
CEGEP Linear Algebra Problems

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CEGEP LEVEL LINEAR ALGEBRA PROBLEMS

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Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 [JH] In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy -plane. By geometrical reasoning, show that there are three possibilities: there is a unique solution, there is no solution, and there are infinitely many solutions.

1.1.2 [JH] Is there a two-unknowns linear system whose solution set is all of \mathbb{R}^2 ?

1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [JH] Use Gauss's Method to find the unique solution for each system.

a.

$$\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$$

b.

$$\begin{aligned} x - z &= 0 \\ 3x + y &= 1 \\ -x + y + z &= 4 \end{aligned}$$

1.2.2 [YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 + x_5 &= 3 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 + 17x_3 - 2x_4 - x_5 &= 1 \end{aligned}$$

- Solve the following system by Gauss-Jordan elimination.
- Find two particular solution to the above system.
- Find a solution to the above system when $x_3 = 1$.

1.2.3 [YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 &= 0 \\ 4x_1 + 17x_3 - 2x_4 &= 0 \\ 9x_1 + 6x_2 + 27x_3 - 4x_4 &= 0 \end{aligned}$$

- Solve the system by Gauss-Jordan elimination.
- Find two particular nontrivial solution to the system.
- Find a solution to the system when $x_1 = 1$.

1.2.4 [JH] Find the coefficients a , b , and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.2.5 [JH] True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.2.6 [JH] For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

1.2.7 [YL] Given the augmented matrix of a linear system:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & \pi & \\ 0 & \sqrt{2} & 4 & 5 & 6 & \\ 0 & 0 & 0 & a^2 - 1 & b^2 - a^2 & \end{array} \right]$$

If possible for what values of a and b the system has

- no solution? Justify.
- exactly one solution? Justify.
- infinitely many solutions? Justify.

1.2.8 [YL] Given the augmented matrix of a linear system

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & -4 & b_1 & \\ 3 & -2 & 4 & 5 & b_2 & \\ 4 & 1 & 5 & 1 & b_3 & \\ 7 & -1 & 9 & 6 & b_4 & \end{array} \right]$$

Determine the restrictions on the b_i 's for the system to be consistent.

1.3.1 Place Holder

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1.2.9 [JH] Prove that, where a, b, \dots, e are real numbers and $a \neq 0$, if

$$ax + by = c$$

has the same solution set as

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

1.2.10 [JH] Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

1.3 Applications of Linear Systems

Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and 2.2 Matrix Inverses and Algebraic Properties

2.1.1 [HE] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix}, \quad \text{where}$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

- $3C - 4D$
- $A - (D + 2C)$
- $A - E$
- AE
- $3BC - 4BD$
- $CB + D$
- GC
- FG
- Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A , B , and C are matrices above.

2.1.2 [YL] A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$.

Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

2.1.3 [YL] A square matrix A is called *idempotent* if $A^2 = A$.

Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A .

2.2.1 [YL] Solve of A given that it satisfies

$$(I - A^T)^{-1} = (\text{tr}(B)B^2)^T$$

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.2.2 [YL] Solve of X given that it satisfies

$$DXD^T = \text{tr}(BC)BC$$

where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}.$$

2.2.3 [YL] Given

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & \frac{1}{2} \end{bmatrix}.$$

- Find A^{-1} .
- Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2.2.4 [YL] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.2.5 [YL] Prove: If AB and BA are both invertible then A and B are both invertible.

2.2.6 [YL] Prove: If B and C are $n \times n$ matrices such that $A = B^T C + C^T B$ is invertible then A^{-1} is symmetric.

2.3 Elementary Matrices

2.3.1 [YL] Write the given matrix as a product of elementary matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

2.3.2 [YL] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.3.3 [YL] Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3 E_2 E_1 A = B$.

2.4 Linear Systems and Matrices

2.4.1 [YL] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

- Find A^{-1} .
- Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Chapter 3

Determinants

3.1 The Laplace Expansion

3.1.1 [YL] Solve for λ .

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1 - \lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

3.2 Determinants and Elementary Operations

3.2.1 [YL] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a + 2d & b + 2e & c + 2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.3 Properties of Determinants

3.3.1 [YL] Let A and B be $n \times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.4 Applications of the Determinant

3.4.1 [YL] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in \mathbb{R}^n* : If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$.

4.3 Cross Product and Planes

4.3.1 Place Holder

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4.4 Areas, Volumes and Distances

4.4.1 Place Holder

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4.5 Geometry of Solutions of Linear Systems

4.5.1 Place Holder

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [JH] Name the zero vector for each of these vector spaces.

- The space of degree three polynomials under the natural operations.
- The space of 2×3 matrices.
- The space $\{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$.
- The space of real-valued functions of one natural number variable.

