CEGEP Linear Algebra Problems

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CEGEP level linear algebra problems

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Contents

Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 [GH] State which of the following equations is a linear equation. If it is not, state why.

a. $x + y + z = 10$

b. $xy + yz + xz = 1$

c. $-3x + 9 = 3y - 5z + x - 7$

d. $\sqrt{5}y + \pi x = -1$

e. $(x-1)(x+1) = 0$

f. $\sqrt{x_1^2 + x_2^2} = 25$

g. $x_1 + y + t = 1$

h. $\frac{1}{x} + 9 = 3\cos(y) - 5z$

i. $\cos(15)y + \frac{x}{4} = -1$

j. $2^x + 2^y = 16$

1.1.2 [GH] Solve the system of linear equations using substitution, comparison and/or elimination.

a. $\begin{aligned} x + y &= -1 \\ 2x - 3y &= 8 \end{aligned}$

b. $\begin{aligned} 2x - 3y &= 3 \\ 3x + 6y &= 8 \end{aligned}$

c. $\begin{aligned} x - y + z &= 1 \\ 2x + 6y - z &= -4 \\ 4x - 5y + 2z &= 0 \end{aligned}$

d. $\begin{aligned} x + y - z &= 1 \\ 2x + y &= 2 \\ y + 2z &= 0 \end{aligned}$

1.1.3 [GH] Convert the given system of linear equations into an augmented matrix.

a. $\begin{aligned} 3x + 4y + 5z &= 7 \\ -x + y - 3z &= 1 \\ 2x - 2y + 3z &= 5 \end{aligned}$

b. $\begin{aligned} 2x + 5y - 6z &= 2 \\ 9x - 8z &= 10 \\ -2x + 4y + z &= -7 \end{aligned}$

c. $\begin{aligned} x_1 + 3x_2 - 4x_3 + 5x_4 &= 17 \\ -x_1 + 4x_3 + 8x_4 &= 1 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 &= 6 \end{aligned}$

d. $\begin{aligned} 3x_1 - 2x_2 &= 4 \\ 2x_1 &= 3 \\ -x_1 + 9x_2 &= 8 \\ 5x_1 - 7x_2 &= 13 \end{aligned}$

1.1.4 [GH] Convert given augmented matrix into a system of linear equations. Use the variables x_1, x_2, ...

a. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array}\right]$

b. $\left[\begin{array}{cc|c} -3 & 4 & 7 \\ 0 & 1 & -2 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & 5 & 7 \end{array}\right]$

d. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

e. $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 7 & 2 \\ 0 & 1 & 3 & 2 & 0 & 5 \end{array}\right]$

1.1.5 [GH] Perform the given row operations on

$$\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}.$$

a. $-1R_1 \to R_1$

b. $R_2 \leftrightarrow R_3$

c. $R_1 + R_2 \to R_2$

d. $2R_2 + R_3 \to R_3$

e. $\frac{1}{2}R_2 \to R_2$

f. $-\frac{5}{2}R_1 + R_3 \to R_3$

1.1.6 [GH] Give the row operation that transforms A into B where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

a. $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

b. $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $B = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

d. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

e. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

1.1.7 [JH] In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy-plane. By geometrical reasoning, show that there are three possibilities:

there is a unique solution, there is no solution, and there are infinitely many solutions.

1.1.8 [**JH**] Is there a two-unknowns linear system whose solution set is all of $\mathbb{R}^2$?

1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [**JH**] Use Gauss's Method to find the unique solution for each system.

a. $$\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$$

b. $$\begin{aligned} x - z &= 0 \\ 3x + y &= 1 \\ -x + y + z &= 4 \end{aligned}$$

1.2.2 [**YL**] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 + x_5 &= 3 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 + 17x_3 - 2x_4 - x_5 &= 1 \end{aligned}$$

a. Solve the following system by Gauss-Jordan elimination.

b. Find two particular solution to the above system.

c. Find a solution to the above system when $x_3 = 1$.

1.2.3 [**YL**] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 &= 0 \\ 4x_1 + 17x_3 - 2x_4 &= 0 \\ 9x_1 + 6x_2 + 27x_3 - 4x_4 &= 0 \end{aligned}$$

a. Solve the system by Gauss-Jordan elimination.

b. Find two particular nontrivial solution to the system.

c. Find a solution to the system when $x_1 = 1$.

1.2.4 [**JH**] Find the coefficients a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.2.5 [**JH**] True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.2.6 [**JH**] For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

1.2.7 [**YL**] Given the augmented matrix of a linear system:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a^2-1 & b^2-a^2 \end{bmatrix}$$

If possible for what values of a and b the system has

a. no solution? Justify.

b. exactly one solution? Justify.

c. infinitely many solutions? Justify.

