CEGEP Linear Algebra Problems

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CEGEP level linear algebra problems

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Contents

Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 [**GH**] State which of the following equations is a linear equation. If it is not, state why.

a. $x + y + z = 10$
b. $xy + yz + xz = 1$
c. $-3x + 9 = 3y - 5z + x - 7$
d. $\sqrt{5}y + \pi x = -1$
e. $(x-1)(x+1) = 0$
f. $\sqrt{x_1^2 + x_2^2} = 25$
g. $x_1 + y + t = 1$
h. $\frac{1}{x} + 9 = 3\cos(y) - 5z$
i. $\cos(15)y + \frac{x}{4} = -1$
j. $2^x + 2^y = 16$

1.1.2 [**GH**] Solve the system of linear equations using substitution, comparison and/or elimination.

a. $$\begin{aligned} x + y &= -1 \\ 2x - 3y &= 8 \end{aligned}$$

b. $$\begin{aligned} 2x - 3y &= 3 \\ 3x + 6y &= 8 \end{aligned}$$

c. $$\begin{aligned} x - y + z &= 1 \\ 2x + 6y - z &= -4 \\ 4x - 5y + 2z &= 0 \end{aligned}$$

d. $$\begin{aligned} x + y - z &= 1 \\ 2x + y \phantom{{}- 2z} &= 2 \\ y + 2z &= 0 \end{aligned}$$

1.1.3 [**GH**] Convert the given system of linear equations into an augmented matrix.

a. $$\begin{aligned} 3x + 4y + 5z &= 7 \\ -x + y - 3z &= 1 \\ 2x - 2y + 3z &= 5 \end{aligned}$$

b. $$\begin{aligned} 2x + 5y - 6z &= 2 \\ 9x \phantom{{}+ 5y} - 8z &= 10 \\ -2x + 4y + z &= -7 \end{aligned}$$

c. $$\begin{aligned} x_1 + 3x_2 - 4x_3 + 5x_4 &= 17 \\ -x_1 \phantom{{}+ 3x_2} + 4x_3 + 8x_4 &= 1 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 &= 6 \end{aligned}$$

d. $$\begin{aligned} 3x_1 - 2x_2 &= 4 \\ 2x_1 \phantom{{}- 2x_2} &= 3 \\ -x_1 + 9x_2 &= 8 \\ 5x_1 - 7x_2 &= 13 \end{aligned}$$

1.1.4 [**GH**] Convert given augmented matrix into a system of linear equations. Use the variables x_1, x_2, ...

a. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array}\right]$

b. $\left[\begin{array}{cc|c} -3 & 4 & 7 \\ 0 & 1 & -2 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & 5 & 7 \end{array}\right]$

d. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

e. $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 7 & 2 \\ 0 & 1 & 3 & 2 & 0 & 5 \end{array}\right]$

1.1.5 [**GH**] Perform the given row operations on

$$\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}.$$

a. $-1R_1 \to R_1$
b. $R_2 \leftrightarrow R_3$
c. $R_1 + R_2 \to R_2$
d. $2R_2 + R_3 \to R_3$
e. $\frac{1}{2}R_2 \to R_2$
f. $-\frac{5}{2}R_1 + R_3 \to R_3$

1.1.6 [**GH**] Give the row operation that transforms A into B where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

a. $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

b. $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $B = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

d. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

e. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

1.1.7 [**JH**] In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy-plane. By geometrical reasoning, show that there are three possibilities:

there is a unique solution, there is no solution, and there are infinitely many solutions.

1.1.8 [JH] Is there a two-unknowns linear system whose solution set is all of $\mathbb{R}^2$?

1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [GH] State whether or not the given matrices are in reduced row echelon form.

a. $\begin{bmatrix}1&0\\0&1\end{bmatrix}$

b. $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

c. $\begin{bmatrix}1&1\\1&1\end{bmatrix}$

d. $\begin{bmatrix}1&0&1\\0&1&2\end{bmatrix}$

e. $\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}$

f. $\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}$

g. $\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}$

h. $\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$

i. $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

j. $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$

k. $\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$

l. $\begin{bmatrix}1&0&0&-5\\0&1&0&7\\0&0&1&3\end{bmatrix}$

m. $\begin{bmatrix}2&0&0&2\\0&2&0&2\\0&0&2&2\end{bmatrix}$

n. $\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$

o. $\begin{bmatrix}0&0&1&-5\\0&0&0&0\\0&0&0&0\end{bmatrix}$

1.2.2 [GH] Use Gauss-Jordan Elimination to put the given matrix into reduced row echelon form.

a. $\begin{bmatrix}1&2\\-3&-5\end{bmatrix}$

b. $\begin{bmatrix}2&-2\\3&-2\end{bmatrix}$

c. $\begin{bmatrix}4&12\\-2&-6\end{bmatrix}$

d. $\begin{bmatrix}-5&7\\10&14\end{bmatrix}$

e. $\begin{bmatrix}-1&1&4\\-2&1&1\end{bmatrix}$

f. $\begin{bmatrix}7&2&3\\3&1&2\end{bmatrix}$

g. $\begin{bmatrix}3&-3&6\\-1&1&-2\end{bmatrix}$

h. $\begin{bmatrix}4&5&-6\\-12&-15&18\end{bmatrix}$

i. $\begin{bmatrix}-2&-4&-8\\-2&-3&-5\\2&3&6\end{bmatrix}$

j. $\begin{bmatrix}2&1&1\\1&1&1\\2&1&2\end{bmatrix}$

k. $\begin{bmatrix}1&2&1\\1&3&1\\-1&-3&0\end{bmatrix}$

l. $\begin{bmatrix}1&2&3\\0&4&5\\1&6&9\end{bmatrix}$

m. $\begin{bmatrix}1&1&1&2\\2&-1&-1&1\\-1&1&1&0\end{bmatrix}$

n. $\begin{bmatrix}2&-1&1&5\\3&1&6&-1\\3&0&5&0\end{bmatrix}$

o. $\begin{bmatrix}1&1&-1&7\\2&1&0&10\\3&2&-1&17\end{bmatrix}$

p. $\begin{bmatrix}4&1&8&15\\1&1&2&7\\3&1&5&11\end{bmatrix}$

1.2.3 [JH] Use Gauss's Method to find the unique solution for each system.

a. $\begin{aligned}2x+3y&=13\\x-y&=-1\end{aligned}$

b. $\begin{aligned}x-z&=0\\3x+y&=1\\-x+y+z&=4\end{aligned}$

1.2.4 [GH] Find the solution to the given linear system. If the system has infinite solutions, give two particular solutions.

a. $\begin{aligned} 2x_1 + 4x_2 &= 2 \\ x_1 + 2x_2 &= 1 \end{aligned}$

b. $\begin{aligned} -x_1 + 5x_2 &= 3 \\ 2x_1 - 10x_2 &= -6 \end{aligned}$

c. $\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + x_2 &= 4 \end{aligned}$

d. $\begin{aligned} -3x_1 + 7x_2 &= -7 \\ 2x_1 - 8x_2 &= 8 \end{aligned}$

e. $\begin{aligned} -2x_1 + 4x_2 + 4x_3 &= 6 \\ x_1 - 3x_2 + 2x_3 &= 1 \end{aligned}$

f. $\begin{aligned} -x_1 + 2x_2 + 2x_3 &= 2 \\ 2x_1 + 5x_2 + x_3 &= 2 \end{aligned}$

g. $\begin{aligned} -x_1 - x_2 + x_3 + x_4 &= 0 \\ -2x_1 - 2x_2 + x_3 &= -1 \end{aligned}$

h. $\begin{aligned} x_1 + x_2 + 6x_3 + 9x_4 &= 0 \\ x_1 + x_3 + 2x_4 &= 3 \end{aligned}$

i. $\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ 3x_1 + 3x_2 + 5x_3 &= 2 \end{aligned}$

j. $\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2 \\ 1x_1 + 2x_2 + 3x_3 &= 1 \\ 3x_1 + 6x_2 + 9x_3 &= 3 \end{aligned}$

k. $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -2x_1 - 3x_2 &= 1 \end{aligned}$

l. $\begin{aligned} 2x_1 + x_2 + 2x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 1 \\ 3x_1 + 2x_2 + 5x_3 &= 3 \end{aligned}$

1.2.5 [**YL**] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 + x_5 &= 3 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 + 17x_3 - 2x_4 - x_5 &= 1 \end{aligned}$$

a. Solve the following system by Gauss-Jordan elimination.

b. Find two particular solution to the above system.

c. Find a solution to the above system when $x_3 = 1$.

1.2.6 [**YL**] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 &= 0 \\ 4x_1 + 17x_3 - 2x_4 &= 0 \\ 9x_1 + 6x_2 + 27x_3 - 4x_4 &= 0 \end{aligned}$$

a. Solve the system by Gauss-Jordan elimination.

b. Find two particular nontrivial solution to the system.

c. Find a solution to the system when $x_1 = 1$.

1.2.7 [**JH**] Find the coefficients a, b, and c so that the graph of $f(x) = ax^2+bx+c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.2.8 [**JH**] True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.2.9 [**JH**] For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

1.2.10 [**GH**] State for which values of k the given system will have exactly 1 solution, infinite solutions, or no solution.

a. $\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= k \end{aligned}$

b. $\begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 + kx_2 &= 1 \end{aligned}$

c. $\begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 + kx_2 &= 2 \end{aligned}$

d. $\begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 + 3x_2 &= k \end{aligned}$

1.2.11 [**YL**] Given the augmented matrix of a linear system:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a^2-1 & b^2-a^2 \end{bmatrix}$$

If possible for what values of a and b the system has

a. no solution? Justify.

b. exactly one solution? Justify.

c. infinitely many solutions? Justify.

1.2.12 [**YL**] Given the augmented matrix of a linear system

$$\begin{bmatrix} 1 & 3 & 1 & -4 & b_1 \\ 3 & -2 & 4 & 5 & b_2 \\ 4 & 1 & 5 & 1 & b_3 \\ 7 & -1 & 9 & 6 & b_4 \end{bmatrix}.$$

Determine the restrictions on the b_i's for the system to be consistent.

1.2.13 [**JH**] Prove that, where $a, b, \ldots, e$ are real numbers and $a \neq 0$, if

$$ax + by = c$$

has the same solution set as

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

1.2.14 [**JH**] Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

1.3 Applications of Linear Systems

1.3.1 Place Holder

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [**JH**] Find the indicated entry of the following matrix.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{bmatrix}$$

a. $a_{2,1}$ **b.** $a_{1,2}$ **c.** $a_{2,2}$ **d.** $a_{3,1}$

2.1.2 [**JH**] Determine the size of each matrix.

a. $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{bmatrix}$ **c.** $\begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$

2.1.3 [**GH**] Simplify the given expression where

$$A = \begin{bmatrix} 1 & -1 \\ 7 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 \\ 5 & 9 \end{bmatrix}$$

a. $A + B$

b. $2A - 3B$

c. $3(A - B) + B$

d. $2(A - B) - (A - 3B)$

2.1.4 [**GH**] The row and column matrix U and V are defined. Find the product UV, where possible.

a. $U = \begin{bmatrix} 1 & -4 \end{bmatrix}$, $V = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

b. $U = \begin{bmatrix} 6 & 2 & -1 & 2 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \\ 9 \\ 5 \end{bmatrix}$

c. $U = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

d. $U = \begin{bmatrix} 2 & -5 \end{bmatrix}$, $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2.1.5 [**GH**] State the dimensions of A and B. State the dimensions of AB and BA, if the product is defined. Then compute the product AB and BA, if possible.

a. $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 7 \\ 4 & 2 & 9 \end{bmatrix}$

c. $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$

d. $A = \begin{bmatrix} -2 & -1 \\ 9 & -5 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 6 & -4 \\ 0 & 6 & -3 \end{bmatrix}$

e. $A = \begin{bmatrix} 2 & 6 \\ 6 & 2 \\ 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 5 & 0 \\ -4 & 4 & -4 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 4 \\ 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -5 & 5 \\ -2 & 1 & 3 & -5 \end{bmatrix}$

g. $A = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $A = \begin{bmatrix} -4 & -1 & 3 \\ 2 & -3 & 5 \\ 1 & 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 & 3 \\ -1 & 1 & -1 \\ 4 & 0 & 2 \end{bmatrix}$

2.1.6 [**GH**] Given a diagonal matrix D and a matrix A, compute the product DA and AD, if possible.

a. $D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}$, $A = \begin{bmatrix} & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

b. $D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

c. $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

d. $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

2.1.7 [**GH**] Given a matrix A compute A^2 and A^3.

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

d. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

e. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

2.1.8 [**HE**] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

a. $3C - 4D$

b. $A - (D + 2C)$

c. $A - E$

d. AE

e. $3BC - 4BD$

f. $CB + D$

g. GC

h. FG

i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A, B, and C are matrices above.

2.1.9 [**GH**] In each part a matrix A is given. Find A^T. State whether A is upper/lower triangular, diagonal, symmetric and/or skew symmetric.

a. $\begin{bmatrix} -9 & 4 & 10 \\ 6 & -3 & -7 \\ -8 & 1 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 2 & -9 \\ 5 & -4 & -10 \\ -6 & 6 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -7 & -4 & -9 \\ -9 & 6 & 3 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$

e. $\begin{bmatrix} 4 & 0 & 0 \\ -2 & -7 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

f. $\begin{bmatrix} -3 & -4 & -5 \\ 0 & -3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$

2.1.10 [**GH**] Find the trace of the given matrix.

a. $\begin{bmatrix} 4 & 1 & 1 \\ -2 & 0 & 0 \\ -1 & -2 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -5 \\ 9 & 5 \end{bmatrix}$

c. $\begin{bmatrix} -10 & 6 & -7 & -9 \\ -2 & 1 & 6 & -9 \\ 0 & 4 & -4 & 0 \\ -3 & -9 & 3 & -10 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 6 & 4 \\ -1 & 8 & -10 \end{bmatrix}$

e. Any skew-symmetric matrix.

f. I_n

2.1.11 [**GH**] Find values for the scalars a and b that satisfy the given equation.

a. $a\begin{bmatrix} -3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

b. $a\begin{bmatrix} 4 \\ 2 \end{bmatrix} + b\begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

c. $a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

d. $a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$

2.1.12 [**GH**] The following statement

$$(A + B)^2 = A^2 + 2AB + B^2$$

is false. We investigate that claim here.

a. Let $A = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} -5 & -5 \\ -2 & 1 \end{bmatrix}$. Compute $A + B$

b. Find $(A + B)^2$ by using the previous part.

c. Compute $A^2 + 2AB + B^2$.

d. Are the results from the two previous parts equal?

e. Carefully expand the expression $(A+B)^2 = (A+B)(A+B)$ and show why this is not equal to $A^2 + 2AB + B^2$.

2.1.13 [**YL**]

a. Prove: If A and B are $n \times n$ matrices then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

b. Prove: If A and B are $n \times n$ matrices then $\text{tr}(AB) = \text{tr}(BA)$.

2.1.14 [**YL**] A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$.
Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

2.1.15 [**YL**] A square matrix A is called *idempotent* if $A^2 = A$.
Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A.

2.2 Matrix Inverses and Algebraic Properties

2.2.1 [GH] Given the matrices A and B below. Find X that satisfies the equation.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}$$

a. $2A + X = B$

b. $A - X = 3B$

c. $3A + 2X = -1B$

d. $A - \frac{1}{2}X = -B$

2.2.2 [GH] Given the matrices A. Find A^{-1}, if possible.

a. $\begin{bmatrix} 1 & 5 \\ -5 & -24 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$

2.2.3 [GH] Given the matrices A and B. Compute $(AB)^{-1}$ and $B^{-1}A^{-1}$.

a. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 5 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 1 \\ 2 & 1 \end{bmatrix}$

2.2.4 [GH] Given the matrices A. Find A^{-1}, if possible.

a. $\begin{bmatrix} 25 & -10 & -4 \\ -18 & 7 & 3 \\ -6 & 2 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -7 \\ 20 & 7 & -48 \end{bmatrix}$

c. $\begin{bmatrix} -4 & 1 & 5 \\ -5 & 1 & 9 \\ -10 & 2 & 19 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -5 & 0 \\ -2 & 15 & 4 \\ 4 & -19 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 25 & -8 & 0 \\ -78 & 25 & 0 \\ 48 & -15 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & 8 \\ -2 & -2 & -3 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i. $\begin{bmatrix} 2 & 3 & 4 \\ -3 & 6 & 9 \\ -1 & 9 & 13 \end{bmatrix}$

j. $\begin{bmatrix} 5 & -1 & 0 \\ 7 & 7 & 1 \\ -2 & -8 & -1 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -19 & -9 & 0 & 4 \\ 33 & 4 & 1 & -7 \\ 4 & 2 & 0 & -1 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 27 & 1 & 0 & 4 \\ 18 & 0 & 1 & 4 \\ 4 & 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -29 & -110 \\ 0 & -3 & -5 & -19 \end{bmatrix}$

n. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

2.2.5 [GH] Prove or disprove: If A and B are $2{\times}2$ invertible matrices then $A + B$ is an invertible matrix.

