CEGEP Linear Algebra Problems

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CEGEP level linear algebra problems

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Contents

Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 [**GH**] State which of the following equations is a linear equation. If it is not, state why.

a. $x + y + z = 10$
b. $xy + yz + xz = 1$
c. $-3x + 9 = 3y - 5z + x - 7$
d. $\sqrt{5}y + \pi x = -1$
e. $(x-1)(x+1) = 0$
f. $\sqrt{x_1^2 + x_2^2} = 25$
g. $x_1 + y + t = 1$
h. $\frac{1}{x} + 9 = 3\cos(y) - 5z$
i. $\cos(15)y + \frac{x}{4} = -1$
j. $2^x + 2^y = 16$

1.1.2 [**GH**] Solve the system of linear equations using substitution, comparison and/or elimination.

a. $\begin{aligned} x + y &= -1 \\ 2x - 3y &= 8 \end{aligned}$

b. $\begin{aligned} 2x - 3y &= 3 \\ 3x + 6y &= 8 \end{aligned}$

c. $\begin{aligned} x - y + z &= 1 \\ 2x + 6y - z &= -4 \\ 4x - 5y + 2z &= 0 \end{aligned}$

d. $\begin{aligned} x + y - z &= 1 \\ 2x + y &= 2 \\ y + 2z &= 0 \end{aligned}$

1.1.3 [**GH**] Convert the given system of linear equations into an augmented matrix.

a. $\begin{aligned} 3x + 4y + 5z &= 7 \\ -x + y - 3z &= 1 \\ 2x - 2y + 3z &= 5 \end{aligned}$

b. $\begin{aligned} 2x + 5y - 6z &= 2 \\ 9x - 8z &= 10 \\ -2x + 4y + z &= -7 \end{aligned}$

c. $\begin{aligned} x_1 + 3x_2 - 4x_3 + 5x_4 &= 17 \\ -x_1 + 4x_3 + 8x_4 &= 1 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 &= 6 \end{aligned}$

d. $\begin{aligned} 3x_1 - 2x_2 &= 4 \\ 2x_1 &= 3 \\ -x_1 + 9x_2 &= 8 \\ 5x_1 - 7x_2 &= 13 \end{aligned}$

1.1.4 [**GH**] Convert given augmented matrix into a system of linear equations. Use the variables x_1, x_2, $\ldots$.

a. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array}\right]$

b. $\left[\begin{array}{cc|c} -3 & 4 & 7 \\ 0 & 1 & -2 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & 5 & 7 \end{array}\right]$

d. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

e. $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 7 & 2 \\ 0 & 1 & 3 & 2 & 0 & 5 \end{array}\right]$

1.1.5 [**GH**] Perform the given row operations on

$$\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}.$$

a. $-1R_1 \to R_1$
b. $R_2 \leftrightarrow R_3$
c. $R_1 + R_2 \to R_2$
d. $2R_2 + R_3 \to R_3$
e. $\frac{1}{2}R_2 \to R_2$
f. $-\frac{5}{2}R_1 + R_3 \to R_3$

1.1.6 [**GH**] Give the row operation that transforms A into B where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

a. $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

b. $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $B = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

d. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

e. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

1.1.7 [**JH**] In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy-plane. By geometrical reasoning, show that there are three possibilities:

there is a unique solution, there is no solution, and there are infinitely many solutions.

1.1.8 [JH] Is there a two-unknowns linear system whose solution set is all of $\mathbb{R}^2$?

1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [GH] State whether or not the given matrices are in reduced row echelon form.

a. $\begin{bmatrix}1&0\\0&1\end{bmatrix}$

b. $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

c. $\begin{bmatrix}1&1\\1&1\end{bmatrix}$

d. $\begin{bmatrix}1&0&1\\0&1&2\end{bmatrix}$

e. $\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}$

f. $\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}$

g. $\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}$

h. $\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$

i. $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

j. $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$

k. $\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$

l. $\begin{bmatrix}1&0&0&-5\\0&1&0&7\\0&0&1&3\end{bmatrix}$

m. $\begin{bmatrix}2&0&0&2\\0&2&0&2\\0&0&2&2\end{bmatrix}$

n. $\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$

o. $\begin{bmatrix}0&0&1&-5\\0&0&0&0\\0&0&0&0\end{bmatrix}$

1.2.2 [GH] Use Gauss-Jordan Elimination to put the given matrix into reduced row echelon form.

a. $\begin{bmatrix}1&2\\-3&-5\end{bmatrix}$

b. $\begin{bmatrix}2&-2\\3&-2\end{bmatrix}$

c. $\begin{bmatrix}4&12\\-2&-6\end{bmatrix}$

d. $\begin{bmatrix}-5&7\\10&14\end{bmatrix}$

e. $\begin{bmatrix}-1&1&4\\-2&1&1\end{bmatrix}$

f. $\begin{bmatrix}7&2&3\\3&1&2\end{bmatrix}$

g. $\begin{bmatrix}3&-3&6\\-1&1&-2\end{bmatrix}$

h. $\begin{bmatrix}4&5&-6\\-12&-15&18\end{bmatrix}$

i. $\begin{bmatrix}-2&-4&-8\\-2&-3&-5\\2&3&6\end{bmatrix}$

j. $\begin{bmatrix}2&1&1\\1&1&1\\2&1&2\end{bmatrix}$

k. $\begin{bmatrix}1&2&1\\1&3&1\\-1&-3&0\end{bmatrix}$

l. $\begin{bmatrix}1&2&3\\0&4&5\\1&6&9\end{bmatrix}$

m. $\begin{bmatrix}1&1&1&2\\2&-1&-1&1\\-1&1&1&0\end{bmatrix}$

n. $\begin{bmatrix}2&-1&1&5\\3&1&6&-1\\3&0&5&0\end{bmatrix}$

o. $\begin{bmatrix}1&1&-1&7\\2&1&0&10\\3&2&-1&17\end{bmatrix}$

p. $\begin{bmatrix}4&1&8&15\\1&1&2&7\\3&1&5&11\end{bmatrix}$

1.2.3 [JH] Use Gauss's Method to find the unique solution for each system.

a. $\begin{aligned}2x+3y&=13\\x-y&=-1\end{aligned}$

b. $\begin{aligned}x-z&=0\\3x+y&=1\\-x+y+z&=4\end{aligned}$

1.2.4 [GH] Find the solution to the given linear system. If the system has infinite solutions, give two particular solutions.

a. $\begin{aligned} 2x_1+4x_2&=2\\ x_1+2x_2&=1\end{aligned}$

b. $\begin{aligned} -x_1+5x_2&=3\\ 2x_1-10x_2&=-6\end{aligned}$

c. $\begin{aligned} x_1+x_2&=3\\ 2x_1+x_2&=4\end{aligned}$

d. $\begin{aligned} -3x_1+7x_2&=-7\\ 2x_1-8x_2&=8\end{aligned}$

e. $\begin{aligned} -2x_1+4x_2+4x_3&=6\\ x_1-3x_2+2x_3&=1\end{aligned}$

f. $\begin{aligned} -x_1+2x_2+2x_3&=2\\ 2x_1+5x_2+x_3&=2\end{aligned}$

g. $\begin{aligned} -x_1-x_2+x_3+x_4&=0\\ -2x_1-2x_2+x_3&=-1\end{aligned}$

h. $\begin{aligned} x_1+x_2+6x_3+9x_4&=0\\ x_1+x_3+2x_4&=3\end{aligned}$

i. $\begin{aligned} x_1+2x_2+2x_3&=1\\ 2x_1+x_2+3x_3&=1\\ 3x_1+3x_2+5x_3&=2\end{aligned}$

j. $\begin{aligned} 2x_1+4x_2+6x_3&=2\\ 1x_1+2x_2+3x_3&=1\\ 3x_1+6x_2+9x_3&=3\end{aligned}$

k. $\begin{aligned} 2x_1+3x_2&=1\\ -2x_1-3x_2&=1\end{aligned}$

l. $\begin{aligned} 2x_1+x_2+2x_3&=0\\ x_1+x_2+3x_3&=1\\ 3x_1+2x_2+5x_3&=3\end{aligned}$

1.2.5 [**YL**] Given

$$\begin{aligned} 3x_1+3x_2+7x_3-3x_4+x_5&=3\\ 2x_1+3x_2+3x_3+x_4-2x_5&=1\\ 4x_1+17x_3-2x_4-x_5&=1\end{aligned}$$

a. Solve the following system by Gauss-Jordan elimination.

b. Find two particular solution to the above system.

c. Find a solution to the above system when $x_3=1$.

1.2.6 [**YL**] Given

$$\begin{aligned} 3x_1+3x_2+7x_3-3x_4&=0\\ 2x_1+3x_2+3x_3+x_4&=0\\ 4x_1+17x_3-2x_4&=0\\ 9x_1+6x_2+27x_3-4x_4&=0\end{aligned}$$

a. Solve the system by Gauss-Jordan elimination.

b. Find two particular nontrivial solution to the system.

c. Find a solution to the system when $x_1=1$.

1.2.7 [**JH**] Find the coefficients a, b, and c so that the graph of $f(x)=ax^2+bx+c$ passes through the points $(1,2)$, $(-1,6)$, and $(2,3)$.

