CEGEP Linear Algebra Problems

An open source collection of
CEGEP level linear algebra problems

Edited by

Yann Lamontagne,
Sylvain Muise

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Contents

Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 [**GH**] State which of the following equations is a linear equation. If it is not, state why.

a. $x + y + z = 10$

b. $xy + yz + xz = 1$

c. $-3x + 9 = 3y - 5z + x - 7$

d. $\sqrt{5}y + \pi x = -1$

e. $(x-1)(x+1) = 0$

f. $\sqrt{x_1^2 + x_2^2} = 25$

g. $x_1 + y + t = 1$

h. $\frac{1}{x} + 9 = 3\cos(y) - 5z$

i. $\cos(15)y + \frac{x}{4} = -1$

j. $2^x + 2^y = 16$

1.1.2 [**GH**] Solve the system of linear equations using substitution, comparison and/or elimination.

a. $\begin{aligned} x + y &= -1 \\ 2x - 3y &= 8 \end{aligned}$

b. $\begin{aligned} 2x - 3y &= 3 \\ 3x + 6y &= 8 \end{aligned}$

c. $\begin{aligned} x - y + z &= 1 \\ 2x + 6y - z &= -4 \\ 4x - 5y + 2z &= 0 \end{aligned}$

d. $\begin{aligned} x + y - z &= 1 \\ 2x + y &= 2 \\ y + 2z &= 0 \end{aligned}$

1.1.3 [**KK**] Graphically, find the point of intersection of the two lines $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect.

1.1.4 [**GH**] Convert the given system of linear equations into an augmented matrix.

a. $\begin{aligned} 3x + 4y + 5z &= 7 \\ -x + y - 3z &= 1 \\ 2x - 2y + 3z &= 5 \end{aligned}$

b. $\begin{aligned} 2x + 5y - 6z &= 2 \\ 9x - 8z &= 10 \\ -2x + 4y + z &= -7 \end{aligned}$

c. $\begin{aligned} x_1 + 3x_2 - 4x_3 + 5x_4 &= 17 \\ -x_1 + 4x_3 + 8x_4 &= 1 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 &= 6 \end{aligned}$

d. $\begin{aligned} 3x_1 - 2x_2 &= 4 \\ 2x_1 &= 3 \\ -x_1 + 9x_2 &= 8 \\ 5x_1 - 7x_2 &= 13 \end{aligned}$

1.1.5 [**GH**] Convert given augmented matrix into a system of linear equations. Use the variables x_1, x_2, ….

a. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array}\right]$

b. $\left[\begin{array}{cc|c} -3 & 4 & 7 \\ 0 & 1 & -2 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & 5 & 7 \end{array}\right]$

d. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

e. $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 7 & 2 \\ 0 & 1 & 3 & 2 & 0 & 5 \end{array}\right]$

1.1.6 [**GH**] Perform the given row operations on

$$\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}.$$

a. $-1R_1 \to R_1$

b. $R_2 \leftrightarrow R_3$

c. $R_1 + R_2 \to R_2$

d. $2R_2 + R_3 \to R_3$

e. $\frac{1}{2}R_2 \to R_2$

f. $-\frac{5}{2}R_1 + R_3 \to R_3$

1.1.7 [**GH**] Give the row operation that transforms A into B where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

a. $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

b. $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $B = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

d. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

e. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

1.1.8 [**JH**] In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy-plane. By geometrical reasoning, show that there are three possibilities: there is a unique solution, there is no solution, and there are infinitely many solutions.

1.1.9 [**JH**] Is there a two-unknowns linear system whose solution set is all of $\mathbb{R}^2$?

1.1.10 [**KK**] You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

a. No solution.

b. A unique solution.

c. An infinite number of solutions.

1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [**GH**] State whether or not the given matrices are in reduced row echelon form.

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

i. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

j. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

m. $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

n. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

o. $\begin{bmatrix} 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.2.2 [**SZ**] State whether the given matrix is in reduced row echelon form, row echelon form only or in neither of those forms.

a. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \end{bmatrix}$

1.2.3 [**KK**] Consider the following augmented matrix in which $\ast$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

a. $\left[\begin{array}{ccccc|c} \blacksquare & \ast & \ast & \ast & \ast & \ast \\ 0 & \blacksquare & \ast & \ast & 0 & \ast \\ 0 & 0 & \blacksquare & \ast & \ast & \ast \\ 0 & 0 & 0 & 0 & \blacksquare & \ast \end{array}\right]$

b. $\left[\begin{array}{ccc|c} \blacksquare & \ast & \ast & \ast \\ 0 & \blacksquare & \ast & \ast \\ 0 & 0 & \blacksquare & \ast \end{array}\right]$

c. $\left[\begin{array}{ccccc|c} \blacksquare & \ast & \ast & \ast & \ast & \ast \\ 0 & \blacksquare & 0 & \ast & 0 & \ast \\ 0 & 0 & 0 & \blacksquare & \ast & \ast \\ 0 & 0 & 0 & 0 & \blacksquare & \ast \end{array}\right]$

d. $\left[\begin{array}{ccccc|c} \blacksquare & \ast & \ast & \ast & \ast & \ast \\ 0 & \blacksquare & \ast & \ast & 0 & \ast \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & \ast & \blacksquare \end{array}\right]$

1.2.4 [**SZ**] The following matrices are in reduced row echelon form. Determine the solution of the corresponding system of linear equations or state that the system is inconsistent.

a. $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 6 & -6 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$

d. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

e. $\left[\begin{array}{cccc|c} 1 & 0 & -8 & 1 & 7 \\ 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

f. $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

1.2.5 [**GH**] Use Gauss-Jordan elimination to put the given matrix into reduced row echelon form.

a. $\begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 12 \\ -2 & -6 \end{bmatrix}$

d. $\begin{bmatrix} -5 & 7 \\ 10 & 14 \end{bmatrix}$

e. $\begin{bmatrix} -1 & 1 & 4 \\ -2 & 1 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 7 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

g. $\begin{bmatrix} 3 & -3 & 6 \\ -1 & 1 & -2 \end{bmatrix}$

h. $\begin{bmatrix} 4 & 5 & -6 \\ -12 & -15 & 18 \end{bmatrix}$

i. $\begin{bmatrix} -2 & -4 & -8 \\ -2 & -3 & -5 \\ 2 & 3 & 6 \end{bmatrix}$

j. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & -3 & 0 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 6 & 9 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$

n. $\begin{bmatrix} 2 & -1 & 1 & 5 \\ 3 & 1 & 6 & -1 \\ 3 & 0 & 5 & 0 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 1 & -1 & 7 \\ 2 & 1 & 0 & 10 \\ 3 & 2 & -1 & 17 \end{bmatrix}$

p. $\begin{bmatrix} 4 & 1 & 8 & 15 \\ 1 & 1 & 2 & 7 \\ 3 & 1 & 5 & 11 \end{bmatrix}$

1.2.6 [**JH**] Use Gauss's Method to find the unique solution for each system.

a. $\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$

b. $\begin{aligned} x - z &= 0 \\ 3x + y &= 1 \\ -x + y + z &= 4 \end{aligned}$

1.2.7 [**SZ**] Solve the following systems of linear equations.

a. $\begin{aligned} -5x + y &= 17 \\ x + y &= 5 \end{aligned}$

b. $\begin{aligned} x + y + z &= 3 \\ 2x - y + z &= 0 \\ -3x + 5y + 7z &= 7 \end{aligned}$

c. $\begin{aligned} 4x - y + z &= 5 \\ 2y + 6z &= 30 \\ x + z &= 5 \end{aligned}$

d. $\begin{aligned} x - 2y + 3z &= 7 \\ -3x + y + 2z &= -5 \\ 2x + 2y + z &= 3 \end{aligned}$

e. $\begin{aligned} 3x - 2y + z &= -5 \\ x + 3y - z &= 12 \\ x + y + 2z &= 0 \end{aligned}$

f. $\begin{aligned} 2x - y + z &= -1 \\ 4x + 3y + 5z &= 1 \\ 5y + 3z &= 4 \end{aligned}$

g. $\begin{aligned} x - y + z &= -4 \\ -3x + 2y + 4z &= -5 \\ x - 5y + 2z &= -18 \end{aligned}$

h. $\begin{aligned} 2x - 4y + z &= -7 \\ x - 2y + 2z &= -2 \\ -x + 4y - 2z &= 3 \end{aligned}$

i. $\begin{aligned} 2x - y + z &= 1 \\ 2x + 2y - z &= 1 \\ 3x + 6y + 4z &= 9 \end{aligned}$

j. $\begin{aligned} x - 3y - 4z &= 3 \\ 3x + 4y - z &= 13 \\ 2x - 19y - 19z &= 2 \end{aligned}$

k. $\begin{aligned} x + y + z &= 4 \\ 2x - 4y - z &= -1 \\ x - y &= 2 \end{aligned}$

l. $\begin{aligned} x - y + z &= 8 \\ 3x + 3y - 9z &= -6 \\ 7x - 2y + 5z &= 39 \end{aligned}$

1.2.8 [**GH**] Find the solution to the given linear system. If the system has infinite solutions, give two particular solutions.

a. $\begin{aligned} 2x_1 + 4x_2 &= 2 \\ x_1 + 2x_2 &= 1 \end{aligned}$

b. $\begin{aligned} -x_1 + 5x_2 &= 3 \\ 2x_1 - 10x_2 &= -6 \end{aligned}$

c. $\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + x_2 &= 4 \end{aligned}$

d. $\begin{aligned} -3x_1 + 7x_2 &= -7 \\ 2x_1 - 8x_2 &= 8 \end{aligned}$

e. $\begin{aligned} -2x_1 + 4x_2 + 4x_3 &= 6 \\ x_1 - 3x_2 + 2x_3 &= 1 \end{aligned}$

f. $\begin{aligned} -x_1 + 2x_2 + 2x_3 &= 2 \\ 2x_1 + 5x_2 + x_3 &= 2 \end{aligned}$

g. $\begin{aligned} -x_1 - x_2 + x_3 + x_4 &= 0 \\ -2x_1 - 2x_2 + x_3 &= -1 \end{aligned}$

h. $\begin{aligned} x_1 + x_2 + 6x_3 + 9x_4 &= 0 \\ x_1 + x_3 + 2x_4 &= 3 \end{aligned}$

i. $\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ 3x_1 + 3x_2 + 5x_3 &= 2 \end{aligned}$

j. $\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2 \\ 1x_1 + 2x_2 + 3x_3 &= 1 \\ 3x_1 + 6x_2 + 9x_3 &= 3 \end{aligned}$

k. $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -2x_1 - 3x_2 &= 1 \end{aligned}$

l. $\begin{aligned} 2x_1 + x_2 + 2x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 1 \\ 3x_1 + 2x_2 + 5x_3 &= 3 \end{aligned}$

1.2.9 [**KK**] Find the solution to the system of equations, $65x + 84y + 16z = 546$, $81x + 105y + 20z = 682$, and $84x + 110y + 21z = 713$.

1.2.10[**YL**] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 + x_5 &= 3 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 + 17x_3 - 2x_4 - x_5 &= 1 \end{aligned}$$

a. Solve the following system by Gauss-Jordan elimination.

b. Find two particular solution to the above system.

c. Find a solution to the above system when $x_3 = 1$.

1.2.11[YL] Given

$$\begin{aligned} 3x_1 + 3x_2 + \ \ 7x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + \ \ 3x_3 + \ \ x_4 &= 0 \\ 4x_1 \qquad\quad + 17x_3 - 2x_4 &= 0 \\ 9x_1 + 6x_2 + 27x_3 - 4x_4 &= 0 \end{aligned}$$

a. Solve the system by Gauss-Jordan elimination.

b. Find two particular nontrivial solution to the system.

c. Find a solution to the system when $x_1 = 1$.

1.2.12 [SZ] Find at least two different row echelon forms for the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 12 & 8 \end{bmatrix}$$

1.2.13 [JH] Find the coefficients a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.2.14 [JH] True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.2.15 [KK] If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, explain why.

1.2.16 [KK] If a system of linear equations has fewer equations than variables and there exist a solution to this system. Is it possible that your solution is the only one? Explain.

1.2.17 [JH] For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - \ \ y &= 1 \\ 3x - 3y &= k \end{aligned}$$

1.2.18 [GH] State for which values of k the given system will have exactly 1 solution, infinite solutions, or no solution.

a. $\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= k \end{aligned}$

b. $\begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 + kx_2 &= 1 \end{aligned}$

c. $\begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 + kx_2 &= 2 \end{aligned}$

d. $\begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 + 3x_2 &= k \end{aligned}$

1.2.19 [KK] Choose h and k such that the augmented matrix shown has each of the following: one solution, no solution and infinitely many solutions.

a. $\left[\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array}\right]$

b. $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array}\right]$

1.2.20[YL] Given the augmented matrix of a linear system:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a^2 - 1 & b^2 - a^2 \end{bmatrix}$$

If possible for what values of a and b the system has

a. no solution? Justify.

b. exactly one solution? Justify.

c. infinitely many solutions? Justify.

1.2.21[YL] Given the augmented matrix of a linear system

$$\begin{bmatrix} 1 & 3 & 1 & -4 & b_1 \\ 3 & -2 & 4 & 5 & b_2 \\ 4 & 1 & 5 & 1 & b_3 \\ 7 & -1 & 9 & 6 & b_4 \end{bmatrix}.$$

Determine the restrictions on the b_i's for the system to be consistent.

1.2.22 [JH] Prove that, where $a, b, \ldots, e$ are real numbers and $a \neq 0$, if

$$ax + by = c$$

has the same solution set as

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

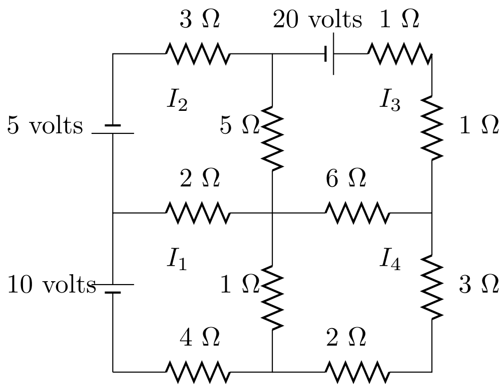
1.2.23 [JH] Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

1.3 Applications of Linear Systems

1.3.1 [**KK**] Consider the following diagram of four circuits.



The jagged lines denote resistors and the numbers next to them give their resistance in ohms, written as Ω. The breaks in the lines having one short line and one long line denote a voltage source which causes the current to flow in the direction which goes from the longer of the two lines toward the shorter along the unbroken part of the circuit. The current in amps in the four circuits is denoted by I_1, I_2, I_3, I_4 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction. Then Kirchhoff's law states:
The sum of the resistance times the amps in the counter clockwise direction around a loop equals the sum of the voltage sources in the same direction around the loop.
In the above diagram, the top left circuit gives the equation

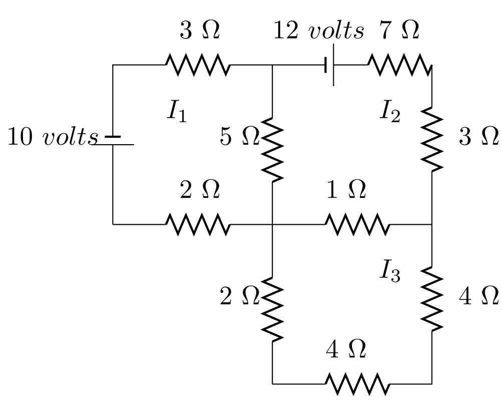
$$2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 = 5$$

For the circuit on the lower left,

$$4I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -10$$

Write equations for each of the other two circuits and then give a solution to the resulting system of equations.

1.3.2 [**KK**] Consider the following diagram of three circuits.



The current in amps in the four circuits is denoted by I_1, I_2, I_3 and it is understood that the motion is in the counter clockwise direction. Solve for I_1, I_2, I_3.

Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [**JH**] Find the indicated entry of the following matrix.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{bmatrix}$$

a. $a_{2,1}$ **b**. $a_{1,2}$ **c**. $a_{2,2}$ **d**. $a_{3,1}$

2.1.2 [**JH**] Determine the size of each matrix.

a. $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ **b**. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{bmatrix}$ **c**. $\begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$

2.1.3 [**GH**] Simplify the given expression where

$$A = \begin{bmatrix} 1 & -1 \\ 7 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 \\ 5 & 9 \end{bmatrix}$$

a. $A + B$

b. $2A - 3B$

c. $3(A - B) + B$

d. $2(A - B) - (A - 3B)$

2.1.4 [**GH**] The row and column matrix U and V are defined. Find the product UV, where possible.

a. $U = \begin{bmatrix} 1 & -4 \end{bmatrix}$, $V = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

b. $U = \begin{bmatrix} 6 & 2 & -1 & 2 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \\ 9 \\ 5 \end{bmatrix}$

c. $U = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

d. $U = \begin{bmatrix} 2 & -5 \end{bmatrix}$, $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2.1.5 [**JH**] Give the size of the product or state "not defined".

a. a 2×3 matrix times a 3×1 matrix

b. a 1×12 matrix times a 12×1 matrix

c. a 2×3 matrix times a 2×1 matrix

d. a 2×2 matrix times a 2×2 matrix

2.1.6 [**GH**] State the dimensions of A and B. State the dimensions of AB and BA, if the product is defined. Then compute the product AB and BA, if possible.

a. $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 7 \\ 4 & 2 & 9 \end{bmatrix}$

c. $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$

d. $A = \begin{bmatrix} -2 & -1 \\ 9 & -5 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 6 & -4 \\ 0 & 6 & -3 \end{bmatrix}$

e. $A = \begin{bmatrix} 2 & 6 \\ 6 & 2 \\ 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 5 & 0 \\ -4 & 4 & -4 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 4 \\ 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -5 & 5 \\ -2 & 1 & 3 & -5 \end{bmatrix}$

g. $A = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $A = \begin{bmatrix} -4 & -1 & 3 \\ 2 & -3 & 5 \\ 1 & 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 & 3 \\ -1 & 1 & -1 \\ 4 & 0 & 2 \end{bmatrix}$

2.1.7 [**GH**] Given a diagonal matrix D and a matrix A, compute the product DA and AD, if possible.

a. $D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}$, $A = \begin{bmatrix} & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

b. $D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

c. $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

d. $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

2.1.8 [GH] Given a matrix A compute A^2 and A^3.

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

d. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

e. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

2.1.9 [HE] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

a. $3C - 4D$

b. $A - (D + 2C)$

c. $A - E$

d. AE

e. $3BC - 4BD$

f. $CB + D$

g. GC

h. FG

i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A, B, and C are matrices above.

2.1.10 [SZ] Use the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 10 & -\frac{11}{2} & 0 \\ \frac{3}{5} & 5 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & -13 \\ -\frac{4}{3} & 0 \\ 6 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

to compute the following or state that the indicated operation is undefined.

a. $7B - 4A$

b. AB

c. BA

d. $E + D$

e. ED

f. $CD + 2I_2A$

g. $A - 4I_2$

h. $A^2 - B^2$

i. $(A + B)(A - B)$

j. $A^2 - 5A - 2I_2$

k. $E^2 + 5E - 36I_3$

l. EDC

m. CDE

n. $ABCEDI_2$

2.1.11 [GH] In each part a matrix A is given. Find A^T. State whether A is upper/lower triangular, diagonal, symmetric and/or skew symmetric.

a. $\begin{bmatrix} -9 & 4 & 10 \\ 6 & -3 & -7 \\ -8 & 1 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 2 & -9 \\ 5 & -4 & -10 \\ -6 & 6 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -7 & -4 & -9 \\ -9 & 6 & 3 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$

e. $\begin{bmatrix} 4 & 0 & 0 \\ -2 & -7 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

f. $\begin{bmatrix} -3 & -4 & -5 \\ 0 & -3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$

2.1.12 [KK] Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find the following if possible. If it is not possible explain why.

a. $-3A^T$

b. $3B - A^T$

c. E^TB

d. EE^T

e. B^TB

f. CA^T

g. D^TBE

2.1.13 [JH] Show that if G has a row of zeros then GH (if defined) has a row of zeros. Does the same statement hold for columns?

2.1.14 [JH] Show that if the first and second rows of G are equal then so are the first and second rows of GH. Generalize.

2.1.15 [JH] Describe the product of two diagonal matrices.

2.1.16 [JH] Show that the product of two upper triangular matrices is upper triangular. Does this also hold for lower triangular matrices?

2.1.17 [KK] Show that the main diagonal of every skew symmetric matrix consists of only zeros.

2.1.18 [GH] Find the trace of the given matrix.

a. $\begin{bmatrix} 4 & 1 & 1 \\ -2 & 0 & 0 \\ -1 & -2 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -5 \\ 9 & 5 \end{bmatrix}$

c. $\begin{bmatrix} -10 & 6 & -7 & -9 \\ -2 & 1 & 6 & -9 \\ 0 & 4 & -4 & 0 \\ -3 & -9 & 3 & -10 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 6 & 4 \\ -1 & 8 & -10 \end{bmatrix}$

e. Any skew-symmetric matrix.

f. I_n

2.1.19 [YL] Consider the matrices:

$$A = \left[a_{ij}\right]_{3\times3}, \quad B = \left[b_{ij}\right]_{2\times3}, \quad C = \left[c_{ij}\right]_{3\times2},$$

$$D = \left[d_{ij}\right]_{2\times2}, \quad E = \left[e_{ij}\right]_{3\times6}$$

where $a_{ij} = i - j$, $b_{ij} = (-1)^i 2 + (-1)^j 3$, $c_{ij} = i + j$, $d_{ij} = (ij)^2$, $e_{ij} = i + j$. Evaluate the following if possible, justify.

a. EA

b. AB^T

c. $C^T AB^T$

d. $\operatorname{tr}(D^2)$

e. $(3BC - 2D)^T$

f. $\operatorname{tr}(E)$

2.1.20 [JH] Find the product of this matrix with its transpose.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2.1.21 [KK] A matrix A is called *idempotent* if $A^2 = A$. Show that the following matrix is idempotent.

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

2.1.22 [GH] Find values for the scalars a and b that satisfy the given equation.

a. $a\begin{bmatrix} -3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

b. $a\begin{bmatrix} 4 \\ 2 \end{bmatrix} + b\begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

c. $a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

d. $a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$

2.1.23 [KK] Let $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$. Find all 2×2 matrices, B such that $AB = 0$.

2.1.24 [YL] Solve for all a, b, c, d such that

$$\begin{bmatrix} 3a + 3b + 7c - 3d & 2a + 3b + 3c + d \\ 4a + 17c - 2d & 9a + 6b + 27c - 4d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

2.2 Algebraic Properties of Matrices

2.2.1 [GH] Given the matrices A and B below. Find X that satisfies the equation.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}$$

a. $2A + X = B$

b. $A - X = 3B$

c. $3A + 2X = -1B$

d. $A - \frac{1}{2}X = -B$

2.2.2 [GH] The following statement

$$(A + B)^2 = A^2 + 2AB + B^2$$

is false. We investigate that claim here.

a. Let $A = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} -5 & -5 \\ -2 & 1 \end{bmatrix}$. Compute $A + B$

b. Find $(A + B)^2$ by using the previous part.

c. Compute $A^2 + 2AB + B^2$.

d. Are the results from the two previous parts equal?

e. Carefully expand the expression $(A+B)^2 = (A+B)(A+B)$ and show why this is not equal to $A^2 + 2AB + B^2$.

2.2.3 [KK] Suppose A and B are square matrices of the same size. Which of the following are necessarily true? Justify.

a. $(A - B)^2 = A^2 - 2AB + B^2$

b. $(AB)^2 = A^2B^2$

c. $(A + B)^2 = A^2 + 2AB + B^2$

d. $(A + B)^2 = A^2 + AB + BA + B^2$

e. $A^2B^2 = A\,(AB)\,B$

f. $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

g. $(A + B)\,(A - B) = A^2 - B^2$

2.2.4 [JH] Prove each, assuming that the operations are defined, where G, H, and J are matrices, where Z is the zero matrix, and where r and s are scalars.

a. Matrix addition is commutative $G + H = H + G$.

b. Matrix addition is associative $G + (H + J) = (G + H) + J$.

c. The zero matrix is an additive identity $G + Z = G$.

d. $0 \cdot G = Z$

e. $(r + s)G = rG + sG$

f. Matrices have an additive inverse $G + (-1) \cdot G = Z$.

g. $r(G + H) = rG + rH$

h. $(rs)G = r(sG)$

2.2.5 [YL] Prove or disprove the following statements

a. *Cancellation Law* If A, B, C are matrices such that $AB = AC$ then $B = C$.

b. *Commutativity* If A, B are square matrices of the same size then $AB = BA$.

c. *Zero Factor Property* If A, B are matrices such that $AB = 0$ then $A = 0$ or $B = 0$

2.2.6 [**KK**] If possible find all k such that $AB = BA$.

a. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix}$

2.2.7 [**JH**]

a. Prove that $H^pH^q = H^{p+q}$ and $(H^p)^q = H^{pq}$ for positive integers p, q.

b. Prove that $(rH)^p = r^p \cdot H^p$ for any positive integer p and scalar $r \in \mathbb{R}$.

2.2.8 [**JH**]

a. Show that $(G + H)^{\mathsf{T}} = G^{\mathsf{T}} + H^{\mathsf{T}}$.

b. Show that $(r \cdot H)^{\mathsf{T}} = r \cdot H^{\mathsf{T}}$.

c. Show that $(GH)^{\mathsf{T}} = H^{\mathsf{T}}G^{\mathsf{T}}$.

d. Show that the matrices HH^{T} and $H^{\mathsf{T}}H$ are symmetric.

2.2.9 [**JH**] Prove that for any square H, the matrix $H + H^{\mathsf{T}}$ is symmetric. Do every symmetric matrix have this form?