2.2.6 [YL] Solve of A given that it satisfies

$$(I - A^T)^{-1} = (\mathrm{tr}(B)B^2)^T$$

where

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.2.7 [YL] Solve of X given that it satisfies

$$DXD^T = \mathrm{tr}(BC)BC$$

where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}.$$

2.2.8 [YL] Given

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & \frac{1}{2} \end{bmatrix}.$$

a. Find A^{-1}.

b. Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2.2.9 [YL] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.2.10 [YL] Prove: If AB and BA are both invertible then A and B are both invertible.

2.2.11 [YL] Prove: If B and C are $n{\times}n$ matrices such that $A = B^TC + C^TB$ is invertible then A^{-1} is symmetric.

2.3 Elementary Matrices

2.3.1 [**YL**] Write the given matrix as a product of elementary matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

2.3.2 [**YL**] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.3.3 [**YL**] Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3E_2E_1A = B$.

2.4 Linear Systems and Matrices

2.4.1 [**YL**] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1}.

b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

2.4.2 [**GH**] Given the matrices A and b below. Find x that satisfies the equation $Ax = b$ by using the inverse of A

a. $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$,
$b = \begin{bmatrix} 21 \\ 13 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -4 \\ 4 & -15 \end{bmatrix}$,
$b = \begin{bmatrix} 21 \\ 77 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 & 12 \\ 0 & 1 & 6 \\ -3 & 0 & 1 \end{bmatrix}$,
$b = \begin{bmatrix} -17 \\ -5 \\ 20 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 0 & -3 \\ 8 & -2 & -13 \\ 12 & -3 & -20 \end{bmatrix}$,
$b = \begin{bmatrix} -34 \\ -159 \\ -243 \end{bmatrix}$

Chapter 3

Determinants

3.1 The Laplace Expansion

3.1.1 [GH] Compute the determinant of the following matrices.

a. $\begin{bmatrix} 10 & 7 \\ 8 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 6 & -1 \\ -7 & 8 \end{bmatrix}$

c. $\begin{bmatrix} -1 & -7 \\ -5 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -10 & -1 \\ -4 & 7 \end{bmatrix}$

3.1.2 [GH] For the following matrices, construct the submatrices used to compute the minors $M_{1,1}$, $M_{1,2}$ and $M_{1,3}$. Compute the cofactors $C_{1,1}$, $C_{1,2}$, and $C_{1,3}$.

a. $\begin{bmatrix} 7 & -3 & 10 \\ 3 & 7 & 6 \\ 1 & 6 & 10 \end{bmatrix}$

b. $\begin{bmatrix} -2 & -9 & 6 \\ -10 & -6 & 8 \\ 0 & -3 & -2 \end{bmatrix}$

c. $\begin{bmatrix} -5 & -3 & 3 \\ -3 & 3 & 10 \\ -9 & 3 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -6 & -4 & 6 \\ -8 & 0 & 0 \\ -10 & 8 & -1 \end{bmatrix}$

3.1.3 [JH] Evaluate the determinant by performing a cofactor expansion

$$\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 0 \end{vmatrix}$$

a. along the first row,

b. along the second row,

c. along the third column.

3.1.4 [GH] Find the determinant of the given matrix using cofactor expansion.

a. $\begin{bmatrix} 3 & 2 & 3 \\ -6 & 1 & -10 \\ -8 & -9 & -9 \end{bmatrix}$

b. $\begin{bmatrix} 8 & -9 & -2 \\ -9 & 9 & -7 \\ 5 & -1 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -1 & 0 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

f. $\begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix}$

3.1.5 [JH] Verify that the determinant of an upper-triangular 3×3 matrix is the product of the main diagonal.

$$\det\left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}\right) = aei$$

Is it the same for lower triangular matrices?

3.1.6 [YL] Solve for λ.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

3.1.7 [JH] True or false: Can we compute a determinant by expanding down the diagonal? Justify.

3.1.8 [JH] Which real numbers θ make

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

equal to zero?

3.2 Determinants and Elementary Operations

3.2.1 [**GH**] A matrix M and $\det(M)$ are given. Matrices A, B and C are obtained by performing operations on M. Determine the determinants of A, B and C and indicate the operations used to obtain A, B and C.

a. $M = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}$, $\det(M) = -41$,

$A = \begin{bmatrix} 18 & 14 & 16 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 96 & 73 & 83 \end{bmatrix}$, $C = \begin{bmatrix} 9 & 1 & 6 \\ 7 & 3 & 3 \\ 8 & 7 & 3 \end{bmatrix}$.

b. $M = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}$, $\det(M) = 45$,

$A = \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & -1 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ 8 & 16 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}$.

c. $M = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, $\det(M) = -16$,

$A = \begin{bmatrix} 0 & 0 & 4 \\ 5 & 1 & 5 \\ 4 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -5 & -1 & -5 \\ -4 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 15 & 3 & 15 \\ 12 & 0 & 6 \\ 0 & 0 & 12 \end{bmatrix}$.

d. $M = \begin{bmatrix} 5 & 4 & 0 \\ 7 & 9 & 3 \\ 1 & 3 & 9 \end{bmatrix}$, $\det(M) = 120$,

$A = \begin{bmatrix} 1 & 3 & 9 \\ 7 & 9 & 3 \\ 5 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 4 & 0 \\ 14 & 18 & 6 \\ 3 & 9 & 27 \end{bmatrix}$, $C = \begin{bmatrix} -5 & -4 & 0 \\ -7 & -9 & -3 \\ -1 & -3 & -9 \end{bmatrix}$.

3.2.2 [**GH**] Find the determinant of the given matrix by using elemetary operations to bring the matrix under triangular form.

a.

b. $\begin{bmatrix} -4 & 3 & -4 \\ -4 & -5 & 3 \\ 3 & -4 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -2 & 1 \\ 5 & 5 & 4 \\ 4 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} -5 & 0 & -4 \\ 2 & 4 & -1 \\ -5 & 0 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

f. $\begin{bmatrix} -5 & 1 & 0 & 0 \\ -3 & -5 & 2 & 5 \\ -2 & 4 & -3 & 4 \\ 5 & 4 & -3 & 3 \end{bmatrix}$

g. $\begin{bmatrix} 2 & -1 & 4 & 4 \\ 3 & -3 & 3 & 2 \\ 0 & 4 & -5 & 1 \\ -2 & -5 & -2 & -5 \end{bmatrix}$

3.2.3 [**YL**] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.2.4 Vandermonde's determinant [**JH**] Prove:

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\right) = (b-a)(c-a)(c-b)$$

3.3 Properties of Determinants and Matrix Inverses

3.3.1 [**JH**] Find the adjoint of the following matrices.

a. $\begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 1 & 8 & 9 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

3.3.2 [**JH**]

a. Find a formula for the adjoint of a 2×2 matrix.

b. Use the above to derive the formula for the inverse of a 2×2 matrix.

3.3.3 [**JH**] Derive a formula for the adjoint of a diagonal matrix.

3.3.4 [**JH**] Prove that the transpose of the adjoint is the adjoint of the transpose.

3.3.5 [**JH**] Prove or disprove: $\operatorname{adj}(\operatorname{adj}(T)) = T$.

3.3.6 [**JH**] Which real numbers x make this matrix singular?

$$\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}$$

3.3.7 [**JH**] Prove: If S and T are $n\times n$ matrix then $\det(TS) = \det(ST)$.

3.3.8 [**JH**] Prove that each statement holds for 2×2 matrices.

a. The determinant of a product is the product of the determinants $\det(ST) = \det(S)\det(T)$.

b. If T is invertible then the determinant of the inverse is the inverse of the determinant $\det(T^{-1}) = (\det(T))^{-1}$.

3.3.9 [**JH**]

a. Suppose that $\det(A) = 3$ and that $\det(B) = 2$. Find $\det(A^2 B^{\mathsf{T}} B^{-2} A^{\mathsf{T}})$.

b. If $\det(A) = 0$ then show that $\det(6A^3 + 5A^2 + 2A) = 0$.

3.3.10 [**JH**]

a. Give a non-identity matrix with the property that $A^{\mathsf{T}} = A^{-1}$.

b. Prove: If $A^{\mathsf{T}} = A^{-1}$ then $\det(A) = \pm 1$.

c. Does the converse to the above hold?

3.3.11 [**JH**] Two matrices H and G are said to be *similar* if there is a nonsingular matrix P such that $H = P^{-1}GP$ Show that similar matrices have the same determinant.

3.3.12 [**JH**] Show that this gives the equation of a line in $\mathbb{R}^2$ thru (x_2, y_2) and (x_3, y_3).

$$\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

3.3.13 [**YL**] Let A and B be $n\times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.3.14 [**JH**] Prove or disprove: The determinant is a linear function, that is $\det(x\cdot T + y\cdot S) = x\cdot\det(T) + y\cdot\det(S)$.

3.4 Applications of the Determinant

3.4.1 [**YL**] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

3.4.2 [**GH**] Given the matrices A and b, evaluate $\det(A)$ and $\det(A_i)$ for all i. Use Cramer's Rule to solve $Ax = b$. If Cramer's Rule cannot be used to find the solution, then state whether or not a solution exists.

a. $A = \begin{bmatrix} 3 & 0 & -3 \\ 5 & 4 & 4 \\ 5 & 5 & -4 \end{bmatrix}$

$b = \begin{bmatrix} 24 \\ 0 \\ 31 \end{bmatrix}$

b. $A = \begin{bmatrix} 9 & 5 \\ -4 & -7 \end{bmatrix}$

$b = \begin{bmatrix} -45 \\ 20 \end{bmatrix}$

c. $A = \begin{bmatrix} -8 & 16 \\ 10 & -20 \end{bmatrix}$

$b = \begin{bmatrix} -48 \\ 60 \end{bmatrix}$

d. $A = \begin{bmatrix} 7 & 14 \\ -2 & -4 \end{bmatrix}$

$b = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

e. $A = \begin{bmatrix} 4 & 9 & 3 \\ -5 & -2 & -13 \\ -1 & 10 & -13 \end{bmatrix}$

$b = \begin{bmatrix} -28 \\ 35 \\ 7 \end{bmatrix}$

f. $A = \begin{bmatrix} 7 & -4 & 25 \\ -2 & 1 & -7 \\ 9 & -7 & 34 \end{bmatrix}$

$b = \begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in* $\mathbb{R}^n$: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$.

4.3 Cross Product and Planes

4.3.1 Place Holder

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4.4 Areas, Volumes and Distances

4.4.1 Place Holder

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4.5 Geometry of Solutions of Linear Systems

4.5.1 Place Holder

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [**JH**] Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.

b. The space of 2×3 matrices.

c. The space $\{\, f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \,\}$.

d. The space of real-valued functions of one natural number variable.

5.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3-2x+x^2$.

b. In the space $\mathcal{M}_{2\times 2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

c. In $\{\, ae^x + be^{-x} \mid a,b \in \mathbb{R} \,\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{\, a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \,\}$ under the usual polynomial addition and scalar multiplication operations.

b. The set of linear polynomials $\{\, a_0 + a_1 x \mid a_0 - 2a_1 = 0 \,\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.

b. The set of 2×2 matrices with real entries where the $2,1$ entry is zero, under the usual matrix operations.

5.1.5 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.

b. The set

$$\{\, (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y - z + w = 0 \,\}$$

under the operations inherited from $\mathbb{R}^4$.

5.1.6 [**JH**] Show that the following are not vector spaces.

a. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x + y + z = 1 \,\}$$

b. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \,\}$$

c. Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\}$$

d. Under the usual polynomial operations,

$$\{\, a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+ \,\}$$

where $\mathbb{R}^+$ is the set of reals greater than zero

e. Under the inherited operations,

$$\{\, (x,\ y) \in \mathbb{R}^2 \mid x + 3y = 4,\ 2x - y = 3 \text{ and } 6x + 4y = 10 \,\}$$

5.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

5.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

b. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5.1.10 [**JH**] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

c. $\{ (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y + w = 1 \}$

d. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 0 \}$

e. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 1 \}$

5.1.11 [**YL**] Let $V = \{ A \mid A \in \mathcal{M}_{2\times2} \text{ and } \det(A) \neq 0 \}$ with the following operations:

$$A + B = AB \text{ and } kA = kA$$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.

a. Does V satisfy closure under vector addition? Justify.

b. Does V contain a zero vector? If so find it. Justify.

c. Does V contains an additive inverse for all of its vectors? Justify.

d. Does V satisfy closure under scalar multiplication? Justify.

5.1.12 [**JH**] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x + y$' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret '$r \cdot x$' as the r-th power of x.

5.1.13 [**JH**] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.14 [**JH**]
Is $\{ (x,\ y) \mid x, y \in \mathbb{R} \}$ a vector space under these operations?

a. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$

b. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x,\ y) = (rx,\ 0)$

5.1.15 [**JH**]
Prove the following:

a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.16 [**JH**] Consider the set

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x + y + z = 1 \right\}$$

under these operations.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \qquad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx - r + 1 \\ ry \\ rz \end{pmatrix}$$

Show that it is a vector space.

5.1.17 [**JH**]
The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.18 [**JH**]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.19 [**JH**]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

5.1.20 [**JH**]
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.

b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [**JH**] Which of these subsets of the vector space of 2×2 matrices are subspaces under the inherited operations? Justify.

a. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

b. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 0 \right\}$

c. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 5 \right\}$

d. $\left\{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 0, c \in \mathbb{R} \right\}$

5.2.2 [**JH**] Is this a subspace of $\mathcal{P}_2$: $\left\{ a_0 + a_1 x + a_2 x^2 \;\middle|\; a_0 + 2a_1 + a_2 = 4 \right\}$? Justify.

5.2.3 [**JH**] The solution set of a homogeneous linear system is a subspace of $\mathbb{R}^n$ where the system has n variables. What about a non-homogeneous linear system; do its solutions form a subspace (under the inherited operations)?

5.2.4 [**JH**]

a. Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ under the inherited operations.

b. What if it doesn't contain the origin?

5.2.5 [**JH**] $\mathbb{R}^3$ has infinitely many subspaces. Do every nontrivial space have infinitely many subspaces?

5.2.6 [**JH**] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.2.7 [**JH**] Determine if each is a subspace of the vector space of real-valued functions of one real variable.

a. The *even* functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x) \text{ for all } x \}$.

b. The *odd* functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x) \text{ for all } x \}$.

5.2.8 [**JH**] Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

5.2.9 [**JH**]

a. Give a set that is closed under scalar multiplication but not addition.

b. Give a set closed under addition but not scalar multiplication.

c. Give a set closed under neither.

5.2.10 [**JH**] Subspaces are subsets and so we naturally consider how 'is a subspace of' interacts with the usual set operations.

a. If A, B are subspaces of a vector space, are their intersection $A \cap B$ be a subspace?

b. Is the union $A \cup B$ a subspace?

c. If A is a subspace, is its complement be a subspace?

5.2.11 [**JH**] Is the relation 'is a subspace of' transitive? That is, if V is a subspace of W and W is a subspace of X, must V be a subspace of X?

