
CEGEP Linear Algebra Problems

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CEGEP LEVEL LINEAR ALGEBRA PROBLEMS

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FEBRUARY 1, 2016

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Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 [GH] State which of the following equations is a linear equation. If it is not, state why.

- | | |
|--------------------------------|--------------------------------------|
| a. $x + y + z = 10$ | f. $\sqrt{x_1^2 + x_2^2} = 25$ |
| b. $xy + yz + xz = 1$ | g. $x_1 + y + t = 1$ |
| c. $-3x + 9 = 3y - 5z + x - 7$ | h. $\frac{1}{x} + 9 = 3\cos(y) - 5z$ |
| d. $\sqrt{5}y + \pi x = -1$ | i. $\cos(15)y + \frac{x}{4} = -1$ |
| e. $(x - 1)(x + 1) = 0$ | j. $2^x + 2^y = 16$ |

1.1.2 [GH] Solve the system of linear equations using substitution, comparison and/or elimination.

- | | |
|------------------|-----------------------|
| a. $x + y = -1$ | $x - y + z = 1$ |
| $2x - 3y = 8$ | c. $2x + 6y - z = -4$ |
| b. $2x - 3y = 3$ | $4x - 5y + 2z = 0$ |
| $3x + 6y = 8$ | $x + y - z = 1$ |
| | d. $2x + y = 2$ |
| | $y + 2z = 0$ |

1.1.3 [GH] Convert the given system of linear equations into an augmented matrix.

- | |
|---------------------------------|
| $3x + 4y + 5z = 7$ |
| a. $-x + y - 3z = 1$ |
| $2x - 2y + 3z = 5$ |
| $2x + 5y - 6z = 2$ |
| b. $9x - 8z = 10$ |
| $-2x + 4y + z = -7$ |
| $x_1 + 3x_2 - 4x_3 + 5x_4 = 17$ |
| c. $-x_1 + 4x_3 + 8x_4 = 1$ |
| $2x_1 + 3x_2 + 4x_3 + 5x_4 = 6$ |
| $3x_1 - 2x_2 = 4$ |
| d. $2x_1 = 3$ |
| $-x_1 + 9x_2 = 8$ |
| $5x_1 - 7x_2 = 13$ |

1.1.4 [GH] Convert given augmented matrix into a system of linear equations. Use the variables x_1, x_2, \dots

- | | |
|--|---|
| a. $\left[\begin{array}{cc c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array} \right]$ | d. $\left[\begin{array}{cccc c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right]$ |
| b. $\left[\begin{array}{cc c} -3 & 4 & 7 \\ 0 & 1 & -2 \end{array} \right]$ | e. $\left[\begin{array}{cccc c} 1 & 0 & 1 & 0 & 7 \\ 0 & 1 & 3 & 2 & 0 & 5 \end{array} \right]$ |
| c. $\left[\begin{array}{cccc c} 1 & 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & 5 & 7 \end{array} \right]$ | |

1.1.5 [GH] Perform the given row operations on

$$\left[\begin{array}{ccc} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{array} \right].$$

- | | |
|--------------------------------|--|
| a. $-1R_1 \rightarrow R_1$ | d. $2R_2 + R_3 \rightarrow R_3$ |
| b. $R_2 \leftrightarrow R_3$ | e. $\frac{1}{2}R_2 \rightarrow R_2$ |
| c. $R_1 + R_2 \rightarrow R_2$ | f. $-\frac{5}{2}R_1 + R_3 \rightarrow R_3$ |

1.1.6 [GH] Give the row operation that transforms A into B where

$$A = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{array} \right].$$

- | | |
|--|--|
| a. $B = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{array} \right]$ | d. $B = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{array} \right]$ |
| b. $B = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{array} \right]$ | e. $B = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{array} \right]$ |
| c. $B = \left[\begin{array}{ccc} 3 & 5 & 7 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{array} \right]$ | |

1.1.7 [JH] In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy -plane. By geometrical reasoning, show that there are three possibilities:

there is a unique solution, there is no solution, and there are infinitely many solutions.

1.1.8 [JH] Is there a two-unknowns linear system whose solution set is all of \mathbb{R}^2 ?

1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [GH] State whether or not the given matrices are in reduced row echelon form.

a.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	h.	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	m.	$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$
b.	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	i.	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	n.	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
c.	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	j.	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	o.	$\begin{bmatrix} 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
d.	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$	k.	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$		
e.	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	l.	$\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$		
f.	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$				
g.	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$				

1.2.2 [GH] Use Gauss-Jordan Elimination to put the given matrix into reduced row echelon form.

a.	$\begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$	h.	$\begin{bmatrix} 4 & 5 & -6 \\ -12 & -15 & 18 \end{bmatrix}$	m.	$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$
b.	$\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}$	i.	$\begin{bmatrix} -2 & -4 & -8 \\ -2 & -3 & -5 \\ 2 & 3 & 6 \end{bmatrix}$	n.	$\begin{bmatrix} 2 & -1 & 1 & 5 \\ 3 & 1 & 6 & -1 \\ 3 & 0 & 5 & 0 \end{bmatrix}$
c.	$\begin{bmatrix} 4 & 12 \\ -2 & -6 \end{bmatrix}$	j.	$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$	o.	$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 2 & 1 & 0 & 10 \\ 3 & 2 & -1 & 17 \end{bmatrix}$
d.	$\begin{bmatrix} -5 & 7 \\ 10 & 14 \end{bmatrix}$	k.	$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & -3 & 0 \end{bmatrix}$	p.	$\begin{bmatrix} 4 & 1 & 8 & 15 \\ 1 & 1 & 2 & 7 \\ 3 & 1 & 5 & 11 \end{bmatrix}$
e.	$\begin{bmatrix} -1 & 1 & 4 \\ -2 & 1 & 1 \end{bmatrix}$	l.	$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 6 & 9 \end{bmatrix}$		
f.	$\begin{bmatrix} 7 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$				
g.	$\begin{bmatrix} 3 & -3 & 6 \\ -1 & 1 & -2 \end{bmatrix}$				

1.2.3 [JH] Use Gauss's Method to find the unique solution for each system.

a.	$\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$	b.	$\begin{aligned} x - z &= 0 \\ 3x + y &= 1 \\ -x + y + z &= 4 \end{aligned}$
----	--	----	--

1.2.4 [GH] Find the solution to the given linear system. If the system has infinite solutions, give two particular solutions.

- a. $2x_1 + 4x_2 = 2$
 $x_1 + 2x_2 = 1$
- b. $-x_1 + 5x_2 = 3$
 $2x_1 - 10x_2 = -6$
- c. $x_1 + x_2 = 3$
 $2x_1 + x_2 = 4$
- d. $-3x_1 + 7x_2 = -7$
 $2x_1 - 8x_2 = 8$
- e. $-2x_1 + 4x_2 + 4x_3 = 6$
 $x_1 - 3x_2 + 2x_3 = 1$
- f. $-x_1 + 2x_2 + 2x_3 = 2$
 $2x_1 + 5x_2 + x_3 = 2$
- g. $-x_1 - x_2 + x_3 + x_4 = 0$
 $-2x_1 - 2x_2 + x_3 = -1$
- h. $x_1 + x_2 + 6x_3 + 9x_4 = 0$
 $x_1 + x_3 + 2x_4 = 3$
 $x_1 + 2x_2 + 2x_3 = 1$
- i. $2x_1 + x_2 + 3x_3 = 1$
 $3x_1 + 3x_2 + 5x_3 = 2$
 $2x_1 + 4x_2 + 6x_3 = 2$
- j. $1x_1 + 2x_2 + 3x_3 = 1$
 $3x_1 + 6x_2 + 9x_3 = 3$
- k. $2x_1 + 3x_2 = 1$
 $-2x_1 - 3x_2 = 1$
 $2x_1 + x_2 + 2x_3 = 0$
- l. $x_1 + x_2 + 3x_3 = 1$
 $3x_1 + 2x_2 + 5x_3 = 3$

1.2.5 [YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 + x_5 &= 3 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 + 17x_3 - 2x_4 - x_5 &= 1 \end{aligned}$$

- a. Solve the following system by Gauss-Jordan elimination.
- b. Find two particular solution to the above system.
- c. Find a solution to the above system when $x_3 = 1$.

