4 \end{bmatrix}$
- e. $\begin{bmatrix} 4 & -3 & 2 & -1 \\ -3 & 6 & -4 & 2 \\ 2 & -4 & 6 & -3 \\ -1 & 2 & -3 & 4 \end{bmatrix}$

3.3.2 This equation

$$0 = \det\left(\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}\right) = 64 - 20x + x^2 = (x-16)(x-4)$$

has roots $x = 16$ and $x = 4$.

3.3.3 $\det(TS) = \det(T) \cdot \det(S) = \det(S) \cdot \det(T) = \det(ST)$.

3.3.4

- a. If it is defined then it is $(3^2) \cdot (2) \cdot (2^{-2}) \cdot (3)$.
- b. Hint: $|6A^3 + 5A^2 + 2A| = |A| \cdot |6A^2 + 5A + 2I|$.

3.3.5

- a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- b. $1 = \det(AA^{-1}) = \det(AA^T) = \det(A) \det(A^T) = (\det(A))^2$
- c. The converse does not hold; here is an example.

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

3.3.6 If $H = P^{-1}GP$ then $\det(H) = \det(P^{-1}) \det(G) \det(P) = \det(P^{-1}) \det(P) \det(G) = \det(P^{-1}P) \det(G) = \det(G)$.

3.3.7 An algebraic check is easy.

$$0 = xy_2 + x_2y_3 + x_3y - x_3y_2 - xy_3 - x_2y = x \cdot (y_2 - y_3) + y \cdot (x_3 - x_2) + x_2y_3 - x_3y_2$$

simplifies to the familiar form

$$y = x \cdot (x_3 - x_2) / (y_3 - y_2) + (x_2y_3 - x_3y_2) / (y_3 - y_2)$$

(the $y_3 - y_2 = 0$ case is easily handled).

3.3.8 Hint: Apply the determinant to both sides $AB = -BA$.

3.4.1 $x_1 = 4$

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}$.

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5.1.1

- a. $0 + 0x + 0x^2 + 0x^3$
- b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- c. The constant function $f(x) = 0$
- d. The constant function $f(n) = 0$

5.1.2

- a. $3 + 2x - x^2$
- b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$
- c. $-3e^x + 2e^{-x}$

5.1.3

- a. $1 + 2x$, $2 - 1x$, and x .
- b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

- a. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- b. $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5.1.5

- a. $(1, 2, 3)$, $(2, 1, 3)$, and $(0, 0, 0)$.
- b. $(1, 1, 1, -1)$, $(1, 0, 1, 0)$ and $(0, 0, 0, 0)$.

5.1.6

For each part the set is called Q . For some parts, there are more than one correct way to show that Q is not a vector space.

- a. It is not closed under addition.

$$(1, 0, 0), (0, 1, 0) \in Q \quad (1, 1, 0) \notin Q$$

- b. It is not closed under addition.

$$(1, 0, 0), (0, 1, 0) \in Q \quad (1, 1, 0) \notin Q$$

- c. It is not closed under addition.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

- d. It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \quad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

- e. The set is empty, violating the existence of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

- a. It is not a vector space.

$$(1 + 1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- b. It is not a vector space.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

- a. Yes.
- b. Yes.
- c. No, this set is not closed under the natural addition operation. The vector of all $1/4$'s is an element of this set but when added to itself the result, the vector of all $1/2$'s, is not an element of the set.
- d. Yes.
- e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11

- a. Closed under vector addition. Hint: Apply determinant properties.

$$\text{b. } \vec{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$$

- c. Every $A \in V$ has an additive inverse A^{-1} .
- d. Yes.
- e. Not closed under scalar multiplication. Since $0\vec{0} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin V$

5.1.12 Check all 10 conditions of the definition of a vector space.

5.1.13 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.14

- a. No since $1 \cdot (0, 1) + 1 \cdot (0, 1) \neq (1 + 1) \cdot (0, 1)$.
- b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0, 1) \neq (0, 1)$.