5.3 Spanning Sets

5.3.1 [**JH**] Determine whether the vector lies in the span of the set.

a. $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

b. $x - x^3$, $\{x^2, 2x + x^2, x + x^3\}$

c. $\begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$, $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \right\}$

5.3.2 [**JH**] Which of these are members of the span $\operatorname{span}(\{\cos^2 x, \sin^2 x\})$ in the vector space of real-valued functions of one real variable?

a. $f(x) = 1$

b. $f(x) = 3 + x^2$

c. $f(x) = \sin x$

d. $f(x) = \cos(2x)$

5.3.3 [**JH**] Which of these sets spans $\mathbb{R}^3$?

a. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$

b. $\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

c. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$

d. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right\}$

e. $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} \right\}$

5.3.4 [**JH**] Express each subspace as a span of a set of vectors.

a. $\{(a \;\; b \;\; c) \mid a - c = 0\}$

b. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a + d = 0 \right\}$

c. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; 2a - c - d = 0 \text{ and } a + 3b = 0 \right\}$

d. $\{a + bx + cx^3 \mid a - 2b + c = 0\}$

e. The subset of $\mathcal{P}_2$ of quadratic polynomials p such that $p(7) = 0$

5.3.5 [**JH**] Find a set that spans the given subspace.

a. The xz-plane in $\mathbb{R}^3$.

b. $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; 3x + 2y + z = 0 \right\}$

c. $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; 2x + y + w = 0 \text{ and } y + 2z = 0 \right\}$

d. $\{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + a_1 = 0 \text{ and } a_2 - a_3 = 0\}$

e. The set $\mathcal{P}_4$ in the space $\mathcal{P}_4$

f. $\mathcal{M}_{2\times2}$ in $\mathcal{M}_{2\times2}$

5.3.6 [**JH**] Show that for any subset S of a vector space, $\operatorname{span}(\operatorname{span}(S)) = \operatorname{span}(S)$. (*Hint.* Members of $\operatorname{span}(S)$ are linear combinations of members of S. Members of $\operatorname{span}(\operatorname{span}(S))$ are linear combinations of linear combinations of members of S.)

5.3.7 [**YL**] Given the following two subspace of $\mathbb{R}^3$: $W_1 = \{x \mid A_1x = 0\}$ and $W_2 = \{x \mid A_2x = 0\}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix}, \; A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.3.8 [**JH**] Prove: $\vec{v} \in \operatorname{span}(\{\vec{v}_1, \ldots, \vec{v}_n\})$ if and only if $\operatorname{span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \operatorname{span}(\{\vec{v}, \vec{v}_1, \ldots, \vec{v}_n\})$

5.3.9 [**JH**] Does the span of a set depend on the enclosing space? That is, if W is a subspace of V and S is a subset of W (and so also a subset of V), might the span of S in W differ from the span of S in V?

5.3.10 [**JH**] Because 'span of' is an operation on sets we naturally consider how it interacts with the usual set operations.

a. If $S \subseteq T$ are subsets of a vector space, is $\operatorname{span}(S) \subseteq \operatorname{span}(T)$?

b. If S, T are subsets of a vector space, is $\operatorname{span}(S \cup T) = \operatorname{span}(S) \cup \operatorname{span}(T)$?

c. If S, T are subsets of a vector space, is $\operatorname{span}(S \cap T) = \operatorname{span}(S) \cap \operatorname{span}(T)$?

d. Is the span of the complement equal to the complement of the span?

5.4 Linear Independence

5.4.1 [**JH**] Determine whether each subset of $\mathbb{R}^3$ is linearly dependent or linearly independent.

a. $\left\{ \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 14 \end{pmatrix} \right\}$

b. $\left\{ \begin{pmatrix} 1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix} \right\}$

c. $\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}$

d. $\left\{ \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \\ -1 \end{pmatrix} \right\}$

5.4.2 [**JH**] Which of these subsets of $\mathcal{P}_2$ are linearly dependent and which are independent?

a. $\{3 - x + 9x^2,\ 5 - 6x + 3x^2,\ 1 + 1x - 5x^2\}$

b. $\{-x^2,\ 1 + 4x^2\}$

c. $\{2 + x + 7x^2,\ 3 - x + 2x^2,\ 4 - 3x^2\}$

d. $\{8 + 3x + 3x^2,\ x + 2x^2,\ 2 + 2x + 2x^2,\ 8 - 2x + 5x^2\}$

5.4.3 [**JH**] Prove that each set $\{f,\ g\}$ is linearly independent in the vector space of all functions from $\mathbb{R}^+$ to $\mathbb{R}$.

a. $f(x) = x$ and $g(x) = 1/x$

b. $f(x) = \cos(x)$ and $g(x) = \sin(x)$

c. $f(x) = e^x$ and $g(x) = \ln(x)$

5.4.4 [**JH**] Which of these subsets of the space of real-valued functions of one real variable are linearly dependent and which are linearly independent?

a. $\{2,\ 4\sin^2(x),\ \cos^2(x)\}$

b. $\{1,\ \sin(x),\ \sin(2x)\}$

c. $\{x,\ \cos(x)\}$

d. $\{(1+x)^2,\ x^2 + 2x,\ 3\}$

e. $\{0,\ x,\ x^2\}$

f. $\{\cos(2x),\ \sin^2(x),\ \cos^2(x)\}$

5.4.5 [**JH**] Is the xy-plane subset of the vector space $\mathbb{R}^3$ linearly independent?

5.4.6 [**YL**] Let $\vec{u} = (1,\ \lambda,\ -\lambda)$, $\vec{v} = (-2\lambda,\ -2,\ 2\lambda)$ and $\vec{w} = (\lambda - 2,\ -5\lambda - 2,\ -2)$.

a. For what value(s) of λ will $\{\vec{u},\ \vec{v}\}$ be linearly dependent.

b. For what value(s) of λ will $\{\vec{u},\ \vec{v},\ \vec{w}\}$ be linearly independent.

5.4.7 [**JH**]

a. Show that if the set $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is linearly independent then so is the set $\{\vec{u},\ \vec{u} + \vec{v},\ \vec{u} + \vec{v} + \vec{w}\}$.

b. What is the relationship between the linear independence or dependence of $\{\vec{u},\ \vec{v},\ \vec{w}\}$ and the independence or dependence of $\{\vec{u} - \vec{v},\ \vec{v} - \vec{w},\ \vec{w} - \vec{u}\}$?

5.4.8 [**JH**]

a. When is a one-element set linearly independent?

b. When is a two-element set linearly independent?

5.4.9 [**JH**] Show that if $\{\vec{x},\ \vec{y},\ \vec{z}\}$ is linearly independent then so are all of its proper subsets: $\{\vec{x},\ \vec{y}\}$, $\{\vec{x},\ \vec{z}\}$, $\{\vec{y},\ \vec{z}\}$, $\{\vec{x}\}$,$\{\vec{y}\}$, $\{\vec{z}\}$. Is the converse also true?

5.4.10 [**JH**]

a. Show that this

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

is a linearly independent subset of $\mathbb{R}^3$.

b. Show that

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

is in the span of S by finding c_1 and c_2 giving a linear relationship.

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

Show that the pair c_1, c_2 is unique.

c. Assume that S is a subset of a vector space and that $\vec{v}$ is in span (S), so that $\vec{v}$ is a linear combination of vectors from S. Prove that if S is linearly independent then a linear combination of vectors from S adding to $\vec{v}$ is unique (that is, unique up to reordering and adding or taking away terms of the form $0 \cdot \vec{s}$). Thus S as a spanning set is minimal in this strong sense: each vector in span (S) is a combination of elements of S a minimum number of times (only once).

d. Prove that it can happen when S is not linearly independent that distinct linear combinations sum to the same vector.

5.4.11 [**JH**]

a. Show that any set of four vectors in $\mathbb{R}^2$ is linearly dependent.

b. Is this true for any set of five? Any set of three?

c. What is the most number of elements that a linearly independent subset of $\mathbb{R}^2$ can have?

5.4.12 [**JH**] Is there a set of four vectors in $\mathbb{R}^3$ such that any three form a linearly independent set?

5.4.13 [**JH**]

a. Prove that a set of two perpendicular nonzero vectors from $\mathbb{R}^n$ is linearly independent when $n > 1$.

b. What if $n = 1$?

c. Generalize to more than two vectors.

5.4.14 [**JH**] Show that, where S is a subspace of V, if a subset T of S is linearly independent in S then T is also linearly independent in V. Is the converse also true?

5.4.15 [**JH**] Show that the nonzero rows of an echelon form matrix form a linearly independent set.

5.4.16 [**JH**] In $\mathbb{R}^4$ what is the largest linearly independent set you can find? The smallest? The largest linearly dependent set? The smallest?

5.4.17 [**JH**]

a. Is the intersection of linearly independent sets independent? Must it be?

b. How does linear independence relate to complementation?

c. Show that the union of two linearly independent sets can be linearly independent.

d. Show that the union of two linearly independent sets need not be linearly independent.

5.4.18 [**JH**]

a. We might conjecture that the union $S \cup T$ of linearly independent sets is linearly independent if and only if their spans have a trivial intersection $\text{span}\,(S) \cap \text{span}\,(T) = \left\{\vec{0}\right\}$. What is wrong with this argument for the 'if' direction of that conjecture? "If the union $S \cup T$ is linearly independent then the only solution to $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m = \vec{0}$ is the trivial one $c_1 = 0, \ldots, d_m = 0$. So any member of the intersection of the spans must be the zero vector because in $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ each scalar is zero."

b. Give an example showing that the conjecture is false.

c. Find linearly independent sets S and T so that the union of $S - (S \cap T)$ and $T - (S \cap T)$ is linearly independent, but the union $S \cup T$ is not linearly independent.

d. Characterize when the union of two linearly independent sets is linearly independent, in terms of the intersection of spans.

5.4.19 [**JH**] With a some calculation we can get formulas to determine whether or not a set of vectors is linearly independent.

a. Show that this subset of $\mathbb{R}^2$

$$\left\{ \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\}$$

is linearly independent if and only if $ad - bc \neq 0$.

b. Show that this subset of $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} a \\ d \\ g \end{pmatrix}, \begin{pmatrix} b \\ e \\ h \end{pmatrix}, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right\}$$

is linearly independent iff $aei + bfg + cdh - hfa - idb - gec \neq 0$.

c. When is this subset of $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} a \\ d \\ g \end{pmatrix}, \begin{pmatrix} b \\ e \\ h \end{pmatrix} \right\}$$

linearly independent?

5.5 Basis

5.5.1 [**JH**] Determine if each is a basis for $\mathbb{R}^3$.

a. $\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle$

b. $\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix} \rangle$

c. $\langle \begin{pmatrix}0\\2\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\5\\0\end{pmatrix} \rangle$

d. $\langle \begin{pmatrix}0\\2\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\3\\0\end{pmatrix} \rangle$

5.5.2 [**JH**] Determine if each is a basis for $\mathcal{P}_2$.

a. $\langle x^2 - x + 1,\ 2x + 1,\ 2x - 1\rangle$

b. $\langle x + x^2,\ x - x^2\rangle$

5.5.3 [**JH**] Represent the vector with respect to the given basis.

a. $\begin{pmatrix}1\\2\end{pmatrix}$, $B = \langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \rangle \subseteq \mathbb{R}^2$

b. $x^2 + x^3$, $D = \langle 1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3\rangle \subseteq \mathcal{P}_3$

5.5.4 [**JH**] Find a basis for $\mathcal{P}_2$, the space of all quadratic polynomials. Must any such basis contain a polynomial of each degree: degree zero, degree one, and degree two?

5.5.5 [**JH**] Find a basis for the solution set of this system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

5.5.6 [**JH**] Find a basis for $\mathcal{M}_{2\times 2}$, the space of 2×2 matrices.

5.5.7 [**JH**] Find a basis for each of the following.

a. The subspace $\{ a_2x^2 + a_1x + a_0 \mid a_2 - 2a_1 = a_0 \}$ of $\mathcal{P}_2$

b. The space of three component vectors whose first and second components add to zero

c. This subspace of the 2×2 matrices

$$\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; c - 2b = 0 \right\}$$

5.5.8 [**JH**] Find the span of each set *(that is, find restriction(s) on the coefficient of the polynomial)* and then find a basis for that span.

a. $\{1 + x,\ 1 + 2x\}$ in $\mathcal{P}_2$

b. $\{2 - 2x,\ 3 + 4x^2\}$ in $\mathcal{P}_2$

5.5.9 [**JH**] Find a basis for each of these subspaces of the space $\mathcal{P}_3$ of cubic polynomials.

a. The subspace of cubic polynomials $p(x)$ such that $p(7) = 0$.

b. The subspace of polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$.

c. The subspace of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, and $p(3) = 0$.

d. The space of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$.

5.5.10 [**YL**] Given

$$W = \{ p(x) = a_0 + a_2x^2 + a_3x^3 \mid p(-1) = 0 \}$$

a subspace of $\mathcal{P}_3$.

a. Find a basis B for $\mathcal{W}$.

b. Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B.

5.5.11 [**JH**] Can a basis contain a zero vector?

5.5.12 [**JH**] Let $\langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ be a basis for a vector space.

a. Show that $\langle c_1\vec{\beta}_1, c_2\vec{\beta}_2, c_3\vec{\beta}_3\rangle$ is a basis when $c_1, c_2, c_3 \neq 0$. What if at least one c_i is 0?

b. Prove that $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\rangle$ is a basis where $\vec{\alpha}_i = \vec{\beta}_1 + \vec{\beta}_i$.

5.6 Dimension

5.6.1 [**YL**] Given

$W = \left\{ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; p(1) = 0 \text{ and } p(-1) = 0 \right\}$

a subspace of P_3. Determine the dimension of W.

Appendix A

Answers to Exercises

Note that either a hint, a final answer or a complete solution is provided.

1.1.1

a. Yes

b. No

c. Yes

d. Yes

e. No

f. No

g. Yes

h. No

i. Yes

j. No

1.1.2

a. $x = 1$, $y = -2$

b. $x = 2$, $y = \frac{1}{3}$

c. $x = -1$, $y = 0$, and $z = 2$.

d. $x = 1$, $y = 0$, and $z = 0$.

1.1.3

a. $\left[\begin{array}{ccc|c} 3 & 4 & 5 & 7 \\ -1 & 1 & -3 & 1 \\ 2 & -2 & 3 & 5 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 2 & 5 & -6 & 2 \\ 9 & 0 & -8 & 10 \\ -2 & 4 & 1 & -7 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 3 & -4 & 5 & 17 \\ -1 & 0 & 4 & 8 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right]$

d. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 0 & 3 \\ -1 & 9 & 8 \\ 5 & -7 & 13 \end{array}\right]$

1.1.4

a. $\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 + 3x_2 &= 9 \end{aligned}$

b. $\begin{aligned} -3x_1 + 4x_2 &= 7 \\ x_2 &= -2 \end{aligned}$

c. $\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 &= 7 \end{aligned}$

d. $\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 5 \\ x_4 &= 3 \end{aligned}$

e. $\begin{aligned} x_1 + x_3 + 7x_5 &= 2 \\ x_2 + 3x_3 + 2x_4 &= 5 \end{aligned}$

1.1.5

a. $\begin{bmatrix} -2 & 1 & -7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 7 \\ 5 & 0 & 3 \\ 0 & 4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -1 & 7 \\ 2 & 3 & 5 \\ 5 & 0 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 8 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 2 & -1 \\ 5 & 0 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 0 & 5/2 & -29/2 \end{bmatrix}$

1.1.6

a. $2R_2 \to R_2$

b. $R_1 + R_2 \to R_2$

c. $2R_3 + R_1 \to R_1$

d. $R_1 \leftrightarrow R_2$

e. $-R_2 + R_3 \leftrightarrow R_3$

1.1.7 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line.
Thus the lines can share one point (giving a unique solution),

share no points (giving no solutions), or share at least two points (which makes them the same line).

1.1.8 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

1.2.1

a. Yes

b. No

c. No

d. Yes

e. Yes

f. Yes

g. No

h. Yes

i. No

j. Yes

k. Yes

l. Yes

m. No

n. Yes

o. Yes

1.2.2

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -7/5 \\ 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$

g. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 1 & \frac{5}{4} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$

i. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

j. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

n. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

p. $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.2.3

a. $x = 2$, $y = 3$

b. $x = -1$, $y = 4$, and $z = -1$.