2.2.10 [**GH**]

a. Prove that for any $n \times n$ matrix A, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

b. Prove that any $n \times n$ can be written as the sum of a symmetric and skew-symmetric matrices.

2.2.11 [**YL**] Prove the following statements

a. If A is a $n\times n$ matrix then $\operatorname{tr}(A^{\mathsf{T}}) = \operatorname{tr}(A)$.

b. If A is a $n\times n$ matrix then $\operatorname{tr}(cA) = c\operatorname{tr}(A)$.

c. If A and B are $n\times n$ matrices then $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

d. If A and B are $n\times n$ matrices then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

2.2.12 [**YL**] A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$.
Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

2.2.13 [**YL**] A square matrix A is called *idempotent* if $A^2 = A$.
Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A.

2.2.14* [**JH**] Find the formula for the n-th power of this matrix.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

2.3 Matrix Inverses and Elementary Matrices

2.3.1 [**GH**] Given the matrices A. Find A^{-1}, if possible.

a. $\begin{bmatrix} 1 & 5 \\ -5 & -24 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$

2.3.2 [**GH**] Given the matrices A and B. Compute $(AB)^{-1}$ and $B^{-1}A^{-1}$.

a. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 5 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 1 \\ 2 & 1 \end{bmatrix}$

2.3.3 [**KK**] Show $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that

$$AB\left(B^{-1}A^{-1}\right) = I \text{ and } B^{-1}A^{-1}(AB) = I$$

2.3.4 [**YL**] Solve for X given that it satisfies

$$DXD^T = \operatorname{tr}(BC)BC$$

where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} \; C = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} \; D = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}.$$

2.3.5 [**YL**] Solve for A given that it satisfies

$$(I - A^T)^{-1} = (\operatorname{tr}(B)B^2)^T$$

where

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.3.6 [**YL**] Find a formula for $\operatorname{tr}(A^{-1})$ if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible.

2.3.7 [**JH**] In real number algebra, there are exactly two numbers, 1 and -1, that are their own multiplicative inverse. Does $H^2 = I$ have exactly two solutions for 2×2 matrices?

2.3.8 [**GH**] Prove or disprove: If A and B are 2×2 invertible matrices then $A + B$ is an invertible matrix.

2.3.9 [**JH**] What is the inverse of rH?

2.3.10 [**JH**] Predict the result of each multiplication by an elementary matrix, and then verify by performing the multiplication.

a. $\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ **b.** $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ **c.** $\begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

2.3.11 [JH] Find

a. a 3×3 matrix that, acting from the left, swaps rows one and two.

b. a 2×2 matrix that, acting from the right, swaps column one and two.

2.3.12 [SZ] Let $A = \begin{bmatrix}a & b & c\\d & e & f\end{bmatrix}$ $E_1 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ $E_2 = \begin{bmatrix}5 & 0\\0 & 1\end{bmatrix}$ $E_3 = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$.
Compute E_1A, E_2A and E_3A. What effect did each of the E_i matrices have on the rows of A? Create E_4 so that its effect on A is to multiply the bottom row by -6.

2.3.13 [JH] Write

$$\begin{bmatrix}1 & 0\\-3 & 3\end{bmatrix}$$

as the product of two elementary reduction matrices.

2.3.14 [GH] Given the matrices A. Find A^{-1}, if possible.

a. $\begin{bmatrix}25 & -10 & -4\\-18 & 7 & 3\\-6 & 2 & 1\end{bmatrix}$

b. $\begin{bmatrix}1 & 0 & 0\\4 & 1 & -7\\20 & 7 & -48\end{bmatrix}$

c. $\begin{bmatrix}-4 & 1 & 5\\-5 & 1 & 9\\-10 & 2 & 19\end{bmatrix}$

d. $\begin{bmatrix}1 & -5 & 0\\-2 & 15 & 4\\4 & -19 & 1\end{bmatrix}$

e. $\begin{bmatrix}25 & -8 & 0\\-78 & 25 & 0\\48 & -15 & 1\end{bmatrix}$

f. $\begin{bmatrix}1 & 0 & 0\\7 & 5 & 8\\-2 & -2 & -3\end{bmatrix}$

g. $\begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$

h. $\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$

i. $\begin{bmatrix}2 & 3 & 4\\-3 & 6 & 9\\-1 & 9 & 13\end{bmatrix}$

j. $\begin{bmatrix}5 & -1 & 0\\7 & 7 & 1\\-2 & -8 & -1\end{bmatrix}$

k. $\begin{bmatrix}1 & 0 & 0 & 0\\-19 & -9 & 0 & 4\\33 & 4 & 1 & -7\\4 & 2 & 0 & -1\end{bmatrix}$

l. $\begin{bmatrix}1 & 0 & 0 & 0\\27 & 1 & 0 & 4\\18 & 0 & 1 & 4\\4 & 0 & 0 & 1\end{bmatrix}$

m. $\begin{bmatrix}1 & 0 & 2 & 8\\0 & 1 & 0 & 0\\0 & -4 & -29 & -110\\0 & -3 & -5 & -19\end{bmatrix}$

n. $\begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}$

o. $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & -4\end{bmatrix}$

2.3.15 [SZ] Find the inverse of the matrix or state that the matrix is not invertible.

a. $\begin{bmatrix}3 & 0 & 4\\2 & -1 & 3\\-3 & 2 & -5\end{bmatrix}$

b. $\begin{bmatrix}4 & 6 & -3\\3 & 4 & -3\\1 & 2 & 6\end{bmatrix}$

c. $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 11\\3 & 4 & 19\end{bmatrix}$

d. $\begin{bmatrix}1 & 0 & -3 & 0\\2 & -2 & 8 & 7\\-5 & 0 & 16 & 0\\1 & 0 & 4 & 1\end{bmatrix}$

2.3.16 [KK] Give an example of a matrix A such that $A^2 = I$, $A \neq I$ and $A \neq -I$.

2.3.17 [YL] Given

$$A = \begin{bmatrix}2 & 2 & 0\\4 & 3 & 0\\3 & 2 & \frac{1}{2}\end{bmatrix}.$$

a. Find A^{-1}.

b. Solve for X where $AX = B$ and

$$B = \begin{bmatrix}1 & 0 & \frac{1}{2} & 0 & 0\\0 & 1 & 0 & 2 & -1\\-4 & 2 & \frac{1}{2} & 0 & 0\end{bmatrix}$$

c. Find $\left(\left(\frac{1}{2}A\right)^{\mathsf{T}}\right)^{-1}$ if possible.

2.3.18 [YL] Write the given matrix as a product of elementary matrices

$$\begin{bmatrix}0 & 0 & 1\\1 & 3 & 0\\2 & 4 & 0\end{bmatrix}$$

2.3.19 [YL] Express

$$A = \begin{bmatrix}1 & 0 & 0\\1 & 0 & 1\\0 & 2 & 1\end{bmatrix}$$

as a product of 4 elementary matrices.

2.3.20 [KK] Given matrices A and B and suppose a row operation is applied to A and the result is B. Find the elementary matrix E such that $EA = B$. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

a. $A = \begin{bmatrix}1 & 2 & 1\\0 & 5 & 1\\2 & -1 & 4\end{bmatrix}$ and $B = \begin{bmatrix}1 & 2 & 1\\2 & -1 & 4\\0 & 5 & 1\end{bmatrix}$

b. $A = \begin{bmatrix}1 & 2 & 1\\0 & 5 & 1\\2 & -1 & 4\end{bmatrix}$ and $B = \begin{bmatrix}1 & 2 & 1\\0 & 5 & 1\\1 & -\frac{1}{2} & 2\end{bmatrix}$

c. $A = \begin{bmatrix}1 & 2 & 1\\0 & 5 & 1\\2 & -1 & 4\end{bmatrix}$ and $B = \begin{bmatrix}1 & 2 & 1\\2 & 4 & 5\\2 & -1 & 4\end{bmatrix}$

2.3.21 [**YL**] Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3 E_2 E_1 A = B$.

2.3.22 [**YL**] Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$\left(X^{\mathsf{T}} E_1 E_2 E_3\right)^{\mathsf{T}} = A$$

solve for X, if possible.

2.3.23 [**JH**] Prove that any matrix row-equivalent to an invertible matrix is also invertible.

2.3.24 [**YL**] Prove: If E_i are elementary matrices and $E_n \cdots E_2 E_1 A$ is invertible then A is invertible.

2.3.25 [**KK**] Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain.

2.3.26 [**JH**] Assume that H is invertible and that HG is the zero matrix. Show that G is a zero matrix.

2.3.27 [**JH**] Prove that if H is invertible then the inverse commutes with a matrix $GH^{-1} = H^{-1}G$ if and only if H itself commutes with that matrix $GH = HG$.

2.3.28 [**YL**] Show that if A is invertible then its inverse is unique.

2.3.29 [**JH**] Prove: If T is an invertible matrix and k is a natural number then $(T^k)^{-1} = (T^{-1})^k$.

2.3.30 [**KK**] Prove: If A is invertible, then $\left(A^{-1}\right)^{-1} = A$.

2.3.31 [**KK**] Show that if A is an invertible $n \times n$ matrix, then so is A^{T} and $\left(A^{\mathsf{T}}\right)^{-1} = \left(A^{-1}\right)^{\mathsf{T}}$.

2.3.32 [**JH**] Show that if T is square and if T^4 is the zero matrix then $(I - T)^{-1} = I + T + T^2 + T^3$.

2.3.33 [**YL**] Prove: If B and C are $n \times n$ matrices such that $A = B^T C + C^T B$ is invertible then A^{-1} is symmetric.

2.3.34 [**YL**] Given a matrix A which satisfy $A^3 + 3A^2 + A + I = 0$ find the inverse of A in terms of A and the identity.

2.3.35 [**YL**] Prove: If AB and BA are both invertible then A and B are both invertible.

2.3.36 [**JH**] Prove or disprove: Nonsingular matrices commute.

2.3.37 [**YL**] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.4 Linear Systems and Matrices

2.4.1 [**YL**] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1}.

b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

2.4.2 [**GH**] Given the matrices A and b below. Find x that satisfies the equation $Ax = b$ by using the inverse of A

a. $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$,
$b = \begin{bmatrix} 21 \\ 13 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -4 \\ 4 & -15 \end{bmatrix}$,
$b = \begin{bmatrix} 21 \\ 77 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 & 12 \\ 0 & 1 & 6 \\ -3 & 0 & 1 \end{bmatrix}$,
$b = \begin{bmatrix} -17 \\ -5 \\ 20 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 0 & -3 \\ 8 & -2 & -13 \\ 12 & -3 & -20 \end{bmatrix}$,
$b = \begin{bmatrix} -34 \\ -159 \\ -243 \end{bmatrix}$

Chapter 3

Determinants

3.1 The Laplace Expansion

3.1.1 [GH] Compute the determinant of the following matrices.

a. $\begin{bmatrix} 10 & 7 \\ 8 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 6 & -1 \\ -7 & 8 \end{bmatrix}$

c. $\begin{bmatrix} -1 & -7 \\ -5 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -10 & -1 \\ -4 & 7 \end{bmatrix}$

3.1.2 [GH] For the following matrices, construct the submatrices used to compute the minors $M_{1,1}$, $M_{1,2}$ and $M_{1,3}$. Compute the cofactors $C_{1,1}$, $C_{1,2}$, and $C_{1,3}$.

a. $\begin{bmatrix} 7 & -3 & 10 \\ 3 & 7 & 6 \\ 1 & 6 & 10 \end{bmatrix}$

b. $\begin{bmatrix} -2 & -9 & 6 \\ -10 & -6 & 8 \\ 0 & -3 & -2 \end{bmatrix}$

c. $\begin{bmatrix} -5 & -3 & 3 \\ -3 & 3 & 10 \\ -9 & 3 & 9 \end{bmatrix}$

d. $\begin{bmatrix} -6 & -4 & 6 \\ -8 & 0 & 0 \\ -10 & 8 & -1 \end{bmatrix}$

3.1.3 [JH] Evaluate the determinant by performing a cofactor expansion

$$\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 0 \end{vmatrix}$$

a. along the first row,

b. along the second row,

c. along the third column.

3.1.4 [KK] Find the determinants of the following matrices.

a. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

3.1.5 [GH] Find the determinant of the given matrix using cofactor expansion.

a. $\begin{bmatrix} 3 & 2 & 3 \\ -6 & 1 & -10 \\ -8 & -9 & -9 \end{bmatrix}$

b. $\begin{bmatrix} 8 & -9 & -2 \\ -9 & 9 & -7 \\ 5 & -1 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -1 & 0 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

f. $\begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix}$

3.1.6 [SZ] Compute the determinant of the given matrix.

a. $\begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

b. $\begin{bmatrix} \frac{1}{x^3} & \frac{\ln(x)}{x^3} \\ -\frac{3}{x^4} & \frac{1-3\ln(x)}{x^4} \end{bmatrix}$

c. $\begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

e. $\begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

3.1.7 [JH] Verify that the determinant of an upper-triangular 3×3 matrix is the product of the main diagonal.

$$\det\left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}\right) = aei$$

Is it the same for lower triangular matrices?

3.1.8 [KK] Find the determinants of the following matrices.

a. $\begin{bmatrix} 1 & -34 \\ 0 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 3 & 14 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 3 & 15 & 0 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3.1.9 [**YL**] Solve for λ.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda-5 \end{vmatrix}$$

3.1.10 [**JH**] True or false: Can we compute a determinant by expanding down the diagonal? Justify.

3.1.11 [**JH**] Which real numbers θ make

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

equal to zero?

3.2 Determinants and Elementary Operations

3.2.1 [**KK**] An operation is done to get from the first matrix to the second. Identify the operation and how the determinant will change.

a. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

b. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$

c. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$

d. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$

e. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$

3.2.2 [**GH**] A matrix M and $\det(M)$ are given. Matrices A, B and C are obtained by performing operations on M. Determine the determinants of A, B and C and indicate the operations used to obtain A, B and C.

a. $M = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}$,
$\det(M) = -41$,

$$A = \begin{bmatrix} 18 & 14 & 16 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 96 & 73 & 83 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & 1 & 6 \\ 7 & 3 & 3 \\ 8 & 7 & 3 \end{bmatrix}.$$

b. $M = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}$,
$\det(M) = 45$,

$$A = \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & -1 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ 8 & 16 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}.$$

c. $M = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$,
$\det(M) = -16$,

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 5 & 1 & 5 \\ 4 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & -1 & -5 \\ -4 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 15 & 3 & 15 \\ 12 & 0 & 6 \\ 0 & 0 & 12 \end{bmatrix}.$$

d. $M = \begin{bmatrix} 5 & 4 & 0 \\ 7 & 9 & 3 \\ 1 & 3 & 9 \end{bmatrix}$,
$\det(M) = 120$,

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 7 & 9 & 3 \\ 5 & 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 4 & 0 \\ 14 & 18 & 6 \\ 3 & 9 & 27 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & -4 & 0 \\ -7 & -9 & -3 \\ -1 & -3 & -9 \end{bmatrix}.$$

3.2.3 [**GH**] Find the determinant of the given matrix by using elemetary operations to bring the matrix under triangular form.

a. $\begin{bmatrix} -4 & 3 & -4 \\ -4 & -5 & 3 \\ 3 & -4 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -2 & 1 \\ 5 & 5 & 4 \\ 4 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} -5 & 0 & -4 \\ 2 & 4 & -1 \\ -5 & 0 & -4 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

e. $\begin{bmatrix} -5 & 1 & 0 & 0 \\ -3 & -5 & 2 & 5 \\ -2 & 4 & -3 & 4 \\ 5 & 4 & -3 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 2 & -1 & 4 & 4 \\ 3 & -3 & 3 & 2 \\ 0 & 4 & -5 & 1 \\ -2 & -5 & -2 & -5 \end{bmatrix}$

3.2.4 [**YL**] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.2.5 Vandermonde's determinant[**JH**] Prove:

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\right) = (b-a)(c-a)(c-b)$$

3.2.6 [**KK**] Let A be an $r \times r$ matrix and suppose there are $r-1$ rows (columns) such that all rows (columns) are linear combinations of these $r-1$ rows (columns). Show $\det(A) = 0$.

3.3 Properties of Determinants and Matrix Inverses

3.3.1 [**JH**] Find the adjoint of the following matrices.

a. $\begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 1 & 8 & 9 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

3.3.2 [**JH**]

a. Find a formula for the adjoint of a 2×2 matrix.

b. Use the above to derive the formula for the inverse of a 2×2 matrix.

3.3.3 [**JH**] Derive a formula for the adjoint of a diagonal matrix.

3.3.4 [**JH**] Prove that the transpose of the adjoint is the adjoint of the transpose.

3.3.5 [**JH**] Prove or disprove: $\operatorname{adj}(\operatorname{adj}(T)) = T$.

3.3.6 [**KK**] Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

a. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

3.3.7 [**JH**] Which real numbers x make this matrix singular?

$$\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}$$

3.3.8 [**KK**] If A, B, and C are each $n\times n$ matrices and ABC is invertible, show why each of A, B, and C are invertible.

3.3.9 [**KK**] Show that if $\det(A) \neq 0$ for A an $n \times n$ matrix, it follows that if $AX = 0$, then $X = 0$.

3.3.10 [**KK**] Suppose A, B are $n \times n$ matrices and that $AB = I$ then show that $BA = I$. *Hint:* First explain why $\det(A), \det(B)$ are both nonzero. Then $(AB)A = A$ and then show $BA(BA - I) = 0$. From this use what is given to conclude $A(BA - I) = 0$.

3.3.11 [**KK**] Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non zero. Is it true that A^{-1} will also be upper triangular? Explain. Could the same be concluded for lower triangular matrices?

3.3.12 [**KK**] Consider the following matrices. Does there exist a value of t for which this matrix fails to have an inverse? Justify.

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$

b. $\begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$

c. $\begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{bmatrix}$

3.3.13 [**KK**] Use the formula for the inverse in terms of the cofactor matrix to find the inverse of the matrix.

a. $\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t\cos t & e^t\sin t \\ 0 & e^t\cos t - e^t\sin t & e^t\cos t + e^t\sin t \end{bmatrix}$

b. $\begin{bmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}$

3.3.14 [**JH**] Prove: If S and T are $n\times n$ matrix then $\det(TS) = \det(ST)$.

3.3.15 [**KK**] Show $\det(aA) = a^n \det(A)$ for an $n\times n$ matrix A and scalar a.

3.3.16 [**JH**] Prove that each statement holds for 2×2 matrices.

a. The determinant of a product is the product of the determinants $\det(ST) = \det(S)\det(T)$.

b. If T is invertible then the determinant of the inverse is the inverse of the determinant $\det(T^{-1}) = (\det(T))^{-1}$.

3.3.17 [**KK**] Prove or disprove: If A and B are square matrices of the same size then $\det(A+B) = \det(A) + \det(B)$.

3.3.18 [**JH**]

a. Suppose that $\det(A) = 3$ and that $\det(B) = 2$. Find $\det(A^2B^{\mathsf{T}}B^{-2}A^{\mathsf{T}})$.

b. If $\det(A) = 0$ then show that $\det(6A^3 + 5A^2 + 2A) = 0$.

3.3.19 [**YL**] Prove: If A is an invertible $n\times n$ matrix then $\det(\operatorname{adj}(A)) = (\det(A))^{n-1}$.

3.3.20 [**YL**] Given

$$A = \begin{bmatrix} 10 & 1 & 1 & 1 & 9 & 3 & 4 & 1 & 0 & 9 \\ 0 & 9 & 1 & 1 & 4 & 9 & 2 & 7 & 7 & 9 \\ 0 & 0 & 8 & 1 & 1 & 4 & 9 & 2 & 7 & 7 \\ 0 & 0 & 0 & 7 & 1 & 1 & 4 & 9 & 2 & 7 \\ 0 & 0 & 0 & 0 & 6 & 1 & 1 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 3 & -2 & 8 \\ 2 & 3 & 1 & -4 \\ -1 & 2 & -1 & 4 \\ 1 & 2 & 2 & -8 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 3 & -2 & 8 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

a. If G is a 4×4 matrix show that BG is not invertible.

b. If F is a 10×10 invertible matrix then evaluate $\det\left(F^{101}\operatorname{adj}(A)(F^{-1})^{101}\right)$.

c. Evaluate $\det\left(2\operatorname{adj}(A) + 3A^{-1}\right)$, if possible.

d. If D is a 10×10 matrix such that $A^{-1}D^2 = I$ then determine $\det(D)$, if possible.

e. Evaluate $\det\left(B^{101}\operatorname{adj}(C) + BC^{2015}\right)$, if possible.

f. If E is a square matrix and $\det\left(\frac{\det(C)}{2}E^{\mathsf{T}}A^3\right) = \pi$ then find $\det(E)$, if possible.

g. Evaluate $\det(\operatorname{adj}(A) + I)$, if possible.

3.3.21 [**JH**]

a. Give a non-identity matrix with the property that $A^{\mathsf{T}} = A^{-1}$.

b. Prove: If $A^{\mathsf{T}} = A^{-1}$ then $\det(A) = \pm 1$.

c. Does the converse to the above hold?

3.3.22 [**JH**] Two matrices H and G are said to be *similar* if there is a nonsingular matrix P such that $H = P^{-1}GP$ Show that similar matrices have the same determinant.

3.3.23 [**KK**] An $n\times n$ matrix is called *nilpotent* if for some positive integer, k it follows $A^k = 0$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = 0$, what are the possible values of $\det(A)$?

3.3.24 [**JH**] Show that this gives the equation of a line in $\mathbb{R}^2$ thru (x_2, y_2) and (x_3, y_3).

$$\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

3.3.25 [**YL**] Let A and B be $n\times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.3.26 [**JH**] Prove or disprove: The determinant is a linear function, that is $\det(x\cdot T + y\cdot S) = x\cdot\det(T) + y\cdot\det(S)$.

3.3.27 [**KK**] True or false. If true, provide a proof. If false, provide a counter example.

a. If A is a 3×3 matrix with a zero determinant, then one column must be a multiple of some other column.

b. If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.

c. For two $n\times n$ matrices A and B, $\det(A+B) = \det(A) + \det(B)$.

d. For an $n\times n$ matrix A, $\det(3A) = 3\det(A)$

e. If A^{-1} exists then $\det\left(A^{-1}\right) = \det(A)^{-1}$.

f. If B is obtained by multiplying a single row of A by 4 then $\det(B) = 4\det(A)$.

g. For A an $n\times n$ matrix, $\det(-A) = (-1)^n \det(A)$.

h. If A is a real $n\times n$ matrix, then $\det\left(A^T A\right) \geq 0$.

i. If $A^k = 0$ for some positive integer k, then $\det(A) = 0$.

j. If $AX = 0$ for some $X \neq 0$, then $\det(A) = 0$.

3.4 Applications of the Determinant

3.4.1 [**YL**] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

3.4.2 [**GH**] Given the matrices A and b, evaluate $\det(A)$ and $\det(A_i)$ for all i. Use Cramer's Rule to solve $Ax = b$. If Cramer's Rule cannot be used to find the solution, then state whether or not a solution exists.

a. $A = \begin{bmatrix} 3 & 0 & -3 \\ 5 & 4 & 4 \\ 5 & 5 & -4 \end{bmatrix}$ $b = \begin{bmatrix} 24 \\ 0 \\ 31 \end{bmatrix}$

b. $A = \begin{bmatrix} 9 & 5 \\ -4 & -7 \end{bmatrix}$ $b = \begin{bmatrix} -45 \\ 20 \end{bmatrix}$

c. $A = \begin{bmatrix} -8 & 16 \\ 10 & -20 \end{bmatrix}$ $b = \begin{bmatrix} -48 \\ 60 \end{bmatrix}$

d. $A = \begin{bmatrix} 7 & 14 \\ -2 & -4 \end{bmatrix}$ $b = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

e. $A = \begin{bmatrix} 4 & 9 & 3 \\ -5 & -2 & -13 \\ -1 & 10 & -13 \end{bmatrix}$ $b = \begin{bmatrix} -28 \\ 35 \\ 7 \end{bmatrix}$

f. $A = \begin{bmatrix} 7 & -4 & 25 \\ -2 & 1 & -7 \\ 9 & -7 & 34 \end{bmatrix}$ $b = \begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$

3.4.3 [**SZ**] Use Cramer's Rule to solve for x_4.

a.
$$\begin{aligned} x_1 - x_3 &= -2 \\ 2x_2 - x_4 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ -x_3 + x_4 &= 1 \end{aligned}$$

b.
$$\begin{aligned} 4x_1 + x_2 &= 4 \\ x_2 - 3x_3 &= 1 \\ 10x_1 + x_3 + x_4 &= 0 \\ -x_2 + x_3 &= -3 \end{aligned}$$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 [**SM**] Find the coordinates of the point which is one third of the way from the point $(1,2)$ to the point $(3,-2)$.

4.1.2 [**SM**] Find the vector equation of the line through $(1,2)$ and $(3,-1)$.

4.1.3 [**SM**] Find the vector equation, and the parametric equations, of the line through $(1,5,-2)$ and $(4,-1,3)$.

4.1.4 [**SM**] Do the lines $\vec{x}=(1,0,1)+t(1,1,-1)$, $t\in\mathbb{R}$ and $\vec{x}=(2,3,4)+s(0,-1,2)$, $s\in\mathbb{R}$ have a point of intersection?