1.2.6 [YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 &= 0 \\ 4x_1 + 17x_3 - 2x_4 &= 0 \\ 9x_1 + 6x_2 + 27x_3 - 4x_4 &= 0 \end{aligned}$$

- a. Solve the system by Gauss-Jordan elimination.
- b. Find two particular nontrivial solution to the system.
- c. Find a solution to the system when $x_1 = 1$.

1.2.7 [JH] Find the coefficients a , b , and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.2.8 [JH] True or false: a system with more unknowns than equations has at least one solution. (As always, to say ‘true’ you must prove it, while to say ‘false’ you must produce a counterexample.)

1.2.9 [JH] For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

1.2.10 [GH] State for which values of k the given system will have exactly 1 solution, infinite solutions, or no solution.

- a. $x_1 + 2x_2 = 1$
 $2x_1 + 4x_2 = k$
- b. $x_1 + 2x_2 = 1$
 $x_1 + kx_2 = 1$
- c. $x_1 + 2x_2 = 1$
 $x_1 + kx_2 = 2$
- d. $x_1 + 2x_2 = 1$
 $x_1 + 3x_2 = k$

1.2.11 [YL] Given the augmented matrix of a linear system:

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a^2 - 1 & b^2 - a^2 \end{array} \right]$$

If possible for what values of a and b the system has

- a. no solution? Justify.
- b. exactly one solution? Justify.
- c. infinitely many solutions? Justify.

1.2.12 [YL] Given the augmented matrix of a linear system

$$\left[\begin{array}{ccccc} 1 & 3 & 1 & -4 & b_1 \\ 3 & -2 & 4 & 5 & b_2 \\ 4 & 1 & 5 & 1 & b_3 \\ 7 & -1 & 9 & 6 & b_4 \end{array} \right]$$

Determine the restrictions on the b_i 's for the system to be consistent.

1.2.13 [JH] Prove that, where a, b, \dots, e are real numbers and $a \neq 0$, if

$$ax + by = c$$

has the same solution set as

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

1.2.14 [JH] Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

1.3 Applications of Linear Systems

1.3.1 Place Holder

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [JH] Find the indicated entry of the following matrix.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{bmatrix}$$

- a. $a_{2,1}$ b. $a_{1,2}$ c. $a_{2,2}$ d. $a_{3,1}$

2.1.2 [JH] Determine the size of each matrix.

a. $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{bmatrix}$ c. $\begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$

2.1.3 [GH] Simplify the given expression where

$$A = \begin{bmatrix} 1 & -1 \\ 7 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 2 \\ 5 & 9 \end{bmatrix}$$

- a. $A + B$ c. $3(A - B) + B$
b. $2A - 3B$ d. $2(A - B) - (A - 3B)$

2.1.4 [GH] The row and column matrix U and V are defined. Find the product UV , where possible.

a. $U = \begin{bmatrix} 1 & -4 \end{bmatrix}, \quad V = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ c. $U = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad V = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
b. $U = \begin{bmatrix} 6 & 2 & -1 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 3 \\ 2 \\ 9 \\ 5 \end{bmatrix}$ d. $U = \begin{bmatrix} 2 & -5 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2.1.5 [GH] State the dimensions of A and B . State the dimensions of AB and BA , if the product is defined. Then compute the product AB and BA , if possible.

a. $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 7 \\ 4 & 2 & 9 \end{bmatrix}$

c. $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$

d. $A = \begin{bmatrix} -2 & -1 \\ 9 & -5 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 6 & -4 \\ 0 & 6 & -3 \end{bmatrix}$

e. $A = \begin{bmatrix} 2 & 6 \\ 6 & 2 \\ 5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 5 & 0 \\ -4 & 4 & -4 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 4 \\ 7 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & -5 & 5 \\ -2 & 1 & 3 & -5 \end{bmatrix}$

g. $A = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $A = \begin{bmatrix} -4 & -1 & 3 \\ 2 & -3 & 5 \\ 1 & 5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 & 3 \\ -1 & 1 & -1 \\ 4 & 0 & 2 \end{bmatrix}$

2.1.6 [GH] Given a diagonal matrix D and a matrix A , compute the product DA and AD , if possible.

a. $D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

b. $D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

c. $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

d. $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

2.1.7 [GH] Given a matrix A compute A^2 and A^3 .

$$\begin{aligned} \text{a. } A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \text{b. } A &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\ \text{c. } A &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{d. } A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \text{e. } A &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

$$\text{a. } \begin{bmatrix} 4 & 1 & 1 \\ -2 & 0 & 0 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 1 & -5 \\ 9 & 5 \end{bmatrix}$$

$$\text{c. } \begin{bmatrix} -10 & 6 & -7 & -9 \\ -2 & 1 & 6 & -9 \\ 0 & 4 & -4 & 0 \\ -3 & -9 & 3 & -10 \end{bmatrix}$$

$$\text{d. } \begin{bmatrix} 2 & 6 & 4 \\ -1 & 8 & -10 \end{bmatrix}$$

e. Any skew-symmetric matrix.

$$\text{f. } I_n$$

2.1.8 [HE] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

$$\text{a. } 3C - 4D$$

$$\text{b. } A - (D + 2C)$$

$$\text{c. } A - E$$

$$\text{d. } AE$$

$$\text{e. } 3BC - 4BD$$

$$\text{f. } CB + D$$

$$\text{g. } GC$$

$$\text{h. } FG$$

i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A , B , and C are matrices above.

2.1.9 [GH] In each part a matrix A is given. Find A^T . State whether A is upper/lower triangular, diagonal, symmetric and/or skew symmetric.

$$\text{a. } \begin{bmatrix} -9 & 4 & 10 \\ 6 & -3 & -7 \\ -8 & 1 & -1 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 4 & 2 & -9 \\ 5 & -4 & -10 \\ -6 & 6 & 9 \end{bmatrix}$$

$$\text{c. } \begin{bmatrix} 4 & -7 & -4 & -9 \\ -9 & 6 & 3 & -9 \end{bmatrix}$$

$$\text{d. } \begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$$

$$\text{e. } \begin{bmatrix} 4 & 0 & 0 \\ -2 & -7 & 0 \\ 4 & -2 & 5 \end{bmatrix}$$

$$\text{f. } \begin{bmatrix} -3 & -4 & -5 \\ 0 & -3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\text{g. } \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\text{h. } \begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$$

$$\text{i. } \begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$$

2.1.11 [GH] Find values for the scalars a and b that satisfy the given equation.

$$\text{a. } a \begin{bmatrix} -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$\text{b. } a \begin{bmatrix} 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$\text{c. } a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\text{d. } a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$$

2.1.12 [GH] The following statement

$$(A + B)^2 = A^2 + 2AB + B^2$$

is false. We investigate that claim here.

a. Let $A = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} -5 & -5 \\ -2 & 1 \end{bmatrix}$. Compute $A + B$

b. Find $(A + B)^2$ by using the previous part.

c. Compute $A^2 + 2AB + B^2$.

d. Are the results from the two previous parts equal?

e. Carefully expand the expression $(A + B)^2 = (A + B)(A + B)$ and show why this is not equal to $A^2 + 2AB + B^2$.