1.2.8 [**JH**] True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.2.9 [**JH**] For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x-y&=1\\ 3x-3y&=k\end{aligned}$$

1.2.10 [**GH**] State for which values of k the given system will have exactly 1 solution, infinite solutions, or no solution.

a. $\begin{aligned} x_1+2x_2&=1\\ 2x_1+4x_2&=k\end{aligned}$

b. $\begin{aligned} x_1+2x_2&=1\\ x_1+kx_2&=1\end{aligned}$

c. $\begin{aligned} x_1+2x_2&=1\\ x_1+kx_2&=2\end{aligned}$

d. $\begin{aligned} x_1+2x_2&=1\\ x_1+3x_2&=k\end{aligned}$

1.2.11 [**YL**] Given the augmented matrix of a linear system:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \pi\\ 0 & \sqrt{2} & 4 & 5 & 6\\ 0 & 0 & 0 & a^2-1 & b^2-a^2\end{bmatrix}$$

If possible for what values of a and b the system has

a. no solution? Justify.

b. exactly one solution? Justify.

c. infinitely many solutions? Justify.

1.2.12 [**YL**] Given the augmented matrix of a linear system

$$\begin{bmatrix} 1 & 3 & 1 & -4 & b_1\\ 3 & -2 & 4 & 5 & b_2\\ 4 & 1 & 5 & 1 & b_3\\ 7 & -1 & 9 & 6 & b_4\end{bmatrix}.$$

Determine the restrictions on the b_i's for the system to be consistent.

1.2.13 [**JH**] Prove that, where $a,b,\dots,e$ are real numbers and $a\neq 0$, if

$$ax+by=c$$

has the same solution set as

$$ax+dy=e$$

then they are the same equation. What if $a=0$?

1.2.14 [**JH**] Show that if $ad-bc\neq 0$ then

$$\begin{aligned} ax+by&=j\\ cx+dy&=k\end{aligned}$$

has a unique solution.

1.3 Applications of Linear Systems

1.3.1 Place Holder

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [**JH**] Find the indicated entry of the following matrix.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{bmatrix}$$

a. $a_{2,1}$ **b.** $a_{1,2}$ **c.** $a_{2,2}$ **d.** $a_{3,1}$

2.1.2 [**JH**] Determine the size of each matrix.

a. $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{bmatrix}$ **c.** $\begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$

2.1.3 [**GH**] Simplify the given expression where

$$A = \begin{bmatrix} 1 & -1 \\ 7 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 \\ 5 & 9 \end{bmatrix}$$

a. $A + B$

b. $2A - 3B$

c. $3(A - B) + B$

d. $2(A - B) - (A - 3B)$

2.1.4 [**GH**] The row and column matrix U and V are defined. Find the product UV, where possible.

a. $U = \begin{bmatrix} 1 & -4 \end{bmatrix}$, $V = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

b. $U = \begin{bmatrix} 6 & 2 & -1 & 2 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \\ 9 \\ 5 \end{bmatrix}$

c. $U = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

d. $U = \begin{bmatrix} 2 & -5 \end{bmatrix}$, $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2.1.5 [**GH**] State the dimensions of A and B. State the dimensions of AB and BA, if the product is defined. Then compute the product AB and BA, if possible.

a. $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 7 \\ 4 & 2 & 9 \end{bmatrix}$

c. $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$

d. $A = \begin{bmatrix} -2 & -1 \\ 9 & -5 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 6 & -4 \\ 0 & 6 & -3 \end{bmatrix}$

e. $A = \begin{bmatrix} 2 & 6 \\ 6 & 2 \\ 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 5 & 0 \\ -4 & 4 & -4 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 4 \\ 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -5 & 5 \\ -2 & 1 & 3 & -5 \end{bmatrix}$

g. $A = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $A = \begin{bmatrix} -4 & -1 & 3 \\ 2 & -3 & 5 \\ 1 & 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 & 3 \\ -1 & 1 & -1 \\ 4 & 0 & 2 \end{bmatrix}$

2.1.6 [**GH**] Given a diagonal matrix D and a matrix A, compute the product DA and AD, if possible.

a. $D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}$, $A = \begin{bmatrix} & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

b. $D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

c. $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

d. $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

2.1.7 [**GH**] Given a matrix A compute A^2 and A^3.

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

d. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

e. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

2.1.8 [**HE**] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

a. $3C - 4D$

b. $A - (D + 2C)$

c. $A - E$

d. AE

e. $3BC - 4BD$

f. $CB + D$

g. GC

h. FG

i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A, B, and C are matrices above.

2.1.9 [**GH**] In each part a matrix A is given. Find A^T. State whether A is upper/lower triangular, diagonal, symmetric and/or skew symmetric.

a. $\begin{bmatrix} -9 & 4 & 10 \\ 6 & -3 & -7 \\ -8 & 1 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 2 & -9 \\ 5 & -4 & -10 \\ -6 & 6 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -7 & -4 & -9 \\ -9 & 6 & 3 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$

e. $\begin{bmatrix} 4 & 0 & 0 \\ -2 & -7 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

f. $\begin{bmatrix} -3 & -4 & -5 \\ 0 & -3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$

2.1.10 [**GH**] Find the trace of the given matrix.

a. $\begin{bmatrix} 4 & 1 & 1 \\ -2 & 0 & 0 \\ -1 & -2 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -5 \\ 9 & 5 \end{bmatrix}$

c. $\begin{bmatrix} -10 & 6 & -7 & -9 \\ -2 & 1 & 6 & -9 \\ 0 & 4 & -4 & 0 \\ -3 & -9 & 3 & -10 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 6 & 4 \\ -1 & 8 & -10 \end{bmatrix}$

e. Any skew-symmetric matrix.

f. I_n

2.1.11 [**GH**] Find values for the scalars a and b that satisfy the given equation.

a. $a \begin{bmatrix} -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

b. $a \begin{bmatrix} 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

c. $a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

d. $a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$

2.1.12 [**GH**] The following statement

$$(A + B)^2 = A^2 + 2AB + B^2$$

is false. We investigate that claim here.

a. Let $A = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} -5 & -5 \\ -2 & 1 \end{bmatrix}$. Compute $A + B$

b. Find $(A + B)^2$ by using the previous part.

c. Compute $A^2 + 2AB + B^2$.

d. Are the results from the two previous parts equal?

e. Carefully expand the expression $(A+B)^2 = (A+B)(A+B)$ and show why this is not equal to $A^2 + 2AB + B^2$.

2.1.13 [**YL**]

a. Prove: If A and B are $n \times n$ matrices then $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

b. Prove: If A and B are $n \times n$ matrices then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

2.1.14 [**YL**] A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$.
Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

2.1.15 [**YL**] A square matrix A is called *idempotent* if $A^2 = A$.
Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A.

2.2 Matrix Inverses and Algebraic Properties

2.2.1 [GH] Given the matrices A and B below. Find X that satisfies the equation.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}$$

a. $2A + X = B$

b. $A - X = 3B$

c. $3A + 2X = -1B$

d. $A - \frac{1}{2}X = -B$

2.2.2 [GH] Given the matrices A. Find A^{-1}, if possible.

a. $\begin{bmatrix} 1 & 5 \\ -5 & -24 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$

2.2.3 [GH] Given the matrices A and B. Compute $(AB)^{-1}$ and $B^{-1}A^{-1}$.

a. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 5 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 1 \\ 2 & 1 \end{bmatrix}$

2.2.4 [GH] Given the matrices A. Find A^{-1}, if possible.

a. $\begin{bmatrix} 25 & -10 & -4 \\ -18 & 7 & 3 \\ -6 & 2 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -7 \\ 20 & 7 & -48 \end{bmatrix}$

c. $\begin{bmatrix} -4 & 1 & 5 \\ -5 & 1 & 9 \\ -10 & 2 & 19 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -5 & 0 \\ -2 & 15 & 4 \\ 4 & -19 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 25 & -8 & 0 \\ -78 & 25 & 0 \\ 48 & -15 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & 8 \\ -2 & -2 & -3 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i. $\begin{bmatrix} 2 & 3 & 4 \\ -3 & 6 & 9 \\ -1 & 9 & 13 \end{bmatrix}$

j. $\begin{bmatrix} 5 & -1 & 0 \\ 7 & 7 & 1 \\ -2 & -8 & -1 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -19 & -9 & 0 & 4 \\ 33 & 4 & 1 & -7 \\ 4 & 2 & 0 & -1 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 27 & 1 & 0 & 4 \\ 18 & 0 & 1 & 4 \\ 4 & 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -29 & -110 \\ 0 & -3 & -5 & -19 \end{bmatrix}$

n. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

2.2.5 [GH] Prove or disprove: If A and B are 2×2 invertible matrices then $A+B$ is an invertible matrix.

2.2.6 [YL] Solve of A given that it satisfies

$$(I - A^T)^{-1} = (\text{tr}(B)B^2)^T$$

where

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.2.7 [YL] Solve of X given that it satisfies

$$DXD^T = \text{tr}(BC)BC$$

where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} \; C = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} \; D = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}.$$

2.2.8 [YL] Given

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & \frac{1}{2} \end{bmatrix}.$$

a. Find A^{-1}.

b. Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2.2.9 [YL] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.2.10 [YL] Prove: If AB and BA are both invertible then A and B are both invertible.