1.2.4

a. $x_1 = 1 - 2t$; $x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$ and $x_1 = -1$, $x_2 = 1$.

b. $x_1 = -3 + 5t$; $x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 3$, $x_2 = 0$ and $x_1 = -8$, $x_2 = -1$.

c. $x_1 = 1$; $x_2 = 2$.

d. $x_1 = 0$; $x_2 = -1$.

e. $x_1 = -11 + 10t$; $x_2 = -4 + 4t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -11$, $x_2 = -4$, $x_3 = 0$ and $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$.

f. $x_1 = -\frac{2}{3} + \frac{8}{9}t$; $x_2 = \frac{2}{3} - \frac{5}{9}t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -\frac{2}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 0$ and $x_1 = \frac{4}{9}$, $x_2 = -\frac{1}{9}$, $x_3 = 1$.

g. $x_1 = 1 - s - t$; $x_2 = s$; $x_3 = 1 - 2t$; $x_4 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ and $x_1 = -2$, $x_2 = 1$, $x_3 = -3$, $x_4 = 2$.

h. $x_1 = 3 - s - 2t$; $x_2 = -3 - 5s - 7t$; $x_3 = s$; $x_4 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 3$, $x_2 = -3$, $x_3 = 0$, $x_4 = 0$ and $x_1 = 0$, $x_2 = -5$, $x_3 = -1$, $x_4 = 1$.

i. $x_1 = \frac{1}{3} - \frac{4}{3}t$; $x_2 = \frac{1}{3} - \frac{1}{3}t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $x_3 = 0$ and $x_1 = -1$, $x_2 = 0$, $x_3 = 1$.

j. $x_1 = 1 - 2s - 3t$; $x_2 = s$; $x_3 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and $x_1 = 8$, $x_2 = 1$, $x_3 = -3$.

k. No solution; the system is inconsistent.

l. No solution; the system is inconsistent.

1.2.5

a. $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (60s - 55t + 30,\ -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3},\ -14s + 13t - 7,\ s,\ t)$ where $s,\ t \in \mathbb{R}$.

b. If $s = t = 0$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) =$

$(30,\ -\frac{38}{3},\ -7,\ 0,\ 0)$.
If $s = 0$ and $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-25,\ \frac{35}{3},\ 6,\ 0,\ 1)$.

c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-\frac{30}{7},\ \frac{316}{21},\ 1,\ \frac{4}{7},\ 0)$.

1.2.6

a. $(x_1,\ x_2,\ x_3,\ x_4) = (60t,\ -\frac{79}{3}t,\ -14t,\ t)$ where $t \in \mathbb{R}$.

b. If $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4) = (60,\ -\frac{79}{3},\ -14,\ 1)$.
If $t = 3$ then $(x_1,\ x_2,\ x_3,\ x_4) = (180,\ -79,\ 42,\ 3)$.

c. If $t = \frac{1}{60}$ then $(x_1,\ x_2,\ x_3,\ x_4) = (1,\ -\frac{79}{180},\ -\frac{14}{60},\ \frac{1}{60})$.

1.2.7 Because $f(1) = 2$, $f(-1) = 6$, and $f(2) = 3$ we get a linear system.

$$\begin{aligned} 1a + 1b + c &= 2 \\ 1a - 1b + c &= 6 \\ 4a + 2b + c &= 3 \end{aligned}$$

After performing Gaussian elimination we obtain

$$\begin{array}{rcrcrcr} a & + & b & + & c & = & 2 \\ & & -2b & & & = & 4 \\ & & & & -3c & = & -9 \end{array}$$

which shows that the solution is $f(x) = 1x^2 - 2x + 3$.

1.2.8 The following system with more unknowns than equations

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

has no solution.

1.2.9 After performing Gaussian elimination the system becomes

$$\begin{aligned} x - y &= 1 \\ 0 &= -3 + k \end{aligned}$$

This system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

1.2.10

a. Never exactly 1 solution; infinite solutions if $k = 2$; no solution if $k \neq 2$.

b. Exactly 1 solution if $k \neq 2$; infinite solutions if $k = 2$; never no solution.

c. Exactly 1 solution if $k \neq 2$; no solution if $k = 2$; never infinite solutions.

d. Exactly 1 solution for all k.

1.2.11

a. Possible if $a = \pm 1$ and $a \neq \pm b$.

b. Not possible.

c. Possible if $a \neq \pm 1$ or $a = \pm b$.

1.2.12 Consistent if $b_3 - b_2 - b_1 = 0$ and $b_4 - 2b_2 - b_1 = 0$.

1.2.13 If $a \neq 0$ then the solution set of the first equation is $\{(x, y) \mid x = (c - by)/a\}$. Taking $y = 0$ gives the solution $(c/a, 0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a) + d \cdot 0 = e$, so $c = e$. Then taking $y = 1$ in $x = (c - by)/a$ gives that $a((c - b)/a) + d \cdot 1 = e$, which gives that $b = d$. Hence they are the same equation.
When $a = 0$ the equations can be different and still have the same solution set: e.g., $0x + 3y = 6$ and $0x + 6y = 12$.

1.2.14 We take three cases: that $a \neq 0$, that $a = 0$ and $c \neq 0$, and that both $a = 0$ and $c = 0$.
For the first, we assume that $a \neq 0$. Then Gaussian elimination

$$\begin{aligned} ax + \qquad\qquad by &= j \\ (-(cb/a) + d)y &= -(cj/a) + k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a) + d \neq 0$; remember that $a \neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a) + d = (ad - bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad - bc \neq 0$.
In the second case, if $a = 0$ but $c \neq 0$, then we swap

$$\begin{aligned} cx + dy &= k \\ by &= j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b \neq 0$ (we use the case assumption that $c \neq 0$ to get a unique x in back substitution). But where $a = 0$ and $c \neq 0$ the condition "$b \neq 0$" is equivalent to the condition "$ad - bc \neq 0$". That finishes the second case.
Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x + by &= j \\ 0x + dy &= k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x, y) will do), but it never has a unique solution. Note that $a = 0$ and $c = 0$ gives that $ad - bc = 0$.

1.3.1 Lorem ipsum dolor sit amet, consectetur adipiscing elit. Aliquam tincidunt cursus volutpat. Quisque non congue sem. Vivamus nec nibh sed est dapibus auctor eu sed nulla. Praesent ornare eleifend nibh a finibus. Proin rutrum neque nec massa tincidunt, non malesuada dolor interdum. Nam a massa sit amet diam efficitur pharetra. Nulla interdum efficitur sem, sit amet commodo orci mattis non. Duis tortor ex, maximus a sapien id, molestie maximus risus.

2.1.1

a. 2

b. 3

c. -1

d. Not defined.

2.1.2

a. 2×3

b. 3×2

c. 2×2

2.1.3

a. $\begin{bmatrix} -2 & -1 \\ 12 & 13 \end{bmatrix}$

b. $\begin{bmatrix} 11 & -8 \\ -1 & -19 \end{bmatrix}$

c. $\begin{bmatrix} 9 & -7 \\ 11 & -6 \end{bmatrix}$

d. $\begin{bmatrix} -2 & 1 \\ 12 & 13 \end{bmatrix}$

2.1.4

a. -22

b. -2

c. 23

d. Not possible.

e. Not possible.

2.1.5

a. $AB = \begin{bmatrix} 8 & 3 \\ 10 & -9 \end{bmatrix}$, $BA = \begin{bmatrix} -3 & 24 \\ 4 & 2 \end{bmatrix}$

b. $AB = \begin{bmatrix} -1 & -2 & 12 \\ 10 & 4 & 32 \end{bmatrix}$, BA is not defined

c. $AB = \begin{bmatrix} 3 & 8 \\ -5 & -8 \\ -8 & -32 \end{bmatrix}$, BA is not defined

d. $AB = \begin{bmatrix} 10 & -18 & 11 \\ -45 & 24 & -21 \\ -15 & 12 & -9 \end{bmatrix}$, $BA = \begin{bmatrix} 52 & -21 \\ 45 & -27 \end{bmatrix}$

e. $AB = \begin{bmatrix} -32 & 34 & -24 \\ -32 & 38 & -8 \\ -16 & 21 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 22 & -14 \\ -4 & -12 \end{bmatrix}$

f. $AB = \begin{bmatrix} -7 & 3 & 7 & -15 \\ -5 & -1 & -17 & 5 \end{bmatrix}$, BA is not defined

g. $AB = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ -2 & 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 & 4 \\ -3 & 6 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

h. $AB = \begin{bmatrix} 21 & -17 & -5 \\ 19 & 5 & 19 \\ 5 & 9 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 19 & 5 & 23 \\ 5 & -7 & -1 \\ -14 & 6 & 18 \end{bmatrix}$

2.1.6

a. $DA = \begin{bmatrix} 2 & 2 & 2 \\ -6 & -6 & -6 \\ -15 & -15 & -15 \end{bmatrix}$, $AD = \begin{bmatrix} 2 & -3 & 5 \\ 4 & -6 & 10 \\ -6 & 9 & -15 \end{bmatrix}$

b. $DA = \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix}$, $AD = \begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix}$

c. $DA = \begin{bmatrix} d_1a & d_1b \\ d_2c & d_2d \end{bmatrix}$, $AD = \begin{bmatrix} d_1a & d_2b \\ d_1c & d_2d \end{bmatrix}$

d. $DA = \begin{bmatrix} d_1a & d_1b & d_1c \\ d_2d & d_2e & d_2f \\ d_3g & d_3h & d_3i \end{bmatrix}$, $AD = \begin{bmatrix} d_1a & d_2b & d_3c \\ d_1d & d_2e & d_3f \\ d_1g & d_2h & d_3i \end{bmatrix}$

2.1.7

a. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$, $A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}$

c. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 125 \end{bmatrix}$

d. $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2.1.8

a. $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$

c. Not possible, since dimension of A and E are not the same.

d. $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$

f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D.

g. $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$

h. $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.1.9

a. $\begin{bmatrix} -9 & 6 & -8 \\ 4 & -3 & 1 \\ 10 & -7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 5 & -6 \\ 2 & -4 & 6 \\ -9 & -10 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -9 \\ -7 & 6 \\ -4 & 3 \\ -9 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$, symmetric

e. $\begin{bmatrix} 4 & -2 & 4 \\ 0 & -7 & -2 \\ 0 & 0 & 5 \end{bmatrix}$, A is lower triangular and A^T is upper triangular.

f. $\begin{bmatrix} -3 & 0 & 0 \\ -4 & -3 & 0 \\ -5 & 5 & -3 \end{bmatrix}$, A is upper triangular and A^T is lower triangular.

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, diagonal.

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$, symmetric.

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$, skew-symmetric.

2.1.10

a. -9

b. 6

c. -23

d. Not defined; the matrix must be square.

e. 0

f. n

2.1.11

a. $a = -1$, $b = 1/2$

b. $a = 5/2 + 3/2t$, $b = t$ where $t \in \mathbb{R}$

c. $a = 5$, $b = 0$

d. No solution.

2.1.12

a. $\begin{bmatrix} 0 & -2 \\ -5 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 10 & 2 \\ 5 & 11 \end{bmatrix}$

c. $\begin{bmatrix} -11 & -15 \\ 37 & 32 \end{bmatrix}$

d. No.

e. $(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2$

2.1.13

a. Hint: Apply the definition of the trace to arbitrary matrices A and B.

b. Hint: Analyse the ij product of the elements of the main diagonal.

2.1.14 Disprove: Show that it is impossible to obtain a nonzero matrix.

2.1.15 Hint: Apply the definition of an idempotent matrix.

2.2.1

a. $X = \begin{bmatrix} -5 & 9 \\ -1 & -14 \end{bmatrix}$

b. $X = \begin{bmatrix} 0 & -22 \\ -7 & 17 \end{bmatrix}$

c. $X = \begin{bmatrix} -5 & -2 \\ -9/2 & -19/2 \end{bmatrix}$

d. $X = \begin{bmatrix} 8 & 12 \\ 10 & 2 \end{bmatrix}$

2.2.2

a. $\begin{bmatrix} -24 & -5 \\ 5 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/7 \end{bmatrix}$

c. $\begin{bmatrix} -4/7 & 5/7 \\ 3/7 & -2/7 \end{bmatrix}$

d. The inverse does not exist.

2.2.3

a. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -7/5 \end{bmatrix}$

b. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -7/10 & 3/10 \\ 29/10 & -11/10 \end{bmatrix}$

2.2.4

a. $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 6 & 10 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 52 & -48 & 7 \\ 8 & -7 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -9 & 4 \\ 5 & -26 & 11 \\ 0 & -2 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 91 & 5 & -20 \\ 18 & 1 & -4 \\ -22 & -1 & 5 \end{bmatrix}$

e. $\begin{bmatrix} 25 & 8 & 0 \\ 78 & 25 & 0 \\ -30 & -9 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 5 & -3 & -8 \\ -4 & 2 & 5 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i. The inverse does not exist.

j. The inverse does not exist.

k. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & -4 \\ -35 & -10 & 1 & -47 \\ -2 & -2 & 0 & -9 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -11 & 1 & 0 & -4 \\ -2 & 0 & 1 & -4 \\ -4 & 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 28 & -2 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 254 & -19 & 110 \\ 0 & -67 & 5 & -29 \end{bmatrix}$

n. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1/4 \end{bmatrix}$

2.2.5 Disprove: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.2.6 $A = \begin{bmatrix} -\frac{3}{4} & 3 \\ 1 & -\frac{3}{4} \end{bmatrix}$

2.2.7 $A = \begin{bmatrix} 0 & -1 \\ -11 & -\frac{17}{2} \end{bmatrix}$

2.2.8

a. $A = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

b. $X = \begin{bmatrix} -\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & \frac{3}{2} & -4 & 2 \end{bmatrix}$

2.2.9 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.2.10 Hint: Use the definition of the inverse of a matrix.

2.2.11 Hint: Apply the definition of symmetric matrices.

2.3.1

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.2

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.3 $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note: The answer is not unique.

2.4.1

a. $A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

b. $x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$

2.4.2

a. $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b. $x = \begin{bmatrix} -7 \\ -7 \end{bmatrix}$

c. $x = \begin{bmatrix} -7 \\ 1 \\ -1 \end{bmatrix}$

d. $x = \begin{bmatrix} -7 \\ -7 \\ 9 \end{bmatrix}$

3.1.1

a. 34

b. 41

c. −44

d. −74

3.1.2

a. $M_{1,1} = \begin{bmatrix} 7 & 6 \\ 6 & 10 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} 3 & 6 \\ 1 & 10 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix}$.
$C_{1,1} = 43$, $C_{1,2} = -24$, $C_{1,3} = 11$.

b. $M_{1,1} = \begin{bmatrix} -6 & 8 \\ -3 & -2 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -10 & 8 \\ 0 & -2 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} 10 & -6 \\ 0 & -3 \end{bmatrix}$.
$C_{1,1} = 36$, $C_{1,2} = -20$, $C_{1,3} = -30$.

c. $M_{1,1} = \begin{bmatrix} 3 & 10 \\ 3 & 9 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -3 & 10 \\ -9 & 9 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} -3 & 3 \\ -9 & 3 \end{bmatrix}$.
$C_{1,1} = -3$, $C_{1,2} = -63$, $C_{1,3} = 18$.

d. $M_{1,1} = \begin{bmatrix} 0 & 0 \\ 8 & -1 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -8 & 0 \\ -10 & -1 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} -8 & 0 \\ -10 & 8 \end{bmatrix}$.
$C_{1,1} = 0$, $C_{1,2} = -8$, $C_{1,3} = -64$.