1.2.8 [**YL**] Given the augmented matrix of a linear system

$$\begin{bmatrix} 1 & 3 & 1 & -4 & b_1 \\ 3 & -2 & 4 & 5 & b_2 \\ 4 & 1 & 5 & 1 & b_3 \\ 7 & -1 & 9 & 6 & b_4 \end{bmatrix}.$$

Determine the restrictions on the b_i's for the system to be consistent.

1.2.9 [**JH**] Prove that, where $a, b, \ldots, e$ are real numbers and $a \neq 0$, if

$$ax + by = c$$

has the same solution set as

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

1.2.10 [**JH**] Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

1.3 Applications of Linear Systems

1.3.1 Place Holder

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [**HE**] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \ C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \ E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \ G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

a. $3C - 4D$

b. $A - (D + 2C)$

c. $A - E$

d. AE

e. $3BC - 4BD$

f. $CB + D$

g. GC

h. FG

i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A, B, and C are matrices above.

2.1.2 [**YL**] A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$.
Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

2.1.3 [**YL**] A square matrix A is called *idempotent* if $A^2 = A$.
Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A.

2.2 Matrix Inverses and Algebraic Properties

2.2.1 [**YL**] Solve of A given that it satisfies

$$(I - A^T)^{-1} = (\mathrm{tr}(B)B^2)^T$$

where

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.2.2 [**YL**] Solve of X given that it satisfies

$$DXD^T = \mathrm{tr}(BC)BC$$

where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} \ C = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} \ D = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}.$$

2.2.3 [**YL**] Given

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & \frac{1}{2} \end{bmatrix}.$$

a. Find A^{-1}.

b. Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2.2.4 [**YL**] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.2.5 [**YL**] Prove: If AB and BA are both invertible then A and B are both invertible.

2.2.6 [**YL**] Prove: If B and C are $n \times n$ matrices such that $A = B^TC + C^TB$ is invertible then A^{-1} is symmetric.

2.3 Elementary Matrices

2.3.1 [**YL**] Write the given matrix as a product of elementary matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

2.3.2 [**YL**] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.3.3 [**YL**] Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3E_2E_1A = B$.

2.4 Linear Systems and Matrices

2.4.1 [**YL**] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1}.

b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Chapter 3

Determinants

3.1 The Laplace Expansion

3.1.1 [**YL**] Solve for λ.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

3.2 Determinants and Elementary Operations

3.2.1 [**YL**] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.3 Properties of Determinants

3.3.1 [**YL**] Let A and B be $n \times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.4 Applications of the Determinant

3.4.1 [**YL**] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in* $\mathbb{R}^n$: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$.

4.3 Cross Product and Planes

4.3.1 Place Holder

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4.4 Areas, Volumes and Distances

4.4.1 Place Holder

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4.5 Geometry of Solutions of Linear Systems

4.5.1 Place Holder

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [**JH**] Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.

b. The space of 2×3 matrices.

c. The space $\{\, f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \,\}$.

d. The space of real-valued functions of one natural number variable.

5.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3 - 2x + x^2$.

b. In the space $\mathcal{M}_{2\times 2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

c. In $\{\, ae^x + be^{-x} \mid a, b \in \mathbb{R} \,\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{\, a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \,\}$ under the usual polynomial addition and scalar multiplication operations.

b. The set of linear polynomials $\{\, a_0 + a_1 x \mid a_0 - 2a_1 = 0 \,\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.

b. The set of 2×2 matrices with real entries where the $2,1$ entry is zero, under the usual matrix operations.

5.1.5 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.

b. The set

$$\{\, (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y - z + w = 0 \,\}$$

under the operations inherited from $\mathbb{R}^4$.

5.1.6 [**JH**] Show that the following are not vector spaces.

a. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x + y + z = 1 \,\}$$

b. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \,\}$$

c. Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

d. Under the usual polynomial operations,

$$\{\, a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+ \,\}$$

where $\mathbb{R}^+$ is the set of reals greater than zero

e. Under the inherited operations,

$$\{\, (x,\ y) \in \mathbb{R}^2 \mid x + 3y = 4,\ 2x - y = 3 \text{ and } 6x + 4y = 10 \,\}$$

5.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

5.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

b. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5.1.10 [**JH**] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

c. $\{(x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x+y+w=1\}$

d. The set of functions $\{f\colon \mathbb{R}\to\mathbb{R} \mid df/dx + 2f = 0\}$

e. The set of functions $\{f\colon \mathbb{R}\to\mathbb{R} \mid df/dx + 2f = 1\}$

5.1.11 [**YL**] Let $V = \{A \mid A \in \mathcal{M}_{2\times 2} \text{ and } \det(A) \neq 0\}$ with the following operations:

$$A + B = AB \text{ and } kA = kA$$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.

a. Does V satisfy closure under vector addition? Justify.

b. Does V contain a zero vector? If so find it. Justify.

c. Does V contains an additive inverse for all of its vectors? Justify.

d. Does V satisfy closure under scalar multiplication? Justify.