2.2.11 [YL] Prove: If B and C are $n\times n$ matrices such that $A = B^TC + C^TB$ is invertible then A^{-1} is symmetric.

2.3 Elementary Matrices

2.3.1 [**YL**] Write the given matrix as a product of elementary matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

2.3.2 [**YL**] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.3.3 [**YL**] Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3E_2E_1A = B$.

2.4 Linear Systems and Matrices

2.4.1 [**YL**] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1}.

b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

2.4.2 [**GH**] Given the matrices A and b below. Find x that satisfies the equation $Ax = b$ by using the inverse of A

a. $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$,
$b = \begin{bmatrix} 21 \\ 13 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -4 \\ 4 & -15 \end{bmatrix}$,
$b = \begin{bmatrix} 21 \\ 77 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 & 12 \\ 0 & 1 & 6 \\ -3 & 0 & 1 \end{bmatrix}$,
$b = \begin{bmatrix} -17 \\ -5 \\ 20 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 0 & -3 \\ 8 & -2 & -13 \\ 12 & -3 & -20 \end{bmatrix}$,
$b = \begin{bmatrix} -34 \\ -159 \\ -243 \end{bmatrix}$

Chapter 3

Determinants

3.1 The Laplace Expansion

3.1.1 [GH] Compute the determinant of the following matrices.

a. $\begin{bmatrix} 10 & 7 \\ 8 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 6 & -1 \\ -7 & 8 \end{bmatrix}$

c. $\begin{bmatrix} -1 & -7 \\ -5 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -10 & -1 \\ -4 & 7 \end{bmatrix}$

3.1.2 [GH] For the following matrices, construct the submatrices used to compute the minors $M_{1,1}$, $M_{1,2}$ and $M_{1,3}$. Compute the cofactors $C_{1,1}$, $C_{1,2}$, and $C_{1,3}$.

a. $\begin{bmatrix} 7 & -3 & 10 \\ 3 & 7 & 6 \\ 1 & 6 & 10 \end{bmatrix}$

b. $\begin{bmatrix} -2 & -9 & 6 \\ -10 & -6 & 8 \\ 0 & -3 & -2 \end{bmatrix}$

c. $\begin{bmatrix} -5 & -3 & 3 \\ -3 & 3 & 10 \\ -9 & 3 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -6 & -4 & 6 \\ -8 & 0 & 0 \\ -10 & 8 & -1 \end{bmatrix}$

3.1.3 [JH] Evaluate the determinant by performing a cofactor expansion

$$\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 0 \end{vmatrix}$$

a. along the first row,

b. along the second row,

c. along the third column.

3.1.4 [GH] Find the determinant of the given matrix using cofactor expansion.

a. $\begin{bmatrix} 3 & 2 & 3 \\ -6 & 1 & -10 \\ -8 & -9 & -9 \end{bmatrix}$

b. $\begin{bmatrix} 8 & -9 & -2 \\ -9 & 9 & -7 \\ 5 & -1 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -1 & 0 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

f. $\begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix}$

3.1.5 [JH] Verify that the determinant of an upper-triangular 3×3 matrix is the product of the main diagonal.

$$\det\left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}\right) = aei$$

Is it the same for lower triangular matrices?

3.1.6 [YL] Solve for λ.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda-5 \end{vmatrix}$$

3.1.7 [JH] True or false: Can we compute a determinant by expanding down the diagonal? Justify.

3.1.8 [JH] Which real numbers θ make

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

equal to zero?

3.2 Determinants and Elementary Operations

3.2.1 [GH] A matrix M and $\det(M)$ are given. Matrices A, B and C are obtained by performing operations on M. Determine the determinants of A, B and C and indicate the operations used to obtain A, B and C.

a. $M = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}$, $\det(M) = -41$,

$$A = \begin{bmatrix} 18 & 14 & 16 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 96 & 73 & 83 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & 1 & 6 \\ 7 & 3 & 3 \\ 8 & 7 & 3 \end{bmatrix}.$$

b. $M = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}$, $\det(M) = 45$,

$$A = \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & -1 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ 8 & 16 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}.$$

c. $M = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, $\det(M) = -16$,

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 5 & 1 & 5 \\ 4 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & -1 & -5 \\ -4 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 15 & 3 & 15 \\ 12 & 0 & 6 \\ 0 & 0 & 12 \end{bmatrix}.$$

d. $M = \begin{bmatrix} 5 & 4 & 0 \\ 7 & 9 & 3 \\ 1 & 3 & 9 \end{bmatrix}$, $\det(M) = 120$,

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 7 & 9 & 3 \\ 5 & 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 4 & 0 \\ 14 & 18 & 6 \\ 3 & 9 & 27 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & -4 & 0 \\ -7 & -9 & -3 \\ -1 & -3 & -9 \end{bmatrix}.$$

3.2.2 [GH] Find the determinant of the given matrix by using elemetary operations to bring the matrix under triangular form.

a.

b. $\begin{bmatrix} -4 & 3 & -4 \\ -4 & -5 & 3 \\ 3 & -4 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -2 & 1 \\ 5 & 5 & 4 \\ 4 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} -5 & 0 & -4 \\ 2 & 4 & -1 \\ -5 & 0 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

f. $\begin{bmatrix} -5 & 1 & 0 & 0 \\ -3 & -5 & 2 & 5 \\ -2 & 4 & -3 & 4 \\ 5 & 4 & -3 & 3 \end{bmatrix}$

g. $\begin{bmatrix} 2 & -1 & 4 & 4 \\ 3 & -3 & 3 & 2 \\ 0 & 4 & -5 & 1 \\ -2 & -5 & -2 & -5 \end{bmatrix}$

3.2.3 [YL] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.2.4 Vandermonde's determinant [JH] Prove:

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\right) = (b-a)(c-a)(c-b)$$

3.3 Properties of Determinants and Matrix Inverses

3.3.1 [**JH**] Find the adjoint of the following matrices.

a. $\begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 1 & 8 & 9 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

3.3.2 [**JH**]

a. Find a formula for the adjoint of a 2×2 matrix.

b. Use the above to derive the formula for the inverse of a 2×2 matrix.

3.3.3 [**JH**] Derive a formula for the adjoint of a diagonal matrix.

3.3.4 [**JH**] Prove that the transpose of the adjoint is the adjoint of the transpose.

3.3.5 [**JH**] Prove or disprove: $\operatorname{adj}(\operatorname{adj}(T)) = T$.

3.3.6 [**JH**] Which real numbers x make this matrix singular?

$$\begin{bmatrix} 12 - x & 4 \\ 8 & 8 - x \end{bmatrix}$$

3.3.7 [**JH**] Prove: If S and T are $n \times n$ matrix then $\det(TS) = \det(ST)$.

3.3.8 [**JH**] Prove that each statement holds for 2×2 matrices.

a. The determinant of a product is the product of the determinants $\det(ST) = \det(S)\det(T)$.

b. If T is invertible then the determinant of the inverse is the inverse of the determinant $\det(T^{-1}) = (\det(T))^{-1}$.

3.3.9 [**JH**]

a. Suppose that $\det(A) = 3$ and that $\det(B) = 2$. Find $\det(A^2 B^{\mathsf{T}} B^{-2} A^{\mathsf{T}})$.

b. If $\det(A) = 0$ then show that $\det(6A^3 + 5A^2 + 2A) = 0$.

3.3.10 [**JH**]

a. Give a non-identity matrix with the property that $A^{\mathsf{T}} = A^{-1}$.

b. Prove: If $A^{\mathsf{T}} = A^{-1}$ then $\det(A) = \pm 1$.

c. Does the converse to the above hold?

3.3.11 [**JH**] Two matrices H and G are said to be *similar* if there is a nonsingular matrix P such that $H = P^{-1}GP$ Show that similar matrices have the same determinant.

3.3.12 [**JH**] Show that this gives the equation of a line in $\mathbb{R}^2$ thru (x_2, y_2) and (x_3, y_3).

$$\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

3.3.13 [**YL**] Let A and B be $n \times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.3.14 [**JH**] Prove or disprove: The determinant is a linear function, that is $\det(x \cdot T + y \cdot S) = x \cdot \det(T) + y \cdot \det(S)$.

3.4 Applications of the Determinant

3.4.1 [**YL**] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

3.4.2 [**GH**] Given the matrices A and b, evaluate $\det(A)$ and $\det(A_i)$ for all i. Use Cramer's Rule to solve $Ax = b$. If Cramer's Rule cannot be used to find the solution, then state whether or not a solution exists.

a. $A = \begin{bmatrix} 3 & 0 & -3 \\ 5 & 4 & 4 \\ 5 & 5 & -4 \end{bmatrix}$

$b = \begin{bmatrix} 24 \\ 0 \\ 31 \end{bmatrix}$

b. $A = \begin{bmatrix} 9 & 5 \\ -4 & -7 \end{bmatrix}$

$b = \begin{bmatrix} -45 \\ 20 \end{bmatrix}$

c. $A = \begin{bmatrix} -8 & 16 \\ 10 & -20 \end{bmatrix}$

$b = \begin{bmatrix} -48 \\ 60 \end{bmatrix}$

d. $A = \begin{bmatrix} 7 & 14 \\ -2 & -4 \end{bmatrix}$

$b = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

e. $A = \begin{bmatrix} 4 & 9 & 3 \\ -5 & -2 & -13 \\ -1 & 10 & -13 \end{bmatrix}$

$b = \begin{bmatrix} -28 \\ 35 \\ 7 \end{bmatrix}$

f. $A = \begin{bmatrix} 7 & -4 & 25 \\ -2 & 1 & -7 \\ 9 & -7 & 34 \end{bmatrix}$

$b = \begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in* $\mathbb{R}^n$: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$.

