CEGEP Linear Algebra Problems

An open source collection of
CEGEP level linear algebra problems

Edited by

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http://obeymath.org/CEGEPLinearAlgebraProblems

Contents

Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 Place Holder

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1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [**JH**] Use Gauss's Method to find the unique solution for each system.

a.
$$\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$$

b.
$$\begin{aligned} x \phantom{{}+y} - z &= 0 \\ 3x + y \phantom{{}+z} &= 1 \\ -x + y + z &= 4 \end{aligned}$$

1.3 Applications of Linear Systems

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [**HE**] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

a. $3C - 4D$
b. $A - (D + 2C)$
c. $A - E$
d. AE
e. $3BC - 4BD$
f. $CB + D$
g. GC
h. FG
i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A, B, and C are matrices above.

2.2 Matrix Inverses and Algebraic Properties

2.2.1 [**YL**] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.3 Elementary Matrices

2.3.1 [**YL**] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.4 Linear Systems and Matrices

2.4.1 [**YL**] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1}.
b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Chapter 3

Determinants

3.1 The Laplace Expansion

3.1.1 [**YL**] Solve for λ.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

3.2 Determinants and Elementary Operations

3.2.1 [**YL**] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.3 Properties of Determinants

3.3.1 [**YL**] Let A and B be $n \times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.4 Applications of the Determinant

3.4.1 [**YL**] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in* $\mathbb{R}^n$: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$.

4.3 Cross Product and Planes

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4.4 Areas, Volumes and Distances

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4.5 Geometry of Solutions of Linear Systems

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [**JH**] Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.
b. The space of 2×3 matrices.
c. The space $\{\, f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\,\}$.
d. The space of real-valued functions of one natural number variable.

5.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3-2x+x^2$.
b. In the space $\mathcal{M}_{2\times2}$,
$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$
c. In $\{\, ae^x + be^{-x} \mid a, b \in \mathbb{R}\,\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{\, a_0 + a_1x \mid a_0, a_1 \in \mathbb{R}\,\}$ under the usual polynomial addition and scalar multiplication operations.
b. The set of linear polynomials $\{\, a_0 + a_1x \mid a_0 - 2a_1 = 0\,\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.
b. The set of 2×2 matrices with real entries where the $2,1$ entry is zero, under the usual matrix operations.

5.1.5 by Jim Hefferon [**JH**] For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.
b. The set
$$\{\, (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x+y-z+w=0\,\}$$
under the operations inherited from $\mathbb{R}^4$.

5.1.6 [**JH**] Show that the following are not vector spaces.

a. Under the operations inherited from $\mathbb{R}^3$, this set
$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x+y+z=1\,\}$$
b. Under the operations inherited from $\mathbb{R}^3$, this set
$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\,\}$$
c. Under the usual matrix operations,
$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$
d. Under the usual polynomial operations,
$$\{\, a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+\,\}$$
where $\mathbb{R}^+$ is the set of reals greater than zero
e. Under the inherited operations,
$$\{\, (x,\ y) \in \mathbb{R}^2 \mid x+3y=4,\ 2x-y=3 \text{ and } 6x+4y=10\,\}$$

5.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

5.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

b. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5.1.10 [**JH**] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

c. $\{ (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y + w = 1 \}$

d. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 0 \}$

e. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 1 \}$

5.1.11 [**JH**] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x + y$' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret '$r \cdot x$' as the r-th power of x.

5.1.12 [**JH**] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.13 [**JH**]
Is $\{ (x,\ y) \mid x, y \in \mathbb{R} \}$ a vector space under these operations?

a. $(x_1,\ y_1)+(x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r\cdot(x, y) = (rx, y)$

b. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x,\ y) = (rx,\ 0)$

5.1.14 [**JH**]
Prove the following:

a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.15 [**JH**]
The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.16 [**JH**]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.17 [**JH**]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

5.1.18 [**JH**]
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.

b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [**JH**]

a. Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ under the inherited operations.

b. What if it doesn't contain the origin?