4.1.5 [**SM**] Let $\vec{x}=(1,2,-2)$ and $\vec{x}=(2,-1,3)$. Determine

- **a.** $2\vec{x}-3\vec{y}$
- **b.** $-3(\vec{x}+2\vec{y})+5\vec{x}$
- **c.** $\vec{z}$ such that $\vec{y}-2\vec{z}=3\vec{x}$
- **d.** $\vec{z}$ such that $\vec{z}-3\vec{x}=2\vec{z}$

4.1.6 [**SM**] Consider the points $P(2,3,1)$, $Q(3,1,-2)$, $R(1,4,0)$, and $S(-5,1,5)$. Determine $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, $\overrightarrow{QR}$, and $\overrightarrow{SR}$, and verify that $\overrightarrow{PQ}+\overrightarrow{QR}=\overrightarrow{PR}=\overrightarrow{PS}+\overrightarrow{SR}$.

4.1.7 [**SM**] Write the vector equations of the line passing through the given point with the given direction vector.

- **a.** point $(3,4)$, direction vector $(-5,1)$
- **b.** point $(2,0,5)$, direction vector $(4,-2,-11)$
- **c.** point $(4,0,1,5,-3)$, direction vector $(-2,0,1,2,-1)$

4.1.8 [**SM**] Write a vector equation for the line that passes through the given points.

- **a.** $(-1,2)$ and $(2,-3)$
- **b.** $(4,1)$ and $(-2,-1)$
- **c.** $(1,3,-5)$ and $(-2,-1,0)$
- **d.** $\left(\frac{1}{2},\frac{1}{4},1\right)$ and $\left(-1,1,\frac{1}{3}\right)$
- **e.** $(1,0,-2,-5)$ and $(-3,2,-1,2)$

4.1.9 [**SM**] Find the midpoint of the line segment joining the given points.

- **a.** $(2,1,1)$ and $(-3,1,-4)$
- **b.** $(2,-1,0,3)$ and $(-3,2,1,-1)$

4.1.10 [**SM**] Find the points that divide the line segment joining the given points into three equal parts.

- **a.** $(2,4,1)$ and $(-1,1,7)$
- **b.** $(-1,1,5)$ and $(4,2,1)$

4.1.11 [**SM**] Given the points P and Q, and the real number r, determine the point R such that $\overrightarrow{PR}=r\overrightarrow{PQ}$. Make a rough sketch to illustrate the idea.

- **a.** $P(1,4,-5)$ and $Q(-3,1,4)$; $r=\frac{1}{4}$
- **b.** $P(2,1,1,6)$ and $Q(8,7,6,0)$; $r=-\frac{1}{3}$
- **c.** $P(2,1,-2)$ and $Q(-3,1,4)$; $r=\frac{4}{3}$

4.1.12 [**SM**] Determine the point of intersections (if any) for each pair of lines.

- **a.** $\vec{x}=(1,2)+t(3,5)$, $t\in\mathbb{R}$ and $\vec{x}=(3,-1)+s(4,1)$, $s\in\mathbb{R}$
- **b.** $\vec{x}=(2,3,4)+t(1,1,1)$, $t\in\mathbb{R}$ and $\vec{x}=(3,2,1)+s(3,1,-1)$, $s\in\mathbb{R}$
- **c.** $\vec{x}=(3,4,5)+t(1,1,1)$, $t\in\mathbb{R}$ and $\vec{x}=(2,4,1)+s(2,3,-2)$, $s\in\mathbb{R}$
- **d.** $\vec{x}=(1,0,1)+t(3,-1,2)$, $t\in\mathbb{R}$ and $\vec{x}=(5,0,7)+s(-2,2,2)$, $s\in\mathbb{R}$

4.1.13 [**SM**] A set of points in $\mathbb{R}^n$ is *collinear* if they all line on one line.

- **a.** By considering directed line segments, give a general method for determining whether a given set of three points is collinear.
- **b.** Determine whether the points $P(1,2,2,1)$, $Q(4,1,4,2)$, and $R(-5,4,-2,-1)$ are collinear. Show how you decide.
- **c.** Determine whether the points $S(1,0,1,2)$, $T(3,-2,3,1)$, and $U(-3,4,-1,5)$ are collinear. Show how you decide.

4.1.14 [**SM**] Show that $a(5,7)+b(3,-10)=(-16,77)$ represents a system of two linear equations in the two variables a and b. Solve and check.

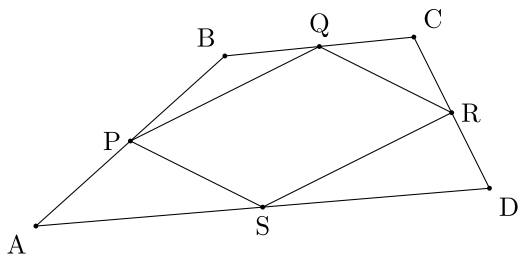
4.1.15 [**SM**] Show that $a(1,-1,0)+b(3,2,1)+c(0,1,4)=$

$(-1,1,19)$ represents a system of three linear equations in the three variables a, b, and c. Solve and check.

4.1.16 [SM] Prove that the two diagonals of a parallelogram bisect each other.

4.1.17 [SM] Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.

4.1.18 [SM] Prove that the quadrilateral $PQRS$, whose vertices are the midpoints of the sides of an arbitrary quadrilateral $ABCD$, is a parallelogram.



4.1.19 [SM] Prove that the line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

4.1.20 [SM] A *median* of a triangle is a line segment from a vertex to the midpoint of the opposite side. Prove that the three medians of any triangle intersect at a common point G that is two-thirds of the distance from each vertex to the midpoint of the opposite side. (The point G is called the *centroid* of the triangle.)

4.1.21 [SM] Find the equations (in vector form and parametric form) of the line passing through the given point and parallel to the given vector.

a. $A(1,-1,-1)$, $\vec{d}=(2,3,-1)$

b. $B(2,-4,5)$, $\vec{d}=3\hat{\imath}-\hat{\jmath}-2\hat{k}$

4.1.22 [SM] Find the equations of the lines through $P(3,-4,7)$ which are parallel to the coordinate axes.

4.1.23 [SM] Find the parametric equations of the line through the given point and parallel to the line with given equations

a. $A(0,0,2)$, $\vec{x}=(1,2,-1)+t(2,3,-3)$, $t\in\mathbb{R}$

b. $B(1,0,0)$, $\begin{cases} x = -1+2t \\ y = -1+3t \\ z = 1-2t \end{cases}$, $t\in\mathbb{R}$

4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality[YL] Prove *without assuming that the law of cosine holds in* $\mathbb{R}^n$: If $\vec{u},\vec{v}\in\mathbb{R}^n$ then $|\vec{u}\cdot\vec{v}|\le\|\vec{u}\|\|\vec{v}\|$.

4.2.2 [SM] If $\vec{v}=(2,3)$, find $\|\vec{v}\|$

4.2.3 [SM] Find the distance between $(-1,3,4)$ and $(2,-5,1)$.

4.2.4 [SM] Find a unit vector in the same direction as $\vec{v}=(2,2,-1)$.

4.2.5 [SM] Find the angle between each pair of vectors.

a. $\vec{u}=(1,4,-2)$ and $\vec{v}=(3,-1,4)$ in $\mathbb{R}^3$

b. $\vec{u}=(2,4,0,1,3)$ and $\vec{v}=(1,1,4,2,0)$ in $\mathbb{R}^5$

4.2.6 [SM] Prove, as a consequence of the triangle inequality, that $\big|\|\vec{x}\|-\|\vec{y}\|\big|\le\|\vec{x}-\vec{y}\|$.

4.2.7 [SM]

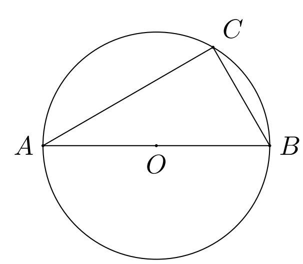
a. Let $\vec{u_1}$, $\vec{u_2}$, and $\vec{u_3}$ be non-zero vectors orthogonal to each other, and let $\vec{w}=a\vec{u_1}+b\vec{u_2}+c\vec{u_3}$. Show that

$$a=\frac{\vec{w}\cdot\vec{u_1}}{\|\vec{u_1}\|^2},\quad b=\frac{\vec{w}\cdot\vec{u_2}}{\|\vec{u_2}\|^2},\quad c=\frac{\vec{w}\cdot\vec{u_3}}{\|\vec{u_3}\|^2}$$

b. Show that $\vec{u_1}=(1,-2,3)$, $\vec{u_2}=(1,2,1)$, and $\vec{u_3}=(-8,2,4)$ are orthogonal to each other, and write $\vec{w}=(13,-4,-7)$ as a linear combination of $\vec{u_1}$, $\vec{u_2}$, and $\vec{u_3}$.

4.2.8 [SM] An *altitude* of a triangle is a line segment from a vertex that is orthogonal to the opposite side. Prove that the three altitudes of a triangle intersect at a common point. (This point is called the *orthocenter* of the triangle.

4.2.9 [SM] Use dot products to prove the Theorem of Thales: If A and B are endpoints of a diameter of a circle, and C is any other point on the circle, then angle $\angle ABC$ is a right angle.



4.2.10 [SM] Let $\vec{v}$ be any non-zero vector in $\mathbb{R}^2$, and $\hat{v}$ be the unit vector in its direction.

a. Show that $\hat{v}$ can be written as $\hat{v}=(\cos\phi,\sin\phi)$, where ϕ is the angle from the positive x-axis to $\vec{v}$. Also show that $\vec{v}=\|\vec{v}\|(\cos\phi,\sin\phi)$.

b. Prove the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ by considering the dot product of the two unit vectors $\vec{e_a} = (\cos\alpha, \sin\alpha)$ and $\vec{e_b} = (\cos\beta, \sin\beta)$.

4.2.11 [SM] Calculate the lengths of the given vectors.

a. $(2, -5)$

b. $(2, 3, -2)$

c. $\left(1, \frac{1}{5}, -3\right)$

d. $(1, -1, 0, 2)$

4.2.12 [SM] Determine the distance from P to Q as given.

a. $P(2, 3)$ and $Q(-4, 1)$

b. $P(1, 1, -2)$ and $Q(-3, 1, 1)$

c. $P(4, -6, 1)$ and $Q(-3, 5, 1)$

d. $P(2, 1, 1, 5)$ and $Q(4, 6, -2, 1)$

4.2.13 [SM] Verify the triangle inequality and the Cauchy-Schwartz inequality for the given vectors.

a. $\vec{x} = (4, 3, 1)$ and $\vec{y} = (2, 1, 5)$

b. $\vec{x} = (1, -1, 2)$ and $\vec{y} = (-3, 2, 4)$

4.2.14 [SM] Determine the angle (in radians) between the vectors $\vec{a}$ and $\vec{b}$ given.

a. $\vec{a} = (2, 1, 4)$ and $\vec{b} = (4, -2, 1)$

b. $\vec{a} = (1, -2, 1)$ and $\vec{b} = (3, 1, 0)$

c. $\vec{a} = (5, 1, 1, -2)$ and $\vec{b} = (2, 3, -2, 1)$

4.2.15 [SM] Determine whether the given pair of vectors is orthogonal.

a. $(1, 3, 2)$, $(2, -2, 2)$

b. $(-3, 1, 7)$, $(2, -1, 1)$

c. $(2, 1, 1)$, $(-1, 4, 2)$

d. $(4, 1, 0, -2)$, $(-1, 4, 3, 0)$

4.2.16 [SM] Determine all values of k for which the vectors are orthogonal.

a. $(3, -1)$, $(2, k)$

b. $(3, -1)$, (k, k^2)

c. $(1, 2, 3)$, $(3, -k, k)$

d. $(1, 2, 3)$, $(k, k, -k)$

4.2.17 [SM] Consider the following statement: "If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ then $\vec{b} = \vec{c}$."

a. If the statement is true, prove it. If the statement is false, provide a counterexample.

b. If we specify $\vec{a} \neq \vec{0}$, does that change the result?

4.2.18 [SM] Prove the parallelogram law for the norm:

$$\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2$$

for all vectors in $\mathbb{R}^n$.

4.2.19 [SM] For each of the given pairs of vectors $\vec{a}$, $\vec{b}$, check that $\vec{a}$ is a unit vector, determine $\text{proj}_{\vec{a}}(\vec{b})$ and $\text{perp}_{\vec{a}}(\vec{b})$, and check your results by verifying that $\text{proj}_{\vec{a}}(\vec{b}) + \text{perp}_{\vec{a}}(\vec{b}) = \vec{b}$ and $\vec{a} \cdot \text{perp}_{\vec{a}}(\vec{b}) = 0$ in each case.

a. $\vec{a} = (0, 1)$ and $\vec{b} = (3, -5)$

b. $\vec{a} = \left(\frac{3}{5}, \frac{4}{5}\right)$ and $\vec{b} = (-4, 6)$

c. $\vec{a} = (0, 1, 0)$ and $\vec{b} = (-3, 5, 2)$

d. $\vec{a} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$ and $\vec{b} = (4, 1, -3)$

4.2.20 [SM] Consider the force represented by the vector $\vec{F} = (10, 18, -6)$, and let $\vec{u} = (2, 6, 3)$.

a. Determine a unit vector in the direction of $\vec{u}$.

b. Determine the projection of $\vec{F}$ onto $\vec{u}$.

c. Determine the component of $\vec{F}$ perpendicular to $\vec{u}$.

4.2.21 [SM] Determine

a. $\text{proj}_{(2,3,-2)}((4, -1, 3))$ and $\text{perp}_{(2,3,-2)}((4, -1, 3))$

b. $\text{proj}_{(1,1,-2)}((4, 1, -2))$ and $\text{perp}_{(1,1,-2)}((4, 1, -2))$

c. $\text{proj}_{(-2,1,-1)}((5, -1, 3))$ and $\text{perp}_{(-2,1,-1)}((5, -1, 3))$

d. $\text{proj}_{(-1,2,1,-3)}((2, -1, 2, 1))$ and $\text{perp}_{(-1,2,1,-3)}((2, -1, 2, 1))$

4.3 Cross Product and Planes

4.3.1 [SM] Find the scalar equation of the plane passing through the point $A(2, 3, -1)$ and having normal vector $\vec{n} = (1, -4, 1)$.

4.3.2 [SM] Find the equation of the plane containing $A(2, 4, -1)$ and parallel to the plane $2x + 3y - 5z = 6$.

4.3.3 [SM] Find the scalar equation of the plane containing the given point with the given normal.

a. point $(-1, 2, -3)$, normal $(2, 4, -1)$

b. point $(2, 5, 4)$, normal $(3, 0, 5)$

c. point $(1, -1, 1)$, normal $(3, -4, 1)$

4.3.4 [SM] Determine the scalar equation of the hyperplane passing through the given point with the given normal.

a. point $(1, 1, -1, -2)$, normal $(3, 1, 4, 1)$

b. point $(2, -2, 0, 1)$, normal $(0, 1, 3, 3)$

4.3.5 [SM] Determine a normal vector for the plane or the hyperplane.

a. $3x_1 - 2x_2 + x_3 = 7$ in $\mathbb{R}^3$

b. $-4x_1 + 3x_2 - 5x_3 - 6 = 0$ in $\mathbb{R}^3$

c. $x_1 - x_2 + 2x_3 - 3x_4 = 5$ in $\mathbb{R}^4$

4.3.6 [SM] Find an equation for the plane through the given point and parallel to the given plane.

a. point $(1, -3, -1)$, plane $2x_1 - 3x_2 + 5x_3 = 17$

b. point $(0, -2, 4)$, plane $x_2 = 0$

4.3.7 [SM] Determine the point of intersection of the given line and plane.

a. $\vec{x} = (2, 3, 1) + t(1, -2, -4)$, $t \in \mathbb{R}$, and $3x_1 - 2x_2 + 5x_3 = 11$

b. $\vec{x} = (1, 1, 2) + t(1, -1, -2)$, $t \in \mathbb{R}$, and $2x_1 + x_2 - x_3 = 5$

4.3.8 [SM] Given the plane $2x_1 - x_2 + 3x_3 = 5$, for each of the following lines, determine if the line is parallel to the plane, orthogonal to the plane, or neither parallel nor orthogonal. If the answer is "neither", determine the angle between the direction vector of the line and the normal vector of the plane.

a. $\vec{x} = (3, 0, 4) + t(-1, 1, 1)$, $t \in \mathbb{R}$

b. $\vec{x} = (1, 1, 2) + t(-2, 1, -3)$, $t \in \mathbb{R}$

c. $\vec{x} = (3, 0, 0) + t(1, 1, 2)$, $t \in \mathbb{R}$

d. $\vec{x} = (-1, -1, 2) + t(4, -2, 6)$, $t \in \mathbb{R}$

e. $\vec{x} = t(0, 3, 1)$, $t \in \mathbb{R}$

4.3.9 [SM] Calculate the following cross products.

a. $(1, -5, 2) \times (-2, 1, 5)$

b. $(2, -3, -5) \times (4, -2, 7)$

c. $(-1, 0, 1) \times (0, 4, 5)$

4.3.10 [SM] Let $\vec{p} = (-1, 4, 2)$, $\vec{q} = (3, 1, -1)$, and $\vec{r} = (2, -3, -1)$. Check by calculation that the following general properties hold.

a. $\vec{p} \times \vec{p} = \vec{0}$

b. $\vec{p} \times \vec{q} = -\vec{q} \times \vec{p}$

c. $\vec{p} \times 3\vec{r} = 3(\vec{p} \times \vec{r})$

d. $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$

e. $\vec{p} \times (\vec{q} \times \vec{r}) \neq (\vec{p} \times \vec{q}) \times \vec{r}$

4.3.11 [SM] Determine the scalar equation of the plane that contains the following points.

a. $(2, 1, 5), (4, -3, 2), (2, 6, -1)$

b. $(3, 1, 4), (-2, 0, 2), (1, 4, -1)$

c. $(-1, 4, 2), (3, 1, -1), (2, -3, -1)$

4.3.12 [SM] Determine the scalar equation of the plane with the given vector equation.

a. $\vec{x} = (1, 4, 7) + s(2, 3, -1) + t(4, 1, 0)$, $s, t \in \mathbb{R}$

b. $\vec{x} = (2, 3, -1) + s(1, 1, 0) + t(-2, 1, 2)$, $s, t \in \mathbb{R}$

c. $\vec{x} = (1, -1, 3) + s(2, -2, 1) + t(0, 3, 1)$, $s, t \in \mathbb{R}$

4.4 Areas, Volumes and Distances

4.4.1 [SM] For the given point and line, find by projection the point on the line that is closest to the given point, and use perp to find the distance from the point to the line.

a. point $(0, 0)$, line $\vec{x} = (1, 4) + t(-2, 2)$, $t \in \mathbb{R}$

b. point $(2, 5)$, line $\vec{x} = (3, 7) + t(1, -4)$, $t \in \mathbb{R}$

c. point $(1, 0, 1)$, line $\vec{x} = (2, 2, -1) + t(1, -2, 1)$, $t \in \mathbb{R}$

d. point $(2, 3, 2)$, line $\vec{x} = (1, 1, -1) + t(1, 4, 1)$, $t \in \mathbb{R}$

4.4.2 [SM] Use a projection (onto or perpendicular to) to find the distance from the point to the plane.

a. point $(2, 3, 1)$, plane $3x_1 - x_2 + 4x_3 = 5$

b. point $(-2, 3, -1)$, plane $2x_1 - 3x_2 - 5x_3 = 5$

c. point $(0, 2, -1)$, plane $2x_1 - x_3 = 5$

d. point $(-1, -1, 1)$, plane $2x_1 - x_2 - x_3 = 4$

4.4.3 [SM] For the given point and hyperplane in $\mathbb{R}^4$, determine by a projection the point in the hyperplane that is closest to the given point.

a. point $(2, 4, 3, 4)$, hyperplane $3x_1 - x_2 + 4x_3 + x_4 = 0$

b. point $(-1, 3, 2, -1)$, hyperplane $x_1 + 2x_2 + x_3 - x_4 = 4$

4.4.4 [SM] Calculate the area of the parallelogram determined by the following vectors. (Hint: For the vectors in $\mathbb{R}^2$, think of them as vectors in $\mathbb{R}^3$ by letting $z = 0$.)

a. $(1, 2, 1)$ and $(2, 3, -1)$

b. $(1, 0, 1)$ and $(1, 1, 4)$

c. $(1, 2)$ and $(-2, 5)$

d. $(-3, 1)$ and $(4, 3)$

4.4.5 [SM] In each case, determine whether the given pair of lines has a point of intersection; if so, determine the scalar equation of the plane containing the lines, and if not, determine the distance between the lines.

a. $\vec{x} = (1, 3, 1) + s(-2, -1, 1)$ and $\vec{x} = (0, 1, 4) + t(3, 0, 1)$, $s, t \in \mathbb{R}$

b. $\vec{x} = (1, 3, 1) + s(-2, -1, 1)$ and $\vec{x} = (0, 1, 7) + t(3, 0, 1)$, $s, t \in \mathbb{R}$

c. $\vec{x} = (2, 1, 4) + s(2, 1, -2)$ and $\vec{x} = (-2, 1, 5) + t(1, 3, 1)$, $s, t \in \mathbb{R}$

d. $\vec{x} = (0, 1, 3) + s(1, -1, 4)$ and $\vec{x} = (0, -1, 5) + t(1, 1, 2)$, $s, t \in \mathbb{R}$

4.4.6 [SM] Find the volume of the parallelepiped determined by the following vectors.

a. $(4, 1, -1)$, $(-1, 5, 2)$, and $(1, 1, 6)$

b. $(-2, 1, 2)$, $(3, 1, 2)$, and $(0, 2, 5)$

4.5 Geometry of Solutions of Linear Systems

4.5.1 [SM] Determine a vector equation of the line of intersection of the given planes.

a. $x + 3y - z = 5$ and $2x - 5y + z = 7$

b. $2x - 3z = 7$ and $y + 2z = 4$

Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [**JH**] Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.

b. The space of 2×3 matrices.

c. The space $\{\, f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\,\}$.

d. The space of real-valued functions of one natural number variable.

5.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3 - 2x + x^2$.

b. In the space $\mathcal{M}_{2\times2}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

c. In $\{\, ae^x + be^{-x} \mid a, b \in \mathbb{R}\,\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{\, a_0 + a_1x \mid a_0, a_1 \in \mathbb{R}\,\}$ under the usual polynomial addition and scalar multiplication operations.

b. The set of linear polynomials $\{\, a_0 + a_1x \mid a_0 - 2a_1 = 0\,\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.

b. The set of 2×2 matrices with real entries where the $2,1$ entry is zero, under the usual matrix operations.

5.1.5 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.

b. The set

$$\{\, (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y - z + w = 0\,\}$$

under the operations inherited from $\mathbb{R}^4$.

5.1.6 [**JH**] Show that the following are not vector spaces.

a. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x + y + z = 1\,\}$$

b. Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\,\}$$

c. Under the usual matrix operations,

$$\left\{ \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

d. Under the usual polynomial operations,

$$\{\, a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+\,\}$$

where $\mathbb{R}^+$ is the set of reals greater than zero

e. Under the inherited operations,

$$\{\, (x,\ y) \in \mathbb{R}^2 \mid x + 3y = 4,\ 2x - y = 3 \text{ and } 6x + 4y = 10\,\}$$

5.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

5.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

5.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}rx\\ry\\rz\end{pmatrix}$

b. $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

5.1.10 [**JH**] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{\begin{bmatrix}a & 0\\0 & b\end{bmatrix}\;\middle|\;a,b\in\mathbb{R}\right\}$$

b. The set of 2×2 matrices

$$\left\{\begin{bmatrix}x & x+y\\x+y & y\end{bmatrix}\;\middle|\;x,y\in\mathbb{R}\right\}$$

c. $\{(x,\ y,\ z,\ w)\in\mathbb{R}^4 \mid x+y+w=1\}$

d. The set of functions $\{f\colon\mathbb{R}\to\mathbb{R} \mid df/dx+2f=0\}$

e. The set of functions $\{f\colon\mathbb{R}\to\mathbb{R} \mid df/dx+2f=1\}$

5.1.11 [**YL**] Let $V=\{A \mid A\in\mathcal{M}_{2\times2} \text{ and } \det(A)\neq 0\}$ with the following operations:

$$A+B=AB \text{ and } kA=kA$$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.

a. Does V satisfy closure under vector addition? Justify.

b. Does V contain a zero vector? If so find it. Justify.

c. Does V contains an additive inverse for all of its vectors? Justify.

d. Does V satisfy closure under scalar multiplication? Justify.

5.1.12 [**JH**] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x+y$' to mean the product of x and y (so that $2+3$ is 6), and we interpret '$r\cdot x$' as the r-th power of x.

5.1.13 [**JH**] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.14 [**JH**]
Is $\{(x,\ y) \mid x,y\in\mathbb{R}\}$ a vector space under these operations?

a. $(x_1,\ y_1)+(x_2,\ y_2)=(x_1+x_2,\ y_1+y_2)$ and $r\cdot(x,y)=(rx,y)$

b. $(x_1,\ y_1)+(x_2,\ y_2)=(x_1+x_2,\ y_1+y_2)$ and $r\cdot(x,\ y)=(rx,\ 0)$

5.1.15 [**JH**]
Prove the following:

a. For any $\vec{v}\in V$, if $\vec{w}\in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v},\vec{s},\vec{t}\in V$ then $\vec{v}+\vec{s}=\vec{v}+\vec{t}$ implies that $\vec{s}=\vec{t}$.

5.1.16 [**JH**] Consider the set

$$\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\;\middle|\;x+y+z=1\right\}$$

under these operations.

$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}x_1+x_2-1\\y_1+y_2\\z_1+z_2\end{pmatrix}\qquad r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}rx-r+1\\ry\\rz\end{pmatrix}$$

Show that it is a vector space.