2.1.13 [YL]

a. Prove: If A and B are $n \times n$ matrices then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

b. Prove: If A and B are $n \times n$ matrices then $\text{tr}(AB) = \text{tr}(BA)$.

2.1.14 [YL] A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$.

Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

2.1.15 [YL] A square matrix A is called *idempotent* if $A^2 = A$.

Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A .

2.1.10 [GH] Find the trace of the given matrix.

2.2 Matrix Inverses and Algebraic Properties

2.2.1 [GH] Given the matrices A and B below. Find X that satisfies the equation.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}$$

- a. $2A + X = B$ c. $3A + 2X = -1B$
 b. $A - X = 3B$ d. $A - \frac{1}{2}X = -B$

2.2.2 [GH] Given the matrices A . Find A^{-1} , if possible.

- a. $\begin{bmatrix} 1 & 5 \\ -5 & -24 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$
 b. $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$ d. $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$

2.2.3 [GH] Given the matrices A and B . Compute $(AB)^{-1}$ and $B^{-1}A^{-1}$.

- a. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 5 \\ 2 & 5 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 1 \\ 2 & 1 \end{bmatrix}$

2.2.4 [GH] Given the matrices A . Find A^{-1} , if possible.

- a. $\begin{bmatrix} 25 & -10 & -4 \\ -18 & 7 & 3 \\ -6 & 2 & 1 \end{bmatrix}$ i. $\begin{bmatrix} 2 & 3 & 4 \\ -3 & 6 & 9 \\ -1 & 9 & 13 \end{bmatrix}$
 b. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -7 \\ 20 & 7 & -48 \end{bmatrix}$ j. $\begin{bmatrix} 5 & -1 & 0 \\ 7 & 7 & 1 \\ -2 & -8 & -1 \end{bmatrix}$
 c. $\begin{bmatrix} -4 & 1 & 5 \\ -5 & 1 & 9 \\ -10 & 2 & 19 \end{bmatrix}$ k. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -19 & -9 & 0 & 4 \\ 33 & 4 & 1 & -7 \\ 4 & 2 & 0 & -1 \end{bmatrix}$
 d. $\begin{bmatrix} 1 & -5 & 0 \\ -2 & 15 & 4 \\ 4 & -19 & 1 \end{bmatrix}$ l. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 27 & 1 & 0 & 4 \\ 18 & 0 & 1 & 4 \\ 4 & 0 & 0 & 1 \end{bmatrix}$
 e. $\begin{bmatrix} 25 & -8 & 0 \\ -78 & 25 & 0 \\ 48 & -15 & 1 \end{bmatrix}$ m. $\begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -29 & -110 \\ 0 & -3 & -5 & -19 \end{bmatrix}$
 f. $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & 8 \\ -2 & -2 & -3 \end{bmatrix}$ n. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
 g. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ o. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$
 h. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.2.5 [GH] Prove or disprove: If A and B are 2×2 invertible matrices then $A + B$ is an invertible matrix.

2.2.6 [YL] Solve for A given that it satisfies

$$(I - A^T)^{-1} = (\text{tr}(B)B^2)^T$$

where

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.2.7 [YL] Solve for X given that it satisfies

$$DXD^T = \text{tr}(BC)BC$$

where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}.$$

2.2.8 [YL] Given

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & \frac{1}{2} \end{bmatrix}.$$

a. Find A^{-1} .

b. Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2.2.9 [YL] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.2.10 [YL] Prove: If AB and BA are both invertible then A and B are both invertible.

2.2.11 [YL] Prove: If B and C are $n \times n$ matrices such that $A = B^T C + C^T B$ is invertible then A^{-1} is symmetric.

2.3 Elementary Matrices

2.3.1 [YL] Write the given matrix as a product of elementary matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

2.3.2 [YL] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.3.3 [YL] Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3 E_2 E_1 A = B$.

2.4 Linear Systems and Matrices

2.4.1 [YL] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1} .

b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

2.4.2 [GH] Given the matrices A and b below. Find x that satisfies the equation $Ax = b$ by using the inverse of A

a. $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix},$

$$b = \begin{bmatrix} 21 \\ 13 \end{bmatrix}$$

b. $A = \begin{bmatrix} 1 & -4 \\ 4 & -15 \end{bmatrix},$

$$b = \begin{bmatrix} 21 \\ 77 \end{bmatrix}$$

c. $A = \begin{bmatrix} 1 & 2 & 12 \\ 0 & 1 & 6 \\ -3 & 0 & 1 \end{bmatrix},$

$$b = \begin{bmatrix} -17 \\ -5 \\ 20 \end{bmatrix}$$

d. $A = \begin{bmatrix} 1 & 0 & -3 \\ 8 & -2 & -13 \\ 12 & -3 & -20 \end{bmatrix},$

$$b = \begin{bmatrix} -34 \\ -159 \\ -243 \end{bmatrix}$$

Chapter 3

Determinants

3.1 The Laplace Expansion

Is it the same for lower triangular matrices?

3.1.1 [GH] Compute the determinant of the following matrices.

a. $\begin{bmatrix} 10 & 7 \\ 8 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 6 & -1 \\ -7 & 8 \end{bmatrix}$

c. $\begin{bmatrix} -1 & -7 \\ -5 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -10 & -1 \\ -4 & 7 \end{bmatrix}$

3.1.2 [GH] For the following matrices, construct the submatrices used to compute the minors $M_{1,1}$, $M_{1,2}$ and $M_{1,3}$. Compute the cofactors $C_{1,1}$, $C_{1,2}$, and $C_{1,3}$.

a. $\begin{bmatrix} 7 & -3 & 10 \\ 3 & 7 & 6 \\ 1 & 6 & 10 \end{bmatrix}$

b. $\begin{bmatrix} -2 & -9 & 6 \\ -10 & -6 & 8 \\ 0 & -3 & -2 \end{bmatrix}$

c. $\begin{bmatrix} -5 & -3 & 3 \\ -3 & 3 & 10 \\ -9 & 3 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -6 & -4 & 6 \\ -8 & 0 & 0 \\ -10 & 8 & -1 \end{bmatrix}$

3.1.3 [GH] Find the determinant of the given matrix using cofactor expansion.

a. $\begin{bmatrix} 3 & 2 & 3 \\ -6 & 1 & -10 \\ -8 & -9 & -9 \end{bmatrix}$

b. $\begin{bmatrix} 8 & -9 & -2 \\ -9 & 9 & -7 \\ 5 & -1 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -1 & 0 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

f. $\begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix}$

3.1.4 [JH] Verify that the determinant of an upper-triangular 3×3 matrix is the product of the main diagonal.

$$\det \left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix} \right) = aei$$

3.1.5 [YL] Solve for λ .