4.3 Cross Product and Planes

4.3.1 Place Holder

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4.4 Areas, Volumes and Distances

4.4.1 Place Holder

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4.5 Geometry of Solutions of Linear Systems

4.5.1 Place Holder

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [**JH**] Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.

b. The space of 2×3 matrices.

c. The space $\{\, f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \,\}$.

d. The space of real-valued functions of one natural number variable.

5.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3 - 2x + x^2$.

b. In the space $\mathcal{M}_{2\times 2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

c. In $\{\, ae^x + be^{-x} \mid a, b \in \mathbb{R} \,\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{\, a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \,\}$ under the usual polynomial addition and scalar multiplication operations.

b. The set of linear polynomials $\{\, a_0 + a_1 x \mid a_0 - 2a_1 = 0 \,\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.

b. The set of 2×2 matrices with real entries where the $2, 1$ entry is zero, under the usual matrix operations.

5.1.5 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.

b. The set

$$\{\, (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y - z + w = 0 \,\}$$

under the operations inherited from $\mathbb{R}^4$.

5.1.6 [**JH**] Show that the following are not vector spaces.

a. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x + y + z = 1 \,\}$$

b. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \,\}$$

c. Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

d. Under the usual polynomial operations,

$$\{\, a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+ \,\}$$

where $\mathbb{R}^+$ is the set of reals greater than zero

e. Under the inherited operations,

$$\{\, (x,\ y) \in \mathbb{R}^2 \mid x + 3y = 4,\ 2x - y = 3 \text{ and } 6x + 4y = 10 \,\}$$

5.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

5.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

b. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5.1.10 [**JH**] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

c. $\left\{ (x,\ y,\ z,\ w) \in \mathbb{R}^4 \;\middle|\; x + y + w = 1 \right\}$

d. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 0 \}$

e. The set of functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 1 \}$

5.1.11 [**YL**] Let $V = \{ A \mid A \in \mathcal{M}_{2\times 2} \text{ and } \det(A) \neq 0 \}$ with the following operations:

$$A + B = AB \text{ and } kA = kA$$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.

a. Does V satisfy closure under vector addition? Justify.

b. Does V contain a zero vector? If so find it. Justify.

c. Does V contains an additive inverse for all of its vectors? Justify.

d. Does V satisfy closure under scalar multiplication? Justify.

5.1.12 [**JH**] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x + y$' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret '$r \cdot x$' as the r-th power of x.

5.1.13 [**JH**] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.14 [**JH**]
Is $\left\{ (x,\ y) \;\middle|\; x, y \in \mathbb{R} \right\}$ a vector space under these operations?

a. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$

b. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x,\ y) = (rx,\ 0)$

5.1.15 [**JH**]
Prove the following:

a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.16 [**JH**]
The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.17 [**JH**]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.18 [**JH**]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

5.1.19 [**JH**]
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.

b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [**JH**] Which of these subsets of the vector space of 2×2 matrices are subspaces under the inherited operations? Justify.

a. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

b. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 0 \right\}$

c. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 5 \right\}$

d. $\left\{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 0, c \in \mathbb{R} \right\}$

5.2.2 [**JH**] Is this a subspace of $\mathcal{P}_2$: $\left\{ a_0 + a_1 x + a_2 x^2 \;\middle|\; a_0 + 2a_1 + a_2 = 4 \right\}$? Justify.

5.2.3 [**JH**]

a. Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ under the inherited operations.

b. What if it doesn't contain the origin?

5.2.4 [**JH**] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.2.5 [**JH**] Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

5.3 Spanning Sets

5.3.1 [**JH**] Determine whether the vector lies in the span of the set.

a. $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

b. $x - x^3$, $\left\{ x^2, 2x + x^2, x + x^3 \right\}$

c. $\begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}, \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \right\}$

5.3.2 [**JH**] Which of these are members of the span $\operatorname{span}\left(\left\{ \cos^2 x,\ \sin^2 x \right\}\right)$ in the vector space of real-valued functions of one real variable?

a. $f(x) = 1$

b. $f(x) = 3 + x^2$

c. $f(x) = \sin x$

d. $f(x) = \cos(2x)$

5.3.3 [**JH**] Which of these sets spans $\mathbb{R}^3$?

a. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$

b. $\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

c. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$

d. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right\}$

e. $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} \right\}$

5.3.4 [**JH**] Express each subspace as a span of a set of vectors.

a. $\left\{ (a \;\; b \;\; c) \;\middle|\; a - c = 0 \right\}$

b. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a + d = 0 \right\}$

c. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; 2a - c - d = 0 \text{ and } a + 3b = 0 \right\}$

d. $\left\{ a + bx + cx^3 \;\middle|\; a - 2b + c = 0 \right\}$

e. The subset of $\mathcal{P}_2$ of quadratic polynomials p such that $p(7) = 0$

5.3.5 [**JH**] Find a set that spans the given subspace.

a. The xz-plane in $\mathbb{R}^3$.

b. $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; 3x + 2y + z = 0 \right\}$

c. $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; 2x + y + w = 0 \text{ and } y + 2z = 0 \right\}$

d. $\left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; a_0 + a_1 = 0 \text{ and } a_2 - a_3 = 0 \right\}$

e. The set $\mathcal{P}_4$ in the space $\mathcal{P}_4$

f. $\mathcal{M}_{2\times 2}$ in $\mathcal{M}_{2\times 2}$

5.3.6 [**JH**] Show that for any subset S of a vector space, $\operatorname{span}(\operatorname{span}(S)) = \operatorname{span}(S)$. (*Hint.* Members of $\operatorname{span}(S)$ are linear combinations of members of S. Members of $\operatorname{span}(\operatorname{span}(S))$ are linear combinations of linear combinations of members of S.)

5.3.7 [**YL**] Given the following two subspace of $\mathbb{R}^3$: $W_1 = \{x \mid A_1x = 0\}$ and $W_2 = \{x \mid A_2x = 0\}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix},\ A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.4 Linear Independence

5.4.1 [**YL**] Let $\vec{u} = (1,\ \lambda,\ -\lambda)$, $\vec{v} = (-2\lambda\ \ -2\ \ 2\lambda)$ and $\vec{w} = (\lambda - 2,\ -5\lambda - 2,\ -2)$.

a. For what value(s) of λ will $\{\vec{u},\ \vec{v}\}$ be linearly dependent.

b. For what value(s) of λ will $\{\vec{u},\ \vec{v},\ \vec{w}\}$ be linearly independent.

5.5 Basis

5.5.1 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_2x^2 + a_3x^3 \;\middle|\; p(-1) = 0 \right\}$$

a subspace of $\mathcal{P}_3$.

a. Find a basis B for $\mathcal{W}$.

b. Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B.

5.6 Dimension

5.6.1 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \;\middle|\; p(1) = 0 \text{ and } p(-1) = 0 \right\}$$

a subspace of P_3. Determine the dimension of W.

Appendix A

Answers to Exercises

Note that either a hint, a final answer or a complete solution is provided.

1.1.1

a. Yes
b. No
c. Yes
d. Yes
e. No
f. No
g. Yes
h. No
i. Yes
j. No

1.1.2

a. $x = 1$, $y = -2$
b. $x = 2$, $y = \frac{1}{3}$
c. $x = -1$, $y = 0$, and $z = 2$.
d. $x = 1$, $y = 0$, and $z = 0$.

1.1.3

a. $\left[\begin{array}{ccc|c} 3 & 4 & 5 & 7 \\ -1 & 1 & -3 & 1 \\ 2 & -2 & 3 & 5 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 2 & 5 & -6 & 2 \\ 9 & 0 & -8 & 10 \\ -2 & 4 & 1 & -7 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 3 & -4 & 5 & 17 \\ -1 & 0 & 4 & 8 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right]$

d. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 0 & 3 \\ -1 & 9 & 8 \\ 5 & -7 & 13 \end{array}\right]$

1.1.4

a. $\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 + 3x_2 &= 9 \end{aligned}$

b. $\begin{aligned} -3x_1 + 4x_2 &= 7 \\ x_2 &= -2 \end{aligned}$

c. $\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 &= 7 \end{aligned}$

d. $\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 5 \\ x_4 &= 3 \end{aligned}$

e. $\begin{aligned} x_1 + x_3 + 7x_5 &= 2 \\ x_2 + 3x_3 + 2x_4 &= 5 \end{aligned}$

1.1.5

a. $\begin{bmatrix} -2 & 1 & -7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 7 \\ 5 & 0 & 3 \\ 0 & 4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -1 & 7 \\ 2 & 3 & 5 \\ 5 & 0 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 8 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 2 & -1 \\ 5 & 0 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 0 & 5/2 & -29/2 \end{bmatrix}$

1.1.6

a. $2R_2 \to R_2$
b. $R_1 + R_2 \to R_2$
c. $2R_3 + R_1 \to R_1$
d. $R_1 \leftrightarrow R_2$
e. $-R_2 + R_3 \leftrightarrow R_3$

1.1.7 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line.
Thus the lines can share one point (giving a unique solution),

share no points (giving no solutions), or share at least two points (which makes them the same line).

