CEGEP Linear Algebra Problems

An open source collection of
CEGEP level linear algebra problems

Edited by

Yann Lamontagne

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http://obeymath.org/CEGEPLinearAlgebraProblems

Contents

Chapter 1

Systems of Linear Equations

1.1 Gaussian and Gauss-Jordan Elimination

1.1.1 (*by Jim Hefferon [JH]*) Use Gauss's Method to find the unique solution for each system.

a.
$$\begin{array}{rcrcr} 2x & + & 3y & = & 13 \\ x & - & y & = & -1 \end{array}$$

b.
$$\begin{array}{rcrcrcr} x & & & - & z & = & 0 \\ 3x & + & y & & & = & 1 \\ -x & + & y & + & z & = & 4 \end{array}$$

Chapter 2

Vector Spaces

2.1 Introduction to Vector Spaces

2.1.1 (*by Jim Hefferon [JH]*) Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.
b. The space of 2×3 matrices.
c. The space $\{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$.
d. The space of real-valued functions of one natural number variable.

2.1.2 (*by Jim Hefferon [JH]*) Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3 - 2x + x^2$.
b. In the space $\mathcal{M}_{2\times 2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

c. In $\{ae^x + be^{-x} \mid a, b \in \mathbb{R}\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

2.1.3 (*by Jim Hefferon [JH]*) For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations.
b. The set of linear polynomials $\{a_0 + a_1 x \mid a_0 - 2a_1 = 0\}$, under the usual polynomial addition and scalar multiplication operations.

2.1.4 (*by Jim Hefferon [JH]*) For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.
b. The set of 2×2 matrices with real entries where the $2,1$ entry is zero, under the usual matrix operations.

2.1.5 (*by Jim Hefferon [JH]*) For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.
b. The set

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$$

under the operations inherited from $\mathbb{R}^4$.

2.1.6 (*by Jim Hefferon [JH]*) Show that each of these is not a vector space.

a. Under the operations inherited from $\mathbb{R}^3$, this set

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\right\}$$

b. Under the operations inherited from $\mathbb{R}^3$, this set

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 = 1 \right\}$$

c. Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

d. Under the usual polynomial operations,

$$\left\{ a_0 + a_1 x + a_2 x^2 \;\middle|\; a_0, a_1, a_2 \in \mathbb{R}^+ \right\}$$

where $\mathbb{R}^+$ is the set of reals greater than zero
e. Under the inherited operations,

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x + 3y = 4 \text{ and } 2x - y = 3 \text{ and } 6x + 4y = 10 \right\}$$

2.2 Subspaces

2.3 Spanning Sets

2.4 Linear Independence

2.5 Basis

2.6 Dimension

Answers to Exercises

1.1.1

a. $x = 2$, $y = 3$

b. $x = -1$, $y = 4$, and $z = -1$.

2.1.1

a. $0 + 0x + 0x^2 + 0x^3$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. The constant function $f(x) = 0$

d. The constant function $f(n) = 0$

2.1.2

a. $3 + 2x - x^2$

b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

c. $-3e^x + 2e^{-x}$

2.1.3

a. $1 + 2x$, $2 - 1x$, and x.

b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

2.1.4

a.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.1.5

a. $(1, 2, 3)$, $(2, 1, 3)$, and $(0, 0, 0)$.

b. $(1, 1, 1, -1)$, $(1, 0, 1, 0)$ and $(0, 0, 0, 0)$.

2.1.6

In each item the set is called Q. For some items, there are other correct ways to show that Q is not a vector space.

a. It is not closed under addition.
$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Q$$

b. It is not closed under addition.
$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Q$$

c. It is not closed under addition.
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

d. It is not closed under scalar multiplication.
$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

e. The set is empty, violating the existance of the zero vector.

References

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