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1 - \lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

3.2 Determinants and Elementary Operations

3.2.1 [GH] A matrix M and $\det(M)$ are given. Matrices A , B and C are obtained by performing operations on M . Determine the determinants of A , B and C and indicate the operations used to obtain A , B and C .

a. $M = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}$,
 $\det(M) = -41$,

$$A = \begin{bmatrix} 18 & 14 & 16 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 96 & 73 & 83 \end{bmatrix},$$

$$C = \begin{bmatrix} 9 & 1 & 6 \\ 7 & 3 & 3 \\ 8 & 7 & 3 \end{bmatrix}.$$

b. $M = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}$,
 $\det(M) = 45$,

$$A = \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & -1 \\ 3 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ 8 & 16 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & 4 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}.$$

c. $M = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$,
 $\det(M) = -16$,

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 5 & 1 & 5 \\ 4 & 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} -5 & -1 & -5 \\ -4 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 15 & 3 & 15 \\ 12 & 0 & 6 \\ 0 & 0 & 12 \end{bmatrix}.$$

d. $M = \begin{bmatrix} 5 & 4 & 0 \\ 7 & 9 & 3 \\ 1 & 3 & 9 \end{bmatrix}$,
 $\det(M) = 120$,

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 7 & 9 & 3 \\ 5 & 4 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 5 & 4 & 0 \\ 14 & 18 & 6 \\ 3 & 9 & 27 \end{bmatrix},$$

$$C = \begin{bmatrix} -5 & -4 & 0 \\ -7 & -9 & -3 \\ -1 & -3 & -9 \end{bmatrix}.$$

3.2.2 [GH] Find the determinant of the given matrix by using elementary operations to bring the matrix under triangular form.

a. $\begin{bmatrix} -4 & 3 & -4 \\ -4 & -5 & 3 \\ 3 & -4 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -2 & 1 \\ 5 & 5 & 4 \\ 4 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} -5 & 0 & -4 \\ 2 & 4 & -1 \\ -5 & 0 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

f. $\begin{bmatrix} -5 & 1 & 0 & 0 \\ -3 & -5 & 2 & 5 \\ -2 & 4 & -3 & 4 \\ 5 & 4 & -3 & 3 \end{bmatrix}$

g. $\begin{bmatrix} 2 & -1 & 4 & 4 \\ 3 & -3 & 3 & 2 \\ 0 & 4 & -5 & 1 \\ -2 & -5 & -2 & -5 \end{bmatrix}$

3.2.3 [YL] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.2.4 Vandermonde's determinant [JH] Prove:

$$\det \left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \right) = (b-a)(c-a)(c-b)$$

3.3 Properties of Determinants and 3.4 Applications of the Determinant Matrix Inverses

3.3.1 [JH] Find the adjoint of the following matrices.

a. $\begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 1 & 8 & 9 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

3.3.2 [JH] Which real numbers x make this matrix singular?

$$\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}$$

3.3.3 [JH] Prove: If S and T are $n \times n$ matrix then $\det(TS) = \det(ST)$.

3.3.4 [JH]

- Suppose that $\det(A) = 3$ and that $\det(B) = 2$. Find $\det(A^2 B^T B^{-2} A^T)$.
- If $\det(A) = 0$ then show that $\det(6A^3 + 5A^2 + 2A) = 0$.

3.3.5 [JH]

- Give a non-identity matrix with the property that $A^T = A^{-1}$.
- Prove: If $A^T = A^{-1}$ then $\det(A) = \pm 1$.
- Does the converse to the above hold?

3.3.6 [JH] Two matrices H and G are said to be *similar* if there is a nonsingular matrix P such that $H = P^{-1}GP$. Show that similar matrices have the same determinant.

3.3.7 [JH] Show that this gives the equation of a line in \mathbb{R}^2 thru (x_2, y_2) and (x_3, y_3) .

$$\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

3.3.8 [YL] Let A and B be $n \times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.4.1 [YL] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

3.4.2 [GH] Given the matrices A and b , evaluate $\det(A)$ and $\det(A_i)$ for all i . Use Cramer's Rule to solve $Ax = b$. If Cramer's Rule cannot be used to find the solution, then state whether or not a solution exists.

a. $A = \begin{bmatrix} 3 & 0 & -3 \\ 5 & 4 & 4 \\ 5 & 5 & -4 \end{bmatrix}$

$$b = \begin{bmatrix} 24 \\ 0 \\ 31 \end{bmatrix}$$

b. $A = \begin{bmatrix} 9 & 5 \\ -4 & -7 \end{bmatrix}$

$$b = \begin{bmatrix} -45 \\ 20 \end{bmatrix}$$

c. $A = \begin{bmatrix} -8 & 16 \\ 10 & -20 \end{bmatrix}$

$$b = \begin{bmatrix} -48 \\ 60 \end{bmatrix}$$

d. $A = \begin{bmatrix} 7 & 14 \\ -2 & -4 \end{bmatrix}$

$$b = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

e. $A = \begin{bmatrix} 4 & 9 & 3 \\ -5 & -2 & -13 \\ -1 & 10 & -13 \end{bmatrix}$

$$b = \begin{bmatrix} -28 \\ 35 \\ 7 \end{bmatrix}$$

f. $A = \begin{bmatrix} 7 & -4 & 25 \\ -2 & 1 & -7 \\ 9 & -7 & 34 \end{bmatrix}$

$$b = \begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in \mathbb{R}^n* : If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$.

4.3 Cross Product and Planes

4.3.1 Place Holder

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4.4 Areas, Volumes and Distances

4.4.1 Place Holder

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4.5 Geometry of Solutions of Linear Systems

4.5.1 Place Holder

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [JH] Name the zero vector for each of these vector spaces.

- The space of degree three polynomials under the natural operations.
- The space of 2×3 matrices.
- The space $\{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$.
- The space of real-valued functions of one natural number variable.

5.1.2 [JH] Find the additive inverse, in the vector space, of the vector.

- In \mathcal{P}_3 , the vector $-3 - 2x + x^2$.
- In the space $\mathcal{M}_{2 \times 2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

- In $\{ae^x + be^{-x} \mid a, b \in \mathbb{R}\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [JH] For each, list three elements and then show it is a vector space.

- The set of linear polynomials $\mathcal{P}_1 = \{a_0 + a_1x \mid a_0, a_1 \in \mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations.
- The set of linear polynomials $\{a_0 + a_1x \mid a_0 - 2a_1 = 0\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [JH] For each, list three elements and then show it is a vector space.

- The set of 2×2 matrices with real entries under the usual matrix operations.
- The set of 2×2 matrices with real entries where the 2, 1 entry is zero, under the usual matrix operations.

5.1.5 [JH] For each, list three elements and then show it is a vector space.

- The set of three-component row vectors with their

usual operations.

- The set

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$$

under the operations inherited from \mathbb{R}^4 .

5.1.6 [JH] Show that the following are not vector spaces.

- Under the operations inherited from \mathbb{R}^3 , this set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$$

- Under the operations inherited from \mathbb{R}^3 , this set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

- Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

- Under the usual polynomial operations,

$$\{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+\}$$

where \mathbb{R}^+ is the set of reals greater than zero

- Under the inherited operations,

$$\{(x, y) \in \mathbb{R}^2 \mid x + 3y = 4, 2x - y = 3 \text{ and } 6x + 4y = 10\}$$

5.1.7 [JH] Is the set of rational numbers a vector space over \mathbb{R} under the usual addition and scalar multiplication operations?

5.1.8 [JH] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \quad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [JH] Prove or disprove that \mathbb{R}^3 is a vector space under these operations.

$$\begin{aligned} \text{a. } \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix} \\ \text{b. } \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

5.1.10 [JH] For each, decide if it is a vector space; the intended operations are the natural ones.

- a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

- b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

- c. $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y + w = 1\}$
 d. The set of functions $\{f: \mathbb{R} \rightarrow \mathbb{R} \mid df/dx + 2f = 0\}$
 e. The set of functions $\{f: \mathbb{R} \rightarrow \mathbb{R} \mid df/dx + 2f = 1\}$

5.1.11 [YL] Let $V = \{A \mid A \in \mathcal{M}_{2 \times 2} \text{ and } \det(A) \neq 0\}$ with the following operations:

$$A + B = AB \text{ and } kA = kA$$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.

- a. Does V satisfy closure under vector addition? Justify.
 b. Does V contain a zero vector? If so find it. Justify.
 c. Does V contains an additive inverse for all of its vectors? Justify.
 d. Does V satisfy closure under scalar multiplication? Justify.

5.1.12 [JH] Show that the set \mathbb{R}^+ of positive reals is a vector space when we interpret ' $x + y$ ' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret ' $r \cdot x$ ' as the r -th power of x .

5.1.13 [JH] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.14 [JH]

Is $\{(x, y) \mid x, y \in \mathbb{R}\}$ a vector space under these operations?