3.1.3

a. $3(+1)\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} + 0(-1)\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + 1(+1)\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -13$

b. $1(-1)\begin{vmatrix}0&1\\3&0\end{vmatrix}+2(+1)\begin{vmatrix}3&1\\-1&0\end{vmatrix}+2(-1)\begin{vmatrix}3&0\\-1&3\end{vmatrix}=-13$

c. $1(+1)\begin{vmatrix}1&2\\-1&3\end{vmatrix}+2(-1)\begin{vmatrix}3&0\\-1&3\end{vmatrix}+0(+1)\begin{vmatrix}3&0\\1&2\end{vmatrix}=-13$

3.1.4

a. -59

b. 250

c. 3

d. 0

e. 0

f. 2

3.1.5 Evaluate the determinant using a cofactor expansion. The same is true for lower triangular matrices.

3.1.6 $\lambda=\frac{3\pm\sqrt{33}}{4}$

3.1.7 False, Here is a determinant whose value

$$\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}=1$$

doesn't equal the result of expanding down the diagonal.

$$1\cdot(+1)\begin{vmatrix}1&0\\0&1\end{vmatrix}+1\cdot(+1)\begin{vmatrix}1&0\\0&1\end{vmatrix}+1\cdot(+1)\begin{vmatrix}1&0\\0&1\end{vmatrix}=3$$

3.1.8 There are no real numbers θ that make the matrix singular because the determinant of the matrix $\cos^2\theta+\sin^2\theta$ is never 0, it equals 1 for all θ.

3.2.1

a. $\det(A)=90$; $2R_1\to R_1$.
$\det(B)=45$; $10R_1+R_3\to R_3$.
$\det(C)=45$; $C=A^T$.

b. $\det(A)=41$; $R_2\leftrightarrow R_3$.
$\det(B)=164$; $-4R_3\to R_3$.
$\det(C)=-41$; $R_2+R_1\to R_1$.

c. $\det(A)=-16$; $R_1\leftrightarrow R_2$ then $R_1\leftrightarrow R_3$.
$\det(B)=-16$; $-R_1\to R_1$ and $-R_2\to R_2$.
$\det(C)=-432$; $C=3M$.

d. $\det(A)=-120$; $R_1\leftrightarrow R_2$ then $R_1\leftrightarrow R_3$ then $R_2\leftrightarrow R_3$.
$\det(B)=720$; $2R_2\to R_2$ and $3R_3\to R_3$.
$\det(C)=-120$; $C=-M$.

3.2.2

a. 15

b. -52

c. 0

d. 1

e. -113

f. 179

3.2.3 $\det(A)=-\frac{5}{12}$

3.2.4 Hint: Use elementary operations to bring the matrix under triangular form.

3.3.1

a. $\begin{bmatrix}0&-1&2\\3&-2&-8\\0&1&1\end{bmatrix}$

b. $\begin{bmatrix}4&1\\-2&3\end{bmatrix}$

c. $\begin{bmatrix}0&-1\\-5&1\end{bmatrix}$

d. $\begin{bmatrix}-24&-12&12\\12&6&-6\\-8&-4&4\end{bmatrix}$

e. $\begin{bmatrix}4&-3&2&-1\\-3&6&-4&2\\2&-4&6&-3\\-1&2&-3&4\end{bmatrix}$

3.3.2

a. $\begin{bmatrix}T_{1,1}&T_{2,1}\\T_{1,2}&T_{2,2}\end{bmatrix}=\begin{bmatrix}|t_{2,2}|&-|t_{1,2}|\\-|t_{2,1}|&|t_{1,1}|\end{bmatrix}=\begin{bmatrix}t_{2,2}&-t_{1,2}\\-t_{2,1}&t_{1,1}\end{bmatrix}$

b. $(1/t_{1,1}t_{2,2}-t_{1,2}t_{2,1})\begin{bmatrix}t_{2,2}&-t_{1,2}\\-t_{2,1}&t_{1,1}\end{bmatrix}$

3.3.3 Consider this diagonal matrix.

$$D=\begin{bmatrix}d_1&0&0&\dots&\\0&d_2&0&&\\0&0&d_3&&\\&&&\ddots&\\&&&&d_n\end{bmatrix}$$

If $i\neq j$ then the i,j minor is an $(n-1)\times(n-1)$ matrix with only $n-2$ nonzero entries, because we have deleted both d_i and d_j. Thus, at least one row or column of the minor is all zeroes, and so the cofactor $D_{i,j}$ is zero. If $i=j$ then the minor is the diagonal matrix with entries d_1, …, d_{i-1}, d_{i+1}, …, d_n. Its determinant is obviously $(-1)^{i+j}=(-1)^{2i}=1$ times the product of those.

$$\operatorname{adj}(D)=\begin{bmatrix}d_2\cdots d_n&0&&0\\0&d_1d_3\cdots d_n&&0\\&&\ddots&\\&&&d_1\cdots d_{n-1}\end{bmatrix}$$

3.3.4 Just note that if $S=T^{\mathsf{T}}$ then the cofactor $S_{j,i}$ equals the cofactor $T_{i,j}$ because $(-1)^{j+i}=(-1)^{i+j}$ and because the minors are the transposes of each other (and the determinant of a transpose equals the determinant of the matrix).

3.3.5 False. A counter example.

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{adj}(T) = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix} \qquad \text{adj}(\text{adj}(T)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.3.6 This equation

$$0 = \det(\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}) = 64 - 20x + x^2 = (x-16)(x-4)$$

has roots $x = 16$ and $x = 4$.

3.3.7 $\det(TS) = \det(T)\cdot\det(S) = \det(S)\cdot\det(T) = \det(ST)$.

3.3.8

a. Plug and chug: the determinant of the product is this

$$\begin{aligned} \det(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix}) &= \det(\begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}) \\ &= acwx + adwz + bcxy + bdyz \\ &\quad -acwx - bcwz - adxy - bdyz \end{aligned}$$

while the product of the determinants is this.

$$\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) \det(\begin{bmatrix} w & x \\ y & z \end{bmatrix}) = (ad-bc)(wz-xy)$$

Verification that they are equal is easy.

b. Use the prior part.

3.3.9

a. If it is defined then it is $(3^2)(2)(2^{-2})(3)$.

b. Hint: $\det 6A^3 + 5A^2 + 2A = \det A \det 6A^2 + 5A + 2I$.

3.3.10

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

b. $1 = \det(AA^{-1}) = \det(AA^{\mathsf{T}}) = \det(A)\det(A^{\mathsf{T}}) = (\det(A))^2$

c. The converse does not hold; here is an example.

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

3.3.11 If $H = P^{-1}GP$ then $\det(H) = \det(P^{-1})\det(G)\det(P) = \det(P^{-1})\det(P)\det(G) = \det(P^{-1}P)\det(G) = \det(G)$.

3.3.12 An algebraic check is easy.

$$0 = xy_2 + x_2y_3 + x_3y - x_3y_2 - xy_3 - x_2y = x\cdot(y_2-y_3) + y\cdot(x_3-x_2) + x_2y_3 - x_3y_2$$

simplifies to the familiar form

$$y = x\cdot(x_3-x_2)/(y_3-y_2) + (x_2y_3 - x_3y_2)/(y_3-y_2)$$

(the $y_3 - y_2 = 0$ case is easily handled).

3.3.13 Hint: Apply the determinant to both sides $AB = -BA$.

3.3.14 Disprove. Recall that constants come out one row at a time.

$$\det(\begin{bmatrix} 2 & 4 \\ 2 & 6 \end{bmatrix}) = 2\cdot\det(\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}) = 2\cdot 2\cdot\det(\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix})$$

This contradicts linearity (here we didn't need S, i.e., we can take S to be the matrix of zeros).

3.4.1 $x_1 = 4$

3.4.2

a. $\det(A) = -123$, $\det(A_1) = -492$, $\det(A_2) = 123$, $\det(A_3) = 492$,
$x = \begin{bmatrix} 4 \\ -1 \\ -4 \end{bmatrix}$.

b. $\det(A) = -43$, $\det(A_1) = 215$, $\det(A_2) = 0$,
$x = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$.

c. $\det(A) = 0$, $\det(A_1) = 0$, $\det(A_2) = 0$, $\det(A_3) = 0$. Infinite solutions exist.

d. $\det(A) = 0$, $\det(A_1) = -56$, $\det(A_2) = 26$. No solution exist.

e. $\det(A) = 0$, $\det(A_1) = 0$, $\det(A_2) = 0$, $\det(A_3) = 0$. Infinite solutions exist.

f. $\det(A) = 0$, $\det(A_1) = 1247$, $\det(A_2) = -49$, $\det(A_3) = -49$. No solution exist.

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$.

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5.1.1

a. $0 + 0x + 0x^2 + 0x^3$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. The constant function $f(x) = 0$

d. The constant function $f(n) = 0$

5.1.2

a. $3 + 2x - x^2$

b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

c. $-3e^x + 2e^{-x}$

5.1.3

a. $1 + 2x$, $2 - 1x$, and x.

b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

a.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

a. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.

b. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

5.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

a. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

b. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

c. It is not closed under addition.
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

d. It is not closed under scalar multiplication.
$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

e. The set is empty, violating the existance of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

a. It is not a vector space.
$$(1+1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b. It is not a vector space.
$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

a. Yes.

b. Yes.

c. No, this set is not closed under the natural addition operation. The vector of all 1/4's is an element of this set but when added to itself the result, the vector of all 1/2's, is not an element of the set.

d. Yes.

e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11

a. Closed under vector addition. Hint: Apply determinant properties.

b. $\vec{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$

c. Every $A \in V$ has an additive inverse A^{-1}.

d. Yes.

e. Not closed under scalar multiplication. Since $0\vec{0} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin V$

5.1.12 Check all 10 conditions of the definition of a vector space.

5.1.13 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.14

a. No since $1 \cdot (0,\ 1) + 1 \cdot (0,\ 1) \neq (1+1) \cdot (0,\ 1)$.

b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0,\ 1) \neq (0,\ 1)$.

5.1.15

a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which implies that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.16 We can combine the argument showing closure under addition with the argument showing closure under scalar multiplication into one single argument showing closure under linear combinations of two vectors. If $r_1, r_2, x_1, x_2, y_1, y_2, z_1, z_2$ are in $\mathbb{R}$ then

$$r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 - r_1 + 1 \\ r_1y_1 \\ r_1z_1 \end{pmatrix} + \begin{pmatrix} r_2x_2 - r_2 + 1 \\ r_2y_2 \\ r_2z_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 - r_1 + r_2x_2 - r_2 + 1 \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix}$$

(note that the definition of addition in this space is that the first components combine as $(r_1x_1 - r_1 + 1) + (r_2x_2 - r_2 + 1) - 1$, so the first component of the last vector does not say '$+2$'). Adding the three components of the last vector gives $r_1(x_1 - 1 + y_1 + z_1) + r_2(x_2 - 1 + y_2 + z_2) + 1 = r_1 \cdot 0 + r_2 \cdot 0 + 1 = 1$.

Most of the other checks of the conditions are easy (although the oddness of the operations keeps them from being routine). Commutativity of addition goes like this.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 - 1 \\ y_2 + y_1 \\ z_2 + z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Associativity of addition has

$$(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2 - 1) + x_3 - 1 \\ (y_1 + y_2) + y_3 \\ (z_1 + z_2) + z_3 \end{pmatrix}$$

while

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + (\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}) = \begin{pmatrix} x_1 + (x_2 + x_3 - 1) - 1 \\ y_1 + (y_2 + y_3) \\ z_1 + (z_2 + z_3) \end{pmatrix}$$

and they are equal. The identity element with respect to this addition operation works this way

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x + 1 - 1 \\ y + 0 \\ z + 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and the additive inverse is similar.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -x + 2 \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} x + (-x + 2) - 1 \\ y - y \\ z - z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The conditions on scalar multiplication are also easy. For the first condition,

$$(r + s) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (r + s)x - (r + s) + 1 \\ (r + s)y \\ (r + s)z \end{pmatrix}$$

while

$$r \begin{pmatrix} x \\ y \\ z \end{pmatrix} + s \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx - r + 1 \\ ry \\ rz \end{pmatrix} + \begin{pmatrix} sx - s + 1 \\ sy \\ sz \end{pmatrix} = \begin{pmatrix} (rx - r + 1) + (sx - s + 1) - 1 \\ ry + sy \\ rz + sz \end{pmatrix}$$

and the two are equal. The second condition compares

$$r \cdot (\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = r \cdot \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} r(x_1 + x_2 - 1) - r + 1 \\ r(y_1 + y_2) \\ r(z_1 + z_2) \end{pmatrix}$$

with

$$r \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} rx_1 - r + 1 \\ ry_1 \\ rz_1 \end{pmatrix} + \begin{pmatrix} rx_2 - r + 1 \\ ry_2 \\ rz_2 \end{pmatrix} = \begin{pmatrix} (rx_1 - r + 1) + (rx_2 - r + 1) - 1 \\ ry_1 + ry_2 \\ rz_1 + rz_2 \end{pmatrix}$$

and they are equal. For the third condition,

$$(rs) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rsx - rs + 1 \\ rsy \\ rsz \end{pmatrix}$$

while

$$r(s \begin{pmatrix} x \\ y \\ z \end{pmatrix}) = r(\begin{pmatrix} sx - s + 1 \\ sy \\ sz \end{pmatrix}) = \begin{pmatrix} r(sx - s + 1) - r + 1 \\ rsy \\ rsz \end{pmatrix}$$

and the two are equal. For scalar multiplication by 1 we have this.

$$1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x - 1 + 1 \\ 1y \\ 1z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus all the conditions on a vector space are met by these two operations.

5.1.17

Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.18
It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.19
Each element of a vector space has one and only one additive inverse.
For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.20
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.

b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.

c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{ k \cdot \vec{v} \mid k \in \mathbb{R} \}$. By the prior item this set is infinite.

5.2.1

a. Yes, we can easily check that it is closed under addition and scalar multiplication.

b. Yes, we can easily check that it is closed under addition and scalar multiplication.

c. No. It is not closed under addition. For instance,

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

is not in the set. (This set is also not closed under scalar multiplication, for instance, it does not contain the zero matrix.)

d. Yes, we can easily check that it is closed under addition and scalar multiplication.

5.2.2 No, it is not closed. In particular, it is not closed under scalar multiplication because it does not contain the zero polynomial.

5.2.3 No, such a set is not closed. For one thing, it does not contain the zero vector.

5.2.4

a. Every such set has the form $\{ r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R} \}$ where either or both of $\vec{v}, \vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.

b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.5 No. The only subspaces of $\mathbb{R}^1$ are the space itself and its trivial subspace. Any subspace S of $\mathbb{R}$ that contains a nonzero member $\vec{v}$ must contain the set of all of its scalar multiples $\{ r \cdot \vec{v} \mid r \in \mathbb{R} \}$. But this set is all of $\mathbb{R}$.

5.2.6 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r\,f'$.

5.2.7

a. This is a subspace. It is closed because if f_1, f_2 are even and c_1, c_2 are scalars then we have this.

$$(c_1 f_1 + c_2 f_2)\,(-x) = c_1\, f_1(-x) + c_2\, f_2(-x) = c_1\, f_1(x) + c_2\, f_2(x) = ($$

b. This is also a subspace; the check is similar to the prior one.

5.2.8 No. Subspaces of $\mathbb{R}^3$ are sets of three-tall vectors, while $\mathbb{R}^2$ is a set of two-tall vectors. Clearly though, $\mathbb{R}^2$ is "just like" this subspace of $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

5.2.9

a. The union of the x-axis and the y-axis in $\mathbb{R}^2$ is one.

b. The set of integers, as a subset of $\mathbb{R}^1$, is one.

c. The subset $\{\vec{v}\}$ of $\mathbb{R}^2$ is one, where $\vec{v}$ is any nonzero vector.

5.2.10

a. Is is a subspace.
Assume that A, B are subspaces of V. Note that their intersection is not empty as both contain the zero vector. If $\vec{w}, \vec{s} \in A \cap B$ and r, s are scalars then $r\vec{v} + s\vec{w} \in A$ because each vector is in A and so a linear combination is in A, and $r\vec{v} + s\vec{w} \in B$ for the same reason. Thus the intersection is closed.

b. In general it is not a subspace. (It is a subspace, only if $A \subseteq B$ or $B \subseteq A$).
Take V to be $\mathbb{R}^3$, take A to be the x-axis, and B to be the y-axis. Note that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in A \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in B \quad \text{but} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin A \cup B$$

as the sum is in neither A nor B.
If $A \subseteq B$ or $B \subseteq A$ then clearly $A \cup B$ is a subspace.
To show that $A \cup B$ is a subspace only if one subspace contains the other, we assume that $A \not\subseteq B$ and $B \not\subseteq A$ and prove that the union is not a subspace. The assumption that A is not a subset of B means that there is an $\vec{a} \in A$ with $\vec{a} \notin B$. The other assumption gives a $\vec{b} \in B$

with $\vec{b} \notin A$. Consider $\vec{a} + \vec{b}$. Note that sum is not an element of A or else $(\vec{a} + \vec{b}) - \vec{a}$ would be in A, which it is not. Similarly the sum is not an element of B. Hence the sum is not an element of $A \cup B$, and so the union is not a subspace.

c. It is not a subspace. As A is a subspace, it contains the zero vector, and therefore the set that is A's complement does not. Without the zero vector, the complement cannot be a vector space.

