CEGEP Linear Algebra Problems

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CEGEP level linear algebra problems

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January 15, 2016

http://obeymath.org/CEGEPLinearAlgebraProblems

Contents

Chapter 1

Systems of Linear Equations

1.1 Gaussian and Gauss-Jordan Elimination

1.1.1 by Jim Hefferon [JH] Use Gauss's Method to find the unique solution for each system.

a.
$$\begin{array}{rcrcr} 2x & + & 3y & = & 13 \\ x & - & y & = & -1 \end{array}$$

b.
$$\begin{array}{rcrcrcr} x & & & - & z & = & 0 \\ 3x & + & y & & & = & 1 \\ -x & + & y & + & z & = & 4 \end{array}$$

Chapter 2

Vector Spaces

2.1 Introduction to Vector Spaces

2.1.1 [**JH**] Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.
b. The space of 2×3 matrices.
c. The space $\{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \}$.
d. The space of real-valued functions of one natural number variable.

2.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3 - 2x + x^2$.
b. In the space $\mathcal{M}_{2\times2}$,
$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$
c. In $\{ ae^x + be^{-x} \mid a, b \in \mathbb{R} \}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

2.1.3 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{ a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \}$ under the usual polynomial addition and scalar multiplication operations.
b. The set of linear polynomials $\{ a_0 + a_1 x \mid a_0 - 2a_1 = 0 \}$, under the usual polynomial addition and scalar multiplication operations.

2.1.4 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.
b. The set of 2×2 matrices with real entries where the $2,1$ entry is zero, under the usual matrix operations.

2.1.5 by Jim Hefferon [**JH**] For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.
b. The set
$$\{ (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y - z + w = 0 \}$$
under the operations inherited from $\mathbb{R}^4$.

2.1.6 [**JH**] Show that the following are not vector spaces.

a. Under the operations inherited from $\mathbb{R}^3$, this set
$$\{ (x,\ y,\ z) \in \mathbb{R}^3 \mid x + y + z = 1 \}$$
b. Under the operations inherited from $\mathbb{R}^3$, this set
$$\{ (x,\ y,\ z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$$
c. Under the usual matrix operations,
$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$
d. Under the usual polynomial operations,
$$\{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+ \}$$
where $\mathbb{R}^+$ is the set of reals greater than zero
e. Under the inherited operations,
$$\{ (x,\ y) \in \mathbb{R}^2 \mid x + 3y = 4,\ 2x - y = 3 \text{ and } 6x + 4y = 10 \}$$

2.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

2.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

2.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

b. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

2.1.10 [**JH**] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

c. $\{(x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y + w = 1\}$

d. The set of functions $\{f : \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 0\}$

e. The set of functions $\{f : \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 1\}$

2.1.11 [**JH**] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x + y$' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret '$r \cdot x$' as the r-th power of x.

2.1.12 [**JH**] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

2.1.13 [**JH**]
Is $\{(x,\ y) \mid x, y \in \mathbb{R}\}$ a vector space under these operations?

a. $(x_1,\ y_1)+(x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r\cdot(x, y) = (rx, y)$

b. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x,\ y) = (rx,\ 0)$

2.1.14 [**JH**]
Prove the following:

a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

2.1.15 [**JH**]
The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

2.1.16 [**JH**]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

2.1.17 [**JH**]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

2.1.18 [**JH**]
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.

b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

c. Prove that any nontrivial vector space is infinite.

2.2 Subspaces

2.3 Spanning Sets

2.4 Linear Independence

2.5 Basis

2.6 Dimension

Answers to Exercises

1.1.1

a. $x = 2$, $y = 3$

b. $x = -1$, $y = 4$, and $z = -1$.

2.1.1

a. $0 + 0x + 0x^2 + 0x^3$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. The constant function $f(x) = 0$

d. The constant function $f(n) = 0$

2.1.2

a. $3 + 2x - x^2$

b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

c. $-3e^x + 2e^{-x}$

2.1.3

a. $1 + 2x$, $2 - 1x$, and x.

b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

2.1.4

a.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.1.5

a. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.

b. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

2.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

a. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

b. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

c. It is not closed under addition.
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

d. It is not closed under scalar multiplication.
$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

e. The set is empty, violating the existance of the zero vector.

2.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

2.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

2.1.9

a. It is not a vector space.
$$(1+1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b. It is not a vector space.
$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

2.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

a. Yes.

b. Yes.

c. No, this set is not closed under the natural addition operation. The vector of all $1/4$'s is an element of this set but when added to itself the result, the vector of all $1/2$'s, is not an element of the set.

d. Yes.

e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

2.1.11 Check all 10 conditions of the definition of a vector space.

2.1.12 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

2.1.13

a. No since $1 \cdot (0,\ 1) + 1 \cdot (0,\ 1) \neq (1 + 1) \cdot (0,\ 1)$.
b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0,\ 1) \neq (0,\ 1)$.

2.1.14

a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.
b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v}+\vec{v}+\vec{s} = -\vec{v}+\vec{v}+\vec{t}$, which implies that $\vec{0}+\vec{s} = \vec{0}+\vec{t}$ and so $\vec{s} = \vec{t}$.

2.1.15
Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

2.1.16
It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

2.1.17
Each element of a vector space has one and only one additive inverse.
For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v}+\vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1+(\vec{v}+\vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

2.1.18
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.
b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.
c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{\, k \cdot \vec{v} \ \mid\ k \in \mathbb{R}\}$. By the prior item this set is infinite.

References

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