1.1.8 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

1.2.1

a. Yes

b. No

c. No

d. Yes

e. Yes

f. Yes

g. No

h. Yes

i. No

j. Yes

k. Yes

l. Yes

m. No

n. Yes

o. Yes

1.2.2

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -7/5 \\ 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$

g. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 1 & \frac{5}{4} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$

i. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

j. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

n. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

p. $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.2.3

a. $x = 2$, $y = 3$

b. $x = -1$, $y = 4$, and $z = -1$.

1.2.4

a. $x_1 = 1 - 2t$; $x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$ and $x_1 = -1$, $x_2 = 1$.

b. $x_1 = -3 + 5t$; $x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 3$, $x_2 = 0$ and $x_1 = -8$, $x_2 = -1$.

c. $x_1 = 1$; $x_2 = 2$.

d. $x_1 = 0$; $x_2 = -1$.

e. $x_1 = -11 + 10t$; $x_2 = -4 + 4t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -11$, $x_2 = -4$, $x_3 = 0$ and $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$.

f. $x_1 = -\frac{2}{3} + \frac{8}{9}t$; $x_2 = \frac{2}{3} - \frac{5}{9}t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -\frac{2}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 0$ and $x_1 = \frac{4}{9}$, $x_2 = -\frac{1}{9}$, $x_3 = 1$.

g. $x_1 = 1 - s - t$; $x_2 = s$; $x_3 = 1 - 2t$; $x_4 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ and $x_1 = -2$, $x_2 = 1$, $x_3 = -3$, $x_4 = 2$.

h. $x_1 = 3 - s - 2t$; $x_2 = -3 - 5s - 7t$; $x_3 = s$; $x_4 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 3$, $x_2 = -3$, $x_3 = 0$, $x_4 = 0$ and $x_1 = 0$, $x_2 = -5$, $x_3 = -1$, $x_4 = 1$.

i. $x_1 = \frac{1}{3} - \frac{4}{3}t$; $x_2 = \frac{1}{3} - \frac{1}{3}t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $x_3 = 0$ and $x_1 = -1$, $x_2 = 0$, $x_3 = 1$.

j. $x_1 = 1 - 2s - 3t$; $x_2 = s$; $x_3 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and $x_1 = 8$, $x_2 = 1$, $x_3 = -3$.

k. No solution; the system is inconsistent.

l. No solution; the system is inconsistent.

1.2.5

a. $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (60s - 55t + 30,\ -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3},\ -14s + 13t - 7,\ s,\ t)$ where $s,\ t \in \mathbb{R}$.

b. If $s = t = 0$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) =$

$(30,\ -\frac{38}{3},\ -7,\ 0,\ 0)$.
If $s = 0$ and $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-25,\ \frac{35}{3},\ 6,\ 0,\ 1)$.

c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-\frac{30}{7},\ \frac{316}{21},\ 1,\ \frac{4}{7},\ 0)$.

1.2.6

a. $(x_1,\ x_2,\ x_3,\ x_4) = (60t,\ -\frac{79}{3}t,\ -14t,\ t)$ where $t \in \mathbb{R}$.

b. If $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4) = (60,\ -\frac{79}{3},\ -14,\ 1)$.
If $t = 3$ then $(x_1,\ x_2,\ x_3,\ x_4) = (180,\ -79,\ 42,\ 3)$.

c. If $t = \frac{1}{60}$ then $(x_1,\ x_2,\ x_3,\ x_4) = (1,\ -\frac{79}{180},\ -\frac{14}{60},\ \frac{1}{60})$.

1.2.7 Because $f(1) = 2$, $f(-1) = 6$, and $f(2) = 3$ we get a linear system.

$$\begin{aligned} 1a + 1b + c &= 2 \\ 1a - 1b + c &= 6 \\ 4a + 2b + c &= 3 \end{aligned}$$

After performing Gaussian elimination we obtain

$$\begin{aligned} a + \quad b + \quad c &= \ \ 2 \\ -2b \qquad &= \ \ 4 \\ -3c &= -9 \end{aligned}$$

which shows that the solution is $f(x) = 1x^2 - 2x + 3$.

1.2.8 The following system with more unknowns than equations

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

has no solution.

1.2.9 After performing Gaussian elimination the system becomes

$$\begin{aligned} x - y &= \quad\ \ 1 \\ 0 &= -3 + k \end{aligned}$$

This system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

1.2.10

a. Never exactly 1 solution; infinite solutions if $k = 2$; no solution if $k \neq 2$.

b. Exactly 1 solution if $k \neq 2$; infinite solutions if $k = 2$; never no solution.

c. Exactly 1 solution if $k \neq 2$; no solution if $k = 2$; never infinite solutions.

d. Exactly 1 solution for all k.

1.2.11

a. Possible if $a = \pm 1$ and $a \neq \pm b$.

b. Not possible.

c. Possible if $a \neq \pm 1$ or $a = \pm b$.

1.2.12 Consistent if $b_3 - b_2 - b_1 = 0$ and $b_4 - 2b_2 - b_1 = 0$.

1.2.13 If $a \neq 0$ then the solution set of the first equation is $\{(x, y) \mid x = (c - by)/a\}$. Taking $y = 0$ gives the solution $(c/a, 0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a) + d \cdot 0 = e$, so $c = e$. Then taking $y = 1$ in $x = (c - by)/a$ gives that $a((c - b)/a) + d \cdot 1 = e$, which gives that $b = d$. Hence they are the same equation.
When $a = 0$ the equations can be different and still have the same solution set: e.g., $0x + 3y = 6$ and $0x + 6y = 12$.

1.2.14 We take three cases: that $a \neq 0$, that $a = 0$ and $c \neq 0$, and that both $a = 0$ and $c = 0$.
For the first, we assume that $a \neq 0$. Then Gaussian elimination

$$\begin{aligned} ax + \qquad\qquad by &= j \\ (-(cb/a) + d)y &= -(cj/a) + k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a) + d \neq 0$; remember that $a \neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a) + d = (ad - bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad - bc \neq 0$.
In the second case, if $a = 0$ but $c \neq 0$, then we swap

$$\begin{aligned} cx + dy &= k \\ by &= j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b \neq 0$ (we use the case assumption that $c \neq 0$ to get a unique x in back substitution). But where $a = 0$ and $c \neq 0$ the condition "$b \neq 0$" is equivalent to the condition "$ad - bc \neq 0$". That finishes the second case.
Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x + by &= j \\ 0x + dy &= k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x, y) will do), but it never has a unique solution. Note that $a = 0$ and $c = 0$ gives that $ad - bc = 0$.

1.3.1 Lorem ipsum dolor sit amet, consectetur adipiscing elit. Aliquam tincidunt cursus volutpat. Quisque non congue sem. Vivamus nec nibh sed est dapibus auctor eu sed nulla. Praesent ornare eleifend nibh a finibus. Proin rutrum neque nec massa tincidunt, non malesuada dolor interdum. Nam a massa sit amet diam efficitur pharetra. Nulla interdum efficitur sem, sit amet commodo orci mattis non. Duis tortor ex, maximus a sapien id, molestie maximus risus.

2.1.1

a. 2

b. 3

c. -1

d. Not defined.

2.1.2

a. 2×3

b. 3×2

c. 2×2

2.1.3

a. $\begin{bmatrix} -2 & -1 \\ 12 & 13 \end{bmatrix}$

b. $\begin{bmatrix} 11 & -8 \\ -1 & -19 \end{bmatrix}$

c. $\begin{bmatrix} 9 & -7 \\ 11 & -6 \end{bmatrix}$

d. $\begin{bmatrix} -2 & 1 \\ 12 & 13 \end{bmatrix}$

2.1.4

a. -22

b. -2

c. 23

d. Not possible.

e. Not possible.