5.1.17 [**JH**]
The definition of vector spaces does not explicitly say that $\vec{0}+\vec{v}=\vec{v}$ (it instead says that $\vec{v}+\vec{0}=\vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.18 [**JH**]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.19 [**JH**]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

5.1.20 [**JH**]
Assume that $\vec{v}\in V$ is not $\vec{0}$.

a. Prove that $r\cdot\vec{v}=\vec{0}$ if and only if $r=0$.

b. Prove that $r_1\cdot\vec{v}=r_2\cdot\vec{v}$ if and only if $r_1=r_2$.

c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [**JH**] Which of these subsets of the vector space of 2×2 matrices are subspaces under the inherited operations? Justify.

a. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

b. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 0 \right\}$

c. $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 5 \right\}$

d. $\left\{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \;\middle|\; a + b = 0, c \in \mathbb{R} \right\}$

5.2.2 [**JH**] Is this a subspace of $\mathcal{P}_2$:
$\{ a_0 + a_1 x + a_2 x^2 \mid a_0 + 2a_1 + a_2 = 4 \}$? Justify.

5.2.3 [**JH**] The solution set of a homogeneous linear system is a subspace of $\mathbb{R}^n$ where the system has n variables. What about a non-homogeneous linear system; do its solutions form a subspace (under the inherited operations)?

5.2.4 [**JH**]

a. Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ under the inherited operations.

b. What if it doesn't contain the origin?

5.2.5 [**JH**] $\mathbb{R}^3$ has infinitely many subspaces. Do every non-trivial space have infinitely many subspaces?

5.2.6 [**JH**] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.2.7 [**JH**] Determine if each is a subspace of the vector space of real-valued functions of one real variable.

a. The *even* functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x) \text{ for all } x \}$.

b. The *odd* functions $\{ f \colon \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x) \text{ for all } x \}$.

5.2.8 [**JH**] Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

5.2.9 [**JH**]

a. Give a set that is closed under scalar multiplication but not addition.

b. Give a set closed under addition but not scalar multiplication.

c. Give a set closed under neither.

5.2.10 [**JH**] Subspaces are subsets and so we naturally consider how 'is a subspace of' interacts with the usual set operations.

a. If A, B are subspaces of a vector space, are their intersection $A \cap B$ be a subspace?

b. Is the union $A \cup B$ a subspace?

c. If A is a subspace, is its complement be a subspace?

5.2.11 [**JH**] Is the relation 'is a subspace of' transitive? That is, if V is a subspace of W and W is a subspace of X, must V be a subspace of X?

5.3 Spanning Sets

5.3.1 [**JH**] Determine whether the vector lies in the span of the set.

a. $\begin{pmatrix}2\\0\\1\end{pmatrix}$, $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

b. $x - x^3$, $\left\{x^2, 2x + x^2, x + x^3\right\}$

c. $\begin{bmatrix}0 & 1\\4 & 2\end{bmatrix}$, $\left\{\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}2 & 0\\2 & 3\end{bmatrix}\right\}$

5.3.2 [**JH**] Which of these are members of the span $\operatorname{span}\left(\left\{\cos^2 x,\ \sin^2 x\right\}\right)$ in the vector space of real-valued functions of one real variable?

a. $f(x) = 1$

b. $f(x) = 3 + x^2$

c. $f(x) = \sin x$

d. $f(x) = \cos(2x)$

5.3.3 [**JH**] Which of these sets spans $\mathbb{R}^3$?

a. $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\3\end{pmatrix}\right\}$

b. $\left\{\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

c. $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}3\\0\\0\end{pmatrix}\right\}$

d. $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\5\end{pmatrix}\right\}$

e. $\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix}, \begin{pmatrix}5\\1\\2\end{pmatrix}, \begin{pmatrix}6\\0\\2\end{pmatrix}\right\}$

5.3.4 [**JH**] Express each subspace as a span of a set of vectors.

a. $\left\{(a\ \ b\ \ c) \;\middle|\; a - c = 0\right\}$

b. $\left\{\begin{bmatrix}a & b\\c & d\end{bmatrix} \;\middle|\; a + d = 0\right\}$

c. $\left\{\begin{bmatrix}a & b\\c & d\end{bmatrix} \;\middle|\; 2a - c - d = 0 \text{ and } a + 3b = 0\right\}$

d. $\left\{a + bx + cx^3 \;\middle|\; a - 2b + c = 0\right\}$

e. The subset of $\mathcal{P}_2$ of quadratic polynomials p such that $p(7) = 0$

5.3.5 [**JH**] Find a set that spans the given subspace.

a. The xz-plane in $\mathbb{R}^3$.

b. $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; 3x + 2y + z = 0\right\}$

c. $\left\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \;\middle|\; 2x + y + w = 0 \text{ and } y + 2z = 0\right\}$

d. $\left\{a_0 + a_1x + a_2x^2 + a_3x^3 \;\middle|\; a_0 + a_1 = 0 \text{ and } a_2 - a_3 = 0\right\}$

e. The set $\mathcal{P}_4$ in the space $\mathcal{P}_4$

f. $\mathcal{M}_{2\times2}$ in $\mathcal{M}_{2\times2}$

5.3.6 [**JH**] Show that for any subset S of a vector space, $\operatorname{span}(\operatorname{span}(S)) = \operatorname{span}(S)$. (*Hint.* Members of $\operatorname{span}(S)$ are linear combinations of members of S. Members of $\operatorname{span}(\operatorname{span}(S))$ are linear combinations of linear combinations of members of S.)

5.3.7 [**YL**] Given the following two subspace of $\mathbb{R}^3$: $W_1 = \{x \mid A_1x = 0\}$ and $W_2 = \{x \mid A_2x = 0\}$ where

$$A_1 = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\-3 & -3 & -3\end{bmatrix},\ A_2 = \begin{bmatrix}5 & 7 & 9\\-5 & -7 & -9\\10 & 14 & 18\end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.3.8 [**JH**] Prove: $\vec{v} \in \operatorname{span}(\{\vec{v}_1, \ldots, \vec{v}_n\})$ if and only if $\operatorname{span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \operatorname{span}(\{\vec{v}, \vec{v}_1, \ldots, \vec{v}_n\})$

5.3.9 [**JH**] Does the span of a set depend on the enclosing space? That is, if W is a subspace of V and S is a subset of W (and so also a subset of V), might the span of S in W differ from the span of S in V?

5.3.10 [**JH**] Because 'span of' is an operation on sets we naturally consider how it interacts with the usual set operations.

a. If $S \subseteq T$ are subsets of a vector space, is $\operatorname{span}(S) \subseteq \operatorname{span}(T)$?

b. If S, T are subsets of a vector space, is $\operatorname{span}(S \cup T) = \operatorname{span}(S) \cup \operatorname{span}(T)$?

c. If S, T are subsets of a vector space, is $\operatorname{span}(S \cap T) = \operatorname{span}(S) \cap \operatorname{span}(T)$?

d. Is the span of the complement equal to the complement of the span?

5.4 Linear Independence

5.4.1 [**JH**] Determine whether each subset of $\mathbb{R}^3$ is linearly dependent or linearly independent.

a. $\left\{ \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 14 \end{pmatrix} \right\}$

b. $\left\{ \begin{pmatrix} 1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix} \right\}$

c. $\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}$

d. $\left\{ \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \\ -1 \end{pmatrix} \right\}$

5.4.2 [**JH**] Which of these subsets of $\mathcal{P}_2$ are linearly dependent and which are independent?

a. $\{3 - x + 9x^2,\ 5 - 6x + 3x^2,\ 1 + 1x - 5x^2\}$

b. $\{-x^2,\ 1 + 4x^2\}$

c. $\{2 + x + 7x^2,\ 3 - x + 2x^2,\ 4 - 3x^2\}$

d. $\{8 + 3x + 3x^2,\ x + 2x^2,\ 2 + 2x + 2x^2,\ 8 - 2x + 5x^2\}$

5.4.3 [**JH**] Prove that each set $\{f,\ g\}$ is linearly independent in the vector space of all functions from $\mathbb{R}^+$ to $\mathbb{R}$.

a. $f(x) = x$ and $g(x) = 1/x$

b. $f(x) = \cos(x)$ and $g(x) = \sin(x)$

c. $f(x) = e^x$ and $g(x) = \ln(x)$

5.4.4 [**JH**] Which of these subsets of the space of real-valued functions of one real variable are linearly dependent and which are linearly independent?

a. $\{2,\ 4\sin^2(x),\ \cos^2(x)\}$

b. $\{1,\ \sin(x),\ \sin(2x)\}$

c. $\{x,\ \cos(x)\}$

d. $\{(1+x)^2,\ x^2 + 2x,\ 3\}$

e. $\{0,\ x,\ x^2\}$

f. $\{\cos(2x),\ \sin^2(x),\ \cos^2(x)\}$

5.4.5 [**JH**] Is the xy-plane subset of the vector space $\mathbb{R}^3$ linearly independent?

5.4.6 [**YL**] Let $\vec{u} = (1,\ \lambda,\ -\lambda)$, $\vec{v} = (-2\lambda,\ -2,\ 2\lambda)$ and $\vec{w} = (\lambda - 2,\ -5\lambda - 2,\ -2)$.

a. For what value(s) of λ will $\{\vec{u},\ \vec{v}\}$ be linearly dependent.

b. For what value(s) of λ will $\{\vec{u},\ \vec{v},\ \vec{w}\}$ be linearly independent.

5.4.7 [**JH**]

a. Show that if the set $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is linearly independent then so is the set $\{\vec{u},\ \vec{u}+\vec{v},\ \vec{u}+\vec{v}+\vec{w}\}$.

b. What is the relationship between the linear independence or dependence of $\{\vec{u},\ \vec{v},\ \vec{w}\}$ and the independence or dependence of $\{\vec{u}-\vec{v},\ \vec{v}-\vec{w},\ \vec{w}-\vec{u}\}$?

5.4.8 [**JH**]

a. When is a one-element set linearly independent?

b. When is a two-element set linearly independent?

5.4.9 [**JH**] Show that if $\{\vec{x},\ \vec{y},\ \vec{z}\}$ is linearly independent then so are all of its proper subsets: $\{\vec{x},\ \vec{y}\}$, $\{\vec{x},\ \vec{z}\}$, $\{\vec{y},\ \vec{z}\}$, $\{\vec{x}\}$,$\{\vec{y}\}$, $\{\vec{z}\}$. Is the converse also true?

5.4.10 [**JH**]

a. Show that this

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

is a linearly independent subset of $\mathbb{R}^3$.

b. Show that

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

is in the span of S by finding c_1 and c_2 giving a linear relationship.

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

Show that the pair c_1, c_2 is unique.

c. Assume that S is a subset of a vector space and that $\vec{v}$ is in span (S), so that $\vec{v}$ is a linear combination of vectors from S. Prove that if S is linearly independent then a linear combination of vectors from S adding to $\vec{v}$ is unique (that is, unique up to reordering and adding or taking away terms of the form $0 \cdot \vec{s}$). Thus S as a spanning set is minimal in this strong sense: each vector in span (S) is a combination of elements of S a minimum number of times (only once).

d. Prove that it can happen when S is not linearly independent that distinct linear combinations sum to the same vector.

5.4.11 [**JH**]

a. Show that any set of four vectors in $\mathbb{R}^2$ is linearly dependent.

b. Is this true for any set of five? Any set of three?

c. What is the most number of elements that a linearly independent subset of $\mathbb{R}^2$ can have?

5.4.12 [**JH**] Is there a set of four vectors in $\mathbb{R}^3$ such that any three form a linearly independent set?

5.4.13 [**JH**]

a. Prove that a set of two perpendicular nonzero vectors from $\mathbb{R}^n$ is linearly independent when $n > 1$.

b. What if $n = 1$?

c. Generalize to more than two vectors.

5.4.14 [**JH**] Show that, where S is a subspace of V, if a subset T of S is linearly independent in S then T is also linearly independent in V. Is the converse also true?

5.4.15 [**JH**] Show that the nonzero rows of an echelon form matrix form a linearly independent set.

5.4.16 [**JH**] In $\mathbb{R}^4$ what is the largest linearly independent set you can find? The smallest? The largest linearly dependent set? The smallest?

5.4.17 [**JH**]

a. Is the intersection of linearly independent sets independent? Must it be?

b. How does linear independence relate to complementation?

c. Show that the union of two linearly independent sets can be linearly independent.

d. Show that the union of two linearly independent sets need not be linearly independent.

5.4.18 [**JH**]

a. We might conjecture that the union $S \cup T$ of linearly independent sets is linearly independent if and only if their spans have a trivial intersection $\text{span}(S) \cap \text{span}(T) = \left\{\vec{0}\right\}$. What is wrong with this argument for the 'if' direction of that conjecture? "If the union $S \cup T$ is linearly independent then the only solution to $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m = \vec{0}$ is the trivial one $c_1 = 0, \ldots, d_m = 0$. So any member of the intersection of the spans must be the zero vector because in $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ each scalar is zero."

b. Give an example showing that the conjecture is false.

c. Find linearly independent sets S and T so that the union of $S - (S \cap T)$ and $T - (S \cap T)$ is linearly independent, but the union $S \cup T$ is not linearly independent.

d. Characterize when the union of two linearly independent sets is linearly independent, in terms of the intersection of spans.

5.4.19 [**JH**] With a some calculation we can get formulas to determine whether or not a set of vectors is linearly independent.

a. Show that this subset of $\mathbb{R}^2$

$$\left\{ \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\}$$

is linearly independent if and only if $ad - bc \neq 0$.

b. Show that this subset of $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} a \\ d \\ g \end{pmatrix}, \begin{pmatrix} b \\ e \\ h \end{pmatrix}, \begin{pmatrix} c \\ f \\ i \end{pmatrix} \right\}$$

is linearly independent iff $aei + bfg + cdh - hfa - idb - gec \neq 0$.

c. When is this subset of $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} a \\ d \\ g \end{pmatrix}, \begin{pmatrix} b \\ e \\ h \end{pmatrix} \right\}$$

linearly independent?

5.5 Basis

5.5.1 [**JH**] Determine if each is a basis for $\mathbb{R}^3$.

a. $\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle$

b. $\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix} \rangle$

c. $\langle \begin{pmatrix}0\\2\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\5\\0\end{pmatrix} \rangle$

d. $\langle \begin{pmatrix}0\\2\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\3\\0\end{pmatrix} \rangle$

5.5.2 [**JH**] Determine if each is a basis for $\mathcal{P}_2$.

a. $\langle x^2 - x + 1,\ 2x + 1,\ 2x - 1\rangle$

b. $\langle x + x^2,\ x - x^2\rangle$

5.5.3 [**JH**] Represent the vector with respect to the given basis.

a. $\begin{pmatrix}1\\2\end{pmatrix}$, $B = \langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \rangle \subseteq \mathbb{R}^2$

b. $x^2 + x^3$, $D = \langle 1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3\rangle \subseteq \mathcal{P}_3$

5.5.4 [**JH**] Find a basis for $\mathcal{P}_2$, the space of all quadratic polynomials. Must any such basis contain a polynomial of each degree: degree zero, degree one, and degree two?

5.5.5 [**JH**] Find a basis for the solution set of this system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

5.5.6 [**JH**] Find a basis for $\mathcal{M}_{2\times2}$, the space of 2×2 matrices.

5.5.7 [**JH**] Find a basis for each of the following.

a. The subspace $\left\{ a_2x^2 + a_1x + a_0 \;\middle|\; a_2 - 2a_1 = a_0 \right\}$ of $\mathcal{P}_2$

b. The space of three component vectors whose first and second components add to zero

c. This subspace of the 2×2 matrices

$$\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; c - 2b = 0 \right\}$$

5.5.8 [**JH**] Find the span of each set *(that is, find restriction(s) on the coefficient of the polynomial)* and then find a basis for that span.

a. $\{1 + x,\ 1 + 2x\}$ in $\mathcal{P}_2$

b. $\{2 - 2x,\ 3 + 4x^2\}$ in $\mathcal{P}_2$

5.5.9 [**JH**] Find a basis for each of these subspaces of the space $\mathcal{P}_3$ of cubic polynomials.

a. The subspace of cubic polynomials $p(x)$ such that $p(7) = 0$.

b. The subspace of polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$.

c. The subspace of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, and $p(3) = 0$.

d. The space of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$.

5.5.10 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_2x^2 + a_3x^3 \;\middle|\; p(-1) = 0 \right\}$$

a subspace of $\mathcal{P}_3$.

a. Find a basis B for $\mathcal{W}$.

b. Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B.

5.5.11 [**JH**] Can a basis contain a zero vector?

5.5.12 [**JH**] Let $\langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ be a basis for a vector space.

a. Show that $\langle c_1\vec{\beta}_1, c_2\vec{\beta}_2, c_3\vec{\beta}_3\rangle$ is a basis when $c_1, c_2, c_3 \neq 0$. What if at least one c_i is 0?

b. Prove that $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\rangle$ is a basis where $\vec{\alpha}_i = \vec{\beta}_1 + \vec{\beta}_i$.

5.5.13 [**JH**] Find one vector $\vec{v}$ that will make each into a basis for the space.

a. $\langle \begin{pmatrix}1\\1\end{pmatrix}, \vec{v}\rangle$ in $\mathbb{R}^2$

b. $\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{v}\rangle$ in $\mathbb{R}^3$

c. $\langle x,\ 1 + x^2,\ \vec{v}\rangle$ in $\mathcal{P}_2$

5.5.14 [**JH**] Where $\langle \vec{\beta}_1, \dots, \vec{\beta}_n\rangle$ is a basis, show that in this equation

$$c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k = c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$$

each of the c_i's is zero. Generalize.

5.5.15 [**JH**] If a subset is not a basis, can linear combinations be not unique? If so, must they be?

5.5.16 [**JH**]

a. Find a basis for the vector space of symmetric 2×2 matrices.

b. Find a basis for the space of symmetric 3×3 matrices.

c. Find a basis for the space of symmetric $n\times n$ matrices.

5.5.17 [**JH**] We can show that every basis for $\mathbb{R}^3$ contains the same number of vectors.

a. Show that no linearly independent subset of $\mathbb{R}^3$ contains more than three vectors.

b. Show that no spanning subset of $\mathbb{R}^3$ contains fewer than three vectors. *Hint:* recall how to calculate the span of a set and show that this method cannot yield all of $\mathbb{R}^3$ when we apply it to fewer than three vectors.

5.5.18 [**JH**] Find a basis for the following vector space:

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x+y+z=1 \right\}$$

is a vector space under these operations.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2-1 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} \qquad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx-r+1 \\ ry \\ rz \end{pmatrix}$$

5.6 Dimension

5.6.1 [**JH**] Find a basis for, and the dimension of, $\mathcal{P}_2$.

5.6.2 [**JH**] Find a basis for, and the dimension of, $\mathcal{M}_{2\times 2}$, the vector space of 2×2 matrices.

5.6.3 [**JH**] Find a basis for, and the dimension of, the solution set of this system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

5.6.4 [**JH**] Find the dimension of the vector space of matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

subject to each condition.

a. $a, b, c, d \in \mathbb{R}$

b. $a - b + 2c = 0$ and $d \in \mathbb{R}$

c. $a + b + c = 0$, $a + b - c = 0$, and $d \in \mathbb{R}$

5.6.5 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; p(1) = 0 \text{ and } p(-1) = 0 \right\}$$

a subspace of $\mathcal{P}_3$. Determine the dimension of W.

5.6.6 [**JH**] Find the dimension of each.

a. The space of cubic polynomials $p(x)$ such that $p(7) = 0$.

b. The space of cubic polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$.

c. The space of cubic polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, and $p(3) = 0$.

d. The space of cubic polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$.

5.6.7 [**JH**] What is the dimension of the span of the set $\{\cos^2\theta,\ \sin^2\theta,\ \cos 2\theta,\ \sin 2\theta\}$? This span is a subspace of the space of all real-valued functions of one real variable.

5.6.8 [**JH**] What is the dimension of the vector space $\mathcal{M}_{3\times 5}$ of 3×5 matrices?

5.6.9 [**JH**] Show that this is a basis for $\mathbb{R}^4$.

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

5.6.10 [**JH**] Where S is a set, the functions $f\colon S \to \mathbb{R}$ form a vector space under the natural operations: the sum $f+g$ is the function given by $f+g\,(s) = f(s) + g(s)$ and the scalar product is $r\cdot f\,(s) = r\cdot f(s)$. What is the dimension of the space resulting for each domain?

a. $S = \{1\}$

b. $S = \{1, 2\}$

c. $S = \{1, \ldots, n\}$

5.6.11 [**JH**] Show that any set of four vectors in $\mathbb{R}^2$ is linearly dependent.

5.6.12 [**JH**] Show that $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \rangle \subset \mathbb{R}^3$ is a basis if and only if there is no plane through the origin containing all three vectors.

5.6.13 [**JH**]

a. Prove that any subspace of a finite dimensional space has a basis.

b. Prove that any subspace of a finite dimensional space is finite dimensional.

5.6.14 [**JH**] Prove that if U and W are both three-dimensional subspaces of $\mathbb{R}^5$ then $U \cap W$ is non-trivial. State a generalization of the above.

5.6.15 [**JH**]

a. Consider first how bases might be related by $\subseteq$. Assume that U, W are subspaces of some vector space and that $U \subseteq W$.
Can there exist bases B_U for U and B_W for W such that $B_U \subseteq B_W$? Must such bases exist?
For any basis B_U for U, must there be a basis B_W for W such that $B_U \subseteq B_W$?
For any basis B_W for W, must there be a basis B_U for U such that $B_U \subseteq B_W$?
For any bases B_U, B_W for U and W, must B_U be a subset of B_W?

b. Is the $\cap$ of bases a basis? For what space?

c. Is the $\cup$ of bases a basis? For what space?

d. What about the complement operation?

5.6.16 [**JH**] Assume U and W are both subspaces of some vector space, and that $U \subseteq W$.

a. Prove that $\dim(U) \leq \dim(W)$.

b. Prove that equality of dimension holds if and only if $U = W$.

Appendix A

Answers to Exercises

Note that either a hint, a final answer or a complete solution is provided.

1.1.1

a. Yes
b. No
c. Yes
d. Yes
e. No
f. No
g. Yes
h. No
i. Yes
j. No

1.1.2

a. $x = 1$, $y = -2$
b. $x = 2$, $y = \frac{1}{3}$
c. $x = -1$, $y = 0$, and $z = 2$.
d. $x = 1$, $y = 0$, and $z = 0$.

1.1.3 $\begin{aligned} 3x + y &= 3 \\ x + 2y &= 1 \end{aligned}$, Solution is: $x = 1, i\ y = 0$

1.1.4

a. $\left[\begin{array}{ccc|c} 3 & 4 & 5 & 7 \\ -1 & 1 & -3 & 1 \\ 2 & -2 & 3 & 5 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 2 & 5 & -6 & 2 \\ 9 & 0 & -8 & 10 \\ -2 & 4 & 1 & -7 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 3 & -4 & 5 & 17 \\ -1 & 0 & 4 & 8 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right]$

d. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 0 & 3 \\ -1 & 9 & 8 \\ 5 & -7 & 13 \end{array}\right]$

1.1.5

a. $\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 + 3x_2 &= 9 \end{aligned}$

b. $\begin{aligned} -3x_1 + 4x_2 &= 7 \\ x_2 &= -2 \end{aligned}$

c. $\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 &= 7 \end{aligned}$

d. $\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 5 \\ x_4 &= 3 \end{aligned}$

e. $\begin{aligned} x_1 + x_3 + 7x_5 &= 2 \\ x_2 + 3x_3 + 2x_4 &= 5 \end{aligned}$

1.1.6

a. $\begin{bmatrix} -2 & 1 & -7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 7 \\ 5 & 0 & 3 \\ 0 & 4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -1 & 7 \\ 2 & 3 & 5 \\ 5 & 0 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 8 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 2 & -1 \\ 5 & 0 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 0 & 5/2 & -29/2 \end{bmatrix}$

1.1.7

a. $2R_2 \to R_2$
b. $R_1 + R_2 \to R_2$
c. $2R_3 + R_1 \to R_1$
d. $R_1 \leftrightarrow R_2$
e. $-R_2 + R_3 \leftrightarrow R_3$

1.1.8 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line.
Thus the lines can share one point (giving a unique solution), share no points (giving no solutions), or share at least two points (which makes them the same line).

1.1.9 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

1.1.10 Each equation can be represented in $\mathbb{R}^2$ as a line.

a. At least one of the line is parallel and does not lie on an other line.

b. All the lines intersect at one and only one point.

c. All the lines are identical, i.e. they all lie on top of each other.

1.2.1

a. Yes
b. No
c. No
d. Yes
e. Yes
f. Yes
g. No
h. Yes
i. No
j. Yes
k. Yes
l. Yes
m. No
n. Yes
o. Yes

1.2.2

a. Reduced row echelon form
b. Neither
c. Row echelon form only
d. Reduced row echelon form
e. Reduced row echelon form
f. Row echelon form only

1.2.3

a. The solution exists but is not unique.
b. A solution exists and is unique.
c. The solution exists but is not unique.
d. There might be a solution. If so, there are infinitely many.

1.2.4

a. $(-2, 7)$
b. $(-3, 20, 19)$
c. $(-3t + 4, -6t - 6, 2, t)$ for all real numbers t
d. Inconsistent
e. $(8s - t + 7, -4s + 3t + 2, s, t)$ for all real numbers s and t
f. $(-9t - 3, 4t + 20, t)$ for all real numbers t

1.2.5

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -7/5 \\ 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$

g. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 1 & \frac{5}{4} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$

i. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

j. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

n. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

p. $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.2.6

a. $x = 2$, $y = 3$

b. $x = -1$, $y = 4$, and $z = -1$.

