CEGEP Linear Algebra Problems

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CEGEP level linear algebra problems

Edited by

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Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 Place Holder

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1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [JH] Use Gauss's Method to find the unique solution for each system.

a.

$$\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$$

b.

$$\begin{aligned} x \qquad - z &= 0 \\ 3x + y \qquad &= 1 \\ -x + y + z &= 4 \end{aligned}$$

1.2.2 [YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 + x_5 &= 3 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 \qquad + 17x_3 - 2x_4 - x_5 &= 1 \end{aligned}$$

a. Solve the following system by Gauss-Jordan elimination.
b. Find two particular solution to the above system.
c. Find a solution to the above system when $x_3 = 1$.

1.2.3 [YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 &= 0 \\ 4x_1 \qquad + 17x_3 - 2x_4 &= 0 \\ 9x_1 + 6x_2 + 27x_3 - 4x_4 &= 0 \end{aligned}$$

a. Solve the system by Gauss-Jordan elimination.
b. Find two particular non-trivial solution to the system.
c. Find a solution to the system when $x_1 = 1$.

1.2.4 [YL] Given the augmented matrix of a linear system:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a^2-1 & b^2-a^2 \end{bmatrix}$$

If possible for what values of a and b the system has

a. no solution? Justify.
b. exactly one solution? Justify.
c. infinitely many solutions? Justify.

1.3 Applications of Linear Systems

1.3.1 Place Holder

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [HE] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

a. $3C - 4D$
b. $A - (D + 2C)$
c. $A - E$
d. AE
e. $3BC - 4BD$
f. $CB + D$
g. GC
h. FG
i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A, B, and C are matrices above.

2.2 Matrix Inverses and Algebraic Properties

2.2.1 [YL] Solve of A given that it satisfies

$$(I - A^T)^{-1} = (\mathrm{tr}(B)B^2)^T$$

where

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.2.2 [YL] Given

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & \frac{1}{2} \end{bmatrix}.$$

a. Find A^{-1}.
b. Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2.2.3 [YL] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.2.4 [YL] Prove: If AB and BA are both invertible then A and B are both invertible.

2.3 Elementary Matrices

2.3.1 [YL] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.4 Linear Systems and Matrices

2.4.1 [YL] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1}.
b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Chapter 3

Determinants

3.1 The Laplace Expansion

3.1.1 [**YL**] Solve for λ.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

3.2 Determinants and Elementary Operations

3.2.1 [**YL**] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.3 Properties of Determinants

3.3.1 [**YL**] Let A and B be $n \times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.4 Applications of the Determinant

3.4.1 [**YL**] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in* $\mathbb{R}^n$: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \bullet \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$.

4.3 Cross Product and Planes

4.3.1 Place Holder

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4.4 Areas, Volumes and Distances

4.4.1 Place Holder

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4.5 Geometry of Solutions of Linear Systems

4.5.1 Place Holder

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [**JH**] Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.
b. The space of 2×3 matrices.
c. The space $\{\, f : [0,1] \to \mathbb{R} \;\mid\; f \text{ is continuous}\,\}$.
d. The space of real-valued functions of one natural number variable.

5.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3 - 2x + x^2$.
b. In the space $\mathcal{M}_{2\times 2}$,
$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$
c. In $\{\, ae^x + be^{-x} \;\mid\; a, b \in \mathbb{R}\,\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{\, a_0 + a_1 x \;\mid\; a_0, a_1 \in \mathbb{R}\,\}$ under the usual polynomial addition and scalar multiplication operations.
b. The set of linear polynomials $\{\, a_0 + a_1 x \;\mid\; a_0 - 2a_1 = 0\,\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.
b. The set of 2×2 matrices with real entries where the $2,1$ entry is zero, under the usual matrix operations.

5.1.5 by Jim Hefferon [**JH**] For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.
b. The set
$$\{\, (x,\ y,\ z,\ w) \in \mathbb{R}^4 \;\mid\; x + y - z + w = 0\,\}$$
under the operations inherited from $\mathbb{R}^4$.

5.1.6 [**JH**] Show that the following are not vector spaces.

a. Under the operations inherited from $\mathbb{R}^3$, this set
$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \;\mid\; x + y + z = 1\,\}$$
b. Under the operations inherited from $\mathbb{R}^3$, this set
$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \;\mid\; x^2 + y^2 + z^2 = 1\,\}$$
c. Under the usual matrix operations,
$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$
d. Under the usual polynomial operations,
$$\{\, a_0 + a_1 x + a_2 x^2 \;\mid\; a_0, a_1, a_2 \in \mathbb{R}^+\,\}$$
where $\mathbb{R}^+$ is the set of reals greater than zero
e. Under the inherited operations,
$$\{\, (x,\ y) \in \mathbb{R}^2 \;\mid\; x + 3y = 4,\ 2x - y = 3 \text{ and } 6x + 4y = 10\,\}$$

5.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

5.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

b. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5.1.10 [**JH**] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

c. $\left\{ (x,\ y,\ z,\ w) \in \mathbb{R}^4 \;\middle|\; x + y + w = 1 \right\}$

d. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 0 \}$

e. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 1 \}$

5.1.11 [**JH**] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x + y$' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret '$r \cdot x$' as the r-th power of x.

5.1.12 [**JH**] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.13 [**JH**]
Is $\left\{ (x,\ y) \;\middle|\; x, y \in \mathbb{R} \right\}$ a vector space under these operations?

a. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$

b. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x,\ y) = (rx,\ 0)$

5.1.14 [**JH**]
Prove the following:

a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.15 [**JH**]
The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.16 [**JH**]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.17 [**JH**]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

5.1.18 [**JH**]
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.

b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [**JH**]

a. Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ under the inherited operations.

b. What if it doesn't contain the origin?