2.1.5

a. $AB = \begin{bmatrix} 8 & 3 \\ 10 & -9 \end{bmatrix}$, $BA = \begin{bmatrix} -3 & 24 \\ 4 & 2 \end{bmatrix}$

b. $AB = \begin{bmatrix} -1 & -2 & 12 \\ 10 & 4 & 32 \end{bmatrix}$, BA is not defined

c. $AB = \begin{bmatrix} 3 & 8 \\ -5 & -8 \\ -8 & -32 \end{bmatrix}$, BA is not defined

d. $AB = \begin{bmatrix} 10 & -18 & 11 \\ -45 & 24 & -21 \\ -15 & 12 & -9 \end{bmatrix}$, $BA = \begin{bmatrix} 52 & -21 \\ 45 & -27 \end{bmatrix}$

e. $AB = \begin{bmatrix} -32 & 34 & -24 \\ -32 & 38 & -8 \\ -16 & 21 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 22 & -14 \\ -4 & -12 \end{bmatrix}$

f. $AB = \begin{bmatrix} -7 & 3 & 7 & -15 \\ -5 & -1 & -17 & 5 \end{bmatrix}$, BA is not defined

g. $AB = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ -2 & 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 & 4 \\ -3 & 6 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

h. $AB = \begin{bmatrix} 21 & -17 & -5 \\ 19 & 5 & 19 \\ 5 & 9 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 19 & 5 & 23 \\ 5 & -7 & -1 \\ -14 & 6 & 18 \end{bmatrix}$

2.1.6

a. $DA = \begin{bmatrix} 2 & 2 & 2 \\ -6 & -6 & -6 \\ -15 & -15 & -15 \end{bmatrix}$, $AD = \begin{bmatrix} 2 & -3 & 5 \\ 4 & -6 & 10 \\ -6 & 9 & -15 \end{bmatrix}$

b. $DA = \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix}$, $AD = \begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix}$

c. $DA = \begin{bmatrix} d_1a & d_1b \\ d_2c & d_2d \end{bmatrix}$, $AD = \begin{bmatrix} d_1a & d_2b \\ d_1c & d_2d \end{bmatrix}$

d. $DA = \begin{bmatrix} d_1a & d_1b & d_1c \\ d_2d & d_2e & d_2f \\ d_3g & d_3h & d_3i \end{bmatrix}$, $AD = \begin{bmatrix} d_1a & d_2b & d_3c \\ d_1d & d_2e & d_3f \\ d_1g & d_2h & d_3i \end{bmatrix}$

2.1.7

a. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$, $A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}$

c. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 125 \end{bmatrix}$

d. $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2.1.8

a. $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$

c. Not possible, since dimension of A and E are not the same.

d. $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$

f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D.

g. $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$

h. $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.1.9

a. $\begin{bmatrix} -9 & 6 & -8 \\ 4 & -3 & 1 \\ 10 & -7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 5 & -6 \\ 2 & -4 & 6 \\ -9 & -10 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -9 \\ -7 & 6 \\ -4 & 3 \\ -9 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$, symmetric

e. $\begin{bmatrix} 4 & -2 & 4 \\ 0 & -7 & -2 \\ 0 & 0 & 5 \end{bmatrix}$, A is lower triangular and A^T is upper triangular.

f. $\begin{bmatrix} -3 & 0 & 0 \\ -4 & -3 & 0 \\ -5 & 5 & -3 \end{bmatrix}$, A is upper triangular and A^T is lower triangular.

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, diagonal.

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$, symmetric.

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$, skew-symmetric.

2.1.10

a. -9

b. 6

c. -23

d. Not defined; the matrix must be square.

e. 0

f. n

2.1.11

a. $a = -1$, $b = 1/2$

b. $a = 5/2 + 3/2t$, $b = t$ where $t \in \mathbb{R}$

c. $a = 5$, $b = 0$

d. No solution.

2.1.12

a. $\begin{bmatrix} 0 & -2 \\ -5 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 10 & 2 \\ 5 & 11 \end{bmatrix}$

c. $\begin{bmatrix} -11 & -15 \\ 37 & 32 \end{bmatrix}$

d. No.

e. $(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2$

2.1.13

a. Hint: Apply the definition of the trace to arbitrary matrices A and B.

b. Hint: Analyse the ij product of the elements of the main diagonal.

2.1.14 Disprove: Show that it is impossible to obtain a nonzero matrix.

2.1.15 Hint: Apply the definition of an idempotent matrix.

2.2.1

a. $X = \begin{bmatrix} -5 & 9 \\ -1 & -14 \end{bmatrix}$

b. $X = \begin{bmatrix} 0 & -22 \\ -7 & 17 \end{bmatrix}$

c. $X = \begin{bmatrix} -5 & -2 \\ -9/2 & -19/2 \end{bmatrix}$

d. $X = \begin{bmatrix} 8 & 12 \\ 10 & 2 \end{bmatrix}$

2.2.2

a. $\begin{bmatrix} -24 & -5 \\ 5 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/7 \end{bmatrix}$

c. $\begin{bmatrix} -4/7 & 5/7 \\ 3/7 & -2/7 \end{bmatrix}$

d. The inverse does not exist.

2.2.3

a. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -7/5 \end{bmatrix}$

b. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -7/10 & 3/10 \\ 29/10 & -11/10 \end{bmatrix}$

2.2.4

a. $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 6 & 10 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 52 & -48 & 7 \\ 8 & -7 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -9 & 4 \\ 5 & -26 & 11 \\ 0 & -2 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 91 & 5 & -20 \\ 18 & 1 & -4 \\ -22 & -1 & 5 \end{bmatrix}$

e. $\begin{bmatrix} 25 & 8 & 0 \\ 78 & 25 & 0 \\ -30 & -9 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 5 & -3 & -8 \\ -4 & 2 & 5 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i. The inverse does not exist.

j. The inverse does not exist.

k. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & -4 \\ -35 & -10 & 1 & -47 \\ -2 & -2 & 0 & -9 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -11 & 1 & 0 & -4 \\ -2 & 0 & 1 & -4 \\ -4 & 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 28 & -2 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 254 & -19 & 110 \\ 0 & -67 & 5 & -29 \end{bmatrix}$

n. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1/4 \end{bmatrix}$

2.2.5 Disprove: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.2.6 $A = \begin{bmatrix} -\frac{3}{4} & 3 \\ 1 & -\frac{3}{4} \end{bmatrix}$

2.2.7 $A = \begin{bmatrix} 0 & -1 \\ -11 & -\frac{17}{2} \end{bmatrix}$

2.2.8

a. $A = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

b. $X = \begin{bmatrix} -\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & \frac{3}{2} & -4 & 2 \end{bmatrix}$

2.2.9 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.2.10 Hint: Use the definition of the inverse of a matrix.

2.2.11 Hint: Apply the definition of symmetric matrices.

2.3.1

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.2

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.3 $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note: The answer is not unique.

2.4.1

a. $A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

b. $x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$

2.4.2

a. $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b. $x = \begin{bmatrix} -7 \\ -7 \end{bmatrix}$

c. $x = \begin{bmatrix} -7 \\ 1 \\ -1 \end{bmatrix}$

d. $x = \begin{bmatrix} -7 \\ -7 \\ 9 \end{bmatrix}$

3.1.1

a. 34

b. 41

c. -44

d. -74

3.1.2

a. $M_{1,1} = \begin{bmatrix} 7 & 6 \\ 6 & 10 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} 3 & 6 \\ 1 & 10 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix}$.
$C_{1,1} = 43$, $C_{1,2} = -24$, $C_{1,3} = 11$.

b. $M_{1,1} = \begin{bmatrix} -6 & 8 \\ -3 & -2 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -10 & 8 \\ 0 & -2 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} 10 & -6 \\ 0 & -3 \end{bmatrix}$.
$C_{1,1} = 36$, $C_{1,2} = -20$, $C_{1,3} = -30$.

c. $M_{1,1} = \begin{bmatrix} 3 & 10 \\ 3 & 9 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -3 & 10 \\ -9 & 9 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} -3 & 3 \\ -9 & 3 \end{bmatrix}$.
$C_{1,1} = -3$, $C_{1,2} = -63$, $C_{1,3} = 18$.

d. $M_{1,1} = \begin{bmatrix} 0 & 0 \\ 8 & -1 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -8 & 0 \\ -10 & -1 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} -8 & 0 \\ -10 & 8 \end{bmatrix}$.
$C_{1,1} = 0$, $C_{1,2} = -8$, $C_{1,3} = -64$.

3.1.3

a. $3(+1)\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} + 0(-1)\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + 1(+1)\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -13$

b. $1(-1)\begin{vmatrix}0 & 1\\ 3 & 0\end{vmatrix}+2(+1)\begin{vmatrix}3 & 1\\ -1 & 0\end{vmatrix}+2(-1)\begin{vmatrix}3 & 0\\ -1 & 3\end{vmatrix}=-13$

c. $1(+1)\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix}+2(-1)\begin{vmatrix}3 & 0\\ -1 & 3\end{vmatrix}+0(+1)\begin{vmatrix}3 & 0\\ 1 & 2\end{vmatrix}=-13$

3.1.4

a. -59

b. 250

c. 3

d. 0

e. 0

f. 2

3.1.5 Evaluate the determinant using a cofactor expansion. The same is true for lower triangular matrices.

3.1.6 $\lambda=\frac{3\pm\sqrt{33}}{4}$

3.1.7 False, Here is a determinant whose value

$$\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=1$$

doesn't equal the result of expanding down the diagonal.

$$1\cdot(+1)\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}+1\cdot(+1)\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}+1\cdot(+1)\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=3$$

3.1.8 There are no real numbers θ that make the matrix singular because the determinant of the matrix $\cos^2\theta+\sin^2\theta$ is never 0, it equals 1 for all θ.