1.2.7

a. $(-2, 7)$

b. $(1, 2, 0)$

c. $(-t + 5, -3t + 15, t)$ for all real numbers t

d. $(2, -1, 1)$

e. $(1, 3, -2)$

f. Inconsistent

g. $(1, 3, -2)$

h. $\left(-3, \frac{1}{2}, 1\right)$

i. $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$

j. $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$ for all real numbers t

k. Inconsistent

l. $(4, -3, 1)$

1.2.8

a. $x_1 = 1 - 2t$; $x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$ and $x_1 = -1$, $x_2 = 1$.

b. $x_1 = -3 + 5t$; $x_2 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = 3$, $x_2 = 0$ and $x_1 = -8$, $x_2 = -1$.

c. $x_1 = 1$; $x_2 = 2$.

d. $x_1 = 0$; $x_2 = -1$.

e. $x_1 = -11 + 10t$; $x_2 = -4 + 4t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -11$, $x_2 = -4$, $x_3 = 0$ and $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$.

f. $x_1 = -\frac{2}{3} + \frac{8}{9}t$; $x_2 = \frac{2}{3} - \frac{5}{9}t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = -\frac{2}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 0$ and $x_1 = \frac{4}{9}$, $x_2 = -\frac{1}{9}$, $x_3 = 1$.

g. $x_1 = 1 - s - t$; $x_2 = s$; $x_3 = 1 - 2t$; $x_4 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ and $x_1 = -2$, $x_2 = 1$, $x_3 = -3$, $x_4 = 2$.

h. $x_1 = 3 - s - 2t$; $x_2 = -3 - 5s - 7t$; $x_3 = s$; $x_4 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 3$, $x_2 = -3$, $x_3 = 0$, $x_4 = 0$ and $x_1 = 0$, $x_2 = -5$, $x_3 = -1$, $x_4 = 1$.

i. $x_1 = \frac{1}{3} - \frac{4}{3}t$; $x_2 = \frac{1}{3} - \frac{1}{3}t$; $x_3 = t$ where $t \in \mathbb{R}$. Possible solutions: $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $x_3 = 0$ and $x_1 = -1$, $x_2 = 0$, $x_3 = 1$.

j. $x_1 = 1 - 2s - 3t$; $x_2 = s$; $x_3 = t$ where $s,\ t \in \mathbb{R}$. Possible solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and $x_1 = 8$, $x_2 = 1$, $x_3 = -3$.

k. No solution; the system is inconsistent.

l. No solution; the system is inconsistent.

1.2.9 $x = 2,\ y = 4,\ z = 5$

1.2.10

a. $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (60s - 55t + 30,\ -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3},\ -14s + 13t - 7,\ s,\ t)$ where $s,\ t \in \mathbb{R}$.

b. If $s = t = 0$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (30,\ -\frac{38}{3},\ -7,\ 0,\ 0)$.
If $s = 0$ and $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-25,\ \frac{35}{3},\ 6,\ 0,\ 1)$.

c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-\frac{30}{7},\ \frac{316}{21},\ 1,\ \frac{4}{7},\ 0)$.

1.2.11

a. $(x_1,\ x_2,\ x_3,\ x_4) = (60t,\ -\frac{79}{3}t,\ -14t,\ t)$ where $t \in \mathbb{R}$.

b. If $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4) = (60,\ -\frac{79}{3},\ -14,\ 1)$.
If $t = 3$ then $(x_1,\ x_2,\ x_3,\ x_4) = (180,\ -79,\ 42,\ 3)$.

c. If $t = \frac{1}{60}$ then $(x_1,\ x_2,\ x_3,\ x_4) = (1,\ -\frac{79}{180},\ -\frac{14}{60},\ \frac{1}{60})$.

1.2.12 $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

1.2.13 Because $f(1) = 2$, $f(-1) = 6$, and $f(2) = 3$ we get a linear system.

$$\begin{aligned} 1a + 1b + c &= 2 \\ 1a - 1b + c &= 6 \\ 4a + 2b + c &= 3 \end{aligned}$$

After performing Gaussian elimination we obtain

$$\begin{aligned} a + \quad b + \quad c &= \ \ 2 \\ -2b \qquad &= \ \ 4 \\ -3c &= -9 \end{aligned}$$

which shows that the solution is $f(x) = 1x^2 - 2x + 3$.

1.2.14 The following system with more unknowns than equations

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

has no solution.

1.2.15 For example, $x + y = 1, 2x + 2y = 2, 3x + 3y = 3$ has an infinitely many solutions.

1.2.16 No. There must be a free variable and since the system is consistent there are infinitely many solutions

1.2.17 After performing Gaussian elimination the system becomes

$$\begin{aligned} x - y &= \quad\ \ 1 \\ 0 &= -3 + k \end{aligned}$$

This system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

1.2.18

a. Never exactly 1 solution; infinite solutions if $k = 2$; no solution if $k \neq 2$.

b. Exactly 1 solution if $k \neq 2$; infinite solutions if $k = 2$; never no solution.

c. Exactly 1 solution if $k \neq 2$; no solution if $k = 2$; never infinite solutions.

d. Exactly 1 solution for all k.

1.2.19

a. If $h \neq 2$ there will be a unique solution for any k. If $h = 2$ and $k \neq 4$, there are no solutions. If $h = 2$ and $k = 4$, then there are infinitely many solutions.

b. If $h \neq 4$, then there is exactly one solution. If $h = 4$ and $k \neq 4$, then there are no solutions. If $h = 4$ and $k = 4$, then there are infinitely many solutions.

1.2.20

a. Possible if $a = \pm 1$ and $a \neq \pm b$.

b. Not possible.

c. Possible if $a \neq \pm 1$ or $a = \pm b$.

1.2.21 Consistent if $b_3 - b_2 - b_1 = 0$ and $b_4 - 2b_2 - b_1 = 0$.

1.2.22 If $a \neq 0$ then the solution set of the first equation is $\{(x, y) \mid x = (c - by)/a\}$. Taking $y = 0$ gives the solution $(c/a, 0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a) + d \cdot 0 = e$, so $c = e$. Then taking $y = 1$ in $x = (c - by)/a$ gives that $a((c - b)/a) + d \cdot 1 = e$, which gives that $b = d$. Hence they are the same equation.
When $a = 0$ the equations can be different and still have the same solution set: e.g., $0x + 3y = 6$ and $0x + 6y = 12$.

1.2.23 We take three cases: that $a \neq 0$, that $a = 0$ and $c \neq 0$, and that both $a = 0$ and $c = 0$.
For the first, we assume that $a \neq 0$. Then Gaussian elimination

$$\begin{aligned} ax + \qquad\qquad by &= j \\ (-(cb/a) + d)y &= -(cj/a) + k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a) + d \neq 0$; remember that $a \neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a) + d = (ad - bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad - bc \neq 0$.
In the second case, if $a = 0$ but $c \neq 0$, then we swap

$$\begin{aligned} cx + dy &= k \\ by &= j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b \neq 0$ (we use the case assumption that $c \neq 0$ to get a unique x in back substitution). But where $a = 0$ and $c \neq 0$ the condition "$b \neq 0$" is equivalent to the condition "$ad - bc \neq 0$". That finishes the second case.
Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x + by &= j \\ 0x + dy &= k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x, y) will do), but it never has a unique solution. Note that $a = 0$ and $c = 0$ gives that $ad - bc = 0$.

1.3.1 The two other equations are

$$\begin{aligned} 6I_3 - 6I_4 + I_3 + I_3 + 5I_3 - 5I_2 &= -20 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \end{aligned}$$

Then the system is

$$\begin{aligned} 2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 &= 5 \\ 4I_1 + I_1 - I_4 + 2I_1 - 2I_2 &= -10 \\ 6I_3 - 6I_4 + I_3 + I_3 + 5I_3 - 5I_2 &= -20 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \end{aligned}$$

The solution is:

$$I_1 = -2.0107, I_2 = -1.2699, I_3 = -2.7355, I_4 = -1.5353$$

1.3.2 The equations obtained are

$$\begin{aligned} 2I_1 + 5I_1 + 3I_1 - 5I_2 &= -10 \\ I_2 + 3I_2 + 7I_2 + 5I_2 - 5I_1 &= -12 \\ 2I_3 + 4I_3 + 4I_3 + I_3 - I_2 &= 0 \end{aligned}$$

Simplifying this yields

$$\begin{aligned} 10I_1 - 5I_2 &= -10 \\ 16I_2 - 5I_1 &= -12 \\ 11I_3 - I_2 &= 0 \end{aligned}$$

Solving the system gives

$$I_1 = -\frac{44}{27}, I_2 = -\frac{34}{27}, I_3 = -\frac{34}{297}$$

Thus all currents flow in the clockwise direction.

2.1.1

a. 2

b. 3

c. -1

d. Not defined.

2.1.2

a. 2×3

b. 3×2

c. 2×2

2.1.3

a. $\begin{bmatrix} -2 & -1 \\ 12 & 13 \end{bmatrix}$

b. $\begin{bmatrix} 11 & -8 \\ -1 & -19 \end{bmatrix}$

c. $\begin{bmatrix} 9 & -7 \\ 11 & -6 \end{bmatrix}$

d. $\begin{bmatrix} -2 & 1 \\ 12 & 13 \end{bmatrix}$

2.1.4

a. -22

b. -2

c. 23

d. Not possible.

e. Not possible.

2.1.5

a. 2×1

b. 1×1

c. Not defined.

d. 2×2

2.1.6

a. $AB = \begin{bmatrix} 8 & 3 \\ 10 & -9 \end{bmatrix}$, $BA = \begin{bmatrix} -3 & 24 \\ 4 & 2 \end{bmatrix}$

b. $AB = \begin{bmatrix} -1 & -2 & 12 \\ 10 & 4 & 32 \end{bmatrix}$, BA is not defined

c. $AB = \begin{bmatrix} 3 & 8 \\ -5 & -8 \\ -8 & -32 \end{bmatrix}$, BA is not defined

d. $AB = \begin{bmatrix} 10 & -18 & 11 \\ -45 & 24 & -21 \\ -15 & 12 & -9 \end{bmatrix}$, $BA = \begin{bmatrix} 52 & -21 \\ 45 & -27 \end{bmatrix}$

e. $AB = \begin{bmatrix} -32 & 34 & -24 \\ -32 & 38 & -8 \\ -16 & 21 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 22 & -14 \\ -4 & -12 \end{bmatrix}$

f. $AB = \begin{bmatrix} -7 & 3 & 7 & -15 \\ -5 & -1 & -17 & 5 \end{bmatrix}$, BA is not defined

g. $AB = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ -2 & 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 & 4 \\ -3 & 6 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

h. $AB = \begin{bmatrix} 21 & -17 & -5 \\ 19 & 5 & 19 \\ 5 & 9 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 19 & 5 & 23 \\ 5 & -7 & -1 \\ -14 & 6 & 18 \end{bmatrix}$

2.1.7

a. $DA = \begin{bmatrix} 2 & 2 & 2 \\ -6 & -6 & -6 \\ -15 & -15 & -15 \end{bmatrix}$, $AD = \begin{bmatrix} 2 & -3 & 5 \\ 4 & -6 & 10 \\ -6 & 9 & -15 \end{bmatrix}$

b. $DA = \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix}$, $AD = \begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix}$

c. $DA = \begin{bmatrix} d_1a & d_1b \\ d_2c & d_2d \end{bmatrix}$, $AD = \begin{bmatrix} d_1a & d_2b \\ d_1c & d_2d \end{bmatrix}$

d. $DA = \begin{bmatrix} d_1a & d_1b & d_1c \\ d_2d & d_2e & d_2f \\ d_3g & d_3h & d_3i \end{bmatrix}$, $AD = \begin{bmatrix} d_1a & d_2b & d_3c \\ d_1d & d_2e & d_3f \\ d_1g & d_2h & d_3i \end{bmatrix}$

2.1.8

a. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$, $A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}$

c. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 125 \end{bmatrix}$

d. $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2.1.9

a. $\begin{bmatrix} 16 & -3 & 2 \\ -3 & 7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 0 & -2 \\ 3 & -13 & -3 \end{bmatrix}$

c. Not possible, since dimension of A and E are not the same.

d. $\begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 36 & 19 & 2 \\ 83 & -22 & 11 \\ 19 & -10 & 3 \end{bmatrix}$

f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D.

g. $\begin{bmatrix} 9 & -7 & 3 \end{bmatrix}$

h. $\begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

2.1.10

a. $7B - 4A = \begin{bmatrix} -4 & -29 \\ -47 & -2 \end{bmatrix}$

b. $AB = \begin{bmatrix} -10 & 1 \\ -20 & -1 \end{bmatrix}$

c. $BA = \begin{bmatrix} -9 & -12 \\ 1 & -2 \end{bmatrix}$

d. $E + D$ is undefined

e. $ED = \begin{bmatrix} \frac{67}{3} & 11 \\ -\frac{178}{3} & -72 \\ -30 & -40 \end{bmatrix}$

$CD + 2I_2A = \begin{bmatrix} \frac{238}{3} & -126 \\ \frac{863}{15} & \frac{361}{5} \end{bmatrix}$

g. $A - 4I_2 = \begin{bmatrix} -3 & 2 \\ 3 & 0 \end{bmatrix}$

h. $A^2 - B^2 = \begin{bmatrix} -8 & 16 \\ 25 & 3 \end{bmatrix}$

i. $(A + B)(A - B) = \begin{bmatrix} -7 & 3 \\ 46 & 2 \end{bmatrix}$

j. $A^2 - 5A - 2I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

k. $E^2 + 5E - 36I_3 = \begin{bmatrix} -30 & 20 & -15 \\ 0 & 0 & -36 \\ 0 & 0 & -36 \end{bmatrix}$

l. $EDC = \begin{bmatrix} \frac{3449}{15} & -\frac{407}{6} & 99 \\ -\frac{9548}{15} & -\frac{101}{3} & -648 \\ -324 & -35 & -360 \end{bmatrix}$

m. CDE is undefined

n. $ABCEDI_2 = \begin{bmatrix} -\frac{90749}{15} & -\frac{28867}{5} \\ -\frac{156601}{15} & -\frac{47033}{5} \end{bmatrix}$

2.1.11

a. $\begin{bmatrix} -9 & 6 & -8 \\ 4 & -3 & 1 \\ 10 & -7 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 5 & -6 \\ 2 & -4 & 6 \\ -9 & -10 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -9 \\ -7 & 6 \\ -4 & 3 \\ -9 & -9 \end{bmatrix}$

d. $\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$, symmetric

e. $\begin{bmatrix} 4 & -2 & 4 \\ 0 & -7 & -2 \\ 0 & 0 & 5 \end{bmatrix}$, A is lower triangular and A^T is upper triangular.

f. $\begin{bmatrix} -3 & 0 & 0 \\ -4 & -3 & 0 \\ -5 & 5 & -3 \end{bmatrix}$, A is upper triangular and A^T is lower triangular.

g. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, diagonal.

h. $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$, symmetric.

i. $\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$, skew-symmetric.

2.1.12

a. $\begin{bmatrix} -3 & -9 & -3 \\ -6 & -6 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 5 & -18 & 5 \\ -11 & 4 & 4 \end{bmatrix}$

c. $\begin{bmatrix} -7 & 1 & 5 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

e. $\begin{bmatrix} 13 & -16 & 1 \\ -16 & 29 & -8 \\ 1 & -8 & 5 \end{bmatrix}$

f. $\begin{bmatrix} 5 & 7 & -1 \\ 5 & 15 & 5 \end{bmatrix}$

g. Not possible.

2.1.13 The i-th row of GH is made up of the products of the i-th row of G with the columns of H. The product of a zero row with a column is zero.
It works for columns if stated correctly: if H has a column of zeros then GH (if defined) has a column of zeros.

2.1.14 The generalization is to go from the first and second rows to the i_1-th and i_2-th rows. Row i of GH is made up of the dot products of row i of G and the columns of H. Thus if rows i_1 and i_2 of G are equal then so are rows i_1 and i_2 of GH.

2.1.15 If the product of two diagonal matrices is defined, if both are $n \times n$ then the product of the diagonals is the diagonal of the products: where G, H are equal-sized diagonal matrices, GH is all zeros except each that i, i entry is $g_{i,i}h_{i,i}$.

2.1.16 A matrix is upper triangular if and only if its i, j entry is zero whenever $i > j$. Thus, if G, H are upper triangular then $h_{i,j}$ and $g_{i,j}$ are zero when $i > j$. An entry in the product $p_{i,j} = g_{i,1}h_{1,j} + \cdots + g_{i,n}h_{n,j}$ is zero unless at least some of the terms are nonzero, that is, unless for at least some of the summands $g_{i,r}h_{r,j}$ both $i \leq r$ and $r \leq j$. Of course, if $i > j$ this cannot happen and so the product of two upper triangular matrices is upper triangular. (A similar argument works for lower triangular matrices.)

2.1.17 f A is skew-symmetric then $A = -A^T$. It follows that $a_{ii} = -a_{ii}$ and so each $a_{ii} = 0$.

2.1.18

a. -9

b. 6

c. -23

d. Not defined; the matrix must be square.

e. 0

f. n

2.1.19

a. Not defined.

b. $\begin{bmatrix} -9 & -3 \\ 0 & 0 \\ -9 & 3 \end{bmatrix}$

c. $\begin{bmatrix} -54 & 6 \\ -72 & 6 \end{bmatrix}$

d. 289

e. $\begin{bmatrix} -83 & 19 \\ -116 & 4 \end{bmatrix}$

f. Not defined.

2.1.20 The product is the identity matrix (recall that $\cos^2\theta + \sin^2\theta = 1$).

2.1.21 Evaluate $A^2 = AA$.

2.1.22

a. $a = -1$, $b = 1/2$

b. $a = 5/2 + 3/2t$, $b = t$ where $t \in \mathbb{R}$

c. $a = 5$, $b = 0$

d. No solution.

2.1.23

$$\begin{aligned}\begin{bmatrix}-1 & -1\\ 3 & 3\end{bmatrix}\begin{bmatrix}x & y\\ z & w\end{bmatrix} &= \begin{bmatrix}-x-z & -w-y\\ 3x+3z & 3w+3y\end{bmatrix}\\ &= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{aligned}$$

Solution is: $w = -y, x = -z$ so the matrices are of the form $\begin{bmatrix}x & y\\ -x & -y\end{bmatrix}$

2.1.24 $(a,\ b,\ c,\ d) = (-29+60t,\ 13-\frac{79}{3}t,\ 7-14t,\ t)$ where $t \in \mathbb{R}$.

2.2.1

a. $X = \begin{bmatrix}-5 & 9\\ -1 & -14\end{bmatrix}$

b. $X = \begin{bmatrix}0 & -22\\ -7 & 17\end{bmatrix}$

c. $X = \begin{bmatrix}-5 & -2\\ -9/2 & -19/2\end{bmatrix}$

d. $X = \begin{bmatrix}8 & 12\\ 10 & 2\end{bmatrix}$

2.2.2

a. $\begin{bmatrix}0 & -2\\ -5 & -1\end{bmatrix}$

b. $\begin{bmatrix}10 & 2\\ 5 & 11\end{bmatrix}$

c. $\begin{bmatrix}-11 & -15\\ 37 & 32\end{bmatrix}$

d. No.

e. $(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2$

2.2.3

a. Not necessarily true. Find a counterexample.

b. Not necessarily true. Find a counterexample.

c. Not necessarily true. Find a counterexample.

d. Necessarily true. Prove.

e. Necessarily true. Prove.

f. Not necessarily true. Find a counterexample.

g. Not necessarily true. Find a counterexample.

2.2.4 First, each of these properties is easy to check in an entry-by-entry way. For example, writing

$$G = \begin{bmatrix}g_{1,1} & \dots & g_{1,n}\\ \vdots & & \vdots\\ g_{m,1} & \dots & g_{m,n}\end{bmatrix} \qquad H = \begin{bmatrix}h_{1,1} & \dots & h_{1,n}\\ \vdots & & \vdots\\ h_{m,1} & \dots & h_{m,n}\end{bmatrix}$$

then, by definition we have

$$G + H = \begin{bmatrix}g_{1,1}+h_{1,1} & \dots & g_{1,n}+h_{1,n}\\ \vdots & & \vdots\\ g_{m,1}+h_{m,1} & \dots & g_{m,n}+h_{m,n}\end{bmatrix}$$

and

$$H + G = \begin{bmatrix}h_{1,1}+g_{1,1} & \dots & h_{1,n}+g_{1,n}\\ \vdots & & \vdots\\ h_{m,1}+g_{m,1} & \dots & h_{m,n}+g_{m,n}\end{bmatrix}$$

and the two are equal since their entries are equal $g_{i,j} + h_{i,j} = h_{i,j} + g_{i,j}$.

2.2.5 All three statements are false, find counter examples.

2.2.6

a.

$$\begin{aligned}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & k\end{bmatrix} &= \begin{bmatrix}7 & 2k+2\\ 15 & 4k+6\end{bmatrix}\\ \begin{bmatrix}1 & 2\\ 3 & k\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} &= \begin{bmatrix}7 & 10\\ 3k+3 & 4k+6\end{bmatrix}\end{aligned}$$

Thus you must have $\begin{bmatrix}3k+3 = 15\\ 2k+2 = 10\end{bmatrix}$, Solution is: $k = 4$

b.

$$\begin{aligned}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & k\end{bmatrix} &= \begin{bmatrix}3 & 2k+2\\ 7 & 4k+6\end{bmatrix}\\ \begin{bmatrix}1 & 2\\ 1 & k\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} &= \begin{bmatrix}7 & 10\\ 3k+1 & 4k+2\end{bmatrix}\end{aligned}$$

However, $7 \neq 3$ and so there is no possible choice of k which will make these matrices commute.

2.2.7 Hint: Expand and rearrange.

2.2.8

a. The i,j entry of $(G+H)^{\mathsf{T}}$ is $g_{j,i} + h_{j,i}$. That is also the i,j entry of $G^{\mathsf{T}} + H^{\mathsf{T}}$.

b. The i,j entry of $(r \cdot H)^{\mathsf{T}}$ is $rh_{j,i}$, which is also the i,j entry of $r \cdot H^{\mathsf{T}}$.

c. The i,j entry of GH^{T} is the j,i entry of GH, which is the dot product of the j-th row of G and the i-th column of H. The i,j entry of $H^{\mathsf{T}}G^{\mathsf{T}}$ is the dot product of the i-th row of H^{T} and the j-th column of G^{T}, which is the dot product of the i-th column of H and the j-th row of G. Dot product is commutative and so these two are equal.

d. By the prior part each equals its transpose, e.g., $(HH^{\mathsf{T}})^{\mathsf{T}} = H^{\mathsf{T}^{\mathsf{T}}}H^{\mathsf{T}} = HH^{\mathsf{T}}$.

2.2.9 For $H + H^{\mathsf{T}}$, the i, j entry is $h_{i,j} + h_{j,i}$ and the j, i entry of is $h_{j,i} + h_{i,j}$. The two are equal and thus $H + H^{\mathsf{T}}$ is symmetric.
Every symmetric matrix does have that form, since we can write $H = (1/2) \cdot (H + H^{\mathsf{T}})$.

2.2.10

a. Verify using the definition of symmetric and skew-symmetric matrices.

b. Hint: use the previous part as the symmetric and skew-symmetric matrices.

2.2.11

a. Hint: The main diagonal is remains the same if a matrix is transposed.

b. Hint: Apply the definition of the trace to arbitrary matrices cA and A.

c. Hint: Apply the definition of the trace to arbitrary matrices A and B.

d. Hint: Analyse the ij product of the elements of the main diagonal.

2.2.12 Disprove: Show that it is impossible to obtain a nonzero matrix.

2.2.13 Hint: Apply the definition of an idempotent matrix.

2.2.14 Let $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. We have

$$F^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad F^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad F^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

In general,

$$F^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

where f_i is the i-th Fibonacci number $f_i = f_{i-1} + f_{i-2}$ and $f_0 = 0$, $f_1 = 1$, which we verify by induction, based on this equation.

$$\begin{bmatrix} f_{i-1} & f_{i-2} \\ f_{i-2} & f_{i-3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_i & f_{i-1} \\ f_{i-1} & f_{i-2} \end{bmatrix}$$

2.3.1

a. $\begin{bmatrix} -24 & -5 \\ 5 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/7 \end{bmatrix}$

c. $\begin{bmatrix} -4/7 & 5/7 \\ 3/7 & -2/7 \end{bmatrix}$

d. The inverse does not exist.

2.3.2

a. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -7/5 \end{bmatrix}$

b. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -7/10 & 3/10 \\ 29/10 & -11/10 \end{bmatrix}$

2.3.3 $(AB)\,B^{-1}A^{-1} = A\left(BB^{-1}\right)A^{-1} = AA^{-1} = I$
$B^{-1}A^{-1}\,(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}B = I$

2.3.4 $A = \begin{bmatrix} 0 & -1 \\ -11 & -\frac{17}{2} \end{bmatrix}$

2.3.5 $A = \begin{bmatrix} -\frac{3}{4} & 3 \\ 1 & -\frac{3}{4} \end{bmatrix}$

2.3.6 $\operatorname{tr} A^{-1} = \frac{a+d}{ad-bc}$

2.3.7 Here are four solutions to $H^2 = I$.

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

2.3.8 Disprove: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.3.9 The proof that the inverse is $r^{-1}H^{-1} = (1/r) \cdot H^{-1}$ (provided, of course, that the matrix is invertible) is easy.

2.3.10

a. The second matrix has its first row multiplied by 3.

$$\begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$$

b. The second matrix has its second row multiplied by 2.

$$\begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix}$$

c. The second matrix undergoes the combination operation of replacing the second row with -2 times the first row added to the second.

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

2.3.11

a. This matrix swaps row one and row three.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

b. This matrix swaps column one and two.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

2.3.12 $E_1A = \begin{bmatrix} d & e & f \\ a & b & c \end{bmatrix}$ E_1 interchanged $R1$ and $R2$ of A.