5.2.11 It is transitive; apply the subspace test. (You must consider the following. Suppose B is a subspace of a vector space V and suppose $A \subseteq B \subseteq V$ is a subspace. From which space does A inherit its operations? The answer is that it doesn't matter A will inherit the same operations in either case.)

5.3.1

a. Yes, solving the linear system arising from

$$r_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + r_2 \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}$$

gives $r_1 = 2$ and $r_2 = 1$.

b. Yes; the linear system arising from $r_1(x^2) + r_2(2x + x^2) + r_3(x + x^3) = x - x^3$

$$\begin{aligned} 2r_2 + r_3 &= 1 \\ r_1 + r_2 &= 0 \\ r_3 &= -1 \end{aligned}$$

gives that $-1(x^2) + 1(2x + x^2) - 1(x + x^3) = x - x^3$.

c. No; any combination of the two given matrices has a zero in the upper right.

5.3.2

a. Yes. It is in that span since $1\cos^2 x + 1\sin^2 x = f(x)$.

b. No. Since $r_1 \cos^2 x + r_2 \sin^2 x = 3 + x^2$ has no scalar solutions that work for all x. For instance, setting x to be 0 and π gives the two equations $r_1 \cdot 1 + r_2 \cdot 0 = 3$ and $r_1 \cdot 1 + r_2 \cdot 0 = 3 + \pi^2$, which are not consistent with each other.

c. No. Consider what happens on setting x to be $\pi/2$ and $3\pi/2$.

d. Yes, $\cos(2x) = 1 \cdot \cos^2(x) - 1 \cdot \sin^2(x)$.

5.3.3

a. Yes, for any $x, y, z \in \mathbb{R}$ this equation

$$r_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + r_2 \begin{pmatrix}0\\2\\0\end{pmatrix} + r_3 \begin{pmatrix}0\\0\\3\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

has the solution $r_1 = x$, $r_2 = y/2$, and $r_3 = z/3$.

b. Yes, the equation

$$r_1 \begin{pmatrix}2\\0\\1\end{pmatrix} + r_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + r_3 \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives rise to this

$$\begin{aligned} 2r_1 + r_2 &= x \\ r_2 &= y \\ r_1 + r_3 &= z \end{aligned}$$

Gaussian elimination gives

$$\begin{aligned} 2r_1 + r_2 &= x \\ r_2 &= y \\ r_3 &= -(1/2)x + (1/2)y + z \end{aligned}$$

so that, given any x, y, and z, we can compute that $r_3 = (-1/2)x + (1/2)y + z$, $r_2 = y$, and $r_1 = (1/2)x - (1/2)y$.

c. No. In particular, we cannot get the vector

$$\begin{pmatrix}0\\0\\1\end{pmatrix}$$

as a linear combination since the two given vectors both have a third component of zero.

d. Yes. The equation

$$r_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + r_2 \begin{pmatrix}3\\1\\0\end{pmatrix} + r_3 \begin{pmatrix}-1\\0\\0\end{pmatrix} + r_4 \begin{pmatrix}2\\1\\5\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 2 & x \\ 0 & 1 & 0 & 1 & y \\ 0 & 0 & 1 & 6 & -x + 3y + z \end{array}\right]$$

We have infinitely many solutions. We can, for example, set r_4 to be zero and solve for r_3, r_2, and r_1 in terms of x, y, and z by the usual methods of back-substitution.

e. No. The equation

$$r_1 \begin{pmatrix}2\\1\\1\end{pmatrix} + r_2 \begin{pmatrix}3\\0\\1\end{pmatrix} + r_3 \begin{pmatrix}5\\1\\2\end{pmatrix} + r_4 \begin{pmatrix}6\\0\\2\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left[\begin{array}{cccc|c} 2 & 3 & 5 & 6 & x \\ 0 & -3/2 & -3/2 & -3 & -(1/2)x + y \\ 0 & 0 & 0 & 0 & -(1/3)x - (1/3)y + z \end{array}\right]$$

This shows that not every vector can be so expressed. Only the vectors satisfying the restriction that $-(1/3)x - (1/3)y + z = 0$ are in the span. (To see that any such vector is indeed expressible, take r_3 and r_4 to be zero and solve for r_1 and r_2 in terms of x, y, and z by back-substitution.)

5.3.4

a. $\{(c\ \ b\ \ c) \mid b, c \in \mathbb{R}\} = \{b(0\ \ 1\ \ 0) + c(1\ \ 0\ \ 1) \mid b, c \in \mathbb{R}\}$ The obvious choice for the set that spans is $\{(0\ \ 1\ \ 0), (1\ \ 0\ \ 1)\}$.

b. $\left\{ \begin{bmatrix} -d & b \\ c & d \end{bmatrix} \;\middle|\; b,c,d \in \mathbb{R} \right\} = \left\{ b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; b,c,d \in \mathbb{R} \right\}$ One set that spans this space consists of those three matrices.

c. The system

$$\begin{aligned} a + 3b \qquad\quad &= 0 \\ 2a \qquad -c-d &= 0 \end{aligned}$$

gives $b = -(c+d)/6$ and $a = (c+d)/2$. So one description is this.

$$\left\{ c\begin{bmatrix} 1/2 & -1/6 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1/2 & -1/6 \\ 0 & 1 \end{bmatrix} \;\middle|\; c,d \in \mathbb{R} \right\}$$

That shows that a set spanning this subspace consists of those two matrices.

d. The $a = 2b - c$ gives that the set $\{(2b-c) + bx + cx^3 \mid b,c \in \mathbb{R}\}$ equals the set $\{b(2+x) + c(-1+x^3) \mid b,c \in \mathbb{R}\}$. So the subspace is the span of the set $\{2+x,\ -1+x^3\}$.

e. The set $\{a + bx + cx^2 \mid a + 7b + 49c = 0\}$ can be parametrized as

$$\{b(-7+x) + c(-49+x^2) \mid b,c \in \mathbb{R}\}$$

and so has the spanning set $\{-7+x,\ -49+x^2\}$.

5.3.5

a. We can parametrize in this way

$$\left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \;\middle|\; x,z \in \mathbb{R} \right\} = \left\{ x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x,z \in \mathbb{R} \right\}$$

giving this for a spanning set.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

b. Here is a parametrization, and the associated spanning set. $\left\{ y\begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; y,z \in \mathbb{R} \right\}$ $\left\{ \begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$

c. $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

d. Parametrize the description as $\{-a_1 + a_1x + a_3x^2 + a_3x^3 \mid a_1, a_3 \in \mathbb{R}\}$ to get $\{-1+x, x^2+x^3\}$.

e. $\{1, x, x^2, x^3, x^4\}$

f. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

5.3.6 We will show mutual containment between the two sets.

The first containment $\text{span}(\text{span}(S)) \supseteq \text{span}(S)$ is an instance of the more general, and obvious, fact that for any subset T of a vector space, $\text{span}(T) \supseteq T$.

For the other containment, that $\text{span}(\text{span}(S)) \subseteq \text{span}(S)$, take m vectors from $\text{span}(S)$, namely $c_{1,1}\vec{s}_{1,1}+\cdots+c_{1,n_1}\vec{s}_{1,n_1}$, $\ldots$, $c_{1,m}\vec{s}_{1,m}+\cdots+c_{1,n_m}\vec{s}_{1,n_m}$, and note that any linear combination of those

$$r_1(c_{1,1}\vec{s}_{1,1}+\cdots+c_{1,n_1}\vec{s}_{1,n_1})+\cdots+r_m(c_{1,m}\vec{s}_{1,m}+\cdots+c_{1,n_m}\vec{s}_{1,n_m})$$

is a linear combination of elements of S

$$= (r_1c_{1,1})\vec{s}_{1,1}+\cdots+(r_1c_{1,n_1})\vec{s}_{1,n_1}+\cdots+(r_mc_{1,m})\vec{s}_{1,m}+\cdots+(r_mc_{1,n_m})\vec{s}_{1,n_m}$$

and so is in $\text{span}(S)$. That is, simply recall that a linear combination of linear combinations (of members of S) is a linear combination (again of members of S).

5.3.7 Hint: For each subspace determine a set of vectors that spans it.
$W_1 \subsetneq W_2$

5.3.8 For 'if', let S be a subset of a vector space V and assume $\vec{v} \in S$ satisfies $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ where $c_1, \ldots, c_n$ are scalars and $\vec{s}_1, \ldots, \vec{s}_n \in S$. We must show that $\text{span}(S \cup \{\vec{v}\}) = \text{span}(S)$.

Containment one way, $\text{span}(S) \subseteq \text{span}(S \cup \{\vec{v}\})$ is obvious. For the other direction, $\text{span}(S \cup \{\vec{v}\}) \subseteq \text{span}(S)$, note that if a vector is in the set on the left then it has the form $d_0\vec{v} + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ where the d's are scalars and the $\vec{t}$'s are in S. Rewrite that as $d_0(c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ and note that the result is a member of the span of S.

The 'only if' is clearly true adding $\vec{v}$ enlarges the span to include at least $\vec{v}$.

5.3.9 The span of a set does not depend on the enclosing space. A linear combination of vectors from S gives the same sum whether we regard the operations as those of W or as those of V, because the operations of W are inherited from V.

5.3.10

a. Always; if $S \subseteq T$ then a linear combination of elements of S is also a linear combination of elements of T.

b. Sometimes (more precisely, if and only if $S \subseteq T$ or $T \subseteq S$).
The answer is not 'always' as is shown by this example from $\mathbb{R}^3$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad T = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

because of this.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \text{span}(S \cup T) \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin \text{span}(S) \cup \text{span}(T)$$

The answer is not 'never' because if either set contains the other then equality is clear. We can characterize equality as happening only when either set contains the other by assuming $S \not\subseteq T$ (implying the existence of a vector $\vec{s} \in S$ with $\vec{s} \notin T$) and $T \not\subseteq S$ (giving a $\vec{t} \in T$ with $\vec{t} \notin S$), noting $\vec{s} + \vec{t} \in \mathrm{span}\,(S \cup T)$, and showing that $\vec{s} + \vec{t} \notin \mathrm{span}\,(S) \cup \mathrm{span}\,(T)$.

c. Sometimes.
Clearly $\mathrm{span}\,(S \cap T) \subseteq \mathrm{span}\,(S) \cap \mathrm{span}\,(T)$ because any linear combination of vectors from $S \cap T$ is a combination of vectors from S and also a combination of vectors from T.
Containment the other way does not always hold. For instance, in $\mathbb{R}^2$, take

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}, \quad T = \left\{ \begin{pmatrix}2\\0\end{pmatrix} \right\}$$

so that $\mathrm{span}\,(S) \cap \mathrm{span}\,(T)$ is the x-axis but $\mathrm{span}\,(S \cap T)$ is the trivial subspace.
Characterizing exactly when equality holds is tough. Clearly equality holds if either set contains the other, but that is not 'only if' by this example in $\mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}, \quad T = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

d. Never, as the span of the complement is a subspace, while the complement of the span is not (it does not contain the zero vector).

5.4.1

a. It is dependent. Considering

$$c_1 \begin{pmatrix}1\\-3\\5\end{pmatrix} + c_2 \begin{pmatrix}2\\2\\4\end{pmatrix} + c_3 \begin{pmatrix}4\\-4\\14\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

gives this linear system.

$$\begin{aligned} c_1 + 2c_2 + 4c_3 &= 0 \\ -3c_1 + 2c_2 - 4c_3 &= 0 \\ 5c_1 + 4c_2 + 14c_3 &= 0 \end{aligned}$$

Gauss's Method

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -3 & 2 & -4 & 0 \\ 5 & 4 & 14 & 0 \end{array}\right]$$

gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yields a free variable, so there are infinitely many solutions. For an example of a particular dependence we can set c_3 to be, say, 1. Then we get $c_2 = -1$ and $c_1 = -2$.

b. It is dependent. The linear system that arises here

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 7 & 7 & 7 & 0 \\ 7 & 7 & 7 & 0 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

has infinitely many solutions. We can get a particular solution by taking c_3 to be, say, 1, and back-substituting to get the resulting c_2 and c_1.

c. It is linearly independent. The system

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & 0 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has only the solution $c_1 = 0$ and $c_2 = 0$. (We could also have gotten the answer by inspection the second vector is obviously not a multiple of the first, and vice versa.)

d. It is linearly dependent. The linear system

$$\left[\begin{array}{cccc|c} 9 & 2 & 3 & 12 & 0 \\ 9 & 0 & 5 & 12 & 0 \\ 0 & 1 & -4 & -1 & 0 \end{array}\right]$$

has more unknowns than equations, and so Gauss's Method must end with at least one variable free (there can't be a contradictory equation because the system is homogeneous, and so has at least the solution of all zeroes). To exhibit a combination, we can do the reduction

$$\left[\begin{array}{cccc|c} 9 & 2 & 3 & 12 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & -3 & -1 & 0 \end{array}\right]$$

and take, say, $c_4 = 1$. Then we have that $c_3 = -1/3$, $c_2 = -1/3$, and $c_1 = -31/27$.

5.4.2

a. This set is independent. Setting up the relation $c_1(3 - x + 9x^2) + c_2(5 - 6x + 3x^2) + c_3(1 + 1x - 5x^2) = 0 + 0x + 0x^2$ gives a linear system

$$\left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ -1 & -6 & 1 & 0 \\ 9 & 3 & -5 & 0 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ 0 & -13 & 4 & 0 \\ 0 & 0 & -128/13 & 0 \end{array}\right]$$

with only one solution: $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

b. This set is independent. We can see this by inspection, straight from the definition of linear independence. Obviously neither is a multiple of the other.

c. This set is linearly independent. The linear system reduces in this way

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 7 & 2 & -3 & 0 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & -5/2 & -2 & 0 \\ 0 & 0 & -51/5 & 0 \end{array}\right]$$

to show that there is only the solution $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

d. This set is linearly dependent. The linear system

$$\left[\begin{array}{cccc|c} 8 & 0 & 2 & 8 & 0 \\ 3 & 1 & 2 & -2 & 0 \\ 3 & 2 & 2 & 5 & 0 \end{array}\right]$$

must, after reduction, end with at least one variable free (there are more variables than equations, and there is no possibility of a contradictory equation because the system is homogeneous). We can take the free variables as parameters to describe the solution set. We can then set the parameter to a nonzero value to get a nontrivial linear relation.

5.4.3 Let Z be the zero function $Z(x) = 0$, which is the additive identity in the vector space under discussion.

a. This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$. Plugging in $x = 1$ and $x = 2$ gives a linear system

$$\begin{aligned} c_1 \cdot 1 + \quad c_2 \cdot 1 &= 0 \\ c_1 \cdot 2 + c_2 \cdot (1/2) &= 0 \end{aligned}$$

with the unique solution $c_1 = 0$, $c_2 = 0$.

b. This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plug in $x = 0$ and $x = \pi/2$ to get

$$\begin{aligned} c_1 \cdot 1 + c_2 \cdot 0 &= 0 \\ c_1 \cdot 0 + c_2 \cdot 1 &= 0 \end{aligned}$$

which obviously gives that $c_1 = 0$, $c_2 = 0$.

c. This set is also linearly independent. Considering $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plugging in $x = 1$ and $x = e$

$$\begin{aligned} c_1 \cdot e + c_2 \cdot 0 &= 0 \\ c_1 \cdot e^e + c_2 \cdot 1 &= 0 \end{aligned}$$

gives that $c_1 = 0$ and $c_2 = 0$.