3.2.1

a. $\det(A)=90$; $2R_1\to R_1$.
$\det(B)=45$; $10R_1+R_3\to R_3$.
$\det(C)=45$; $C=A^T$.

b. $\det(A)=41$; $R_2\leftrightarrow R_3$.
$\det(B)=164$; $-4R_3\to R_3$.
$\det(C)=-41$; $R_2+R_1\to R_1$.

c. $\det(A)=-16$; $R_1\leftrightarrow R_2$ then $R_1\leftrightarrow R_3$.
$\det(B)=-16$; $-R_1\to R_1$ and $-R_2\to R_2$.
$\det(C)=-432$; $C=3M$.

d. $\det(A)=-120$; $R_1\leftrightarrow R_2$ then $R_1\leftrightarrow R_3$ then $R_2\leftrightarrow R_3$.
$\det(B)=720$; $2R_2\to R_2$ and $3R_3\to R_3$.
$\det(C)=-120$; $C=-M$.

3.2.2

a. 15

b. -52

c. 0

d. 1

e. -113

f. 179

3.2.3 $\det(A)=-\frac{5}{12}$

3.2.4 Hint: Use elementary operations to bring the matrix under triangular form.

3.3.1

a. $\begin{bmatrix}0 & -1 & 2\\ 3 & -2 & -8\\ 0 & 1 & 1\end{bmatrix}$

b. $\begin{bmatrix}4 & 1\\ -2 & 3\end{bmatrix}$

c. $\begin{bmatrix}0 & -1\\ -5 & 1\end{bmatrix}$

d. $\begin{bmatrix}-24 & -12 & 12\\ 12 & 6 & -6\\ -8 & -4 & 4\end{bmatrix}$

e. $\begin{bmatrix}4 & -3 & 2 & -1\\ -3 & 6 & -4 & 2\\ 2 & -4 & 6 & -3\\ -1 & 2 & -3 & 4\end{bmatrix}$

3.3.2

a. $\begin{bmatrix}T_{1,1} & T_{2,1}\\ T_{1,2} & T_{2,2}\end{bmatrix}=\begin{bmatrix}|t_{2,2}| & -|t_{1,2}|\\ -|t_{2,1}| & |t_{1,1}|\end{bmatrix}=\begin{bmatrix}t_{2,2} & -t_{1,2}\\ -t_{2,1} & t_{1,1}\end{bmatrix}$

b. $(1/t_{1,1}t_{2,2}-t_{1,2}t_{2,1})\begin{bmatrix}t_{2,2} & -t_{1,2}\\ -t_{2,1} & t_{1,1}\end{bmatrix}$

3.3.3 Consider this diagonal matrix.

$$D=\begin{bmatrix}d_1 & 0 & 0 & \dots & \\ 0 & d_2 & 0 & & \\ 0 & 0 & d_3 & & \\ & & & \ddots & \\ & & & & d_n\end{bmatrix}$$

If $i\neq j$ then the i,j minor is an $(n-1)\times(n-1)$ matrix with only $n-2$ nonzero entries, because we have deleted both d_i and d_j. Thus, at least one row or column of the minor is all zeroes, and so the cofactor $D_{i,j}$ is zero. If $i=j$ then the minor is the diagonal matrix with entries d_1, …, d_{i-1}, d_{i+1}, …, d_n. Its determinant is obviously $(-1)^{i+j}=(-1)^{2i}=1$ times the product of those.

$$\operatorname{adj}(D)=\begin{bmatrix}d_2\cdots d_n & 0 & & 0\\ 0 & d_1d_3\cdots d_n & & 0\\ & & \ddots & \\ & & & d_1\cdots d_{n-1}\end{bmatrix}$$

3.3.4 Just note that if $S=T^{\mathsf{T}}$ then the cofactor $S_{j,i}$ equals the cofactor $T_{i,j}$ because $(-1)^{j+i}=(-1)^{i+j}$ and because the minors are the transposes of each other (and the determinant of a transpose equals the determinant of the matrix).

3.3.5 False. A counter example.

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{adj}(T) = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix} \qquad \text{adj}(\text{adj}(T)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.3.6 This equation

$$0 = \det(\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}) = 64 - 20x + x^2 = (x-16)(x-4)$$

has roots $x = 16$ and $x = 4$.

3.3.7 $\det(TS) = \det(T)\cdot\det(S) = \det(S)\cdot\det(T) = \det(ST)$.

3.3.8

a. Plug and chug: the determinant of the product is this

$$\begin{aligned} \det(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix}) &= \det(\begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}) \\ &= acwx + adwz + bcxy + bdyz \\ &\quad -acwx - bcwz - adxy - bdyz \end{aligned}$$

while the product of the determinants is this.

$$\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\det(\begin{bmatrix} w & x \\ y & z \end{bmatrix}) = (ad-bc)(wz-xy)$$

Verification that they are equal is easy.

b. Use the prior part.

3.3.9

a. If it is defined then it is $(3^2)(2)(2^{-2})(3)$.

b. Hint: $\det 6A^3 + 5A^2 + 2A = \det A \det 6A^2 + 5A + 2I$.

3.3.10

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

b. $1 = \det(AA^{-1}) = \det(AA^{\mathsf{T}}) = \det(A)\det(A^{\mathsf{T}}) = (\det(A))^2$

c. The converse does not hold; here is an example.

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

3.3.11 If $H = P^{-1}GP$ then $\det(H) = \det(P^{-1})\det(G)\det(P) = \det(P^{-1})\det(P)\det(G) = \det(P^{-1}P)\det(G) = \det(G)$.

3.3.12 An algebraic check is easy.

$$0 = xy_2 + x_2y_3 + x_3y - x_3y_2 - xy_3 - x_2y = x\cdot(y_2-y_3) + y\cdot(x_3-x_2) + x_2y_3 - x_3y_2$$

simplifies to the familiar form

$$y = x\cdot(x_3-x_2)/(y_3-y_2) + (x_2y_3 - x_3y_2)/(y_3-y_2)$$

(the $y_3 - y_2 = 0$ case is easily handled).

3.3.13 Hint: Apply the determinant to both sides $AB = -BA$.

3.3.14 Disprove. Recall that constants come out one row at a time

$$\det(\begin{bmatrix} 2 & 4 \\ 2 & 6 \end{bmatrix}) = 2\cdot\det(\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}) = 2\cdot 2\cdot\det(\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix})$$

This contradicts linearity (here we didn't need S, i.e., we can take S to be the matrix of zeros).

3.4.1 $x_1 = 4$

3.4.2

a. $\det(A) = -123$, $\det(A_1) = -492$, $\det(A_2) = 123$, $\det(A_3) = 492$,
$x = \begin{bmatrix} 4 \\ -1 \\ -4 \end{bmatrix}$.

b. $\det(A) = -43$, $\det(A_1) = 215$, $\det(A_2) = 0$,
$x = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$.

c. $\det(A) = 0$, $\det(A_1) = 0$, $\det(A_2) = 0$, $\det(A_3) = 0$. Infinite solutions exist.

d. $\det(A) = 0$, $\det(A_1) = -56$, $\det(A_2) = 26$. No solution exist.

e. $\det(A) = 0$, $\det(A_1) = 0$, $\det(A_2) = 0$, $\det(A_3) = 0$. Infinite solutions exist.

f. $\det(A) = 0$, $\det(A_1) = 1247$, $\det(A_2) = -49$, $\det(A_3) = -49$. No solution exist.

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$.

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5.1.1

a. $0 + 0x + 0x^2 + 0x^3$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. The constant function $f(x) = 0$

d. The constant function $f(n) = 0$

5.1.2

a. $3 + 2x - x^2$

b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

c. $-3e^x + 2e^{-x}$

5.1.3

a. $1 + 2x$, $2 - 1x$, and x.

b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

a.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b.

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

a. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.

b. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

5.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

a. It is not closed under addition.

$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

b. It is not closed under addition.

$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

c. It is not closed under addition.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

d. It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

e. The set is empty, violating the existance of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

a. It is not a vector space.

$$(1+1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b. It is not a vector space.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

a. Yes.

b. Yes.

c. No, this set is not closed under the natural addition operation. The vector of all 1/4's is an element of this set but when added to itself the result, the vector of all 1/2's, is not an element of the set.

d. Yes.

e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11

a. Closed under vector addition. Hint: Apply determinant properties.

b. $\vec{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$

c. Every $A \in V$ has an additive inverse A^{-1}.

d. Yes.

e. Not closed under scalar multiplication. Since $0\vec{0} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin V$

5.1.12 Check all 10 conditions of the definition of a vector space.

5.1.13 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.14

a. No since $1 \cdot (0,\ 1) + 1 \cdot (0,\ 1) \neq (1+1) \cdot (0,\ 1)$.

b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0,\ 1) \neq (0,\ 1)$.

5.1.15

a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which implies that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.16

Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.17

It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.18

Each element of a vector space has one and only one additive inverse.

For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.19

Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.

b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.

c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

5.2.1

a. Yes, we can easily check that it is closed under addition and scalar multiplication.

b. Yes, we can easily check that it is closed under addition and scalar multiplication.

c. No. It is not closed under addition. For instance,

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

is not in the set. (This set is also not closed under scalar multiplication, for instance, it does not contain the zero matrix.)

d. Yes, we can easily check that it is closed under addition and scalar multiplication.