$E_2A = \begin{bmatrix} 5a & 5b & 5c \\ d & e & f \end{bmatrix}$ E_2 multiplied $R1$ of A by 5.
$E_3A = \begin{bmatrix} a-2d & b-2e & c-2f \\ d & e & f \end{bmatrix}$ E_3 replaced $R1$ in A with $R1-2R2$.
$E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix}$

2.3.13

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

2.3.14

a. $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 6 & 10 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 52 & -48 & 7 \\ 8 & -7 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -9 & 4 \\ 5 & -26 & 11 \\ 0 & -2 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 91 & 5 & -20 \\ 18 & 1 & -4 \\ -22 & -1 & 5 \end{bmatrix}$

e. $\begin{bmatrix} 25 & 8 & 0 \\ 78 & 25 & 0 \\ -30 & -9 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 5 & -3 & -8 \\ -4 & 2 & 5 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i. The inverse does not exist.

j. The inverse does not exist.

k. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & -4 \\ -35 & -10 & 1 & -47 \\ -2 & -2 & 0 & -9 \end{bmatrix}$

l. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -11 & 1 & 0 & -4 \\ -2 & 0 & 1 & -4 \\ -4 & 0 & 0 & 1 \end{bmatrix}$

m. $\begin{bmatrix} 1 & 28 & -2 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 254 & -19 & 110 \\ 0 & -67 & 5 & -29 \end{bmatrix}$

n. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

o. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1/4 \end{bmatrix}$

2.3.15

a. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

b. $\begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

c. not invertible

d. $\begin{bmatrix} 16 & 0 & 3 & 0 \\ -90 & -\frac{1}{2} & -\frac{35}{2} & \frac{7}{2} \\ 5 & 0 & 1 & 0 \\ -36 & 0 & -7 & 1 \end{bmatrix}$

2.3.16 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.3.17

a. $A = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

b. $X = \begin{bmatrix} -\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & \frac{3}{2} & -4 & 2 \end{bmatrix}$

c. $\begin{bmatrix} -3 & 4 & 2 \\ 2 & -2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$

2.3.18

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.19

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The answer is not unique.

2.3.20

a. $E = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

b. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

c. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

2.3.21 $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Note: The answer is not unique.

2.3.22 $X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix}$

2.3.23 Assume that B is row equivalent to A and that A is invertible. Because they are row-equivalent, there is a sequence of elementary row operations to reduce one to the other. We can do that reduction with elementary matrices, for instance, A can change by row operations to B as $B = E_n \cdots E_1 A$. This equation gives B as a product of elementary which are invertible matrices then, B is also invertible.

2.3.24 Hint: A matrix is invertible if and only if it can be expressed as a product of elementary matrices.

2.3.25 Yes $B = C$. Multiply $AB = AC$ on the left by A^{-1}.

2.3.26 The associativity of matrix multiplication gives $H^{-1}(HG) = H^{-1}Z = Z$ and also $H^{-1}(HG) = (H^{-1}H)G = IG = G$.

2.3.27 Multiply both sides of the first equation by H.

2.3.28 Suppose that that there exists two inverse then show that they are equal using the fact that they are both the inverse of A.

2.3.29 $T^k(T^{-1})^k = (TT\cdots T)\cdot(T^{-1}T^{-1}\cdots T^{-1}) = T^{k-1}(TT^{-1})(T^{-1})^{k-1} = \cdots = I$.

2.3.30 Since $A^{-1}A = AA^{-1} = I$ and inverses are unique then it follows that $\left(A^{-1}\right)^{-1} = A$.

2.3.31

$$\begin{aligned} A^{\mathsf{T}}\left(A^{-1}\right)^{\mathsf{T}} &= \left(A^{-1}A\right)^{\mathsf{T}} = I^{\mathsf{T}} = I \\ \left(A^{-1}\right)^{\mathsf{T}}A^{\mathsf{T}} &= \left(AA^{-1}\right)^{\mathsf{T}} = I^{\mathsf{T}} = I \end{aligned}$$

2.3.32 Checking that when $I - T$ is multiplied on both sides by that expression (assuming that T^4 is the zero matrix) then the result is the identity matrix is easy. The obvious generalization is that if T^n is the zero matrix then $(I-T)^{-1} = I + T + T^2 + \cdots + T^{n-1}$; the check again is easy.

2.3.33 Hint: Apply the definition of symmetric matrices.

2.3.34 Show that $A^{-1} = -A^2 - 3A - I$.

2.3.35 Hint: Use the definition of the inverse of a matrix.

2.3.36 It is false; these two don't commute.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

2.3.37 Hint: Show that the homogeneous system $Ax = 0$ has only the trivial solution.

2.4.1

a. $A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

b. $x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$

2.4.2

a. $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b. $x = \begin{bmatrix} -7 \\ -7 \end{bmatrix}$

c. $x = \begin{bmatrix} -7 \\ 1 \\ -1 \end{bmatrix}$

d. $x = \begin{bmatrix} -7 \\ -7 \\ 9 \end{bmatrix}$

3.1.1

a. 34

b. 41

c. -44

d. -74

3.1.2

a. $M_{1,1} = \begin{bmatrix} 7 & 6 \\ 6 & 10 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} 3 & 6 \\ 1 & 10 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix}$.
$C_{1,1} = 43$, $C_{1,2} = -24$, $C_{1,3} = 11$.

b. $M_{1,1} = \begin{bmatrix} -6 & 8 \\ -3 & -2 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -10 & 8 \\ 0 & -2 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} 10 & -6 \\ 0 & -3 \end{bmatrix}$.
$C_{1,1} = 36$, $C_{1,2} = -20$, $C_{1,3} = -30$.

c. $M_{1,1} = \begin{bmatrix} 3 & 10 \\ 3 & 9 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -3 & 10 \\ -9 & 9 \end{bmatrix}$, $M_{1,3} = \begin{bmatrix} -3 & 3 \\ -9 & 3 \end{bmatrix}$.
$C_{1,1} = -3$, $C_{1,2} = -63$, $C_{1,3} = 18$.

d. $M_{1,1} = \begin{bmatrix} 0 & 0 \\ 8 & -1 \end{bmatrix}$, $M_{1,2} = \begin{bmatrix} -8 & 0 \\ -10 & -1 \end{bmatrix}$, $M_{1,3} =$

$\begin{bmatrix} -8 & 0 \\ -10 & 8 \end{bmatrix}$.
$C_{1,1} = 0$, $C_{1,2} = -8$, $C_{1,3} = -64$.

3.1.3

a. $3(+1)\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} + 0(-1)\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + 1(+1)\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -13$

b. $1(-1)\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} + 2(+1)\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} + 2(-1)\begin{vmatrix} 3 & 0 \\ -1 & 3 \end{vmatrix} = -13$

c. $1(+1)\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} + 2(-1)\begin{vmatrix} 3 & 0 \\ -1 & 3 \end{vmatrix} + 0(+1)\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = -13$

3.1.4

a. 31

b. 375

c. -2

3.1.5

a. -59

b. 250

c. 3

d. 0

e. 0

f. 2

3.1.6

a. x^2

b. $\dfrac{1}{x^7}$

c. -12

d. 0

e. $20i + 43j + 4k$

f. -2

3.1.7 Evaluate the determinant using a cofactor expansion. The same is true for lower triangular matrices.

3.1.8

a. 2

b. -40

c. -24

3.1.9 $\lambda = \frac{3\pm\sqrt{33}}{4}$

3.1.10 False, Here is a determinant whose value

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

doesn't equal the result of expanding down the diagonal.

$$1\cdot(+1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 1\cdot(+1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 1\cdot(+1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

3.1.11 There are no real numbers θ that make the matrix singular because the determinant of the matrix $\cos^2\theta + \sin^2\theta$ is never 0, it equals 1 for all θ.

3.2.1

a. The transpose was applied and it does not change the determinant.

b. Two rows were switched and so the resulting determinant is -1 times the first.

c. The determinant is unchanged since the operation is adding first row to the second.

d. The second row was multiplied by 2 so the determinant of the result is 2 times the original determinant.

e. Two columns were switched so the determinant of the second is -1 times the determinant of the first.

3.2.2

a. $\det(A) = 90$; $2R_1 \to R_1$.
$\det(B) = 45$; $10R_1 + R_3 \to R_3$.
$\det(C) = 45$; $C = A^T$.

b. $\det(A) = 41$; $R_2 \leftrightarrow R_3$.
$\det(B) = 164$; $-4R_3 \to R_3$.
$\det(C) = -41$; $R_2 + R_1 \to R_1$.

c. $\det(A) = -16$; $R_1 \leftrightarrow R_2$ then $R_1 \leftrightarrow R_3$.
$\det(B) = -16$; $-R_1 \to R_1$ and $-R_2 \to R_2$.
$\det(C) = -432$; $C = 3M$.

d. $\det(A) = -120$; $R_1 \leftrightarrow R_2$ then $R_1 \leftrightarrow R_3$ then $R_2 \leftrightarrow R_3$.
$\det(B) = 720$; $2R_2 \to R_2$ and $3R_3 \to R_3$.
$\det(C) = -120$; $C = -M$.

3.2.3

a. 15

b. -52

c. 0

d. 1

e. -113

f. 179

3.2.4 $\det(A) = -\frac{5}{12}$

3.2.5 Hint: Use elementary operations to bring the matrix under triangular form.

3.2.6 If the determinant is nonzero, then it will remain nonzero with row operations applied to the matrix. However, by assumption, you can obtain a row of zeros by doing row operations. Thus the determinant must have been zero.

3.3.1

a. $\begin{bmatrix} 0 & -1 & 2 \\ 3 & -2 & -8 \\ 0 & 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 0 & -1 \\ -5 & 1 \end{bmatrix}$

d. $\begin{bmatrix} -24 & -12 & 12 \\ 12 & 6 & -6 \\ -8 & -4 & 4 \end{bmatrix}$

e. $\begin{bmatrix} 4 & -3 & 2 & -1 \\ -3 & 6 & -4 & 2 \\ 2 & -4 & 6 & -3 \\ -1 & 2 & -3 & 4 \end{bmatrix}$

3.3.2

a. $\begin{bmatrix} T_{1,1} & T_{2,1} \\ T_{1,2} & T_{2,2} \end{bmatrix} = \begin{bmatrix} \left|t_{2,2}\right| & -\left|t_{1,2}\right| \\ -\left|t_{2,1}\right| & \left|t_{1,1}\right| \end{bmatrix} = \begin{bmatrix} t_{2,2} & -t_{1,2} \\ -t_{2,1} & t_{1,1} \end{bmatrix}$

b. $(1/t_{1,1}t_{2,2} - t_{1,2}t_{2,1}) \begin{bmatrix} t_{2,2} & -t_{1,2} \\ -t_{2,1} & t_{1,1} \end{bmatrix}$

3.3.3 Consider this diagonal matrix.

$$D = \begin{bmatrix} d_1 & 0 & 0 & \ldots & \\ 0 & d_2 & 0 & & \\ 0 & 0 & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix}$$

If $i \neq j$ then the i,j minor is an $(n-1)\times(n-1)$ matrix with only $n-2$ nonzero entries, because we have deleted both d_i and d_j. Thus, at least one row or column of the minor is all zeroes, and so the cofactor $D_{i,j}$ is zero. If $i = j$ then the minor is the diagonal matrix with entries d_1, …, d_{i-1}, d_{i+1}, …, d_n. Its determinant is obviously $(-1)^{i+j} = (-1)^{2i} = 1$ times the product of those.

$$\text{adj}(D) = \begin{bmatrix} d_2 \cdots d_n & 0 & & 0 \\ 0 & d_1 d_3 \cdots d_n & & 0 \\ & & \ddots & \\ & & & d_1 \cdots d_{n-1} \end{bmatrix}$$

3.3.4 Just note that if $S = T^{\mathsf{T}}$ then the cofactor $S_{j,i}$ equals the cofactor $T_{i,j}$ because $(-1)^{j+i} = (-1)^{i+j}$ and because the minors are the transposes of each other (and the determinant of a transpose equals the determinant of the matrix).

3.3.5 False. A counter example.

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{adj}(T) = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix} \qquad \text{adj}(\text{adj}(T)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.3.6

a. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix} = -13$ and so it has an inverse given by

$$\frac{1}{-13} \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^{\mathsf{T}}$$

$$= \frac{1}{-13} \begin{bmatrix} -1 & 3 & -6 \\ 3 & -9 & 5 \\ -4 & -1 & 2 \end{bmatrix}^{\mathsf{T}}$$

$$= \begin{bmatrix} \frac{1}{13} & -\frac{3}{13} & \frac{4}{13} \\ -\frac{3}{13} & \frac{9}{13} & \frac{1}{13} \\ \frac{6}{13} & -\frac{5}{13} & -\frac{2}{13} \end{bmatrix}$$

b. $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 7$ so it has an inverse. This inverse is

$$\frac{1}{7} \begin{bmatrix} 1 & 3 & -6 \\ -2 & 1 & 5 \\ 2 & -1 & 2 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} & -\frac{1}{7} \\ -\frac{6}{7} & \frac{5}{7} & \frac{2}{7} \end{bmatrix}$$

c. $\begin{vmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 3$ so it has an inverse which is $\begin{bmatrix} 1 & 0 & -3 \\ -\frac{2}{3} & \frac{1}{3} & \frac{5}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

d. $\begin{vmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} = 2$ and so it has an inverse. The inverse is $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{9}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$

3.3.7 This equation

$$0 = \det\left(\begin{bmatrix} 12-x & 4 \\ 8 & 8-x \end{bmatrix}\right) = 64 - 20x + x^2 = (x-16)(x-4)$$

has roots $x = 16$ and $x = 4$.

3.3.8 This follows because $\det(ABC) = \det(A)\det(B)\det(C)$ and if this product is nonzero, then each determinant in the product is nonzero and so each of these matrices is invertible.

3.3.9 If $\det(A) \neq 0$, then A^{-1} exists and by multiplying both sides on the left by A^{-1}. The result is $X = 0$.

3.3.10 $1 = \det(A)\det(B)$. Hence both A and B have inverses. Given any X,

$$A(BA - I)X = (AB)AX - AX = AX - AX = 0$$

and so it follows that $(BA - I)X = 0$. Since X is arbitrary, it follows that $BA = I$.

3.3.11 The given condition is what it takes for the determinant to be non zero. Recall that the determinant of an upper triangular matrix is just the product of the entries on the main diagonal.

3.3.12

a. The determinant is equal to 1, so the matrix is invertible for all t.

b. The determinant is equal to $5e^{-t}$ hence it never equal to zero, so the matrix is invertible for all t.

c. The determinant is equal to t^3+2 hence it has no inverse when $t = -\sqrt[3]{2}$.

3.3.13

a. $\begin{vmatrix} e^t & 0 & 0 \\ 0 & e^t\cos t & e^t \sin t \\ 0 & e^t\cos t - e^t \sin t & e^t \cos t + e^t \sin t \end{vmatrix} = e^{3t}$. Hence the inverse is

$$e^{-3t}\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t}\cos t + e^{2t}\sin t & -\left(e^{2t}\cos t - e^{2t}\sin\right)t \\ 0 & -e^{2t}\sin t & e^{2t}\cos(t) \end{bmatrix}^{\mathsf{T}}$$

$$= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t}(\cos t + \sin t) & -(\sin t)\,e^{-t} \\ 0 & -e^{-t}(\cos t - \sin t) & (\cos t)\,e^{-t} \end{bmatrix}$$

b. $\begin{bmatrix} \frac{1}{2}e^{-t} & 0 & \frac{1}{2}e^{-t} \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t & -\sin t & \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t & \cos t & -\frac{1}{2}\cos t - \frac{1}{2}\sin t \end{bmatrix}$

3.3.14 $\det(TS) = \det(T)\cdot\det(S) = \det(S)\cdot\det(T) = \det(ST)$.

3.3.15 $\det(aA) = \det(aIA) = \det(aI)\det(A) = a^n\det(A)$. The diagonal matrix which has a down the main diagonal has determinant equal to a^n.

3.3.16

a. Plug and chug: the determinant of the product is this

$$\begin{aligned}\det(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} w & x \\ y & z\end{bmatrix}) &= \det(\begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz\end{bmatrix}) \\ &= acwx + adwz + bcxy + bdyz \\ &\quad -acwx - bcwz - adxy - bdyz\end{aligned}$$

while the product of the determinants is this.

$$\det(\begin{bmatrix} a & b \\ c & d\end{bmatrix})\det(\begin{bmatrix} w & x \\ y & z\end{bmatrix}) = (ad-bc)(wz-xy)$$

Verification that they are equal is easy.

b. Use the prior part.

3.3.17 The statement is false. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}$.

3.3.18

a. If it is defined then it is $(3^2)(2)(2^{-2})(3)$.

b. Hint: $\det 6A^3 + 5A^2 + 2A = \det A \det 6A^2 + 5A + 2I$.

3.3.19 Hint: Use the identity $\det(A)A^{-1} = \mathrm{adj}(A)$.

3.3.20

a. Hint: take the determinant of BG.

b. $(10!)^9$

c. $\frac{(2(10!)+3)^{10}}{10!}$

d. $\pm\sqrt{10!}$

e. 0

f. $\frac{2^{10}\pi}{(96)^{10}(10!)^3}$

g. $\frac{(10+10!)(9+10!)(8+10!)(7+10!)(6+10!)(5+10!)(4+10!)(3+10!)(2+10!)(1+10!)}{(10!)^{10}}$

3.3.21

a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$.

b. $1 = \det(AA^{-1}) = \det(AA^{\mathsf{T}}) = \det(A)\det(A^{\mathsf{T}}) = (\det(A))^2$

c. The converse does not hold; here is an example.

$$\begin{bmatrix} 3 & 1 \\ 2 & 1\end{bmatrix}$$

3.3.22 If $H = P^{-1}GP$ then $\det(H) = \det(P^{-1})\det(G)\det(P) = \det(P^{-1})\det(P)\det(G) = \det(P^{-1}P)\det(G) = \det(G)$.

3.3.23 0 since $0 = \det(0) = \det\left(A^k\right) = (\det(A))^k$.

3.3.24 An algebraic check is easy.

$$0 = xy_2 + x_2y_3 + x_3y - x_3y_2 - xy_3 - x_2y = x\cdot(y_2-y_3) + y\cdot(x_3-x_2) + x_2y_3 -$$

simplifies to the familiar form

$$y = x\cdot(x_3-x_2)/(y_3-y_2) + (x_2y_3 - x_3y_2)/(y_3-y_2)$$

(the $y_3 - y_2 = 0$ case is easily handled).

3.3.25 Hint: Apply the determinant to both sides $AB = -BA$.

3.3.26 Disprove. Recall that constants come out one row at a time.

$$\det(\begin{bmatrix} 2 & 4 \\ 2 & 6\end{bmatrix}) = 2\cdot\det(\begin{bmatrix} 1 & 2 \\ 2 & 6\end{bmatrix}) = 2\cdot 2\cdot\det(\begin{bmatrix} 1 & 2 \\ 1 & 3\end{bmatrix})$$

This contradicts linearity (here we didn't need S, i.e., we can take S to be the matrix of zeros).

3.3.27

a. False. Consider $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 5 & 4 \\ 0 & 3 & 3 \end{bmatrix}$

b. True.

c. False.

d. False.

e. True.

f. True.

g. True.

h. True.

i. True.

j. True.

3.4.1 $x_1 = 4$

3.4.2

a. $\det(A) = -123$, $\det(A_1) = -492$, $\det(A_2) = 123$, $\det(A_3) = 492$,
$x = \begin{bmatrix} 4 \\ -1 \\ -4 \end{bmatrix}$.

b. $\det(A) = -43$, $\det(A_1) = 215$, $\det(A_2) = 0$,
$x = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$.

c. $\det(A) = 0$, $\det(A_1) = 0$, $\det(A_2) = 0$, $\det(A_3) = 0$. Infinite solutions exist.

d. $\det(A) = 0$, $\det(A_1) = -56$, $\det(A_2) = 26$. No solution exist.

e. $\det(A) = 0$, $\det(A_1) = 0$, $\det(A_2) = 0$, $\det(A_3) = 0$. Infinite solutions exist.

f. $\det(A) = 0$, $\det(A_1) = 1247$, $\det(A_2) = -49$, $\det(A_3) = -49$. No solution exist.

3.4.3

a. 4

b. -1

4.1.1 $\left(\frac{5}{3}, \frac{2}{3}\right)$

4.1.2 $\vec{x} = (1, 2) + t(2, -3)$, $t \in \mathbb{R}$

4.1.3 $\vec{x} = (1, 5, -2) + t(3, -6, 5)$, and $\begin{cases} x = 1 + 3t \\ y = 5 - 6t \\ z = -2 + 5t \end{cases}$, $t \in \mathbb{R}$.

4.1.4 No, they are skew lines.

4.1.5

a. $(-4, 7, -13)$

b. $(-10, 10, -22)$

c. $(-1/2, 7/2, 9/2)$

d. $(-3, -6, 6)$

4.1.6 $\overrightarrow{PQ} = (1, -2, -3)$, $\overrightarrow{PR} = (-1, 1, -1)$, $\overrightarrow{PS} = (-7, -2, 4)$, $\overrightarrow{QR} = (-2, 3, 2)$, $\overrightarrow{SR} = (6, 3, -5)$

4.1.7 Note that alternative correct answers are possible.

a. $\vec{x} = (3, 4) + t(-5, 1)$, $t \in \mathbb{R}$

b. $\vec{x} = (2, 0, 5) + t(4, -2, -11)$, $t \in \mathbb{R}$

c. $\vec{x} = (4, 0, 1, 5, -3) + t(-2, 0, 1, 2, -1)$, $t \in \mathbb{R}$

4.1.8 Note that alternative correct answers are possible.

a. $\vec{x} = (-1, 2) + t(3, -5)$, $t \in \mathbb{R}$

b. $\vec{x} = (4, 1) + t(-6, -2)$, $t \in \mathbb{R}$

c. $\vec{x} = (1, 3, -5) + t(-3, -4, 5)$, $t \in \mathbb{R}$

d. $\vec{x} = \left(\frac{1}{2}, \frac{1}{4}, 1\right) + t\left(-\frac{3}{2}, \frac{3}{4}, -\frac{2}{3}\right)$, $t \in \mathbb{R}$

e. $\vec{x} = (1, 0, -2, -5) + t(-4, 2, 1, 7)$, $t \in \mathbb{R}$

4.1.9

a. $\left(-\frac{1}{2}, 1, -\frac{3}{2}\right)$

b. $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$

4.1.10

a. $(1, 3, 3)$ and $(0, 2, 5)$

b. $\left(\frac{2}{3}, \frac{4}{3}, \frac{11}{3}\right)$ and $\left(\frac{7}{3}, \frac{5}{3}, \frac{7}{3}\right)$

4.1.11

a. $\left(0, \frac{13}{4}, -\frac{11}{4}\right)$

b. $\left(0, -1, -\frac{2}{3}, 8\right)$

c. $\left(-\frac{14}{3}, 1, 6\right)$

4.1.12

a. $\left(-\frac{25}{17}, -\frac{36}{17}\right)$

b. $(0, 1, 2)$

c. no point of intersection

d. $(7, -2, 5)$

4.1.13

a. $\overrightarrow{AB} = k\overrightarrow{AC}$ for some $k \in \mathbb{R}$

b. since $-2\overrightarrow{PQ} = \overrightarrow{PR}$, the points are collinear

c. S, T, and U are not collinear because $\overrightarrow{SU} \neq k\overrightarrow{ST}$ for any $k \in \mathbb{R}$

4.1.14 We have $\begin{cases} 5a + 3b = -16 \\ 7a - 10b = 77 \end{cases}$ which gives that $a = 1$ and $b = -7$.

4.1.15 We have $\begin{cases} a + 3b = -1 \\ -a + 2b + c = 1 \\ b + 4c = 19 \end{cases}$ which implies that $a = 2$, $b = -1$ and $c = 5$.

4.1.16 Suppose the vertices are labeled A, B, C and D. Show that $\frac{1}{2}\overrightarrow{AC} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{4}BD$.

4.1.17 If A, B, and C are the vertices of the triangle, P is the midpoint of AC, and Q is the midpoint of BC, show that $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB}$.

4.1.18 Use Exercise 4.1.17 twice.

4.1.19 Use Exercises 4.1.18 and 4.1.16.

4.1.20 In the triangle ABC with midpoints PQR, show that the point that is two thirds of the distance from A to P is given by $\frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$. Then show that $\frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$ is two thirds of the distance from B to Q and two thirds of the distance from C to R.

4.1.21

a. $\vec{x} = (1,-1,-1) + t(2,3,-1)$, $\begin{cases} x = 1+2t \\ y = -1+3t \\ z = -1-t \end{cases}$, $t \in \mathbb{R}$

b. $\vec{x} = (2,-4,5) + t(3,-1,-2)$, $\begin{cases} x = 2+3t \\ y = -4-t \\ z = 5-2t \end{cases}$, $t \in \mathbb{R}$

4.1.22 To the x-axis: $\vec{x} = (3,-4,7) + t(1,0,0)$, $t \in \mathbb{R}$. To the y-axis: $\vec{x} = (3,-4,7) + t(0,1,0)$, $t \in \mathbb{R}$. To the z-axis: $\vec{x} = (3,-4,7) + t(0,0,1)$, $t \in \mathbb{R}$.

4.1.23

a. $\begin{cases} x = 2t \\ y = 3t \\ z = 2-3t \end{cases}$, $t \in \mathbb{R}$.

b. $\begin{cases} x = 1+2t \\ y = 3t \\ z = -2t \end{cases}$, $t \in \mathbb{R}$.

4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$.