5.4.4

a. This set is dependent. The familiar relation $\sin^2(x) + \cos^2(x) = 1$ shows that $2 = c_1 \cdot (4\sin^2(x)) + c_2 \cdot (\cos^2(x))$ is satisfied by $c_1 = 1/2$ and $c_2 = 2$.

b. This set is independent. Consider the relationship $c_1 \cdot 1 + c_2 \cdot \sin(x) + c_3 \cdot \sin(2x) = 0$ (that '0' is the zero function). Taking three suitable points such as $x = \pi$, $x = \pi/2$, $x = \pi/4$ gives a system

$$\begin{aligned} c_1 \qquad\qquad\qquad &= 0 \\ c_1 + \qquad c_2 \qquad &= 0 \\ c_1 + (\sqrt{2}/2)c_2 + c_3 &= 0 \end{aligned}$$

whose only solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

c. By inspection, this set is independent. Any dependence $\cos(x) = c \cdot x$ is not possible since the cosine function is not a multiple of the identity function

d. By inspection, we spot that there is a dependence. Because $(1 + x)^2 = x^2 + 2x + 1$, we get that $c_1 \cdot (1 + x)^2 + c_2 \cdot (x^2 + 2x) = 3$ is satisfied by $c_1 = 3$ and $c_2 = -3$.

e. This set is dependent, because it contains the zero object in the vector space, the zero polynomial.

f. This set is dependent. The easiest way to see that is to recall the trigonometric relationship $\cos^2(x) - \sin^2(x) = \cos(2x)$.

5.4.5 No. Here are two members of the plane where the second is a multiple of the first.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

(Another reason that the answer is "no" is the the zero vector is a member of the plane and no set containing the zero vector is linearly independent.)

5.4.6

a. $\lambda = 1$

b. $\lambda \neq -1,\ -\frac{1}{2},\ 1$

5.4.7

a. Assume that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, so that any relationship $d_0\vec{u} + d_1\vec{v} + d_2\vec{w} = \vec{0}$ leads to the conclusion that $d_0 = 0$, $d_1 = 0$, and $d_2 = 0$.
Consider the relationship $c_1(\vec{u}) + c_2(\vec{u} + \vec{v}) + c_3(\vec{u} + \vec{v} + \vec{w}) = \vec{0}$. Rewrite it to get $(c_1 + c_2 + c_3)\vec{u} + (c_2 + c_3)\vec{v} + (c_3)\vec{w} = \vec{0}$. Taking d_0 to be $c_1 + c_2 + c_3$, taking d_1 to be $c_2 + c_3$, and taking d_2 to be c_3 we have this system.

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

Conclusion: the c's are all zero, and so the set is linearly independent.

b. The second set is dependent

$$1 \cdot (\vec{u} - \vec{v}) + 1 \cdot (\vec{v} - \vec{w}) + 1 \cdot (\vec{w} - \vec{u}) = \vec{0}$$

whether or not the first set is independent.

5.4.8

a. A singleton set $\{\vec{v}\}$ is linearly independent if and only if $\vec{v} \neq \vec{0}$. For the 'if' direction, with $\vec{v} \neq \vec{0}$, we consider the relationship $c \cdot \vec{v} = \vec{0}$ and noting that the only solution is the trivial one: $c = 0$. For the 'only if' direction, it is evident from the definition.

b. A set with two elements is linearly independent if and only if neither member is a multiple of the other (note that if one is the zero vector then it is a multiple of the other). This is an equivalent statement: a set is linearly dependent if and only if one element is a multiple of the other.
The proof is easy. A set $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if and only if there is a relationship $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ with either $c_1 \neq 0$ or $c_2 \neq 0$ (or both). That holds if and only if $\vec{v}_1 = (-c_2/c_1)\vec{v}_2$ or $\vec{v}_2 = (-c_1/c_2)\vec{v}_1$ (or both).

5.4.9 Hint: Prove by contradiction. The converse (the 'only if' statement) does not hold. An example is to consider the vector space $\mathbb{R}^2$ and these vectors.

$$\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \vec{y} = \begin{pmatrix}0\\1\end{pmatrix}, \quad \vec{z} = \begin{pmatrix}1\\1\end{pmatrix}$$

5.4.10

a. The linear system arising from

$$c_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + c_2 \begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

has the unique solution $c_1 = 0$ and $c_2 = 0$.

b. The linear system arising from

$$c_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + c_2 \begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix}$$

has the unique solution $c_1 = 8/3$ and $c_2 = -1/3$.

c. Suppose that S is linearly independent. Suppose that we have both $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{v} = d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ (where the vectors are members of S). Now,

$$c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{v} = d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$$

can be rewritten in this way.

$$c_1\vec{s}_1 + \cdots + c_n\vec{s}_n - d_1\vec{t}_1 - \cdots - d_m\vec{t}_m = \vec{0}$$

Possibly some of the $\vec{s}$'s equal some of the $\vec{t}$'s; we can combine the associated coefficients (i.e., if $\vec{s}_i = \vec{t}_j$ then $\cdots + c_i\vec{s}_i + \cdots - d_j\vec{t}_j - \cdots$ can be rewritten as $\cdots + (c_i - d_j)\vec{s}_i + \cdots$). That equation is a linear relationship among distinct (after the combining is done) members of the set S. We've assumed that S is linearly independent, so all of the coefficients are zero. If i is such that $\vec{s}_i$ does not equal any $\vec{t}_j$ then c_i is zero. If j is such that $\vec{t}_j$ does not equal any $\vec{s}_i$ then d_j is zero. In the final case, we have that $c_i - d_j = 0$ and so $c_i = d_j$.
Therefore, the original two sums are the same, except perhaps for some $0 \cdot \vec{s}_i$ or $0 \cdot \vec{t}_j$ terms that we can neglect.

d. This set is not linearly independent:

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix} \right\} \subset \mathbb{R}^2$$

and these two linear combinations give the same result

$$\begin{pmatrix}0\\0\end{pmatrix} = 2 \cdot \begin{pmatrix}1\\0\end{pmatrix} - 1 \cdot \begin{pmatrix}2\\0\end{pmatrix} = 4 \cdot \begin{pmatrix}1\\0\end{pmatrix} - 2 \cdot \begin{pmatrix}2\\0\end{pmatrix}$$

Thus, a linearly dependent set might have indistinct sums.
In fact, this stronger statement holds: if a set is linearly dependent then it must have the property that there are two distinct linear combinations that sum to the same vector. Briefly, where $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ then multiplying both sides of the relationship by two gives another relationship. If the first relationship is nontrivial then the second is also.

5.4.11

a. For any $a_{1,1}, \ldots, a_{2,4}$,

$$c_1 \begin{pmatrix}a_{1,1}\\a_{2,1}\end{pmatrix} + c_2 \begin{pmatrix}a_{1,2}\\a_{2,2}\end{pmatrix} + c_3 \begin{pmatrix}a_{1,3}\\a_{2,3}\end{pmatrix} + c_4 \begin{pmatrix}a_{1,4}\\a_{2,4}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

yields a linear system

$$\begin{aligned} a_{1,1}c_1 + a_{1,2}c_2 + a_{1,3}c_3 + a_{1,4}c_4 &= 0 \\ a_{2,1}c_1 + a_{2,2}c_2 + a_{2,3}c_3 + a_{2,4}c_4 &= 0 \end{aligned}$$

that has infinitely many solutions (Gauss's Method leaves at least two variables free). Hence there are nontrivial linear relationships among the given members of $\mathbb{R}^2$.

b. Any set five vectors is a superset of a set of four vectors, and so is linearly dependent.
With three vectors from $\mathbb{R}^2$, the argument from the prior item still applies, with the slight change that Gauss's Method now only leaves at least one variable free (but that still gives infinitely many solutions).

c. The prior part shows that no three-element subset of $\mathbb{R}^2$ is independent. We know that there are two-element subsets of $\mathbb{R}^2$ that are independent. The following one is

$$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

and so the answer is two.

5.4.12 Yes; here is one.

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

5.4.13

a. Assume that $\vec{v}$ and $\vec{w}$ are perpendicular nonzero vectors in $\mathbb{R}^n$, with $n > 1$. With the linear relationship $c\vec{v}+d\vec{w} = \vec{0}$, apply $\vec{v}$ to both sides to conclude that $c\cdot\|\vec{v}\|^2+d\cdot 0 = 0$. Because $\vec{v} \neq \vec{0}$ we have that $c = 0$. A similar application of $\vec{w}$ shows that $d = 0$.

b. Two vectors in $\mathbb{R}^1$ are perpendicular if and only if at least one of them is zero.

c. The right generalization is to look at a set $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq \mathbb{R}^k$ of vectors that are *mutually orthogonal* (also called *pairwise perpendicular*): if $i \neq j$ then $\vec{v}_i$ is perpendicular to $\vec{v}_j$. Mimicking the proof of the first item above shows that such a set of nonzero vectors is linearly independent.

5.4.14 It is both 'if' and 'only if'.
Let T be a subset of the subspace S of the vector space V. The assertion that any linear relationship $c_1\vec{t}_1+\cdots+c_n\vec{t}_n = \vec{0}$ among members of T must be the trivial relationship $c_1 = 0$, $\dots$, $c_n = 0$ is a statement that holds in S if and only if it holds in V, because the subspace S inherits its addition and scalar multiplication operations from V.

5.4.15 Hint: Use the definition of linear independence to show that there only exists the trivial linear combination giving the zero vector.

5.4.16 In $\mathbb{R}^4$ the biggest linearly independent set has four vectors. There are many examples of such sets, this is one.

$$\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

To see that no set with five or more vectors can be independent, set up

$$c_1\begin{pmatrix}a_{1,1}\\a_{2,1}\\a_{3,1}\\a_{4,1}\end{pmatrix}+c_2\begin{pmatrix}a_{1,2}\\a_{2,2}\\a_{3,2}\\a_{4,2}\end{pmatrix}+c_3\begin{pmatrix}a_{1,3}\\a_{2,3}\\a_{3,3}\\a_{4,3}\end{pmatrix}+c_4\begin{pmatrix}a_{1,4}\\a_{2,4}\\a_{3,4}\\a_{4,4}\end{pmatrix}+c_5\begin{pmatrix}a_{1,5}\\a_{2,5}\\a_{3,5}\\a_{4,5}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

and note that the resulting linear system

$$\begin{aligned}
a_{1,1}c_1 + a_{1,2}c_2 + a_{1,3}c_3 + a_{1,4}c_4 + a_{1,5}c_5 &= 0\\
a_{2,1}c_1 + a_{2,2}c_2 + a_{2,3}c_3 + a_{2,4}c_4 + a_{2,5}c_5 &= 0\\
a_{3,1}c_1 + a_{3,2}c_2 + a_{3,3}c_3 + a_{3,4}c_4 + a_{3,5}c_5 &= 0\\
a_{4,1}c_1 + a_{4,2}c_2 + a_{4,3}c_3 + a_{4,4}c_4 + a_{4,5}c_5 &= 0
\end{aligned}$$

has four equations and five unknowns, so Gauss's Method must end with at least one c variable free, so there are infinitely many solutions, and so the above linear relationship among the four-tall vectors has more solutions than just the trivial solution.
The smallest linearly independent set is the empty set.
The biggest linearly dependent set is $\mathbb{R}^4$. The smallest is $\left\{\vec{0}\right\}$.

5.4.17

a. The intersection of two linearly independent sets $S \cap T$ must be linearly independent as it is a subset of the linearly independent set S (as well as the linearly independent set T also, of course).

b. The complement of a linearly independent set is linearly dependent as it contains the zero vector.

c. A simple example in $\mathbb{R}^2$ is these two sets.

$$S = \left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \qquad T = \left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

A somewhat subtler example, again in $\mathbb{R}^2$, is these two.

$$S = \left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \qquad T = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

d. We must produce an example. One, in $\mathbb{R}^2$, is

$$S = \left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \qquad T = \left\{\begin{pmatrix}2\\0\end{pmatrix}\right\}$$

since the linear dependence of $S_1 \cup S_2$ is easy to see.

5.4.18

a. The vectors $\vec{s}_1, \dots, \vec{s}_n, \vec{t}_1, \dots, \vec{t}_m$ are distinct. But we could have that the union $S \cup T$ is linearly independent with some $\vec{s}_i$ equal to some $\vec{t}_j$.

b. One example in $\mathbb{R}^2$ is these two.

$$S = \left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \qquad T = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

c. An example from $\mathbb{R}^2$ is these sets.

$$S = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \qquad T = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$$

d. The union of two linearly independent sets $S \cup T$ is linearly independent if and only if their spans of S and $T - (S \cap T)$ have a trivial intersection $\operatorname{span}(S) \cap \operatorname{span}(T - (S \cap T)) = \left\{\vec{0}\right\}$. To prove that, assume that S and T are linearly independent subsets of some vector space.
For the 'only if' direction, assume that the intersection of the spans is trivial $\operatorname{span}(S) \cap \operatorname{span}(T - (S \cap T)) = \left\{\vec{0}\right\}$. Consider the set $S \cup (T - (S \cap T)) = S \cup T$ and consider the linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m = \vec{0}$. Subtracting gives $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = -d_1\vec{t}_1 - \cdots - d_m\vec{t}_m$. The left side of that equation sums to a vector in $\operatorname{span}(S)$, and the right side is a vector in $\operatorname{span}(T - (S \cap T))$. Therefore, since the intersection of the spans is trivial, both sides equal the zero vector. Because S is linearly independent, all of the c's are zero. Because T is linearly independent so also is $T - (S \cap T)$ linearly independent, and therefore all of the d's are zero. Thus, the original linear relationship among members of

$S \cup T$ only holds if all of the coefficients are zero. Hence, $S \cup T$ is linearly independent.

For the 'if' half we can make the same argument in reverse. Suppose that the union $S \cup T$ is linearly independent. Consider a linear relationship among members of S and $T-(S\cap T)$. $c_1\vec{s}_1+\cdots+c_n\vec{s}_n+d_1\vec{t}_1+\cdots+d_m\vec{t}_m=\vec{0}$ Note that no $\vec{s}_i$ is equal to a $\vec{t}_j$ so that is a combination of distinct vectors. So the only solution is the trivial one $c_1=0,\ \ldots,\ d_m=0$. Since any vector $\vec{v}$ in the intersection of the spans $\operatorname{span}(S)\cap\operatorname{span}(T-(S\cap T))$ we can write $\vec{v}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n=-d_1\vec{t}_1-\cdots-d_m\vec{t}_m$, and it must be the zero vector because each scalar is zero.

5.4.19

a. Assuming first that $a\neq 0$,

$$x\begin{pmatrix}a\\c\end{pmatrix}+y\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

gives

$$\begin{aligned}ax+by&=0\\cx+dy&=0\end{aligned}$$

and Gaussian elimination

$$\begin{aligned}ax+\qquad\qquad by&=0\\(-(c/a)b+d)y&=0\end{aligned}$$

which has a solution if and only if $0\neq-(c/a)b+d=(-cb+ad)/d$ (we've assumed in this case that $a\neq 0$, and so back substitution yields a unique solution).

The $a=0$ case is also not hard break it into the $c\neq 0$ and $c=0$ subcases and note that in these cases $ad-bc=0\cdot d-bc$.

b. The equation

$$c_1\begin{pmatrix}a\\d\\g\end{pmatrix}+c_2\begin{pmatrix}b\\e\\h\end{pmatrix}+c_3\begin{pmatrix}c\\f\\i\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

expresses a homogeneous linear system. We proceed by writing it in matrix form and applying Gauss's Method. We first reduce the matrix to upper-triangular. Assume that $a\neq 0$. With that, we can clear down the first column.

$$\left[\begin{array}{ccc|c}1&b/a&c/a&0\\0&(ae-bd)/a&(af-cd)/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right]$$

Then we get a 1 in the second row, second column entry. (Assuming for the moment that $ae-bd\neq 0$, in order to do the row reduction step.)

$$\left[\begin{array}{ccc|c}1&b/a&c/a&0\\0&1&(af-cd)/(ae-bd)&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right]$$

Then, under the assumptions, we perform the row operation $((ah-bg)/a)\rho_2+\rho_3$ to get this.

$$\left[\begin{array}{ccc|c}1&b/a&c/a&0\\0&1&(af-cd)/(ae-bd)&0\\0&0&(aei+bgf+cdh-hfa-idb-gec)/(ae-bd)&0\end{array}\right]$$

Therefore, the original system is nonsingular if and only if the above $3,3$ entry is nonzero (this fraction is defined because of the $ae-bd\neq 0$ assumption). It equals zero if and only if the numerator is zero.