5.1.12 [**JH**] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x+y$' to mean the product of x and y (so that $2+3$ is 6), and we interpret '$r\cdot x$' as the r-th power of x.

5.1.13 [**JH**] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.14 [**JH**]
Is $\{(x,\ y) \mid x, y \in \mathbb{R}\}$ a vector space under these operations?

a. $(x_1,\ y_1)+(x_2,\ y_2) = (x_1+x_2,\ y_1+y_2)$ and $r\cdot(x,y) = (rx, y)$

b. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1+x_2,\ y_1+y_2)$ and $r\cdot(x,\ y) = (rx,\ 0)$

5.1.15 [**JH**]
Prove the following:

a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v}+\vec{s} = \vec{v}+\vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.16 [**JH**]
The definition of vector spaces does not explicitly say that $\vec{0}+\vec{v} = \vec{v}$ (it instead says that $\vec{v}+\vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.17 [**JH**]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.18 [**JH**]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

5.1.19 [**JH**]
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. Prove that $r\cdot\vec{v} = \vec{0}$ if and only if $r = 0$.

b. Prove that $r_1\cdot\vec{v} = r_2\cdot\vec{v}$ if and only if $r_1 = r_2$.

c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [**JH**]

a. Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ under the inherited operations.

b. What if it doesn't contain the origin?

5.2.2 [**JH**] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.3 Spanning Sets

5.3.1 [**YL**] Given the following two subspace of $\mathbb{R}^3$: $W_1 = \{x \mid A_1 x = 0\}$ and $W_2 = \{x \mid A_2 x = 0\}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix},\ A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.4 Linear Independence

5.4.1 [**YL**] Let $\vec{u} = (1,\ \lambda,\ -\lambda)$, $\vec{v} = (-2\lambda\ \ -2\ \ 2\lambda)$ and $\vec{w} = (\lambda - 2,\ -5\lambda - 2,\ -2)$.

a. For what value(s) of λ will $\{\vec{u},\ \vec{v}\}$ be linearly dependent.

b. For what value(s) of λ will $\{\vec{u},\ \vec{v},\ \vec{w}\}$ be linearly independent.

5.5 Basis

5.5.1 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_2x^2 + a_3x^3 \ \middle|\ p(-1) = 0 \right\}$$

a subspace of $\mathcal{P}_3$.

a. Find a basis B for $\mathcal{W}$.

b. Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B.

5.6 Dimension

5.6.1 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \ \middle|\ p(1) = 0 \text{ and } p(-1) = 0 \right\}$$

a subspace of P_3. Determine the dimension of W.

Appendix A

Answers to Exercises

1.1.1

a. Yes
b. No
c. Yes
d. Yes
e. No
f. No
g. Yes
h. No
i. Yes
j. No

1.1.2

a. $x = 1$, $y = -2$
b. $x = 2$, $y = \frac{1}{3}$
c. $x = -1$, $y = 0$, and $z = 2$.
d. $x = 1$, $y = 0$, and $z = 0$.

1.1.3

a. $\left[\begin{array}{ccc|c} 3 & 4 & 5 & 7 \\ -1 & 1 & -3 & 1 \\ 2 & -2 & 3 & 5 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 2 & 5 & -6 & 2 \\ 9 & 0 & -8 & 10 \\ -2 & 4 & 1 & -7 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 3 & -4 & 5 & 17 \\ -1 & 0 & 4 & 8 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right]$

d. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 0 & 3 \\ -1 & 9 & 8 \\ 5 & -7 & 13 \end{array}\right]$

1.1.4

a. $\begin{array}{rcl} x_1 + 2x_2 &=& 3 \\ -x_1 + 3x_2 &=& 9 \end{array}$

b. $\begin{array}{rcr} -3x_1 + 4x_2 &=& 7 \\ x_2 &=& -2 \end{array}$

c. $\begin{array}{rcl} x_1 + x_2 - x_3 - x_4 &=& 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 &=& 7 \end{array}$

d. $\begin{array}{rcr} x_1 &=& 2 \\ x_2 &=& -1 \\ x_3 &=& 5 \\ x_4 &=& 3 \end{array}$

e. $\begin{array}{rcl} x_1 + x_3 + 7x_5 &=& 2 \\ x_2 + 3x_3 + 2x_4 &=& 5 \end{array}$