- a. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$
 b. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, 0)$

5.1.15 [JH]

Prove the following:

- a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of \vec{v} , then \vec{v} is an additive inverse of \vec{w} . So a vector is an additive inverse of any additive inverse of itself.
 b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.16 [JH]

The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.17 [JH]

Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.18 [JH]

In a vector space every element has an additive inverse. Is the additive inverse unique (*Can some elements have two or more*)?

5.1.19 [JH]

Assume that $\vec{v} \in V$ is not $\vec{0}$.

- a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.
 b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.
 c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [JH]

- a. Prove that every point, line, or plane thru the origin in \mathbb{R}^3 is a subspace of \mathbb{R}^3 under the inherited operations.
 b. What if it doesn't contain the origin?

5.2.2 [JH] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.3 Spanning Sets

5.3.1 [YL] Given the following two subspace of \mathbb{R}^3 : $W_1 = \{x \mid A_1 x = 0\}$ and $W_2 = \{x \mid A_2 x = 0\}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.4 Linear Independence

5.4.1 [YL] Let $\vec{u} = (1, \lambda, -\lambda)$, $\vec{v} = (-2\lambda \ -2 \ 2\lambda)$ and $\vec{w} = (\lambda - 2, -5\lambda - 2, -2)$.

- For what value(s) of λ will $\{\vec{u}, \vec{v}\}$ be linearly dependent.
- For what value(s) of λ will $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent.

5.5 Basis

5.5.1 [YL] Given

$$W = \{p(x) = a_0 + a_2x^2 + a_3x^3 \mid p(-1) = 0\}$$

a subspace of \mathcal{P}_3 .

- Find a basis B for W .
- Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B .

5.6 Dimension

5.6.1 [YL] Given

$$W = \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mid p(1) = 0 \text{ and } p(-1) = 0\}$$

a subspace of \mathcal{P}_3 . Determine the dimension of W .

Appendix A

Answers to Exercises

Note that either a hint, a final answer or a complete solution is provided.

1.1.1

- a. Yes
- b. No
- c. Yes
- d. Yes
- e. No
- f. No
- g. Yes
- h. No
- i. Yes
- j. No

1.1.2

- a. $x = 1, y = -2$
- b. $x = 2, y = \frac{1}{3}$
- c. $x = -1, y = 0$, and $z = 2$.
- d. $x = 1, y = 0$, and $z = 0$.

1.1.3

- a. $\left[\begin{array}{ccc|c} 3 & 4 & 5 & 7 \\ -1 & 1 & -3 & 1 \\ 2 & -2 & 3 & 5 \end{array} \right]$
- b. $\left[\begin{array}{ccc|c} 2 & 5 & -6 & 2 \\ 9 & 0 & -8 & 10 \\ -2 & 4 & 1 & -7 \end{array} \right]$
- c. $\left[\begin{array}{cccc|c} 1 & 3 & -4 & 5 & 17 \\ -1 & 0 & 4 & 8 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{array} \right]$
- d. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 0 & 3 \\ -1 & 9 & 8 \\ 5 & -7 & 13 \end{array} \right]$

1.1.4

- a. $\begin{array}{l} x_1 + 2x_2 = 3 \\ -x_1 + 3x_2 = 9 \end{array}$
- b. $\begin{array}{l} -3x_1 + 4x_2 = 7 \\ x_2 = -2 \end{array}$

c. $\begin{array}{l} x_1 + x_2 - x_3 - x_4 = 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 = 7 \end{array}$

d. $\begin{array}{l} x_1 = 2 \\ x_2 = -1 \\ x_3 = 5 \\ x_4 = 3 \end{array}$

e. $\begin{array}{l} x_1 + x_3 + 7x_5 = 2 \\ x_2 + 3x_3 + 2x_4 = 5 \end{array}$

1.1.5

a. $\begin{bmatrix} -2 & 1 & -7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 7 \\ 5 & 0 & 3 \\ 0 & 4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -1 & 7 \\ 2 & 3 & 5 \\ 5 & 0 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 8 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 2 & -1 \\ 5 & 0 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 0 & 5/2 & -29/2 \end{bmatrix}$

1.1.6

- a. $2R_2 \rightarrow R_2$
- b. $R_1 + R_2 \rightarrow R_2$
- c. $2R_3 + R_1 \rightarrow R_1$
- d. $R_1 \leftrightarrow R_2$
- e. $-R_2 + R_3 \leftrightarrow R_3$

1.1.7 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line. Thus the lines can share one point (giving a unique solution),

share no points (giving no solutions), or share at least two points (which makes them the same line).

1.1.8 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

1.2.1

- a. Yes
- b. No
- c. No
- d. Yes
- e. Yes
- f. Yes
- g. No
- h. Yes
- i. No
- j. Yes
- k. Yes
- l. Yes
- m. No
- n. Yes
- o. Yes

1.2.2

- a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- c. $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$
- d. $\begin{bmatrix} 1 & -7/5 \\ 0 & 0 \end{bmatrix}$
- e. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \end{bmatrix}$
- f. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$
- g. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
- h. $\begin{bmatrix} 1 & \frac{5}{4} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$
- i. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- j. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- k. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

n. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

p. $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.2.3

- a. $x = 2, y = 3$
- b. $x = -1, y = 4$, and $z = -1$.

1.2.4

- a. $x_1 = 1 - 2t; x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 1, x_2 = 0$ and $x_1 = -1, x_2 = 1$.
- b. $x_1 = -3 + 5t; x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 3, x_2 = 0$ and $x_1 = -8, x_2 = -1$.
- c. $x_1 = 1; x_2 = 2$.
- d. $x_1 = 0; x_2 = -1$.
- e. $x_1 = -11 + 10t; x_2 = -4 + 4t; x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -11, x_2 = -4, x_3 = 0$ and $x_1 = -1, x_2 = 0$ and $x_3 = 1$.
- f. $x_1 = -\frac{2}{3} + \frac{8}{9}t; x_2 = \frac{2}{3} - \frac{5}{9}t; x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -\frac{2}{3}, x_2 = \frac{2}{3}, x_3 = 0$ and $x_1 = \frac{4}{9}, x_2 = -\frac{1}{9}, x_3 = 1$.
- g. $x_1 = 1 - s - t; x_2 = s; x_3 = 1 - 2t; x_4 = t$ where $s, t \in \mathbb{R}$. Possible solutions: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ and $x_1 = -2, x_2 = 1, x_3 = -3, x_4 = 2$.
- h. $x_1 = 3 - s - 2t; x_2 = -3 - 5s - 7t; x_3 = s; x_4 = t$ where $s, t \in \mathbb{R}$. Possible solutions: $x_1 = 3, x_2 = -3, x_3 = 0, x_4 = 0$ and $x_1 = 0, x_2 = -5, x_3 = -1, x_4 = 1$.
- i. $x_1 = \frac{1}{3} - \frac{4}{3}t; x_2 = \frac{1}{3} - \frac{1}{3}t; x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = \frac{1}{3}, x_2 = \frac{1}{3}, x_3 = 0$ and $x_1 = -1, x_2 = 0, x_3 = 1$.
- j. $x_1 = 1 - 2s - 3t; x_2 = s; x_3 = t$ where $s, t \in \mathbb{R}$. Possible solutions: $x_1 = 1, x_2 = 0, x_3 = 0$ and $x_1 = 8, x_2 = 1, x_3 = -3$.
- k. No solution; the system is inconsistent.
- l. No solution; the system is inconsistent.

1.2.5

- a. $(x_1, x_2, x_3, x_4, x_5) = (60s - 55t + 30, -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3}, -14s + 13t - 7, s, t)$ where $s, t \in \mathbb{R}$.