5.2.2 [**JH**] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.3 Spanning Sets

5.3.1 [**YL**] Given the following two subspace of $\mathbb{R}^3$: $W_1 = \{x \mid A_1 x = 0\}$ and $W_2 = \{x \mid A_2 x = 0\}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix},\ A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.4 Linear Independence

5.4.1 [**YL**] Let $\vec{u} = (1,\ \lambda,\ -\lambda)$, $\vec{v} = (-2\lambda\ \ -2\ \ 2\lambda)$ and $\vec{w} = (\lambda - 2,\ -5\lambda - 2,\ -2)$.

a. For what value(s) of λ will $\{\vec{u},\ \vec{v}\}$ be linearly dependent.

b. For what value(s) of λ will $\{\vec{u},\ \vec{v},\ \vec{w}\}$ be linearly independent.

5.5 Basis

5.5.1 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_2 x^2 + a_3 x^3 \;\middle|\; p(-1) = 0 \right\}$$

a subspace of $\mathcal{P}_3$.

a. Find a basis B for $\mathcal{W}$.
b. Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B.

5.6 Dimension

5.6.1 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; p(1) = 0 \text{ and } p(-1) = 0 \right\}$$

a subspace of P_3. Determine the dimension of W.

Answers to Exercises

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1.2.1

a. $x = 2$, $y = 3$

b. $x = -1$, $y = 4$, and $z = -1$.

1.2.2

a. $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (60s - 55t + 30,\ -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3},\ -14s + 13t - 7,\ s,\ t)$ where $s,\ t \in \mathbb{R}$.

b. If $s = t = 0$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (30,\ -\frac{38}{3},\ -7,\ 0,\ 0)$.
If $s = 0$ and $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-25,\ \frac{35}{3},\ 6,\ 0,\ 1)$.

c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-\frac{30}{7},\ \frac{316}{21},\ 1,\ \frac{4}{7},\ 0)$.

1.2.3

a. $(x_1,\ x_2,\ x_3,\ x_4) = (60t,\ -\frac{79}{3}t,\ -14t,\ t)$ where $t \in \mathbb{R}$.

b. If $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4) = (60,\ -\frac{79}{3},\ -14,\ 1)$.
If $t = 3$ then $(x_1,\ x_2,\ x_3,\ x_4) = (180,\ -79,\ 42,\ 3)$.

c. If $t = \frac{1}{60}$ then $(x_1,\ x_2,\ x_3,\ x_4) = (1,\ -\frac{79}{180},\ -\frac{14}{60},\ \frac{1}{60})$.

1.2.4

a. Possible if $a = \pm 1$ and $a \neq \pm b$.

b. Not possible.

c. Possible if $a \neq \pm 1$ or $a = \pm b$.

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2.1.1

a. $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$

c. Not possible, since dimension of A and E are not the same.

d. $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$

f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D.

g. $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$

h. $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.2.1 $A = \begin{bmatrix} -\frac{3}{4} & 3 \\ 1 & -\frac{3}{4} \end{bmatrix}$

2.2.2

a. $A = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

b. $X = \begin{bmatrix} -\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & \frac{3}{2} & -4 & 2 \end{bmatrix}$

2.2.3 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.2.4 Hint: Use the definition of the inverse of a matrix.

2.3.1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.4.1

a. $A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

b. $x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$

3.1.1 $\lambda = \frac{3 \pm \sqrt{33}}{4}$

3.2.1 $\det(A) = -\frac{5}{12}$

3.3.1 Hint: Apply the determinant to both sides $AB = -BA$.

3.4.1 $x_1 = 4$

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$.

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5.1.1

a. $0 + 0x + 0x^2 + 0x^3$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. The constant function $f(x) = 0$

d. The constant function $f(n) = 0$

5.1.2

a. $3 + 2x - x^2$

b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

c. $-3e^x + 2e^{-x}$

5.1.3

a. $1 + 2x$, $2 - 1x$, and x.

b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

a.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

a. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.

b. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

5.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

a. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

b. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

c. It is not closed under addition.
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

d. It is not closed under scalar multiplication.
$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

e. The set is empty, violating the existance of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

a. It is not a vector space.
$$(1+1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b. It is not a vector space.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

a. Yes.
b. Yes.
c. No, this set is not closed under the natural addition operation. The vector of all 1/4's is an element of this set but when added to itself the result, the vector of all 1/2's, is not an element of the set.
d. Yes.
e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11 Check all 10 conditions of the definition of a vector space.

5.1.12 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.13

a. No since $1 \cdot (0,\ 1) + 1 \cdot (0,\ 1) \neq (1 + 1) \cdot (0,\ 1)$.
b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0,\ 1) \neq (0,\ 1)$.

5.1.14

a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.
b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which implies that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.15
Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.16
It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.17
Each element of a vector space has one and only one additive inverse.
For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.18
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.
b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.
c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

5.2.1

a. Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of $\vec{v}, \vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.
b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.2 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r\,f'$.

5.3.1 Hint: For each subspace determine a set of vectors that spans it.
$W_1 \subsetneq W_2$

5.4.1

a. $\lambda = 1$
b. $\lambda \neq -1,\ -\frac{1}{2},\ 1$

5.5.1

a. $B = \left\{1 + x^3,\ x^2 + x^3\right\}$
b. $(p(x))_B = (-2, 2)$

5.6.1 $\left\{-1 + x^2,\ -x + x^3\right\}$ is a basis of W, therefore W is of dimension 2.

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