5.3 Spanning Sets

5.3.1 [**YL**] Given the following two subspace of $\mathbb{R}^3$: $W_1 = \{ x \mid A_1 x = 0 \}$ and $W_2 = \{ x \mid A_2 x = 0 \}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix},\ A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.4 Linear Independence

5.4.1 [**YL**] Let $\vec{u} = (1,\ \lambda,\ -\lambda)$, $\vec{v} = (-2\lambda\ \ -2\ \ 2\lambda)$ and $\vec{w} = (\lambda - 2,\ -5\lambda - 2,\ -2)$.

a. For what value(s) of λ will $\{\vec{u},\ \vec{v}\}$ be linearly dependent.

b. For what value(s) of λ will $\{\vec{u},\ \vec{v},\ \vec{w}\}$ be linearly independent.

5.5 Basis

5.5.1 [**YL**] Given

$$W = \{ p(x) = a_0 + a_2 x^2 + a_3 x^3 \mid p(-1) = 0 \}$$

a subspace of $\mathcal{P}_3$.

a. Find a basis B for $\mathcal{W}$.

b. Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B.

5.6 Dimension

5.6.1 [**YL**] Given

$W = \{\, p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \;\big|\; p(1) = 0 \text{ and } p(-1) = 0 \,\}$

a subspace of P_3. Determine the dimension of W.

Answers to Exercises

1.1.1 Lorem ipsum dolor sit amet, consectetur adipiscing elit. Aliquam tincidunt cursus volutpat. Quisque non congue sem. Vivamus nec nibh sed est dapibus auctor eu sed nulla. Praesent ornare eleifend nibh a finibus. Proin rutrum neque nec massa tincidunt, non malesuada dolor interdum. Nam a massa sit amet diam efficitur pharetra. Nulla interdum efficitur sem, sit amet commodo orci mattis non. Duis tortor ex, maximus a sapien id, molestie maximus risus.

1.2.1

a. $x = 2$, $y = 3$

b. $x = -1$, $y = 4$, and $z = -1$.

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2.1.1

a. $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$

c. Not possible, since dimension of A and E are not the same.

d. $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$

f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D.

g. $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$

h. $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.2.1 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.3.1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.4.1

a. $A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

b. $x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$

3.1.1 $\lambda = \frac{3\pm\sqrt{33}}{4}$

3.2.1 $\det(A) = -\frac{5}{12}$

3.3.1 Hint: Apply the determinant to both sides $AB = -BA$.

3.4.1 $x_1 = 4$

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$.

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5.1.1

a. $0 + 0x + 0x^2 + 0x^3$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. The constant function $f(x) = 0$

d. The constant function $f(n) = 0$

5.1.2

a. $3 + 2x - x^2$

b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

c. $-3e^x + 2e^{-x}$

5.1.3

a. $1 + 2x$, $2 - 1x$, and x.

b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

a.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

a. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.

b. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

5.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

a. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

b. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

c. It is not closed under addition.
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

d. It is not closed under scalar multiplication.
$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

e. The set is empty, violating the existance of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

a. It is not a vector space.
$$(1 + 1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b. It is not a vector space.
$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

a. Yes.

b. Yes.

c. No, this set is not closed under the natural addition operation. The vector of all 1/4's is an element of this set but when added to itself the result, the vector of all 1/2's, is not an element of the set.

d. Yes.

e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11 Check all 10 conditions of the definition of a vector space.

5.1.12 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.13

a. No since $1 \cdot (0,\ 1) + 1 \cdot (0,\ 1) \neq (1 + 1) \cdot (0,\ 1)$.

b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0,\ 1) \neq (0,\ 1)$.

5.1.14

a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v}+\vec{v}+\vec{s} = -\vec{v}+\vec{v}+\vec{t}$, which implies that $\vec{0}+\vec{s} = \vec{0}+\vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.15
Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.16
It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.17
Each element of a vector space has one and only one additive inverse.
For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.18
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.
b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.
c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

5.2.1

a. Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of $\vec{v}, \vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v}+s_1\vec{w})+(r_2\vec{v}+s_2\vec{w}) = (r_1+r_2)\vec{v}+(s_1+s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.
b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.3.1 Hint: For each subspace determine a set of vectors that spans it.
$W_1 \subsetneq W_2$

5.4.1

a. $\lambda = 1$
b. $\lambda \neq -1,\ -\frac{1}{2},\ 1$

5.5.1

a. $B = \left\{1 + x^3,\ x^2 + x^3\right\}$
b. $(p(x))_B = (-2, 2)$

5.6.1 $\left\{-1 + x^2,\ -x + x^3\right\}$ is a basis of W, therefore W is of dimension 2.

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