1.1.5

a. $\begin{bmatrix} -2 & 1 & -7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 7 \\ 5 & 0 & 3 \\ 0 & 4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -1 & 7 \\ 2 & 3 & 5 \\ 5 & 0 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 8 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 2 & -1 \\ 5 & 0 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 0 & 5/2 & -29/2 \end{bmatrix}$

1.1.6

a. $2R_2 \to R_2$
b. $R_1 + R_2 \to R_2$
c. $2R_3 + R_1 \to R_1$
d. $R_1 \leftrightarrow R_2$
e. $-R_2 + R_3 \leftrightarrow R_3$

1.1.7 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line.
Thus the lines can share one point (giving a unique solution), share no points (giving no solutions), or share at least two points (which makes them the same line).

1.1.8 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

1.2.1

a. $x = 2$, $y = 3$

b. $x = -1$, $y = 4$, and $z = -1$.

1.2.2

a. $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (60s - 55t + 30,\ -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3},\ -14s + 13t - 7,\ s,\ t)$ where $s,\ t \in \mathbb{R}$.

b. If $s = t = 0$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (30,\ -\frac{38}{3},\ -7,\ 0,\ 0)$.
If $s = 0$ and $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-25,\ \frac{35}{3},\ 6,\ 0,\ 1)$.

c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-\frac{30}{7},\ \frac{316}{21},\ 1,\ \frac{4}{7},\ 0)$.

1.2.3

a. $(x_1,\ x_2,\ x_3,\ x_4) = (60t,\ -\frac{79}{3}t,\ -14t,\ t)$ where $t \in \mathbb{R}$.

b. If $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4) = (60,\ -\frac{79}{3},\ -14,\ 1)$.
If $t = 3$ then $(x_1,\ x_2,\ x_3,\ x_4) = (180,\ -79,\ 42,\ 3)$.

c. If $t = \frac{1}{60}$ then $(x_1,\ x_2,\ x_3,\ x_4) = (1,\ -\frac{79}{180},\ -\frac{14}{60},\ \frac{1}{60})$.

1.2.4 Because $f(1) = 2$, $f(-1) = 6$, and $f(2) = 3$ we get a linear system.

$$\begin{aligned} 1a + 1b + c &= 2 \\ 1a - 1b + c &= 6 \\ 4a + 2b + c &= 3 \end{aligned}$$

After performing Gaussian elimination we obtain

$$\begin{aligned} a + \quad b + \quad c &= 2 \\ -2b \qquad &= 4 \\ -3c &= -9 \end{aligned}$$

which shows that the solution is $f(x) = 1x^2 - 2x + 3$.

1.2.5 The following system with more unknowns than equations

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

has no solution.

1.2.6 After performing Gaussian elimination the system becomes

$$\begin{aligned} x - y &= 1 \\ 0 &= -3 + k \end{aligned}$$

This system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

1.2.7

a. Possible if $a = \pm 1$ and $a \neq \pm b$.

b. Not possible.

c. Possible if $a \neq \pm 1$ or $a = \pm b$.

1.2.8 Consistent if $b_3 - b_2 - b_1 = 0$ and $b_4 - 2b_2 - b_1 = 0$.

1.2.9 If $a \neq 0$ then the solution set of the first equation is $\{\,(x, y) \mid x = (c - by)/a\,\}$. Taking $y = 0$ gives the solution $(c/a, 0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a) + d \cdot 0 = e$, so $c = e$. Then taking $y = 1$ in $x = (c - by)/a$ gives that $a((c - b)/a) + d \cdot 1 = e$, which gives that $b = d$. Hence they are the same equation.
When $a = 0$ the equations can be different and still have the same solution set: e.g., $0x + 3y = 6$ and $0x + 6y = 12$.