- b. If $s = t = 0$ then $(x_1, x_2, x_3, x_4, x_5) = (30, -\frac{38}{3}, -7, 0, 0)$.
 If $s = 0$ and $t = 1$ then $(x_1, x_2, x_3, x_4, x_5) = (-25, \frac{35}{3}, 6, 0, 1)$.
 c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1, x_2, x_3, x_4, x_5) = (-\frac{30}{7}, \frac{316}{21}, 1, \frac{4}{7}, 0)$.

1.2.6

- a. $(x_1, x_2, x_3, x_4) = (60t, -\frac{79}{3}t, -14t, t)$ where $t \in \mathbb{R}$.
 b. If $t = 1$ then $(x_1, x_2, x_3, x_4) = (60, -\frac{79}{3}, -14, 1)$.
 If $t = 3$ then $(x_1, x_2, x_3, x_4) = (180, -79, 42, 3)$.
 c. If $t = \frac{1}{60}$ then $(x_1, x_2, x_3, x_4) = (1, -\frac{79}{180}, -\frac{14}{60}, \frac{1}{60})$.

1.2.7 Because $f(1) = 2$, $f(-1) = 6$, and $f(2) = 3$ we get a linear system.

$$\begin{aligned} 1a + 1b + c &= 2 \\ 1a - 1b + c &= 6 \\ 4a + 2b + c &= 3 \end{aligned}$$

After performing Gaussian elimination we obtain

$$\begin{aligned} a + b + c &= 2 \\ -2b &= 4 \\ -3c &= -9 \end{aligned}$$

which shows that the solution is $f(x) = 1x^2 - 2x + 3$.

1.2.8 The following system with more unknowns than equations

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

has no solution.

1.2.9 After performing Gaussian elimination the system becomes

$$\begin{aligned} x - y &= 1 \\ 0 &= -3 + k \end{aligned}$$

This system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

1.2.10

- a. Never exactly 1 solution; infinite solutions if $k = 2$; no solution if $k \neq 2$.
 b. Exactly 1 solution if $k \neq 2$; infinite solutions if $k = 2$; never no solution.
 c. Exactly 1 solution if $k \neq 2$; no solution if $k = 2$; never infinite solutions.
 d. Exactly 1 solution for all k .

1.2.11

- a. Possible if $a = \pm 1$ and $a \neq \pm b$.
 b. Not possible.
 c. Possible if $a \neq \pm 1$ or $a = \pm b$.

1.2.12 Consistent if $b_3 - b_2 - b_1 = 0$ and $b_4 - 2b_2 - b_1 = 0$.

1.2.13 If $a \neq 0$ then the solution set of the first equation is $\{(x, y) \mid x = (c - by)/a\}$. Taking $y = 0$ gives the solution $(c/a, 0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a) + d \cdot 0 = e$, so $c = e$. Then taking $y = 1$ in $x = (c - by)/a$ gives that $a((c - b)/a) + d \cdot 1 = e$, which gives that $b = d$. Hence they are the same equation.

When $a = 0$ the equations can be different and still have the same solution set: e.g., $0x + 3y = 6$ and $0x + 6y = 12$.

1.2.14 We take three cases: that $a \neq 0$, that $a = 0$ and $c \neq 0$, and that both $a = 0$ and $c = 0$.

For the first, we assume that $a \neq 0$. Then Gaussian elimination

$$\begin{aligned} ax + by &= j \\ -(cb/a) + d)y &= -(cj/a) + k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a) + d \neq 0$; remember that $a \neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a) + d = (ad - bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad - bc \neq 0$.

In the second case, if $a = 0$ but $c \neq 0$, then we swap

$$\begin{aligned} cx + dy &= k \\ by &= j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b \neq 0$ (we use the case assumption that $c \neq 0$ to get a unique x in back substitution). But where $a = 0$ and $c \neq 0$ the condition " $b \neq 0$ " is equivalent to the condition " $ad - bc \neq 0$ ". That finishes the second case.

Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x + by &= j \\ 0x + dy &= k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x, y) will do), but it never has a unique solution. Note that $a = 0$ and $c = 0$ gives that $ad - bc = 0$.

1.3.1 Lorem ipsum dolor sit amet, consectetur adipiscing elit. Aliquam tincidunt cursus volutpat. Quisque non congue sem. Vivamus nec nibh sed est dapibus auctor eu sed nulla. Praesent ornare eleifend nibh a finibus. Proin rutrum neque nec massa tincidunt, non malesuada dolor interdum. Nam a massa sit amet diam efficitur pharetra. Nulla interdum efficitur sem, sit amet commodo orci mattis non. Duis tortor ex, maximus a sapien id, molestie maximus risus.

2.1.1

- a. 2
- b. 3
- c. -1
- d. Not defined.

2.1.2

- a. 2×3
- b. 3×2
- c. 2×2

2.1.3

- a. $\begin{bmatrix} -2 & -1 \\ 12 & 13 \end{bmatrix}$
- b. $\begin{bmatrix} 11 & -8 \\ -1 & -19 \end{bmatrix}$
- c. $\begin{bmatrix} 9 & -7 \\ 11 & -6 \end{bmatrix}$
- d. $\begin{bmatrix} -2 & 1 \\ 12 & 13 \end{bmatrix}$

2.1.4

- a. -22
- b. -2
- c. 23
- d. Not possible.
- e. Not possible.

2.1.5

- a. $AB = \begin{bmatrix} 8 & 3 \\ 10 & -9 \end{bmatrix}, BA = \begin{bmatrix} -3 & 24 \\ 4 & 2 \end{bmatrix}$
- b. $AB = \begin{bmatrix} -1 & -2 & 12 \\ 10 & 4 & 32 \end{bmatrix}, BA$ is not defined
- c. $AB = \begin{bmatrix} 3 & 8 \\ -5 & -8 \\ -8 & -32 \end{bmatrix}, BA$ is not defined
- d. $AB = \begin{bmatrix} 10 & -18 & 11 \\ -45 & 24 & -21 \\ -15 & 12 & -9 \end{bmatrix}, BA = \begin{bmatrix} 52 & -21 \\ 45 & -27 \end{bmatrix}$
- e. $AB = \begin{bmatrix} -32 & 34 & -24 \\ -32 & 38 & -8 \\ -16 & 21 & 4 \end{bmatrix}, BA = \begin{bmatrix} 22 & -14 \\ -4 & -12 \end{bmatrix}$
- f. $AB = \begin{bmatrix} -7 & 3 & 7 & -15 \\ -5 & -1 & -17 & 5 \end{bmatrix}, BA$ is not defined
- g. $AB = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ -2 & 0 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 & 4 \\ -3 & 6 & 1 \\ -1 & 2 & 1 \end{bmatrix}$
- h. $AB = \begin{bmatrix} 21 & -17 & -5 \\ 19 & 5 & 19 \\ 5 & 9 & 4 \end{bmatrix}, BA = \begin{bmatrix} 19 & 5 & 23 \\ 5 & -7 & -1 \\ -14 & 6 & 18 \end{bmatrix}$

2.1.6

- a. $DA = \begin{bmatrix} 2 & 2 & 2 \\ -6 & -6 & -6 \\ -15 & -15 & -15 \end{bmatrix}, AD = \begin{bmatrix} 2 & -3 & 5 \\ 4 & -6 & 10 \\ -6 & 9 & -15 \end{bmatrix}$
- b. $DA = \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix}, AD = \begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix}$
- c. $DA = \begin{bmatrix} d_1a & d_1b \\ d_2c & d_2d \end{bmatrix}, AD = \begin{bmatrix} d_1a & d_2b \\ d_1c & d_2d \end{bmatrix}$
- d. $DA = \begin{bmatrix} d_1a & d_1b & d_1c \\ d_2d & d_2e & d_2f \\ d_3g & d_3h & d_3i \end{bmatrix}, AD = \begin{bmatrix} d_1a & d_2b & d_3c \\ d_1d & d_2e & d_3f \\ d_1g & d_2h & d_3i \end{bmatrix}$