4.2.2 $\sqrt{13}$

4.2.3 $\sqrt{82}$

4.2.4 $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

4.2.5

a. $\arccos\left(-\dfrac{9}{546}\right) \approx 112.65°$

b. $\arccos\left(\dfrac{4}{165}\right) \approx 71.86°$

4.2.6 Hint: $\vec{x} = \vec{x} - \vec{y} + \vec{y}$

4.2.7

a. Find a, b, and c by taking the dot product of $\vec{w}$ with $\vec{u_1}$, $\vec{u_2}$, and $\vec{u_3}$, respectively.

b. $\vec{w} = -\frac{1}{3}\vec{u_2} - \frac{5}{3}\vec{u_3}$

4.2.8 If A, B, and C are the vertices of the triangle, let H be the point of intersection of the altitudes from A and B. Then prove that $\overrightarrow{CH}$ is orthogonal to $\overrightarrow{AB}$.

4.2.9 Hint: Let O be the centre of the circle, and express everything in terms of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

4.2.10

a. If $\vec{v} = (v_1, v_2)$, show that $v_1 = \|\vec{v}\|\cos\phi$ and that $v_2 = \|\vec{v}\|\sin\phi$. This shows that $\vec{v} = \|\vec{v}\|(\cos\phi, \sin\phi)$, and since $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$, the required formula follows.

b. The angle between the vectors $\vec{e_a}$ and $\vec{e_b}$ is exactly $\alpha-\beta$, so by the definition of the dot product we get $\cos(\alpha-\beta) = \dfrac{\vec{e_a} \bullet \vec{e_b}}{\|\vec{e_a}\|\|\vec{e_b}\|}$. The denominator here is one since they are both unit vectors, and working out the top gives the required result.

4.2.11

a. $\sqrt{29}$

b. $\sqrt{17}$

c. $\frac{\sqrt{251}}{5}$

d. $\sqrt{6}$

4.2.12

a. $2\sqrt{10}$

b. 5

c. $\sqrt{170}$

d. $3\sqrt{6}$

4.2.13

4.2.14

a. ≈ 1.074 radians

b. ≈ 1.180 radians

c. ≈ 1.441 radians

4.2.15

a. Yes

b. Yes

c. No

d. Yes

4.2.16

a. $k = 6$

b. $k = 0$ or $k = 3$

c. $k = -3$

d. any $k \in \mathbb{R}$

4.2.17

a. The statement is false: Let $\vec{a}$ be any non-zero vector, let $\vec{b}$ be any non-zero vector that is orthogonal to $\vec{a}$, and let $\vec{c} = -\vec{b}$. Then the antecedent of the statement is true, since both sides are equal to 0, while the consequent is false.

b. No.

4.2.18 Expand the left side of the equation by using the fact that $\|\vec{v}\|^2 = \vec{v} \bullet \vec{v}$ for any vector $\vec{v}$ to get to the right side.

4.2.19

a. $\text{proj}_{\vec{a}}(\vec{b}) = (0, -5)$, $\text{perp}_{\vec{a}}(\vec{b}) = (3, 0)$

b. $\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{36}{25}, \frac{48}{25}\right)$, $\text{perp}_{\vec{a}}(\vec{b}) = \left(-\frac{136}{25}, \frac{102}{25}\right)$

c. $\text{proj}_{\vec{a}}(\vec{b}) = (0, 5, 0)$, $\text{perp}_{\vec{a}}(\vec{b}) = (-3, 0, 2)$

d. $\text{proj}_{\vec{a}}(\vec{b}) = \left(-\frac{4}{9}, \frac{8}{9}, -\frac{8}{9}\right)$, $\text{perp}_{\vec{a}}(\vec{b}) = \left(\frac{40}{9}, \frac{1}{9}, -\frac{19}{9}\right)$

4.2.20

a. $\hat{u} = \left(\frac{2}{7}, \frac{6}{7}, \frac{3}{7}\right)$

b. $\left(\frac{220}{49}, \frac{660}{49}, \frac{330}{49}\right)$

c. $\left(\frac{270}{49}, \frac{222}{49}, -\frac{624}{49}\right)$

4.2.21

a. $\left(-\frac{2}{17}, -\frac{3}{17}, \frac{2}{17}\right)$, $\left(\frac{70}{17}, -\frac{14}{17}, \frac{49}{17}\right)$

b. $\left(\frac{3}{2}, \frac{3}{2}, -3\right)$, $\left(\frac{5}{2}, -\frac{1}{2}, 1\right)$

c. $\left(\frac{14}{3}, -\frac{7}{3}, \frac{7}{3}\right)$, $\left(\frac{1}{3}, \frac{4}{3}, \frac{2}{3}\right)$

d. $\left(\frac{1}{3}, -\frac{2}{3}, -\frac{1}{3}, 1\right)$, $\left(\frac{5}{3}, -\frac{1}{3}, \frac{7}{3}, 0\right)$

4.3.1 $x - 4y + z = -11$

4.3.2 $2x + 3y - 5 = 21$

4.3.3

a. $2x_1 + 4x_2 - x_3 = 9$

b. $3x_1 + 5x_3 = 26$

c. $3x_1 - 4x_2 + x_3 = 8$

4.3.4

a. $3x_1 + x_2 + 4x_3 + x_4 = -2$

b. $x_2 + 3x_3 + 3x_4 = 1$

4.3.5

a. $\vec{n} = (3, -2, 1)$

b. $\vec{n} = (-4, 3, -5)$

c. $\vec{n} = (1, -1, 2, -3)$

4.3.6

a. $2x_1 - 3x_2 + 5x_3 = 6$

b. $x_2 = -2$

4.3.7

a. $\left(\frac{20}{13}, \frac{51}{13}, \frac{37}{13}\right)$

b. $\left(\frac{7}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

4.3.8

a. The line is parallel to the plane.

b. The line is orthogonal to the plane.

c. The line is neither parallel nor orthogonal to the plane, $\theta \approx 0.702$ radians.

d. The line is orthogonal to the plane.

e. The line is parallel to the plane.

4.3.9

a. $(-27, -9, -9)$

b. $(-31, -34, 8)$

c. $(-4, 5, -4)$

4.3.10

4.3.11

a. $39x + 12y + 10z = 140$

b. $11x - 21y - 17z = -56$

c. $-12x + 3y - 19z = -14$

4.3.12

a. $x - 4y - 10z = -85$

b. $2x - 2y + 3z = -5$

c. $-5x - 2y + 6z = 15$

4.4.1

a. $\left(\frac{5}{2}, \frac{5}{2}\right)$, $\frac{5}{\sqrt{2}}$

b. $\left(\frac{58}{17}, \frac{91}{17}\right)$, $\frac{6}{\sqrt{17}}$

c. $\left(\frac{17}{6}, \frac{1}{3}, -\frac{1}{6}\right)$, $\sqrt{\frac{29}{6}}$

d. $\left(\frac{5}{3}, \frac{11}{3}, -\frac{1}{3}\right)$, $\sqrt{6}$

4.4.2

a. $\frac{2}{\sqrt{26}}$

b. $\frac{13}{\sqrt{38}}$

c. $\frac{4}{\sqrt{5}}$

d. $\sqrt{6}$

4.4.3

a. $\frac{1}{3}(0, 14, 1, 10)$

b. $\frac{1}{7}(-11, 13, 10, -3)$

4.4.4

a. $\sqrt{35}$

b. $\sqrt{11}$

c. 9

d. 13

4.4.5

a. Point of intersection: $(-3, 1, 3)$, $-x + 5y + 3z = 17$

b. No point of intersection, $\frac{9}{\sqrt{35}}$

c. No point of intersection, $\frac{23}{3\sqrt{10}}$

d. Point of intersection $(1, 0, 7)$, $3x - y - z = -4$

4.4.6

a. 126

b. 5

4.5.1

a. $\vec{x} = \left(\frac{46}{11}, \frac{3}{11}, 0\right) + t(-2, -3, -11)$, $t \in \mathbb{R}$

b. $\vec{x} = \left(\frac{7}{2}, 4, 0\right) + t(3, -4, 2)$, $t \in \mathbb{R}$

5.1.1

a. $0 + 0x + 0x^2 + 0x^3$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. The constant function $f(x) = 0$

d. The constant function $f(n) = 0$

5.1.2

a. $3 + 2x - x^2$

b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

c. $-3e^x + 2e^{-x}$

5.1.3

a. $1 + 2x$, $2 - 1x$, and x.

b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

a.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b.

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

a. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.

b. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

5.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

a. It is not closed under addition.

$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

b. It is not closed under addition.

$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

c. It is not closed under addition.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

d. It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

e. The set is empty, violating the existance of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

a. It is not a vector space.

$$(1+1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b. It is not a vector space.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5.1.10 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

a. Yes.

b. Yes.

c. No, this set is not closed under the natural addition operation. The vector of all 1/4's is an element of this set but when added to itself the result, the vector of all 1/2's, is not an element of the set.

d. Yes.

e. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11

a. Closed under vector addition. Hint: Apply determinant properties.

b. $\vec{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$

c. Every $A \in V$ has an additive inverse A^{-1}.

d. Yes.

e. Not closed under scalar multiplication. Since $0\vec{0} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin V$

5.1.12 Check all 10 conditions of the definition of a vector space.

5.1.13 It is not a vector space since it is not closed under addition, as $(x^2) + (1 + x - x^2)$ is not in the set.

5.1.14

a. No since $1 \cdot (0,\ 1) + 1 \cdot (0,\ 1) \neq (1+1) \cdot (0,\ 1)$.

b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0,\ 1) \neq (0,\ 1)$.

5.1.15

a. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

b. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which implies that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

5.1.16 We can combine the argument showing closure under addition with the argument showing closure under scalar multiplication into one single argument showing closure under linear combinations of two vectors. If $r_1, r_2, x_1, x_2, y_1, y_2, z_1, z_2$ are in $\mathbb{R}$ then

$$r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 - r_1 + 1 \\ r_1y_1 \\ r_1z_1 \end{pmatrix} + \begin{pmatrix} r_2x_2 - r_2 + 1 \\ r_2y_2 \\ r_2z_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 - r_1 + r_2x_2 - r_2 + 1 \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix}$$

(note that the definition of addition in this space is that the first components combine as $(r_1x_1 - r_1 + 1) + (r_2x_2 - r_2 + 1) - 1$, so the first component of the last vector does not say '$+2$'). Adding the three components of the last vector gives $r_1(x_1 - 1 + y_1 + z_1) + r_2(x_2 - 1 + y_2 + z_2) + 1 = r_1 \cdot 0 + r_2 \cdot 0 + 1 = 1$.

Most of the other checks of the conditions are easy (although the oddness of the operations keeps them from being routine). Commutativity of addition goes like this.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 - 1 \\ y_2 + y_1 \\ z_2 + z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Associativity of addition has

$$(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2 - 1) + x_3 - 1 \\ (y_1 + y_2) + y_3 \\ (z_1 + z_2) + z_3 \end{pmatrix}$$

while

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + (\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}) = \begin{pmatrix} x_1 + (x_2 + x_3 - 1) - 1 \\ y_1 + (y_2 + y_3) \\ z_1 + (z_2 + z_3) \end{pmatrix}$$

and they are equal. The identity element with respect to this addition operation works this way

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x + 1 - 1 \\ y + 0 \\ z + 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and the additive inverse is similar.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -x + 2 \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} x + (-x + 2) - 1 \\ y - y \\ z - z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The conditions on scalar multiplication are also easy. For the first condition,

$$(r + s) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (r + s)x - (r + s) + 1 \\ (r + s)y \\ (r + s)z \end{pmatrix}$$

while

$$r \begin{pmatrix} x \\ y \\ z \end{pmatrix} + s \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx - r + 1 \\ ry \\ rz \end{pmatrix} + \begin{pmatrix} sx - s + 1 \\ sy \\ sz \end{pmatrix} = \begin{pmatrix} (rx - r + 1) + (sx - s + 1) - 1 \\ ry + sy \\ rz + sz \end{pmatrix}$$

and the two are equal. The second condition compares

$$r \cdot (\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = r \cdot \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} r(x_1 + x_2 - 1) - r + 1 \\ r(y_1 + y_2) \\ r(z_1 + z_2) \end{pmatrix}$$

with

$$r \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} rx_1 - r + 1 \\ ry_1 \\ rz_1 \end{pmatrix} + \begin{pmatrix} rx_2 - r + 1 \\ ry_2 \\ rz_2 \end{pmatrix} = \begin{pmatrix} (rx_1 - r + 1) + (rx_2 - r + 1) - 1 \\ ry_1 + ry_2 \\ rz_1 + rz_2 \end{pmatrix}$$

and they are equal. For the third condition,

$$(rs) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rsx - rs + 1 \\ rsy \\ rsz \end{pmatrix}$$

while

$$r(s \begin{pmatrix} x \\ y \\ z \end{pmatrix}) = r(\begin{pmatrix} sx - s + 1 \\ sy \\ sz \end{pmatrix}) = \begin{pmatrix} r(sx - s + 1) - r + 1 \\ rsy \\ rsz \end{pmatrix}$$

and the two are equal. For scalar multiplication by 1 we have this.

$$1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x - 1 + 1 \\ 1y \\ 1z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus all the conditions on a vector space are met by these two operations.

5.1.17

Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$.

5.1.18
It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.19
Each element of a vector space has one and only one additive inverse.
For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

5.1.20
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.

b. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.

c. A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{ k \cdot \vec{v} \mid k \in \mathbb{R} \}$. By the prior item this set is infinite.

5.2.1

a. Yes, we can easily check that it is closed under addition and scalar multiplication.

b. Yes, we can easily check that it is closed under addition and scalar multiplication.

c. No. It is not closed under addition. For instance,

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

is not in the set. (This set is also not closed under scalar multiplication, for instance, it does not contain the zero matrix.)

d. Yes, we can easily check that it is closed under addition and scalar multiplication.

5.2.2 No, it is not closed. In particular, it is not closed under scalar multiplication because it does not contain the zero polynomial.

5.2.3 No, such a set is not closed. For one thing, it does not contain the zero vector.

5.2.4

a. Every such set has the form $\{ r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R} \}$ where either or both of $\vec{v}, \vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ is clear.

b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.5 No. The only subspaces of $\mathbb{R}^1$ are the space itself and its trivial subspace. Any subspace S of $\mathbb{R}$ that contains a nonzero member $\vec{v}$ must contain the set of all of its scalar multiples $\{ r \cdot \vec{v} \mid r \in \mathbb{R} \}$. But this set is all of $\mathbb{R}$.

5.2.6 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r\, f'$.

5.2.7

a. This is a subspace. It is closed because if f_1, f_2 are even and c_1, c_2 are scalars then we have this.

$$(c_1 f_1 + c_2 f_2)(-x) = c_1\, f_1(-x) + c_2\, f_2(-x) = c_1\, f_1(x) + c_2\, f_2(x) = ($$

b. This is also a subspace; the check is similar to the prior one.

5.2.8 No. Subspaces of $\mathbb{R}^3$ are sets of three-tall vectors, while $\mathbb{R}^2$ is a set of two-tall vectors. Clearly though, $\mathbb{R}^2$ is "just like" this subspace of $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

5.2.9

a. The union of the x-axis and the y-axis in $\mathbb{R}^2$ is one.

b. The set of integers, as a subset of $\mathbb{R}^1$, is one.

c. The subset $\{\vec{v}\}$ of $\mathbb{R}^2$ is one, where $\vec{v}$ is any nonzero vector.

5.2.10

a. Is is a subspace.
Assume that A, B are subspaces of V. Note that their intersection is not empty as both contain the zero vector. If $\vec{w}, \vec{s} \in A \cap B$ and r, s are scalars then $r\vec{v} + s\vec{w} \in A$ because each vector is in A and so a linear combination is in A, and $r\vec{v} + s\vec{w} \in B$ for the same reason. Thus the intersection is closed.

b. In general it is not a subspace. (It is a subspace, only if $A \subseteq B$ or $B \subseteq A$).
Take V to be $\mathbb{R}^3$, take A to be the x-axis, and B to be the y-axis. Note that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in A \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in B \quad \text{but} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin A \cup B$$

as the sum is in neither A nor B.
If $A \subseteq B$ or $B \subseteq A$ then clearly $A \cup B$ is a subspace.
To show that $A \cup B$ is a subspace only if one subspace contains the other, we assume that $A \not\subseteq B$ and $B \not\subseteq A$ and prove that the union is not a subspace. The assumption that A is not a subset of B means that there is an $\vec{a} \in A$ with $\vec{a} \notin B$. The other assumption gives a $\vec{b} \in B$

with $\vec{b} \notin A$. Consider $\vec{a}+\vec{b}$. Note that sum is not an element of A or else $(\vec{a}+\vec{b})-\vec{a}$ would be in A, which it is not. Similarly the sum is not an element of B. Hence the sum is not an element of $A \cup B$, and so the union is not a subspace.

c. It is not a subspace. As A is a subspace, it contains the zero vector, and therefore the set that is A's complement does not. Without the zero vector, the complement cannot be a vector space.

5.2.11 It is transitive; apply the subspace test. (You must consider the following. Suppose B is a subspace of a vector space V and suppose $A \subseteq B \subseteq V$ is a subspace. From which space does A inherit its operations? The answer is that it doesn't matter A will inherit the same operations in either case.)

5.3.1

a. Yes, solving the linear system arising from

$$r_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + r_2 \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}$$

gives $r_1 = 2$ and $r_2 = 1$.

b. Yes; the linear system arising from $r_1(x^2)+r_2(2x+x^2)+r_3(x+x^3) = x-x^3$

$$\begin{aligned} 2r_2 + r_3 &= 1 \\ r_1 + r_2 \quad &= 0 \\ r_3 &= -1 \end{aligned}$$

gives that $-1(x^2)+1(2x+x^2)-1(x+x^3) = x-x^3$.

c. No; any combination of the two given matrices has a zero in the upper right.

5.3.2

a. Yes. It is in that span since $1\cos^2 x + 1\sin^2 x = f(x)$.

b. No. Since $r_1 \cos^2 x + r_2 \sin^2 x = 3 + x^2$ has no scalar solutions that work for all x. For instance, setting x to be 0 and π gives the two equations $r_1 \cdot 1 + r_2 \cdot 0 = 3$ and $r_1 \cdot 1 + r_2 \cdot 0 = 3+\pi^2$, which are not consistent with each other.

c. No. Consider what happens on setting x to be $\pi/2$ and $3\pi/2$.

d. Yes, $\cos(2x) = 1 \cdot \cos^2(x) - 1 \cdot \sin^2(x)$.

5.3.3

a. Yes, for any $x, y, z \in \mathbb{R}$ this equation

$$r_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + r_2 \begin{pmatrix}0\\2\\0\end{pmatrix} + r_3 \begin{pmatrix}0\\0\\3\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

has the solution $r_1 = x$, $r_2 = y/2$, and $r_3 = z/3$.

b. Yes, the equation

$$r_1 \begin{pmatrix}2\\0\\1\end{pmatrix} + r_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + r_3 \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives rise to this

$$\begin{aligned} 2r_1 + r_2 \quad &= x \\ r_2 \quad &= y \\ r_1 \quad + r_3 &= z \end{aligned}$$

Gaussian elimination gives

$$\begin{aligned} 2r_1 + r_2 \quad &= x \\ r_2 \quad &= y \\ r_3 &= -(1/2)x + (1/2)y + z \end{aligned}$$

so that, given any x, y, and z, we can compute that $r_3 = (-1/2)x + (1/2)y + z$, $r_2 = y$, and $r_1 = (1/2)x - (1/2)y$.

c. No. In particular, we cannot get the vector

$$\begin{pmatrix}0\\0\\1\end{pmatrix}$$

as a linear combination since the two given vectors both have a third component of zero.

d. Yes. The equation

$$r_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + r_2 \begin{pmatrix}3\\1\\0\end{pmatrix} + r_3 \begin{pmatrix}-1\\0\\0\end{pmatrix} + r_4 \begin{pmatrix}2\\1\\5\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 2 & x \\ 0 & 1 & 0 & 1 & y \\ 0 & 0 & 1 & 6 & -x+3y+z \end{array}\right]$$

We have infinitely many solutions. We can, for example, set r_4 to be zero and solve for r_3, r_2, and r_1 in terms of x, y, and z by the usual methods of back-substitution.

e. No. The equation

$$r_1 \begin{pmatrix}2\\1\\1\end{pmatrix} + r_2 \begin{pmatrix}3\\0\\1\end{pmatrix} + r_3 \begin{pmatrix}5\\1\\2\end{pmatrix} + r_4 \begin{pmatrix}6\\0\\2\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left[\begin{array}{cccc|c} 2 & 3 & 5 & 6 & x \\ 0 & -3/2 & -3/2 & -3 & -(1/2)x+y \\ 0 & 0 & 0 & 0 & -(1/3)x-(1/3)y+z \end{array}\right]$$

This shows that not every vector can be so expressed. Only the vectors satisfying the restriction that $-(1/3)x-(1/3)y+z=0$ are in the span. (To see that any such vector is indeed expressible, take r_3 and r_4 to be zero and solve for r_1 and r_2 in terms of x, y, and z by back-substitution.)

5.3.4

a. $\{(c\ \ b\ \ c) \mid b, c \in \mathbb{R}\} = \{b(0\ \ 1\ \ 0) + c(1\ \ 0\ \ 1) \mid b, c \in \mathbb{R}\}$ The obvious choice for the set that spans is $\{(0\ \ 1\ \ 0), (1\ \ 0\ \ 1)\}$.

b. $\left\{\begin{bmatrix}-d & b\\ c & d\end{bmatrix} \;\middle|\; b,c,d\in\mathbb{R}\right\} = \left\{b\begin{bmatrix}0&1\\0&0\end{bmatrix}+c\begin{bmatrix}0&0\\1&0\end{bmatrix}+d\begin{bmatrix}-1&0\\0&1\end{bmatrix} \;\middle|\; b,c,d\in\mathbb{R}\right\}$ One set that spans this space consists of those three matrices.

c. The system

$$\begin{aligned} a+3b \qquad &=0\\ 2a \qquad -c-d&=0\end{aligned}$$

gives $b=-(c+d)/6$ and $a=(c+d)/2$. So one description is this.

$$\left\{c\begin{bmatrix}1/2 & -1/6\\ 1 & 0\end{bmatrix}+d\begin{bmatrix}1/2 & -1/6\\ 0 & 1\end{bmatrix} \;\middle|\; c,d\in\mathbb{R}\right\}$$

That shows that a set spanning this subspace consists of those two matrices.

d. The $a=2b-c$ gives that the set $\{(2b-c)+bx+cx^3 \mid b,c\in\mathbb{R}\}$ equals the set $\{b(2+x)+c(-1+x^3) \mid b,c\in\mathbb{R}\}$. So the subspace is the span of the set $\{2+x,\ -1+x^3\}$.

e. The set $\{a+bx+cx^2 \mid a+7b+49c=0\}$ can be parametrized as

$$\{b(-7+x)+c(-49+x^2) \mid b,c\in\mathbb{R}\}$$

and so has the spanning set $\{-7+x,\ -49+x^2\}$.

5.3.5

a. We can parametrize in this way

$$\left\{\begin{pmatrix}x\\0\\z\end{pmatrix} \;\middle|\; x,z\in\mathbb{R}\right\}=\left\{x\begin{pmatrix}1\\0\\0\end{pmatrix}+z\begin{pmatrix}0\\0\\1\end{pmatrix} \;\middle|\; x,z\in\mathbb{R}\right\}$$

giving this for a spanning set.

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

b. Here is a parametrization, and the associated spanning set. $\left\{y\begin{pmatrix}-2/3\\1\\0\end{pmatrix}+z\begin{pmatrix}-1/3\\0\\1\end{pmatrix} \;\middle|\; y,z\in\mathbb{R}\right\}$ $\left\{\begin{pmatrix}-2/3\\1\\0\end{pmatrix},\begin{pmatrix}-1/3\\0\\1\end{pmatrix}\right\}$

c. $\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix},\begin{pmatrix}-1/2\\0\\0\\1\end{pmatrix}\right\}$

d. Parametrize the description as $\{-a_1+a_1x+a_3x^2+a_3x^3 \mid a_1,a_3\in\mathbb{R}\}$ to get $\{-1+x, x^2+x^3\}$.

e. $\{1,x,x^2,x^3,x^4\}$

f. $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

5.3.6 We will show mutual containment between the two sets.
The first containment $\operatorname{span}(\operatorname{span}(S))\supseteq\operatorname{span}(S)$ is an instance of the more general, and obvious, fact that for any subset T of a vector space, $\operatorname{span}(T)\supseteq T$.
For the other containment, that $\operatorname{span}(\operatorname{span}(S))\subseteq\operatorname{span}(S)$, take m vectors from $\operatorname{span}(S)$, namely $c_{1,1}\vec{s}_{1,1}+\cdots+c_{1,n_1}\vec{s}_{1,n_1}$, $\ldots$, $c_{1,m}\vec{s}_{1,m}+\cdots+c_{1,n_m}\vec{s}_{1,n_m}$, and note that any linear combination of those

$$r_1(c_{1,1}\vec{s}_{1,1}+\cdots+c_{1,n_1}\vec{s}_{1,n_1})+\cdots+r_m(c_{1,m}\vec{s}_{1,m}+\cdots+c_{1,n_m}\vec{s}_{1,n_m})$$

is a linear combination of elements of S

$$=(r_1c_{1,1})\vec{s}_{1,1}+\cdots+(r_1c_{1,n_1})\vec{s}_{1,n_1}+\cdots+(r_mc_{1,m})\vec{s}_{1,m}+\cdots+(r_mc_{1,n_m})\vec{s}$$

and so is in $\operatorname{span}(S)$. That is, simply recall that a linear combination of linear combinations (of members of S) is a linear combination (again of members of S).