5.2.2 No, it is not closed. In particular, it is not closed under scalar multiplication because it does not contain the zero polynomial.

5.2.3

a. Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of $\vec{v}, \vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.

b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.4 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r\,f'$.

5.2.5 No. Subspaces of $\mathbb{R}^3$ are sets of three-tall vectors, while $\mathbb{R}^2$ is a set of two-tall vectors. Clearly though, $\mathbb{R}^2$ is "just like" this subspace of $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

5.3.1

a. Yes, solving the linear system arising from

$$r_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

gives $r_1 = 2$ and $r_2 = 1$.

b. Yes; the linear system arising from $r_1(x^2) + r_2(2x + x^2) + r_3(x + x^3) = x - x^3$

$$\begin{aligned} 2r_2 + r_3 &= 1 \\ r_1 + \quad r_2 \qquad &= 0 \\ r_3 &= -1 \end{aligned}$$

gives that $-1(x^2) + 1(2x + x^2) - 1(x + x^3) = x - x^3$.

c. No; any combination of the two given matrices has a zero in the upper right.

5.3.2

a. Yes. It is in that span since $1\cos^2 x + 1\sin^2 x = f(x)$.

b. No. Since $r_1\cos^2 x + r_2\sin^2 x = 3 + x^2$ has no scalar solutions that work for all x. For instance, setting x to be 0 and π gives the two equations $r_1 \cdot 1 + r_2 \cdot 0 = 3$ and $r_1 \cdot 1 + r_2 \cdot 0 = 3 + \pi^2$, which are not consistent with each other.

c. No. Consider what happens on setting x to be $\pi/2$ and $3\pi/2$.

d. Yes, $\cos(2x) = 1\cdot\cos^2(x) - 1\cdot\sin^2(x)$.

5.3.3

a. Yes, for any $x, y, z \in \mathbb{R}$ this equation

$$r_1\begin{pmatrix}1\\0\\0\end{pmatrix}+r_2\begin{pmatrix}0\\2\\0\end{pmatrix}+r_3\begin{pmatrix}0\\0\\3\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

has the solution $r_1 = x$, $r_2 = y/2$, and $r_3 = z/3$.

b. Yes, the equation

$$r_1\begin{pmatrix}2\\0\\1\end{pmatrix}+r_2\begin{pmatrix}1\\1\\0\end{pmatrix}+r_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives rise to this

$$\begin{aligned}2r_1+r_2&=x\\ r_2&=y\\ r_1+r_3&=z\end{aligned}$$

Gaussian elimination gives

$$\begin{aligned}2r_1+r_2&=x\\ r_2&=y\\ r_3&=-(1/2)x+(1/2)y+z\end{aligned}$$

so that, given any x, y, and z, we can compute that $r_3 = (-1/2)x+(1/2)y+z$, $r_2 = y$, and $r_1 = (1/2)x-(1/2)y$.

c. No. In particular, we cannot get the vector

$$\begin{pmatrix}0\\0\\1\end{pmatrix}$$

as a linear combination since the two given vectors both have a third component of zero.

d. Yes. The equation

$$r_1\begin{pmatrix}1\\0\\1\end{pmatrix}+r_2\begin{pmatrix}3\\1\\0\end{pmatrix}+r_3\begin{pmatrix}-1\\0\\0\end{pmatrix}+r_4\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left[\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\0&0&1&6&-x+3y+z\end{array}\right]$$

We have infinitely many solutions. We can, for example, set r_4 to be zero and solve for r_3, r_2, and r_1 in terms of x, y, and z by the usual methods of back-substitution.

e. No. The equation

$$r_1\begin{pmatrix}2\\1\\1\end{pmatrix}+r_2\begin{pmatrix}3\\0\\1\end{pmatrix}+r_3\begin{pmatrix}5\\1\\2\end{pmatrix}+r_4\begin{pmatrix}6\\0\\2\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left[\begin{array}{cccc|c}2&3&5&6&x\\0&-3/2&-3/2&-3&-(1/2)x+y\\0&0&0&0&-(1/3)x-(1/3)y+z\end{array}\right]$$

This shows that not every vector can be so expressed. Only the vectors satisfying the restriction that $-(1/3)x-(1/3)y+z=0$ are in the span. (To see that any such vector is indeed expressible, take r_3 and r_4 to be zero and solve for r_1 and r_2 in terms of x, y, and z by back-substitution.)

5.3.4

a. $\{(c\ \ b\ \ c) \mid b, c \in \mathbb{R}\} = \{b(0\ \ 1\ \ 0) + c(1\ \ 0\ \ 1) \mid b, c \in \mathbb{R}\}$ The obvious choice for the set that spans is $\{(0\ \ 1\ \ 0), (1\ \ 0\ \ 1)\}$.

b. $\left\{\begin{bmatrix}-d&b\\c&d\end{bmatrix} \;\middle|\; b,c,d\in\mathbb{R}\right\} = \left\{b\begin{bmatrix}0&1\\0&0\end{bmatrix}+c\begin{bmatrix}0&0\\1&0\end{bmatrix}+d\begin{bmatrix}-1&0\\0&1\end{bmatrix} \;\middle|\; b,c,d\in\mathbb{R}\right\}$ One set that spans this space consists of those three matrices.

c. The system

$$\begin{aligned}a+3b&=0\\ 2a-c-d&=0\end{aligned}$$

gives $b = -(c+d)/6$ and $a = (c+d)/2$. So one description is this.

$$\left\{c\begin{bmatrix}1/2&-1/6\\1&0\end{bmatrix}+d\begin{bmatrix}1/2&-1/6\\0&1\end{bmatrix} \;\middle|\; c,d\in\mathbb{R}\right\}$$

That shows that a set spanning this subspace consists of those two matrices.

d. The $a = 2b - c$ gives that the set $\{(2b-c)+bx+cx^3 \mid b,c\in\mathbb{R}\}$ equals the set $\{b(2+x)+c(-1+x^3) \mid b,c\in\mathbb{R}\}$. So the subspace is the span of the set $\{2+x,\ -1+x^3\}$.

e. The set $\{a+bx+cx^2 \mid a+7b+49c=0\}$ can be parametrized as

$$\{b(-7+x)+c(-49+x^2) \mid b,c\in\mathbb{R}\}$$

and so has the spanning set $\{-7+x,\ -49+x^2\}$.

5.3.5

a. We can parametrize in this way

$$\left\{\begin{pmatrix}x\\0\\z\end{pmatrix} \;\middle|\; x,z\in\mathbb{R}\right\}=\left\{x\begin{pmatrix}1\\0\\0\end{pmatrix}+z\begin{pmatrix}0\\0\\1\end{pmatrix} \;\middle|\; x,z\in\mathbb{R}\right\}$$

giving this for a spanning set.

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

b. Here is a parametrization, and the associated spanning set. $\left\{ y \begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\}$
$\left\{ \begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$

c. $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

d. Parametrize the description as $\left\{ -a_1 + a_1 x + a_3 x^2 + a_3 x^3 \;\middle|\; a_1, a_3 \in \mathbb{R} \right\}$ to get $\left\{ -1 + x, x^2 + x^3 \right\}$.

e. $\left\{ 1, x, x^2, x^3, x^4 \right\}$

f. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

5.3.6 We will show mutual containment between the two sets.
The first containment $\operatorname{span}(\operatorname{span}(S)) \supseteq \operatorname{span}(S)$ is an instance of the more general, and obvious, fact that for any subset T of a vector space, $\operatorname{span}(T) \supseteq T$.
For the other containment, that $\operatorname{span}(\operatorname{span}(S)) \subseteq \operatorname{span}(S)$, take m vectors from $\operatorname{span}(S)$, namely $c_{1,1}\vec{s}_{1,1} + \cdots + c_{1,n_1}\vec{s}_{1,n_1}$, $\ldots$, $c_{1,m}\vec{s}_{1,m} + \cdots + c_{1,n_m}\vec{s}_{1,n_m}$, and note that any linear combination of those

$$r_1(c_{1,1}\vec{s}_{1,1} + \cdots + c_{1,n_1}\vec{s}_{1,n_1}) + \cdots + r_m(c_{1,m}\vec{s}_{1,m} + \cdots + c_{1,n_m}\vec{s}_{1,n_m})$$

is a linear combination of elements of S

$$= (r_1 c_{1,1})\vec{s}_{1,1} + \cdots + (r_1 c_{1,n_1})\vec{s}_{1,n_1} + \cdots + (r_m c_{1,m})\vec{s}_{1,m} + \cdots + (r_m c_{1,n_m})\vec{s}_{1,n_m}$$

and so is in $\operatorname{span}(S)$. That is, simply recall that a linear combination of linear combinations (of members of S) is a linear combination (again of members of S).

5.3.7 Hint: For each subspace determine a set of vectors that spans it.
$W_1 \subsetneq W_2$

5.4.1

a. $\lambda = 1$

b. $\lambda \neq -1,\ -\frac{1}{2},\ 1$

5.5.1

a. $B = \left\{ 1 + x^3,\ x^2 + x^3 \right\}$

b. $(p(x))_B = (-2, 2)$

5.6.1 $\left\{ -1 + x^2,\ -x + x^3 \right\}$ is a basis of W, therefore W is of dimension 2.

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