2.1.7

- a. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- b. $A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}$
- c. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}, A^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 125 \end{bmatrix}$
- d. $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- e. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2.1.8

- a. $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$
- b. $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$
- c. Not possible, since dimension of A and E are not the same.
- d. $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$
- e. $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$
- f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D .
- g. $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$
- h. $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.1.9

- a. $\begin{bmatrix} -9 & 6 & -8 \\ 4 & -3 & 1 \\ 10 & -7 & -1 \end{bmatrix}$
- b. $\begin{bmatrix} 4 & 5 & -6 \\ 2 & -4 & 6 \\ -9 & -10 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -9 \\ -7 & 6 \\ -4 & 3 \\ -9 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$, symmetric

e. $\begin{bmatrix} 4 & -2 & 4 \\ 0 & -7 & -2 \\ 0 & 0 & 5 \end{bmatrix}$, A is lower triangular and A^T is upper triangular.

f. $\begin{bmatrix} -3 & 0 & 0 \\ -4 & -3 & 0 \\ -5 & 5 & -3 \end{bmatrix}$, A is upper triangular and A^T is lower triangular.

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, diagonal.

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$, symmetric.

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$, skew-symmetric.

2.1.10

- a. -9
- b. 6
- c. -23
- d. Not defined; the matrix must be square.
- e. 0
- f. n

2.1.11

- a. $a = -1, b = 1/2$
- b. $a = 5/2 + 3/2t, b = t$ where $t \in \mathbb{R}$
- c. $a = 5, b = 0$
- d. No solution.

2.1.12

- a. $\begin{bmatrix} 0 & -2 \\ -5 & -1 \end{bmatrix}$
- b. $\begin{bmatrix} 10 & 2 \\ 5 & 11 \end{bmatrix}$
- c. $\begin{bmatrix} -11 & -15 \\ 37 & 32 \end{bmatrix}$
- d. No.
- e. $(A+B) = AA+AB+BA+BB = A^2+AB+BA+B^2$

2.1.13

- a. Hint: Apply the definition of the trace to arbitrary matrices A and B .
- b. Hint: Analyse the ij product of the elements of the main diagonal.

2.1.14 Disprove: Show that it is impossible to obtain a nonzero matrix.

2.1.15 Hint: Apply the definition of an idempotent matrix.

2.2.1

a. $X = \begin{bmatrix} -5 & 9 \\ -1 & -14 \end{bmatrix}$

b. $X = \begin{bmatrix} 0 & -22 \\ -7 & 17 \end{bmatrix}$

c. $X = \begin{bmatrix} -5 & -2 \\ -9/2 & -19/2 \end{bmatrix}$

d. $X = \begin{bmatrix} 8 & 12 \\ 10 & 2 \end{bmatrix}$

2.2.2

a. $\begin{bmatrix} -24 & -5 \\ 5 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/7 \end{bmatrix}$

c. $\begin{bmatrix} -4/7 & 5/7 \\ 3/7 & -2/7 \end{bmatrix}$

d. The inverse does not exist.

2.2.3

a. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -7/5 \end{bmatrix}$

b. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -7/10 & 3/10 \\ 29/10 & -11/10 \end{bmatrix}$

2.2.4

a. $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 6 & 10 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 52 & -48 & 7 \\ 8 & -7 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -9 & 4 \\ 5 & -26 & 11 \\ 0 & -2 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 91 & 5 & -20 \\ 18 & 1 & -4 \\ -22 & -1 & 5 \end{bmatrix}$

e. $\begin{bmatrix} 25 & 8 & 0 \\ 78 & 25 & 0 \\ -30 & -9 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 5 & -3 & -8 \\ -4 & 2 & 5 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$\text{h. } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i. The inverse does not exist.

j. The inverse does not exist.

$$\text{k. } \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & -4 \\ -35 & -10 & 1 & -47 \\ -2 & -2 & 0 & -9 \end{bmatrix}$$

$$\text{l. } \begin{bmatrix} 1 & 0 & 0 & 0 \\ -11 & 1 & 0 & -4 \\ -2 & 0 & 1 & -4 \\ -4 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{m. } \begin{bmatrix} 1 & 28 & -2 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 254 & -19 & 110 \\ 0 & -67 & 5 & -29 \end{bmatrix}$$

$$\text{n. } \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{o. } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1/4 \end{bmatrix}$$

$$\text{2.2.5 Disprove: } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\text{2.2.6 } A = \begin{bmatrix} -\frac{3}{4} & 3 \\ 1 & -\frac{3}{4} \end{bmatrix}$$

$$\text{2.2.7 } A = \begin{bmatrix} 0 & -1 \\ -11 & -\frac{17}{2} \end{bmatrix}$$

2.2.8

$$\text{a. } A = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$$

$$\text{b. } X = \begin{bmatrix} -\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & \frac{3}{2} & -4 & 2 \end{bmatrix}$$

2.2.9 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.2.10 Hint: Use the definition of the inverse of a matrix.

2.2.11 Hint: Apply the definition of symmetric matrices.

2.3.1

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.2

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

$$\text{2.3.3 } E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.4.1

$$\text{a. } A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\text{b. } x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$$

2.4.2

$$\text{a. } x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{b. } x = \begin{bmatrix} -7 \\ -7 \end{bmatrix}$$

$$\text{c. } x = \begin{bmatrix} -7 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{d. } x = \begin{bmatrix} -7 \\ -7 \\ 9 \end{bmatrix}$$

3.1.1

a. 34

b. 41

c. -44

d. -74

3.1.2

$$\text{a. } M_{1,1} = \begin{bmatrix} 7 & 6 \\ 6 & 10 \end{bmatrix}, \quad M_{1,2} = \begin{bmatrix} 3 & 6 \\ 1 & 10 \end{bmatrix}, \quad M_{1,3} = \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix}.$$

$$C_{1,1} = 43, \quad C_{1,2} = -24, \quad C_{1,3} = 11.$$

$$\text{b. } M_{1,1} = \begin{bmatrix} -6 & 8 \\ -3 & -2 \end{bmatrix}, \quad M_{1,2} = \begin{bmatrix} -10 & 8 \\ 0 & -2 \end{bmatrix}, \quad M_{1,3} = \begin{bmatrix} 10 & -6 \\ 0 & -3 \end{bmatrix}.$$

$$C_{1,1} = 36, \quad C_{1,2} = -20, \quad C_{1,3} = -30.$$

$$\text{c. } M_{1,1} = \begin{bmatrix} 3 & 10 \\ 3 & 9 \end{bmatrix}, \quad M_{1,2} = \begin{bmatrix} -3 & 10 \\ -9 & 9 \end{bmatrix}, \quad M_{1,3} = \begin{bmatrix} -3 & 3 \\ -9 & 3 \end{bmatrix}.$$

$$C_{1,1} = -3, \quad C_{1,2} = -63, \quad C_{1,3} = 18.$$

$$\text{d. } M_{1,1} = \begin{bmatrix} 0 & 0 \\ 8 & -1 \end{bmatrix}, \quad M_{1,2} = \begin{bmatrix} -8 & 0 \\ -10 & -1 \end{bmatrix}, \quad M_{1,3} =$$

$$\begin{bmatrix} -8 & 0 \\ -10 & 8 \end{bmatrix}.$$

$C_{1,1} = 0, C_{1,2} = -8, C_{1,3} = -64.$

3.1.3

- a. -59
- b. 250
- c. 3
- d. 0
- e. 0
- f. 2

3.1.4 Evaluate the determinant using a cofactor expansion. The same is true for lower triangular matrices.

3.1.5 $\lambda = \frac{3 \pm \sqrt{33}}{4}$

3.2.1

- a. $\det(A) = 90; 2R_1 \rightarrow R_1.$
 $\det(B) = 45; 10R_1 + R_3 \rightarrow R_3.$
 $\det(C) = 45; C = A^T.$
- b. $\det(A) = 41; R_2 \leftrightarrow R_3.$
 $\det(B) = 164; -4R_3 \rightarrow R_3.$
 $\det(C) = -41; R_2 + R_1 \rightarrow R_1.$
- c. $\det(A) = -16; R_1 \leftrightarrow R_2$ then $R_1 \leftrightarrow R_3.$
 $\det(B) = -16; -R_1 \rightarrow R_1$ and $-R_2 \rightarrow R_2.$
 $\det(C) = -432; C = 3M.$
- d. $\det(A) = -120; R_1 \leftrightarrow R_2$ then $R_1 \leftrightarrow R_3$ then $R_2 \leftrightarrow R_3.$
 $\det(B) = 720; 2R_2 \rightarrow R_2$ and $3R_3 \rightarrow R_3.$
 $\det(C) = -120; C = -M.$