5.1.15

- a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of \vec{v} so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore \vec{v} is also the additive inverse of \vec{w} .
- b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of \vec{v} is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which implies that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.16

Addition is commutative, so in any vector space, for any vector \vec{v} we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.17

It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.18

Each element of a vector space has one and only one additive inverse.

For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of \vec{v} then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.19

Assume that $\vec{v} \in V$ is not $\vec{0}$.

- a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.
- b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.
- c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

5.2.1

- a. Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of \vec{v}, \vec{w} may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.
- b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.2 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r f'$.

5.3.1 Hint: For each subspace determine a set of vectors that spans it.

$$W_1 \subsetneq W_2$$

5.4.1

- a. $\lambda = 1$
- b. $\lambda \neq -1, -\frac{1}{2}, 1$

5.5.1

- a. $B = \{1 + x^3, x^2 + x^3\}$
- b. $(p(x))_B = (-2, 2)$

5.6.1 $\{-1 + x^2, -x + x^3\}$ is a basis of W , therefore W is of dimension 2.

The problems contained in this textbook are taken directly or based on content from the following open source references.

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Index

additive inverse, [15](#), [16](#)
augmented matrix, [1](#), [3](#)

basis, [17](#)

Cauchy-Schwartz Inequality, [13](#)
co-factor expansion, [9](#)
cofactor, [9](#)
cofactor expansion, [9](#)
consistent system, [3](#)
consistent, inconsistent, [3](#)
coordinate vector, [17](#)
Cramer's Rule, [11](#)

determinant, [9](#), [10](#)
diagonal matrix, [5](#), [6](#), [16](#)
dimension, [17](#)

elementary matrix, [8](#)
elementary operation, [10](#)
elementary row operation, [1](#)

function space, [15](#), [16](#)
function subspace, [16](#)

Gauss-Jordan Elimination, [3](#)
Gauss-Jordan elimination, [2](#)
Gaussian Elimination, [2](#)
Gaussian elimination, [2](#), [3](#)
geometric interpretation, [1](#)

homogeneous system, [3](#)

idempotent matrix, [6](#)
ij product, [5](#)
inconsistent, [3](#)
infinite solutions, [3](#)

Laplace expansion, [9](#)
line, plane, [16](#)
linear dependence, [16](#)
linear equation, [1](#)
linear independence, [16](#)
linear system, [8](#)

matrix addition, [5–7](#)
matrix associativity, [5](#), [6](#)
matrix dimension, [5](#)
matrix element, [5](#)
matrix entry, [5](#)

matrix equation, [7](#), [8](#)
matrix inverse, [7](#), [8](#)
matrix multiplication, [6](#)
matrix power, [5](#), [6](#)
matrix product, [5](#), [6](#)
matrix size, [5](#)
matrix space, [15](#), [16](#)
minor, [9](#)

nilpotent matrix, [6](#)
no solution, unique solution, infinitely many solution, [3](#)

particular solution, [2](#), [3](#)
ph, [4](#), [13](#)
place holder, [4](#), [13](#)
polynomial space, [15](#), [17](#)
positive real numbers, [16](#)

quadratic equation, parabola, [3](#)

rational numbers, [15](#)
reduced row echelon form, [2](#)
row-equivalent, [8](#)

scalar multiplication, [6](#)
skew-symmetric matrix, [6](#)
spaning set, [16](#)
subspace, [16](#)
symmetric equation, [7](#)
symmetric matrix, [6](#)
system of linear equation, [8](#)
system of linear equations, [1](#)

trace of a matrix, [6](#)
triangular matrix, [6](#)
trivial subspace, [16](#)

unique solution, [3](#)
unique solution, inconsistent, infinitely many solutions, [3](#)

vector space, [15](#), [16](#)

zero vector, [15](#), [16](#)