We next worry about the assumptions. First, if $a\neq 0$ but $ae-bd=0$ then we swap row 2 and row 3

$$\left[\begin{array}{ccc|c}1&b/a&c/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\\0&0&(af-cd)/a&0\end{array}\right]$$

and conclude that the system is nonsingular if and only if either $ah-bg=0$ or $af-cd=0$. That's the same as asking that their product be zero:

$$\begin{aligned}ahaf-ahcd-bgaf+bgcd&=0\\ahaf-ahcd-bgaf+aegc&=0\\a(haf-hcd-bgf+egc)&=0\end{aligned}$$

(in going from the first line to the second we've applied the case assumption that $ae-bd=0$ by substituting ae for bd). Since we are assuming that $a\neq 0$, we have that $haf-hcd-bgf+egc=0$. With $ae-bd=0$ we can rewrite this to fit the form we need: in this $a\neq 0$ and $ae-bd=0$ case, the given system is nonsingular when $haf-hcd-bgf+egc-i(ae-bd)=0$, as required.

The remaining cases have the same character. Do the $a=0$ but $d\neq 0$ case and the $a=0$ and $d=0$ but $g\neq 0$ case by first swapping rows and then going on as above. The $a=0$, $d=0$, and $g=0$ case is easy a set with a zero vector is linearly dependent, and the formula comes out to equal zero.

c. It is linearly dependent if and only if either vector is a multiple of the other. That is, it is not independent iff

$$\begin{pmatrix}a\\d\\g\end{pmatrix}=r\cdot\begin{pmatrix}b\\e\\h\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}b\\e\\h\end{pmatrix}=s\cdot\begin{pmatrix}a\\d\\g\end{pmatrix}$$

(or both) for some scalars r and s. Eliminating r and s in order to restate this condition only in terms of the given letters a, b, d, e, g, h, we have that it is not independent, it is dependent iff $ae-bd=ah-gb=dh-ge$

5.5.1 Each set is a basis if and only if we can express each vector in the space in a unique way as a linear combination of the given vectors.

a. Yes this is a basis. The relation

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}3\\2\\1\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives

$$\left[\begin{array}{ccc|c}1&3&0&x\\2&2&0&y\\3&1&1&z\end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & x \\ 0 & -4 & 0 & -2x+y \\ 0 & 0 & 1 & x-2y+z \end{array}\right]$$

which has the unique solution $c_3 = x - 2y + z$, $c_2 = x/2 - y/4$, and $c_1 = -x/2 + 3y/4$.

b. This is not a basis. Setting it up as in the prior part

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives a linear system whose solution

$$\left[\begin{array}{cc|c} 1 & 3 & x \\ 2 & 2 & y \\ 3 & 1 & z \end{array}\right]$$

gives

$$\left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & -4 & -2x+y \\ 0 & 0 & x-2y+z \end{array}\right]$$

is possible if and only if the three-tall vector's components x, y, and z satisfy $x - 2y + z = 0$. For instance, we can find the coefficients c_1 and c_2 that work when $x = 1$, $y = 1$, and $z = 1$. However, there are no c's that work for $x = 1$, $y = 1$, and $z = 2$. Thus this is not a basis; it does not span the space.

c. Yes, this is a basis. Setting up the relationship leads to this reduction

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & x \\ 2 & 1 & 5 & y \\ -1 & 1 & 0 & z \end{array}\right]$$

gives

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & z \\ 0 & 3 & 5 & y+2z \\ 0 & 0 & 1/3 & x-y/3-2z/3 \end{array}\right]$$

which has a unique solution for each triple of components x, y, and z.

d. No, this is not a basis. The reduction of

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & x \\ 2 & 1 & 3 & y \\ -1 & 1 & 0 & z \end{array}\right]$$

gives

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & z \\ 0 & 3 & 3 & y+2z \\ 0 & 0 & 0 & x-y/3-2z/3 \end{array}\right]$$

which does not have a solution for each triple x, y, and z. Instead, the span of the given set includes only those vectors where $x = y/3 + 2z/3$.

5.5.2

a. This is a basis for $\mathcal{P}_2$. To show that it spans the space we consider a generic $a_2x^2 + a_1x + a_0 \in \mathcal{P}_2$ and look for scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that $a_2x^2 + a_1x + a_0 = c_1 \cdot (x^2 - x + 1) + c_2 \cdot (2x + 1) + c_3(2x - 1)$. Gauss's Method on the linear system

$$\begin{aligned} c_1 \qquad\qquad &= a_2 \\ 2c_2 + 2c_3 &= a_1 \\ c_2 - c_3 &= a_0 \end{aligned}$$

shows that given the a_i's we can compute the c_j's as $c_1 = a_2$, $c_2 = (1/4)a_1 + (1/2)a_0$, and $c_3 = (1/4)a_1 - (1/2)a_0$. Thus each element of $\mathcal{P}_2$ is a combination of the given three.

To prove that the set of the given three is linearly independent we can set up the equation $0x^2 + 0x + 0 = c_1 \cdot (x^2 - x + 1) + c_2 \cdot (2x + 1) + c_3(2x - 1)$ and solve, and it will give that $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

b. This is not a basis. It does not span the space since no combination of the two $c_1 \cdot (x + x^2) + c_2 \cdot (x - x^2)$ will sum to the polynomial $3 \in \mathcal{P}_2$.

5.5.3

a. We solve

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

with

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 1 \end{array}\right]$$

and conclude that $c_2 = 1/2$ and so $c_1 = 3/2$. Thus, the representation is this.

$$\mathrm{Rep}_B(\begin{pmatrix} 1 \\ 2 \end{pmatrix}) = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}_B$$

b. The relationship $c_1 \cdot (1) + c_2 \cdot (1 + x) + c_3 \cdot (1 + x + x^2) + c_4 \cdot (1 + x + x^2 + x^3) = x^2 + x^3$ is easily solved by inspection to give that $c_4 = 1$, $c_3 = 0$, $c_2 = -1$, and $c_1 = 0$.

$$\mathrm{Rep}_D(x^2 + x^3) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}_D$$

5.5.4 A natural basis is $\langle 1,\ x,\ x^2 \rangle$. There are bases for $\mathcal{P}_2$ that do not contain any polynomials of degree one or degree zero. One is $\langle 1 + x + x^2,\ x + x^2,\ x^2 \rangle$. (Every basis has at least one polynomial of degree two, though.)

5.5.5 The reduction

$$\left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{array}\right]$$

Gaussian elimination gives

$$\left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

gives that the only condition is that $x_1 = 4x_2 - 3x_3 + x_4$. The solution set is

$$\left\{ \begin{pmatrix} 4x_2 - 3x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_2, x_3, x_4 \in \mathbb{R} \right\}$$
$$= \left\{ x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_3, x_4 \in \mathbb{R} \right\}$$

and so the obvious candidate for the basis is this.

$$\langle \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

We've shown that this spans the space, and showing it is also linearly independent is routine.

5.5.6 There are many bases. This is a natural one.

$$\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$$

5.5.7 For each, many answers are possible.

a. One way to proceed is to parametrize by expressing the a_2 as a combination of the other two $a_2 = 2a_1 + a_0$. Then $a_2x^2 + a_1x + a_0$ is $(2a_1 + a_0)x^2 + a_1x + a_0$ and

$$\{\, (2a_1 + a_0)x^2 + a_1x + a_0 \mid a_1, a_0 \in \mathbb{R} \,\}$$
$$= \{\, a_1 \cdot (2x^2 + x) + a_0 \cdot (x^2 + 1) \mid a_1, a_0 \in \mathbb{R} \,\}$$

suggests $\langle 2x^2 + x, x^2 + 1 \rangle$. This only shows that it spans, but checking that it is linearly independent is routine.

b. Parametrize $\{\, (a\ \ b\ \ c) \mid a + b = 0 \,\}$ to get $\{\, (-b\ \ b\ \ c) \mid b, c \in \mathbb{R} \,\}$, which suggests using the sequence $\langle (-1\ \ 1\ \ 0), (0\ \ 0\ \ 1) \rangle$. We've shown that it spans, and checking that it is linearly independent is easy.

c. Rewriting

$$\left\{ \begin{bmatrix} a & b \\ 0 & 2b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} = \left\{ a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

suggests this for the basis.

$$\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \rangle$$

5.5.8

a. Asking which $a_0 + a_1x + a_2x^2$ can be expressed as $c_1 \cdot (1 + x) + c_2 \cdot (1 + 2x)$ gives rise to three linear equations, describing the coefficients of x^2, x, and the constants.

$$\begin{aligned} c_1 + c_2 &= a_0 \\ c_1 + 2c_2 &= a_1 \\ 0 &= a_2 \end{aligned}$$

Gauss's Method with back-substitution shows, provided that $a_2 = 0$, that $c_2 = -a_0 + a_1$ and $c_1 = 2a_0 - a_1$. Thus, with $a_2 = 0$, we can compute appropriate c_1 and c_2 for any a_0 and a_1. So the span is the entire set of linear polynomials $\{\, a_0 + a_1x \mid a_0, a_1 \in \mathbb{R} \,\}$. Parametrizing that set $\{\, a_0 \cdot 1 + a_1 \cdot x \mid a_0, a_1 \in \mathbb{R} \,\}$ suggests a basis $\langle 1, x \rangle$ (we've shown that it spans; checking linear independence is easy).

b. With

$$a_0 + a_1x + a_2x^2 = c_1 \cdot (2 - 2x) + c_2 \cdot (3 + 4x^2) = (2c_1 + 3c_2) + (-2c_1)x + ($$

we get this system.

$$\begin{aligned} 2c_1 + 3c_2 &= a_0 \\ -2c_1 &= a_1 \\ 4c_2 &= a_2 \end{aligned}$$

and Gaussian elimination gives

$$\begin{aligned} 2c_1 + 3c_2 &= a_0 \\ 3c_2 &= a_0 + a_1 \\ 0 &= (-4/3)a_0 - (4/3)a_1 + a_2 \end{aligned}$$

Thus, the only quadratic polynomials $a_0 + a_1x + a_2x^2$ with associated c's are the ones such that $0 = (-4/3)a_0 - (4/3)a_1 + a_2$. Hence the span is this.

$$\{\, (-a_1 + (3/4)a_2) + a_1x + a_2x^2 \mid a_1, a_2 \in \mathbb{R} \,\}$$

Parametrizing gives $\{\, a_1 \cdot (-1 + x) + a_2 \cdot ((3/4) + x^2) \mid a_1, a_2 \in$ which suggests $\langle -1 + x, (3/4) + x^2 \rangle$ (checking that it is linearly independent is routine).

5.5.9

a. The subspace is this.

$$\{\, a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + 7a_1 + 49a_2 + 343a_3 = 0 \,\}$$

Rewriting $a_0 = -7a_1 - 49a_2 - 343a_3$ gives this.

$$\{\, (-7a_1 - 49a_2 - 343a_3) + a_1x + a_2x^2 + a_3x^3 \mid a_1, a_2, a_3 \in \mathbb{R} \,\}$$

On breaking out the parameters, this suggests $\langle -7 + x, -49 + x^2, -343 + x^3 \rangle$ for the basis (it is easily verified).

b. The given subspace is the collection of cubics $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_0 + 7a_1 + 49a_2 + 343a_3 = 0$ and $a_0 + 5a_1 + 25a_2 + 125a_3 = 0$. Gauss's Method

$$\begin{aligned} a_0 + 7a_1 + 49a_2 + 343a_3 &= 0 \\ a_0 + 5a_1 + 25a_2 + 125a_3 &= 0 \end{aligned}$$

gives

$$\begin{aligned} a_0 + \quad 7a_1 + 49a_2 + 343a_3 &= 0 \\ -2a_1 - 24a_2 - 218a_3 &= 0 \end{aligned}$$

gives that $a_1 = -12a_2 - 109a_3$ and that $a_0 = 35a_2 + 420a_3$. Rewriting $(35a_2 + 420a_3) + (-12a_2 - 109a_3)x + a_2x^2 + a_3x^3$ as $a_2 \cdot (35 - 12x + x^2) + a_3 \cdot (420 - 109x + x^3)$ suggests this for a basis $\langle 35 - 12x + x^2, 420 - 109x + x^3\rangle$. The above shows that it spans the space. Checking it is linearly independent is routine. (*Comment.* A worthwhile check is to verify that both polynomials in the basis have both seven and five as roots.)

c. Here there are three conditions on the cubics, that $a_0 + 7a_1 + 49a_2 + 343a_3 = 0$, that $a_0 + 5a_1 + 25a_2 + 125a_3 = 0$, and that $a_0 + 3a_1 + 9a_2 + 27a_3 = 0$. Gauss's Method

$$\begin{aligned} a_0 + 7a_1 + 49a_2 + 343a_3 &= 0 \\ a_0 + 5a_1 + 25a_2 + 125a_3 &= 0 \\ a_0 + 3a_1 + \quad 9a_2 + \quad 27a_3 &= 0 \end{aligned}$$

gives

$$\begin{aligned} a_0 + \quad 7a_1 + 49a_2 + 343a_3 &= 0 \\ -2a_1 - 24a_2 - 218a_3 &= 0 \\ 8a_2 + 120a_3 &= 0 \end{aligned}$$

yields the single free variable a_3, with $a_2 = -15a_3$, $a_1 = 71a_3$, and $a_0 = -105a_3$. The parametrization is this.

$$\begin{aligned} &\{(-105a_3) + (71a_3)x + (-15a_3)x^2 + (a_3)x^3 \mid a_3 \in \mathbb{R}\} \\ &\quad = \{a_3 \cdot (-105 + 71x - 15x^2 + x^3) \mid a_3 \in \mathbb{R}\} \end{aligned}$$

Therefore, a natural candidate for the basis is $\langle -105 + 71x - 15x^2 + x^3\rangle$. It spans the space by the work above. It is clearly linearly independent because it is a one-element set (with that single element not the zero object of the space). Thus, any cubic through the three points $(7, 0)$, $(5, 0)$, and $(3, 0)$ is a multiple of this one. (*Comment.* As in the prior question, a worthwhile check is to verify that plugging seven, five, and three into this polynomial yields zero each time.)

d. This is the trivial subspace of $\mathcal{P}_3$. Thus, the basis is empty $\langle\rangle$.

5.5.10

a. $B = \{1 + x^3,\ x^2 + x^3\}$

b. $(p(x))_B = (-2, 2)$

5.5.11 No linearly independent set contains a zero vector.

5.5.12

a. To show that it is linearly independent, note that if $d_1(c_1\vec{\beta}_1) + d_2(c_2\vec{\beta}_2) + d_3(c_3\vec{\beta}_3) = \vec{0}$ then $(d_1c_1)\vec{\beta}_1 + (d_2c_2)\vec{\beta}_2 + (d_3c_3)\vec{\beta}_3 = \vec{0}$, which in turn implies that each d_ic_i is zero. But with $c_i \neq 0$ that means that each d_i is zero. Showing that it spans the space is much the same; because $\langle\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ is a basis, and so spans the space, we can for any $\vec{v}$ write $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$, and then $\vec{v} = (d_1/c_1)(c_1\vec{\beta}_1) + (d_2/c_2)(c_2\vec{\beta}_2) + (d_3/c_3)(c_3\vec{\beta}_3)$. If any of the scalars are zero then the result is not a basis, because it is not linearly independent.

b. Showing that $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ is linearly independent is easy. To show that it spans the space, assume that $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$. Then, we can represent the same $\vec{v}$ with respect to $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ in this way $\vec{v} = (1/2)(d_1 - d_2 - d_3)(2\vec{\beta}_1) + d_2(\vec{\beta}_1 + \vec{\beta}_2) + d_3(\vec{\beta}_1 + \vec{\beta}_3)$.

5.6.1 $\{-1 + x^2,\ -x + x^3\}$ is a basis of W, therefore W is of dimension 2.

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