1.2.10 We take three cases: that $a \neq 0$, that $a = 0$ and $c \neq 0$, and that both $a = 0$ and $c = 0$.
For the first, we assume that $a \neq 0$. Then Gaussian elimination

$$\begin{aligned} ax + \qquad\qquad by &= j \\ (-(cb/a) + d)y &= -(cj/a) + k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a) + d \neq 0$; remember that $a \neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a) + d = (ad - bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad - bc \neq 0$.
In the second case, if $a = 0$ but $c \neq 0$, then we swap

$$\begin{aligned} cx + dy &= k \\ by &= j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b \neq 0$ (we use the case assumption that $c \neq 0$ to get a unique x in back substitution). But where $a = 0$ and $c \neq 0$ the condition "$b \neq 0$" is equivalent to the condition "$ad - bc \neq 0$". That finishes the second case.
Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x + by &= j \\ 0x + dy &= k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x, y) will do), but it never has a unique solution. Note that $a = 0$ and $c = 0$ gives that $ad - bc = 0$.

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2.1.1

a. $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$

c. Not possible, since dimension of A and E are not the same.

d. $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$

f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D.

g. $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$

h. $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.1.2 Disprove: Show that it is impossible to obtain a nonzero matrix.

2.1.3 Hint: Apply the definition of an idempotent matrix.

2.2.1 $A = \begin{bmatrix} -\frac{3}{4} & 3 \\ 1 & -\frac{3}{4} \end{bmatrix}$

2.2.2 $A = \begin{bmatrix} 0 & -1 \\ -11 & -\frac{17}{2} \end{bmatrix}$

2.2.3

a. $A = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

b. $X = \begin{bmatrix} -\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & \frac{3}{2} & -4 & 2 \end{bmatrix}$

2.2.4 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.2.5 Hint: Use the definition of the inverse of a matrix.

2.2.6 Hint: Apply the definition of symmetric matrices.

2.3.1

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.2

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.3 $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note: The answer is not unique.

2.4.1

a. $A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

b. $x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$

3.1.1 $\lambda = \frac{3 \pm \sqrt{33}}{4}$

3.2.1 $\det(A) = -\frac{5}{12}$

3.3.1 Hint: Apply the determinant to both sides $AB = -BA$.

3.4.1 $x_1 = 4$

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$.

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5.1.1

- **a**. $0 + 0x + 0x^2 + 0x^3$
- **b**. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- **c**. The constant function $f(x) = 0$
- **d**. The constant function $f(n) = 0$

5.1.2

- **a**. $3 + 2x - x^2$
- **b**. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$
- **c**. $-3e^x + 2e^{-x}$

5.1.3

- **a**. $1 + 2x$, $2 - 1x$, and x.
- **b**. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

- **a**.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **b**.

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

- **a**. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.
- **b**. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

5.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

- **a**. It is not closed under addition.

$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

- **b**. It is not closed under addition.

$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

- **c**. It is not closed under addition.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

- **d**. It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

- **e**. The set is empty, violating the existance of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

- **a**. It is not a vector space.

$$(1+1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- **b**. It is not a vector space.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

- **a**. Yes.
- **b**. Yes.
- **c**. No, this set is not closed under the natural addition operation. The vector of all 1/4's is an element of this set but when added to itself the result, the vector of all 1/2's, is not an element of the set.
- **d**. Yes.
- **e**. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11

- **a**. Closed under vector addition. Hint: Apply determinant properties.
- **b**. $\vec{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$
- **c**. Every $A \in V$ has an additive inverse A^{-1}.
- **d**. Yes.
- **e**. Not closed under scalar multiplication. Since $0\vec{0} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin V$

5.1.12 Check all 10 conditions of the definition of a vector space.

5.1.13 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.14

- **a**. No since $1 \cdot (0,\ 1) + 1 \cdot (0,\ 1) \neq (1+1) \cdot (0,\ 1)$.
- **b**. No since the same calculation as the prior part shows a

condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0,\ 1) \neq (0,\ 1)$.

5.1.15

a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v}+\vec{v}+\vec{s} = -\vec{v}+\vec{v}+\vec{t}$, which implies that $\vec{0}+\vec{s} = \vec{0}+\vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.16

Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.17

It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.18

Each element of a vector space has one and only one additive inverse.

For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.19

Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.

b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.

c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

5.2.1

a. Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of $\vec{v}, \vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v}+s_1\vec{w})+(r_2\vec{v}+s_2\vec{w}) = (r_1+r_2)\vec{v}+(s_1+s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.

b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.2 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r\, f'$.

5.3.1 Hint: For each subspace determine a set of vectors that spans it.
$W_1 \subsetneq W_2$

5.4.1

a. $\lambda = 1$

b. $\lambda \neq -1,\ -\frac{1}{2},\ 1$

5.5.1

a. $B = \left\{ 1 + x^3,\ x^2 + x^3 \right\}$

b. $(p(x))_B = (-2, 2)$

5.6.1 $\left\{ -1 + x^2,\ -x + x^3 \right\}$ is a basis of W, therefore W is of dimension 2.

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