3.2.2

- a. 15
- b. -52
- c. 0
- d. 1
- e. -113
- f. 179

3.2.3 $\det(A) = -\frac{5}{12}$

3.2.4 Hint: Use elementary operations to bring the matrix under triangular form.

3.3.1

- a. $\begin{bmatrix} 0 & -1 & 2 \\ 3 & -2 & -8 \\ 0 & 1 & 1 \end{bmatrix}$
- b. $\begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$
- c. $\begin{bmatrix} 0 & -1 \\ -5 & 1 \end{bmatrix}$

d. $\begin{bmatrix} -24 & -12 & 12 \\ 12 & 6 & -6 \\ -8 & -4 & 4 \end{bmatrix}$

e. $\begin{bmatrix} 4 & -3 & 2 & -1 \\ -3 & 6 & -4 & 2 \\ 2 & -4 & 6 & -3 \\ -1 & 2 & -3 & 4 \end{bmatrix}$

3.3.2 This equation

$$0 = \det\left(\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}\right) = 64 - 20x + x^2 = (x-16)(x-4)$$

has roots $x = 16$ and $x = 4.$

3.3.3 $\det(TS) = \det(T) \cdot \det(S) = \det(S) \cdot \det(T) = \det(ST).$

3.3.4

- a. If it is defined then it is $(3^2)(2)(2^{-2})(3).$
- b. Hint: $\det 6A^3 + 5A^2 + 2A = \det A \det 6A^2 + 5A + 2I.$

3.3.5

- a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$
- b. $1 = \det(AA^{-1}) = \det(AA^T) = \det(A) \det(A^T) = (\det(A))^2$
- c. The converse does not hold; here is an example.

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

3.3.6 If $H = P^{-1}GP$ then $\det(H) = \det(P^{-1})\det(G)\det(P) = \det(P^{-1})\det(P)\det(G) = \det(P^{-1}P)\det(G) = \det(G).$

3.3.7 An algebraic check is easy.

$$0 = xy_2 + x_2y_3 + x_3y - x_3y_2 - xy_3 - x_2y = x \cdot (y_2 - y_3) + y \cdot (x_3 - x_2) + x_2y_3 - x_3y_2$$

simplifies to the familiar form

$$y = x \cdot (x_3 - x_2) / (y_3 - y_2) + (x_2y_3 - x_3y_2) / (y_3 - y_2)$$

(the $y_3 - y_2 = 0$ case is easily handled).

3.3.8 Hint: Apply the determinant to both sides $AB = -BA.$

3.4.1 $x_1 = 4$

3.4.2

- a. $\det(A) = -123, \det(A_1) = -492, \det(A_2) = 123, \det(A_3) = 492,$
 $x = \begin{bmatrix} 4 \\ -1 \\ -4 \end{bmatrix}.$
- b. $\det(A) = -43, \det(A_1) = 215, \det(A_2) = 0,$
 $x = \begin{bmatrix} -5 \\ 0 \end{bmatrix}.$

- c. $\det(A) = 0$, $\det(A_1) = 0$, $\det(A_2) = 0$, $\det(A_3) = 0$. Infinite solutions exist.
- d. $\det(A) = 0$, $\det(A_1) = -56$, $\det(A_2) = 26$. No solution exist.
- e. $\det(A) = 0$, $\det(A_1) = 0$, $\det(A_2) = 0$, $\det(A_3) = 0$. Infinite solutions exist.
- f. $\det(A) = 0$, $\det(A_1) = 1247$, $\det(A_2) = -49$, $\det(A_3) = -49$. No solution exist.

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}$.

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5.1.1

- a. $0 + 0x + 0x^2 + 0x^3$
- b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- c. The constant function $f(x) = 0$
- d. The constant function $f(n) = 0$

5.1.2

- a. $3 + 2x - x^2$

- b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$
- c. $-3e^x + 2e^{-x}$

5.1.3

- a. $1 + 2x$, $2 - 1x$, and x .
- b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

- a. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- b. $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5.1.5

- a. $(1, 2, 3)$, $(2, 1, 3)$, and $(0, 0, 0)$.
- b. $(1, 1, 1, -1)$, $(1, 0, 1, 0)$ and $(0, 0, 0, 0)$.

5.1.6

For each part the set is called Q . For some parts, there are more than one correct way to show that Q is not a vector space.

- a. It is not closed under addition.

$$(1, 0, 0), (0, 1, 0) \in Q \quad (1, 1, 0) \notin Q$$

- b. It is not closed under addition.

$$(1, 0, 0), (0, 1, 0) \in Q \quad (1, 1, 0) \notin Q$$

- c. It is not closed under addition.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

- d. It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \quad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

- e. The set is empty, violating the existence of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

- a. It is not a vector space.

$$(1 + 1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- b. It is not a vector space.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each “yes” answer, you must give a check of all the conditions given in the definition of a vector space. For each “no” answer, give a specific example of the failure of one of the conditions.

- Yes.
- Yes.
- No, this set is not closed under the natural addition operation. The vector of all $1/4$'s is an element of this set but when added to itself the result, the vector of all $1/2$'s, is not an element of the set.
- Yes.
- No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11

- Closed under vector addition. Hint: Apply determinant properties.
- $\vec{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$
- Every $A \in V$ has an additive inverse A^{-1} .
- Yes.
- Not closed under scalar multiplication. Since $0\vec{0} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin V$

5.1.12 Check all 10 conditions of the definition of a vector space.

5.1.13 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.14

- No since $1 \cdot (0, 1) + 1 \cdot (0, 1) \neq (1 + 1) \cdot (0, 1)$.
- No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0, 1) \neq (0, 1)$.

5.1.15

- Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of \vec{v} so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore \vec{v} is also the additive inverse of \vec{w} .
- Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of \vec{v} is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which implies that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.16

Addition is commutative, so in any vector space, for any vector \vec{v} we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.17

It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.18

Each element of a vector space has one and only one additive inverse.

For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of \vec{v} then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.19

Assume that $\vec{v} \in V$ is not $\vec{0}$.

- One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.
- Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.
- A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

5.2.1

- Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of \vec{v}, \vec{w} may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.
- No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.2 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r f'$.

5.3.1 Hint: For each subspace determine a set of vectors that spans it.

$$W_1 \subsetneq W_2$$

5.4.1

- $\lambda = 1$
- $\lambda \neq -1, -\frac{1}{2}, 1$

5.5.1

- $B = \{1 + x^3, x^2 + x^3\}$
- $(p(x))_B = (-2, 2)$

5.6.1 $\{-1 + x^2, -x + x^3\}$ is a basis of W , therefore W is of dimension 2.

The problems contained in this textbook are taken directly or based on content from the following open source references.

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