5.1.2 [JH] Find the additive inverse, in the vector space, of the vector.

- In \mathcal{P}_3 , the vector $-3 - 2x + x^2$.
- In the space $\mathcal{M}_{2 \times 2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

- In $\{ae^x + be^{-x} \mid a, b \in \mathbb{R}\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [JH] For each, list three elements and then show it is a vector space.

- The set of linear polynomials $\mathcal{P}_1 = \{a_0 + a_1x \mid a_0, a_1 \in \mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations.
- The set of linear polynomials $\{a_0 + a_1x \mid a_0 - 2a_1 = 0\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [JH] For each, list three elements and then show it is a vector space.

- The set of 2×2 matrices with real entries under the usual matrix operations.
- The set of 2×2 matrices with real entries where the 2, 1 entry is zero, under the usual matrix operations.

5.1.5 [JH] For each, list three elements and then show it is a vector space.

- The set of three-component row vectors with their

usual operations.

- The set

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$$

under the operations inherited from \mathbb{R}^4 .

5.1.6 [JH] Show that the following are not vector spaces.

- Under the operations inherited from \mathbb{R}^3 , this set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$$

- Under the operations inherited from \mathbb{R}^3 , this set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

- Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

- Under the usual polynomial operations,

$$\{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+\}$$

where \mathbb{R}^+ is the set of reals greater than zero

- Under the inherited operations,

$$\{(x, y) \in \mathbb{R}^2 \mid x + 3y = 4, 2x - y = 3 \text{ and } 6x + 4y = 10\}$$

5.1.7 [JH] Is the set of rational numbers a vector space over \mathbb{R} under the usual addition and scalar multiplication operations?

5.1.8 [JH] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \quad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [JH] Prove or disprove that \mathbb{R}^3 is a vector space under these operations.

$$\begin{aligned} \text{a. } \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix} \\ \text{b. } \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

5.1.10 [JH] For each, decide if it is a vector space; the intended operations are the natural ones.

- a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

- b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

- c. $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y + w = 1\}$
 d. The set of functions $\{f: \mathbb{R} \rightarrow \mathbb{R} \mid df/dx + 2f = 0\}$
 e. The set of functions $\{f: \mathbb{R} \rightarrow \mathbb{R} \mid df/dx + 2f = 1\}$

5.1.11 [YL] Let $V = \{A \mid A \in \mathcal{M}_{2 \times 2} \text{ and } \det(A) \neq 0\}$ with the following operations:

$$A + B = AB \text{ and } kA = kA$$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.

- a. Does V satisfy closure under vector addition? Justify.
 b. Does V contain a zero vector? If so find it. Justify.
 c. Does V contains an additive inverse for all of its vectors? Justify.
 d. Does V satisfy closure under scalar multiplication? Justify.

5.1.12 [JH] Show that the set \mathbb{R}^+ of positive reals is a vector space when we interpret ‘ $x + y$ ’ to mean the product of x and y (so that $2 + 3$ is 6), and we interpret ‘ $r \cdot x$ ’ as the r -th power of x .

5.1.13 [JH] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.14 [JH]

Is $\{(x, y) \mid x, y \in \mathbb{R}\}$ a vector space under these operations?

- a. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$
 b. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, 0)$

5.1.15 [JH]

Prove the following:

- a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of \vec{v} , then \vec{v} is an additive inverse of \vec{w} . So a vector is an additive inverse of any additive inverse of itself.
 b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.16 [JH]

The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.17 [JH]

Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.18 [JH]

In a vector space every element has an additive inverse. Is the additive inverse unique (*Can some elements have two or more*)?

5.1.19 [JH]

Assume that $\vec{v} \in V$ is not $\vec{0}$.

- a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.
 b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.
 c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [JH]

- a. Prove that every point, line, or plane thru the origin in \mathbb{R}^3 is a subspace of \mathbb{R}^3 under the inherited operations.
 b. What if it doesn't contain the origin?

5.2.2 [JH] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.3 Spanning Sets

5.3.1 [YL] Given the following two subspace of \mathbb{R}^3 : $W_1 = \{x \mid A_1 x = 0\}$ and $W_2 = \{x \mid A_2 x = 0\}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.4 Linear Independence

5.4.1 [YL] Let $\vec{u} = (1, \lambda, -\lambda)$, $\vec{v} = (-2\lambda \ -2 \ 2\lambda)$ and $\vec{w} = (\lambda - 2, -5\lambda - 2, -2)$.

- For what value(s) of λ will $\{\vec{u}, \vec{v}\}$ be linearly dependent.
- For what value(s) of λ will $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent.

