CEGEP Linear Algebra Problems

An open source collection of
CEGEP level linear algebra problems

Edited by

Yann Lamontagne, add your name here

January 19, 2016

Contents

Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 Place Holder

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1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [JH] Use Gauss's Method to find the unique solution for each system.

a.
$$2x + 3y = 13$$
$$x - y = -1$$

b.
$$x \qquad - z = 0$$
$$3x + y \qquad = 1$$
$$-x + y + z = 4$$

1.3 Applications of Linear Systems

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

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2.2 Matrix Inverses and Algebraic Properties

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2.3 Elementary Matrices

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2.4 Linear Systems and Matrices

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Chapter 3

Determinants

3.1 The Laplace Expansion

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3.2 Determinants and Elementary Operations

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3.3 Determinants and Elementary Matrices

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3.4 Applications of the Determinant

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Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in $\mathbb{R}^n$.*: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$.

4.3 Cross Product and Planes

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4.4 Areas, Volumes and Distances

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4.5 Geometry of Solutions of Linear Systems

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [**JH**] Name the zero vector for each of these vector spaces.

    a. The space of degree three polynomials under the natural operations.

    b. The space of 2×3 matrices.

    c. The space $\{\, f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \,\}$.

    d. The space of real-valued functions of one natural number variable.

5.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

    a. In $\mathcal{P}_3$, the vector $-3 - 2x + x^2$.

    b. In the space $\mathcal{M}_{2 \times 2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

    c. In $\{\, ae^x + be^{-x} \mid a, b \in \mathbb{R} \,\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [**JH**] For each, list three elements and then show it is a vector space.

    a. The set of linear polynomials $\mathcal{P}_1 = \{\, a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \,\}$ under the usual polynomial addition and scalar multiplication operations.

    b. The set of linear polynomials $\{\, a_0 + a_1 x \mid a_0 - 2a_1 = 0 \,\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [**JH**] For each, list three elements and then show it is a vector space.

    a. The set of 2×2 matrices with real entries under the usual matrix operations.

    b. The set of 2×2 matrices with real entries where the $2, 1$ entry is zero, under the usual matrix operations.

5.1.5 by Jim Hefferon [**JH**] For each, list three elements and then show it is a vector space.

    a. The set of three-component row vectors with their usual operations.

    b. The set

$$\{\, (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y - z + w = 0 \,\}$$

    under the operations inherited from $\mathbb{R}^4$.

5.1.6 [**JH**] Show that the following are not vector spaces.

    a. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x + y + z = 1 \,\}$$

    b. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \,\}$$

    c. Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

    d. Under the usual polynomial operations,

$$\{\, a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+ \,\}$$

    where $\mathbb{R}^+$ is the set of reals greater than zero

    e. Under the inherited operations,

$$\{\, (x,\ y) \in \mathbb{R}^2 \mid x + 3y = 4,\ 2x - y = 3 \text{ and } 6x + 4y = 10 \,\}$$

5.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

5.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

b. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5.1.10 [JH] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

c. $\left\{ (x,\ y,\ z,\ w) \in \mathbb{R}^4 \;\middle|\; x+y+w=1 \right\}$

d. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \;\mid\; df/dx + 2f = 0 \}$

e. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \;\mid\; df/dx + 2f = 1 \}$

5.1.11 [JH] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x + y$' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret '$r \cdot x$' as the r-th power of x.

5.1.12 [JH] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.13 [JH]
Is $\left\{ (x,\ y) \;\middle|\; x, y \in \mathbb{R} \right\}$ a vector space under these operations?

a. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$

b. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x,\ y) = (rx,\ 0)$

5.1.14 [JH]
Prove the following:

a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.15 [JH]
The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.16 [JH]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.17 [JH]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

5.1.18 [JH]
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.

b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

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5.3 Spanning Sets

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5.4 Linear Independence

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5.5 Basis

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5.6 Dimension

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Answers to Exercises

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1.2.1

    a. $x = 2$, $y = 3$

    b. $x = -1$, $y = 4$, and $z = -1$.

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$.

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5.1.1

   a. $0 + 0x + 0x^2 + 0x^3$

   b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   c. The constant function $f(x) = 0$

   d. The constant function $f(n) = 0$

5.1.2

   a. $3 + 2x - x^2$

   b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

   c. $-3e^x + 2e^{-x}$

5.1.3

   a. $1 + 2x$, $2 - 1x$, and x.

   b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

   a.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   b.

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

   a. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.

   b. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

5.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

   a. It is not closed under addition.

$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

   b. It is not closed under addition.

$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

   c. It is not closed under addition.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

   d. It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

   e. The set is empty, violating the existance of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

   a. It is not a vector space.

$$(1 + 1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

   b. It is not a vector space.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

   a. Yes.

   b. Yes.

c. No, this set is not closed under the natural addition operation. The vector of all $1/4$'s is an element of this set but when added to itself the result, the vector of all $1/2$'s, is not an element of the set.

d. Yes.

e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11 Check all 10 conditions of the definition of a vector space.

5.1.12 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.13

a. No since $1 \cdot (0,\ 1) + 1 \cdot (0,\ 1) \neq (1 + 1) \cdot (0,\ 1)$.

b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0,\ 1) \neq (0,\ 1)$.

5.1.14

a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which implies that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.15
Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.16
It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.17
Each element of a vector space has one and only one additive inverse.
For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.18
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.

b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.

c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{ k \cdot \vec{v} \mid k \in \mathbb{R} \}$. By the prior item this set is infinite.

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