5.5 Basis

5.5.1 [YL] Given

$$W = \{p(x) = a_0 + a_2x^2 + a_3x^3 \mid p(-1) = 0\}$$

a subspace of \mathcal{P}_3 .

- Find a basis B for W .
- Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B .

5.6 Dimension

5.6.1 [YL] Given

$$W = \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mid p(1) = 0 \text{ and } p(-1) = 0\}$$

a subspace of \mathcal{P}_3 . Determine the dimension of W .

Answers to Exercises

1.1.1 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line.

Thus the lines can share one point (giving a unique solution), share no points (giving no solutions), or share at least two points (which makes them the same line).

1.1.2 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

1.2.1

- a. $x = 2, y = 3$
- b. $x = -1, y = 4$, and $z = -1$.

1.2.2

- a. $(x_1, x_2, x_3, x_4, x_5) = (60s - 55t + 30, -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3}, -14s + 13t - 7, s, t)$ where $s, t \in \mathbb{R}$.
- b. If $s = t = 0$ then $(x_1, x_2, x_3, x_4, x_5) = (30, -\frac{38}{3}, -7, 0, 0)$.
If $s = 0$ and $t = 1$ then $(x_1, x_2, x_3, x_4, x_5) = (-25, \frac{35}{3}, 6, 0, 1)$.
- c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1, x_2, x_3, x_4, x_5) = (-\frac{30}{7}, \frac{316}{21}, 1, \frac{4}{7}, 0)$.

1.2.3

- a. $(x_1, x_2, x_3, x_4) = (60t, -\frac{79}{3}t, -14t, t)$ where $t \in \mathbb{R}$.
- b. If $t = 1$ then $(x_1, x_2, x_3, x_4) = (60, -\frac{79}{3}, -14, 1)$.
If $t = 3$ then $(x_1, x_2, x_3, x_4) = (180, -79, 42, 3)$.
- c. If $t = \frac{1}{60}$ then $(x_1, x_2, x_3, x_4) = (1, -\frac{79}{180}, -\frac{14}{60}, \frac{1}{60})$.

1.2.4 Because $f(1) = 2$, $f(-1) = 6$, and $f(2) = 3$ we get a linear system.

$$\begin{aligned} 1a + 1b + c &= 2 \\ 1a - 1b + c &= 6 \\ 4a + 2b + c &= 3 \end{aligned}$$

After performing Gaussian elimination we obtain

$$\begin{aligned} a + b + c &= 2 \\ -2b &= 4 \\ -3c &= -9 \end{aligned}$$

which shows that the solution is $f(x) = 1x^2 - 2x + 3$.

1.2.5 The following system with more unknowns than equations

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

has no solution.

1.2.6 After performing Gaussian elimination the system becomes

$$\begin{aligned} x - y &= 1 \\ 0 &= -3 + k \end{aligned}$$

This system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

1.2.7

- a. Possible if $a = \pm 1$ and $a \neq \pm b$.
- b. Not possible.
- c. Possible if $a \neq \pm 1$ or $a = \pm b$.

1.2.8 Consistent if $b_3 - b_2 - b_1 = 0$ and $b_4 - 2b_2 - b_1 = 0$.

1.2.9 If $a \neq 0$ then the solution set of the first equation is $\{(x, y) \mid x = (c - by)/a\}$. Taking $y = 0$ gives the solution $(c/a, 0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a) + d \cdot 0 = e$, so $c = e$. Then taking $y = 1$ in $x = (c - by)/a$ gives that $a((c - b)/a) + d \cdot 1 = e$, which gives that $b = d$. Hence they are the same equation.

When $a = 0$ the equations can be different and still have the same solution set: e.g., $0x + 3y = 6$ and $0x + 6y = 12$.

1.2.10 We take three cases: that $a \neq 0$, that $a = 0$ and $c \neq 0$, and that both $a = 0$ and $c = 0$.

For the first, we assume that $a \neq 0$. Then Gaussian elimination

$$\begin{aligned} ax + by &= j \\ -(cb/a) + d)y &= -(cj/a) + k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a) + d \neq 0$; remember that $a \neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a) + d = (ad - bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad - bc \neq 0$.

In the second case, if $a = 0$ but $c \neq 0$, then we swap

$$\begin{aligned} cx + dy &= k \\ by &= j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b \neq 0$ (we use the case assumption that $c \neq 0$ to get a unique x in back substitution). But where $a = 0$ and $c \neq 0$ the condition “ $b \neq 0$ ” is equivalent to the condition “ $ad - bc \neq 0$ ”. That finishes the second case.

Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x + by &= j \\ 0x + dy &= k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x, y) will do), but it never has a unique solution. Note that $a = 0$ and $c = 0$ gives that $ad - bc = 0$.

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2.1.1

- $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$
- $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$
- Not possible, since dimension of A and E are not the same.
- $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$
- $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$
- Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D .
- $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$
- $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.1.2 Disprove: Show that it is impossible to obtain a nonzero matrix.

2.1.3 Hint: Apply the definition of an idempotent matrix.

$$\mathbf{2.2.1} \quad A = \begin{bmatrix} -\frac{3}{4} & 3 \\ 1 & -\frac{3}{4} \end{bmatrix}$$

$$\mathbf{2.2.2} \quad A = \begin{bmatrix} 0 & -1 \\ -11 & -\frac{17}{2} \end{bmatrix}$$

2.2.3

$$\begin{aligned} \text{a. } A &= \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix} \\ \text{b. } X &= \begin{bmatrix} -\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & \frac{3}{2} & -4 & 2 \end{bmatrix} \end{aligned}$$

2.2.4 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.2.5 Hint: Use the definition of the inverse of a matrix.

2.2.6 Hint: Apply the definition of symmetric matrices.

2.3.1

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.2

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

$$\mathbf{2.3.3} \quad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.4.1

$$\begin{aligned} \text{a. } A^{-1} &= \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \\ \text{b. } x &= \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix} \end{aligned}$$

$$\mathbf{3.1.1} \quad \lambda = \frac{3 \pm \sqrt{33}}{4}$$

$$\mathbf{3.2.1} \quad \det(A) = -\frac{5}{12}$$

3.3.1 Hint: Apply the determinant to both sides $AB = -BA$.

$$\mathbf{3.4.1} \quad x_1 = 4$$

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}$.

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5.1.1

- $0 + 0x + 0x^2 + 0x^3$
- $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- The constant function $f(x) = 0$
- The constant function $f(n) = 0$

5.1.2

- $3 + 2x - x^2$
- $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$
- $-3e^x + 2e^{-x}$

5.1.3

- $1 + 2x$, $2 - 1x$, and x .
- $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

-

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

-

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

- $(1, 2, 3)$, $(2, 1, 3)$, and $(0, 0, 0)$.
- $(1, 1, 1, -1)$, $(1, 0, 1, 0)$ and $(0, 0, 0, 0)$.

5.1.6

For each part the set is called Q . For some parts, there are more than one correct way to show that Q is not a vector space.

- It is not closed under addition.

$$(1, 0, 0), (0, 1, 0) \in Q \quad (1, 1, 0) \notin Q$$

- It is not closed under addition.

$$(1, 0, 0), (0, 1, 0) \in Q \quad (1, 1, 0) \notin Q$$

- It is not closed under addition.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

- It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \quad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

- The set is empty, violating the existence of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The ‘+’ operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

- It is not a vector space.

$$(1 + 1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- It is not a vector space.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each “yes” answer, you must give a check of all the conditions given in the definition of a vector space. For each “no” answer, give a specific example of the failure of one of the conditions.

- Yes.
- Yes.

- c. No, this set is not closed under the natural addition operation. The vector of all $1/4$'s is an element of this set but when added to itself the result, the vector of all $1/2$'s, is not an element of the set.
- d. Yes.
- e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11

- a. Closed under vector addition. Hint: Apply determinant properties.
- b. $\vec{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$
- c. Every $A \in V$ has an additive inverse A^{-1} .
- d. Yes.
- e. Not closed under scalar multiplication. Since $0\vec{0} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin V$

5.1.12 Check all 10 conditions of the definition of a vector space.

5.1.13 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.14

- a. No since $1 \cdot (0, 1) + 1 \cdot (0, 1) \neq (1 + 1) \cdot (0, 1)$.
- b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0, 1) \neq (0, 1)$.

5.1.15

- a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of \vec{v} so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore \vec{v} is also the additive inverse of \vec{w} .
- b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of \vec{v} is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which implies that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.16

Addition is commutative, so in any vector space, for any vector \vec{v} we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.17

It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.18

Each element of a vector space has one and only one additive inverse.

For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of \vec{v} then consider $\vec{w}_1 +$

$\vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.19

Assume that $\vec{v} \in V$ is not $\vec{0}$.

- a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.
- b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.
- c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

5.2.1

- a. Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of \vec{v}, \vec{w} may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.
- b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.2 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r f'$.

5.3.1 Hint: For each subspace determine a set of vectors that spans it.

$W_1 \subsetneq W_2$

5.4.1

- a. $\lambda = 1$
- b. $\lambda \neq -1, -\frac{1}{2}, 1$

5.5.1

- a. $B = \{1 + x^3, x^2 + x^3\}$
- b. $(p(x))_B = (-2, 2)$

5.6.1 $\{-1 + x^2, -x + x^3\}$ is a basis of W , therefore W is of dimension 2.

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