5.3.7 Hint: For each subspace determine a set of vectors that spans it.
$W_1\subsetneq W_2$

5.3.8 For 'if', let S be a subset of a vector space V and assume $\vec{v}\in S$ satisfies $\vec{v}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n$ where $c_1,\ldots,c_n$ are scalars and $\vec{s}_1,\ldots,\vec{s}_n\in S$. We must show that $\operatorname{span}(S\cup\{\vec{v}\})=\operatorname{span}(S)$.
Containment one way, $\operatorname{span}(S)\subseteq\operatorname{span}(S\cup\{\vec{v}\})$ is obvious. For the other direction, $\operatorname{span}(S\cup\{\vec{v}\})\subseteq\operatorname{span}(S)$, note that if a vector is in the set on the left then it has the form $d_0\vec{v}+d_1\vec{t}_1+\cdots+d_m\vec{t}_m$ where the d's are scalars and the $\vec{t}$'s are in S. Rewrite that as $d_0(c_1\vec{s}_1+\cdots+c_n\vec{s}_n)+d_1\vec{t}_1+\cdots+d_m\vec{t}_m$ and note that the result is a member of the span of S.
The 'only if' is clearly true adding $\vec{v}$ enlarges the span to include at least $\vec{v}$.

5.3.9 The span of a set does not depend on the enclosing space. A linear combination of vectors from S gives the same sum whether we regard the operations as those of W or as those of V, because the operations of W are inherited from V.

5.3.10

a. Always; if $S\subseteq T$ then a linear combination of elements of S is also a linear combination of elements of T.

b. Sometimes (more precisely, if and only if $S\subseteq T$ or $T\subseteq S$).
The answer is not 'always' as is shown by this example from $\mathbb{R}^3$

$$S=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\},\quad T=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

because of this.

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\in\operatorname{span}(S\cup T)\qquad\begin{pmatrix}1\\1\\1\end{pmatrix}\notin\operatorname{span}(S)\cup\operatorname{span}(T)$$

The answer is not 'never' because if either set contains the other then equality is clear. We can characterize equality as happening only when either set contains the other by assuming $S \not\subseteq T$ (implying the existence of a vector $\vec{s} \in S$ with $\vec{s} \notin T$) and $T \not\subseteq S$ (giving a $\vec{t} \in T$ with $\vec{t} \notin S$), noting $\vec{s} + \vec{t} \in \operatorname{span}(S \cup T)$, and showing that $\vec{s} + \vec{t} \notin \operatorname{span}(S) \cup \operatorname{span}(T)$.

c. Sometimes.
Clearly $\operatorname{span}(S \cap T) \subseteq \operatorname{span}(S) \cap \operatorname{span}(T)$ because any linear combination of vectors from $S \cap T$ is a combination of vectors from S and also a combination of vectors from T.
Containment the other way does not always hold. For instance, in $\mathbb{R}^2$, take

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad T = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

so that $\operatorname{span}(S) \cap \operatorname{span}(T)$ is the x-axis but $\operatorname{span}(S \cap T)$ is the trivial subspace.
Characterizing exactly when equality holds is tough. Clearly equality holds if either set contains the other, but that is not 'only if' by this example in $\mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad T = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

d. Never, as the span of the complement is a subspace, while the complement of the span is not (it does not contain the zero vector).

5.4.1

a. It is dependent. Considering

$$c_1 \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ -4 \\ 14 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives this linear system.

$$\begin{aligned} c_1 + 2c_2 + 4c_3 &= 0 \\ -3c_1 + 2c_2 - 4c_3 &= 0 \\ 5c_1 + 4c_2 + 14c_3 &= 0 \end{aligned}$$

Gauss's Method

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -3 & 2 & -4 & 0 \\ 5 & 4 & 14 & 0 \end{array}\right]$$

gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yields a free variable, so there are infinitely many solutions. For an example of a particular dependence we can set c_3 to be, say, 1. Then we get $c_2 = -1$ and $c_1 = -2$.

b. It is dependent. The linear system that arises here

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 7 & 7 & 7 & 0 \\ 7 & 7 & 7 & 0 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

has infinitely many solutions. We can get a particular solution by taking c_3 to be, say, 1, and back-substituting to get the resulting c_2 and c_1.

c. It is linearly independent. The system

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & 0 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has only the solution $c_1 = 0$ and $c_2 = 0$. (We could also have gotten the answer by inspection the second vector is obviously not a multiple of the first, and vice versa.)

d. It is linearly dependent. The linear system

$$\left[\begin{array}{cccc|c} 9 & 2 & 3 & 12 & 0 \\ 9 & 0 & 5 & 12 & 0 \\ 0 & 1 & -4 & -1 & 0 \end{array}\right]$$

has more unknowns than equations, and so Gauss's Method must end with at least one variable free (there can't be a contradictory equation because the system is homogeneous, and so has at least the solution of all zeroes). To exhibit a combination, we can do the reduction

$$\left[\begin{array}{cccc|c} 9 & 2 & 3 & 12 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & -3 & -1 & 0 \end{array}\right]$$

and take, say, $c_4 = 1$. Then we have that $c_3 = -1/3$, $c_2 = -1/3$, and $c_1 = -31/27$.

5.4.2

a. This set is independent. Setting up the relation $c_1(3 - x + 9x^2) + c_2(5 - 6x + 3x^2) + c_3(1 + 1x - 5x^2) = 0 + 0x + 0x^2$ gives a linear system

$$\left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ -1 & -6 & 1 & 0 \\ 9 & 3 & -5 & 0 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ 0 & -13 & 4 & 0 \\ 0 & 0 & -128/13 & 0 \end{array}\right]$$

with only one solution: $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

b. This set is independent. We can see this by inspection, straight from the definition of linear independence. Obviously neither is a multiple of the other.

c. This set is linearly independent. The linear system reduces in this way

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 7 & 2 & -3 & 0 \end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & -5/2 & -2 & 0 \\ 0 & 0 & -51/5 & 0 \end{array}\right]$$

to show that there is only the solution $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

d. This set is linearly dependent. The linear system

$$\left[\begin{array}{cccc|c} 8 & 0 & 2 & 8 & 0 \\ 3 & 1 & 2 & -2 & 0 \\ 3 & 2 & 2 & 5 & 0 \end{array}\right]$$

must, after reduction, end with at least one variable free (there are more variables than equations, and there is no possibility of a contradictory equation because the system is homogeneous). We can take the free variables as parameters to describe the solution set. We can then set the parameter to a nonzero value to get a nontrivial linear relation.

5.4.3 Let Z be the zero function $Z(x) = 0$, which is the additive identity in the vector space under discussion.

a. This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$. Plugging in $x = 1$ and $x = 2$ gives a linear system

$$\begin{aligned} c_1 \cdot 1 + c_2 \cdot 1 &= 0 \\ c_1 \cdot 2 + c_2 \cdot (1/2) &= 0 \end{aligned}$$

with the unique solution $c_1 = 0$, $c_2 = 0$.

b. This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plug in $x = 0$ and $x = \pi/2$ to get

$$\begin{aligned} c_1 \cdot 1 + c_2 \cdot 0 &= 0 \\ c_1 \cdot 0 + c_2 \cdot 1 &= 0 \end{aligned}$$

which obviously gives that $c_1 = 0$, $c_2 = 0$.

c. This set is also linearly independent. Considering $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plugging in $x = 1$ and $x = e$

$$\begin{aligned} c_1 \cdot e + c_2 \cdot 0 &= 0 \\ c_1 \cdot e^e + c_2 \cdot 1 &= 0 \end{aligned}$$

gives that $c_1 = 0$ and $c_2 = 0$.

5.4.4

a. This set is dependent. The familiar relation $\sin^2(x) + \cos^2(x) = 1$ shows that $2 = c_1 \cdot (4\sin^2(x)) + c_2 \cdot (\cos^2(x))$ is satisfied by $c_1 = 1/2$ and $c_2 = 2$.

b. This set is independent. Consider the relationship $c_1 \cdot 1 + c_2 \cdot \sin(x) + c_3 \cdot \sin(2x) = 0$ (that '0' is the zero function). Taking three suitable points such as $x = \pi$, $x = \pi/2$, $x = \pi/4$ gives a system

$$\begin{aligned} c_1 \phantom{+ (\sqrt{2}/2)c_2 + c_3} &= 0 \\ c_1 + \phantom{(\sqrt{2}/2)} c_2 &= 0 \\ c_1 + (\sqrt{2}/2)c_2 + c_3 &= 0 \end{aligned}$$

whose only solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

c. By inspection, this set is independent. Any dependence $\cos(x) = c \cdot x$ is not possible since the cosine function is not a multiple of the identity function

d. By inspection, we spot that there is a dependence. Because $(1 + x)^2 = x^2 + 2x + 1$, we get that $c_1 \cdot (1 + x)^2 + c_2 \cdot (x^2 + 2x) = 3$ is satisfied by $c_1 = 3$ and $c_2 = -3$.

e. This set is dependent, because it contains the zero object in the vector space, the zero polynomial.

f. This set is dependent. The easiest way to see that is to recall the trigonometric relationship $\cos^2(x) - \sin^2(x) = \cos(2x)$.

5.4.5 No. Here are two members of the plane where the second is a multiple of the first.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

(Another reason that the answer is "no" is the the zero vector is a member of the plane and no set containing the zero vector is linearly independent.)

5.4.6

a. $\lambda = 1$

b. $\lambda \neq -1, -\frac{1}{2}, 1$

5.4.7

a. Assume that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, so that any relationship $d_0\vec{u} + d_1\vec{v} + d_2\vec{w} = \vec{0}$ leads to the conclusion that $d_0 = 0$, $d_1 = 0$, and $d_2 = 0$.
Consider the relationship $c_1(\vec{u})+c_2(\vec{u}+\vec{v})+c_3(\vec{u}+\vec{v}+\vec{w}) = \vec{0}$. Rewrite it to get $(c_1+c_2+c_3)\vec{u}+(c_2+c_3)\vec{v}+(c_3)\vec{w} = \vec{0}$. Taking d_0 to be $c_1 + c_2 + c_3$, taking d_1 to be $c_2 + c_3$, and taking d_2 to be c_3 we have this system.

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

Conclusion: the c's are all zero, and so the set is linearly independent.

b. The second set is dependent

$$1 \cdot (\vec{u} - \vec{v}) + 1 \cdot (\vec{v} - \vec{w}) + 1 \cdot (\vec{w} - \vec{u}) = \vec{0}$$

whether or not the first set is independent.

5.4.8

a. A singleton set $\{\vec{v}\}$ is linearly independent if and only if $\vec{v} \neq \vec{0}$. For the 'if' direction, with $\vec{v} \neq \vec{0}$, we consider the relationship $c \cdot \vec{v} = \vec{0}$ and noting that the only solution is the trivial one: $c = 0$. For the 'only if' direction, it is evident from the definition.

b. A set with two elements is linearly independent if and only if neither member is a multiple of the other (note that if one is the zero vector then it is a multiple of the other). This is an equivalent statement: a set is linearly dependent if and only if one element is a multiple of the other.

The proof is easy. A set $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if and only if there is a relationship $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ with either $c_1 \neq 0$ or $c_2 \neq 0$ (or both). That holds if and only if $\vec{v}_1 = (-c_2/c_1)\vec{v}_2$ or $\vec{v}_2 = (-c_1/c_2)\vec{v}_1$ (or both).

5.4.9 Hint: Prove by contradiction. The converse (the 'only if' statement) does not hold. An example is to consider the vector space $\mathbb{R}^2$ and these vectors.

$$\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \vec{y} = \begin{pmatrix}0\\1\end{pmatrix}, \quad \vec{z} = \begin{pmatrix}1\\1\end{pmatrix}$$

5.4.10

a. The linear system arising from

$$c_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + c_2 \begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

has the unique solution $c_1 = 0$ and $c_2 = 0$.

b. The linear system arising from

$$c_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + c_2 \begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix}$$

has the unique solution $c_1 = 8/3$ and $c_2 = -1/3$.

c. Suppose that S is linearly independent. Suppose that we have both $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{v} = d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ (where the vectors are members of S). Now,

$$c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{v} = d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$$

can be rewritten in this way.

$$c_1\vec{s}_1 + \cdots + c_n\vec{s}_n - d_1\vec{t}_1 - \cdots - d_m\vec{t}_m = \vec{0}$$

Possibly some of the $\vec{s}$'s equal some of the $\vec{t}$'s; we can combine the associated coefficients (i.e., if $\vec{s}_i = \vec{t}_j$ then $\cdots + c_i\vec{s}_i + \cdots - d_j\vec{t}_j - \cdots$ can be rewritten as $\cdots + (c_i - d_j)\vec{s}_i + \cdots$). That equation is a linear relationship among distinct (after the combining is done) members of the set S. We've assumed that S is linearly independent, so all of the coefficients are zero. If i is such that $\vec{s}_i$ does not equal any $\vec{t}_j$ then c_i is zero. If j is such that $\vec{t}_j$ does not equal any $\vec{s}_i$ then d_j is zero. In the final case, we have that $c_i - d_j = 0$ and so $c_i = d_j$.

Therefore, the original two sums are the same, except perhaps for some $0 \cdot \vec{s}_i$ or $0 \cdot \vec{t}_j$ terms that we can neglect.

d. This set is not linearly independent:

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix} \right\} \subset \mathbb{R}^2$$

and these two linear combinations give the same result

$$\begin{pmatrix}0\\0\end{pmatrix} = 2 \cdot \begin{pmatrix}1\\0\end{pmatrix} - 1 \cdot \begin{pmatrix}2\\0\end{pmatrix} = 4 \cdot \begin{pmatrix}1\\0\end{pmatrix} - 2 \cdot \begin{pmatrix}2\\0\end{pmatrix}$$

Thus, a linearly dependent set might have indistinct sums.

In fact, this stronger statement holds: if a set is linearly dependent then it must have the property that there are two distinct linear combinations that sum to the same vector. Briefly, where $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ then multiplying both sides of the relationship by two gives another relationship. If the first relationship is nontrivial then the second is also.

5.4.11

a. For any $a_{1,1}, \ldots, a_{2,4}$,

$$c_1 \begin{pmatrix}a_{1,1}\\a_{2,1}\end{pmatrix} + c_2 \begin{pmatrix}a_{1,2}\\a_{2,2}\end{pmatrix} + c_3 \begin{pmatrix}a_{1,3}\\a_{2,3}\end{pmatrix} + c_4 \begin{pmatrix}a_{1,4}\\a_{2,4}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

yields a linear system

$$\begin{aligned} a_{1,1}c_1 + a_{1,2}c_2 + a_{1,3}c_3 + a_{1,4}c_4 &= 0 \\ a_{2,1}c_1 + a_{2,2}c_2 + a_{2,3}c_3 + a_{2,4}c_4 &= 0 \end{aligned}$$

that has infinitely many solutions (Gauss's Method leaves at least two variables free). Hence there are nontrivial linear relationships among the given members of $\mathbb{R}^2$.

b. Any set five vectors is a superset of a set of four vectors, and so is linearly dependent.

With three vectors from $\mathbb{R}^2$, the argument from the prior item still applies, with the slight change that Gauss's Method now only leaves at least one variable free (but that still gives infinitely many solutions).

c. The prior part shows that no three-element subset of $\mathbb{R}^2$ is independent. We know that there are two-element subsets of $\mathbb{R}^2$ that are independent. The following one is

$$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

and so the answer is two.

5.4.12 Yes; here is one.

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

5.4.13

a. Assume that $\vec{v}$ and $\vec{w}$ are perpendicular nonzero vectors in $\mathbb{R}^n$, with $n > 1$. With the linear relationship $c\vec{v}+d\vec{w} = \vec{0}$, apply $\vec{v}$ to both sides to conclude that $c \cdot \|\vec{v}\|^2 + d \cdot 0 = 0$. Because $\vec{v} \neq \vec{0}$ we have that $c = 0$. A similar application of $\vec{w}$ shows that $d = 0$.

b. Two vectors in $\mathbb{R}^1$ are perpendicular if and only if at least one of them is zero.

c. The right generalization is to look at a set $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq \mathbb{R}^k$ of vectors that are *mutually orthogonal* (also called *pairwise perpendicular*): if $i \neq j$ then $\vec{v}_i$ is perpendicular to $\vec{v}_j$. Mimicking the proof of the first item above shows that such a set of nonzero vectors is linearly independent.

5.4.14 It is both 'if' and 'only if'.
Let T be a subset of the subspace S of the vector space V. The assertion that any linear relationship $c_1\vec{t}_1+\dots+c_n\vec{t}_n = \vec{0}$ among members of T must be the trivial relationship $c_1 = 0$, $\dots$, $c_n = 0$ is a statement that holds in S if and only if it holds in V, because the subspace S inherits its addition and scalar multiplication operations from V.

5.4.15 Hint: Use the definition of linear independence to show that there only exists the trivial linear combination giving the zero vector.

5.4.16 In $\mathbb{R}^4$ the biggest linearly independent set has four vectors. There are many examples of such sets, this is one.

$$\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

To see that no set with five or more vectors can be independent, set up

$$c_1 \begin{pmatrix}a_{1,1}\\a_{2,1}\\a_{3,1}\\a_{4,1}\end{pmatrix} + c_2 \begin{pmatrix}a_{1,2}\\a_{2,2}\\a_{3,2}\\a_{4,2}\end{pmatrix} + c_3 \begin{pmatrix}a_{1,3}\\a_{2,3}\\a_{3,3}\\a_{4,3}\end{pmatrix} + c_4 \begin{pmatrix}a_{1,4}\\a_{2,4}\\a_{3,4}\\a_{4,4}\end{pmatrix} + c_5 \begin{pmatrix}a_{1,5}\\a_{2,5}\\a_{3,5}\\a_{4,5}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

and note that the resulting linear system

$$\begin{aligned}
a_{1,1}c_1 + a_{1,2}c_2 + a_{1,3}c_3 + a_{1,4}c_4 + a_{1,5}c_5 &= 0 \\
a_{2,1}c_1 + a_{2,2}c_2 + a_{2,3}c_3 + a_{2,4}c_4 + a_{2,5}c_5 &= 0 \\
a_{3,1}c_1 + a_{3,2}c_2 + a_{3,3}c_3 + a_{3,4}c_4 + a_{3,5}c_5 &= 0 \\
a_{4,1}c_1 + a_{4,2}c_2 + a_{4,3}c_3 + a_{4,4}c_4 + a_{4,5}c_5 &= 0
\end{aligned}$$

has four equations and five unknowns, so Gauss's Method must end with at least one c variable free, so there are infinitely many solutions, and so the above linear relationship among the four-tall vectors has more solutions than just the trivial solution.
The smallest linearly independent set is the empty set.
The biggest linearly dependent set is $\mathbb{R}^4$. The smallest is $\left\{\vec{0}\right\}$.

5.4.17

a. The intersection of two linearly independent sets $S \cap T$ must be linearly independent as it is a subset of the linearly independent set S (as well as the linearly independent set T also, of course).

b. The complement of a linearly independent set is linearly dependent as it contains the zero vector.

c. A simple example in $\mathbb{R}^2$ is these two sets.

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix} \right\} \qquad T = \left\{ \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

A somewhat subtler example, again in $\mathbb{R}^2$, is these two.

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix} \right\} \qquad T = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

d. We must produce an example. One, in $\mathbb{R}^2$, is

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix} \right\} \qquad T = \left\{ \begin{pmatrix}2\\0\end{pmatrix} \right\}$$

since the linear dependence of $S_1 \cup S_2$ is easy to see.

5.4.18

a. The vectors $\vec{s}_1, \dots, \vec{s}_n, \vec{t}_1, \dots, \vec{t}_m$ are distinct. But we could have that the union $S \cup T$ is linearly independent with some $\vec{s}_i$ equal to some $\vec{t}_j$.

b. One example in $\mathbb{R}^2$ is these two.

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix} \right\} \qquad T = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

c. An example from $\mathbb{R}^2$ is these sets.

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} \qquad T = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$$

d. The union of two linearly independent sets $S \cup T$ is linearly independent if and only if their spans of S and $T - (S \cap T)$ have a trivial intersection $\operatorname{span}(S) \cap \operatorname{span}(T - (S \cap T)) = \left\{\vec{0}\right\}$. To prove that, assume that S and T are linearly independent subsets of some vector space.
For the 'only if' direction, assume that the intersection of the spans is trivial $\operatorname{span}(S) \cap \operatorname{span}(T - (S \cap T)) = \left\{\vec{0}\right\}$. Consider the set $S \cup (T - (S \cap T)) = S \cup T$ and consider the linear relationship $c_1\vec{s}_1 + \dots + c_n\vec{s}_n + d_1\vec{t}_1 + \dots + d_m\vec{t}_m = \vec{0}$. Subtracting gives $c_1\vec{s}_1 + \dots + c_n\vec{s}_n = -d_1\vec{t}_1 - \dots - d_m\vec{t}_m$. The left side of that equation sums to a vector in $\operatorname{span}(S)$, and the right side is a vector in $\operatorname{span}(T - (S \cap T))$. Therefore, since the intersection of the spans is trivial, both sides equal the zero vector. Because S is linearly independent, all of the c's are zero. Because T is linearly independent so also is $T - (S \cap T)$ linearly independent, and therefore all of the d's are zero. Thus, the original linear relationship among members of

$S \cup T$ only holds if all of the coefficients are zero. Hence, $S \cup T$ is linearly independent.

For the 'if' half we can make the same argument in reverse. Suppose that the union $S \cup T$ is linearly independent. Consider a linear relationship among members of S and $T-(S\cap T)$. $c_1\vec{s}_1+\cdots+c_n\vec{s}_n+d_1\vec{t}_1+\cdots+d_m\vec{t}_m=\vec{0}$ Note that no $\vec{s}_i$ is equal to a $\vec{t}_j$ so that is a combination of distinct vectors. So the only solution is the trivial one $c_1=0,\ \ldots,\ d_m=0$. Since any vector $\vec{v}$ in the intersection of the spans $\operatorname{span}(S)\cap\operatorname{span}(T-(S\cap T))$ we can write $\vec{v}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n=-d_1\vec{t}_1-\cdots-d_m\vec{t}_m$, and it must be the zero vector because each scalar is zero.

5.4.19

a. Assuming first that $a\neq 0$,

$$x\begin{pmatrix}a\\c\end{pmatrix}+y\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

gives

$$\begin{aligned}ax+by&=0\\cx+dy&=0\end{aligned}$$

and Gaussian elimination

$$\begin{aligned}ax+\qquad\qquad by&=0\\(-(c/a)b+d)y&=0\end{aligned}$$

which has a solution if and only if $0\neq-(c/a)b+d=(-cb+ad)/d$ (we've assumed in this case that $a\neq 0$, and so back substitution yields a unique solution).

The $a=0$ case is also not hard break it into the $c\neq 0$ and $c=0$ subcases and note that in these cases $ad-bc=0\cdot d-bc$.

b. The equation

$$c_1\begin{pmatrix}a\\d\\g\end{pmatrix}+c_2\begin{pmatrix}b\\e\\h\end{pmatrix}+c_3\begin{pmatrix}c\\f\\i\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

expresses a homogeneous linear system. We proceed by writing it in matrix form and applying Gauss's Method. We first reduce the matrix to upper-triangular. Assume that $a\neq 0$. With that, we can clear down the first column.

$$\left[\begin{array}{ccc|c}1&b/a&c/a&0\\0&(ae-bd)/a&(af-cd)/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right]$$

Then we get a 1 in the second row, second column entry. (Assuming for the moment that $ae-bd\neq 0$, in order to do the row reduction step.)

$$\left[\begin{array}{ccc|c}1&b/a&c/a&0\\0&1&(af-cd)/(ae-bd)&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right]$$

Then, under the assumptions, we perform the row operation $((ah-bg)/a)\rho_2+\rho_3$ to get this.

$$\left[\begin{array}{ccc|c}1&b/a&c/a&0\\0&1&(af-cd)/(ae-bd)&0\\0&0&(aei+bgf+cdh-hfa-idb-gec)/(ae-bd)&0\end{array}\right]$$

Therefore, the original system is nonsingular if and only if the above $3,3$ entry is nonzero (this fraction is defined because of the $ae-bd\neq 0$ assumption). It equals zero if and only if the numerator is zero.

We next worry about the assumptions. First, if $a\neq 0$ but $ae-bd=0$ then we swap row 2 and row 3

$$\left[\begin{array}{ccc|c}1&b/a&c/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\\0&0&(af-cd)/a&0\end{array}\right]$$

and conclude that the system is nonsingular if and only if either $ah-bg=0$ or $af-cd=0$. That's the same as asking that their product be zero:

$$\begin{aligned}ahaf-ahcd-bgaf+bgcd&=0\\ahaf-ahcd-bgaf+aegc&=0\\a(haf-hcd-bgf+egc)&=0\end{aligned}$$

(in going from the first line to the second we've applied the case assumption that $ae-bd=0$ by substituting ae for bd). Since we are assuming that $a\neq 0$, we have that $haf-hcd-bgf+egc=0$. With $ae-bd=0$ we can rewrite this to fit the form we need: in this $a\neq 0$ and $ae-bd=0$ case, the given system is nonsingular when $haf-hcd-bgf+egc-i(ae-bd)=0$, as required.

The remaining cases have the same character. Do the $a=0$ but $d\neq 0$ case and the $a=0$ and $d=0$ but $g\neq 0$ case by first swapping rows and then going on as above. The $a=0$, $d=0$, and $g=0$ case is easy a set with a zero vector is linearly dependent, and the formula comes out to equal zero.

c. It is linearly dependent if and only if either vector is a multiple of the other. That is, it is not independent iff

$$\begin{pmatrix}a\\d\\g\end{pmatrix}=r\cdot\begin{pmatrix}b\\e\\h\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}b\\e\\h\end{pmatrix}=s\cdot\begin{pmatrix}a\\d\\g\end{pmatrix}$$

(or both) for some scalars r and s. Eliminating r and s in order to restate this condition only in terms of the given letters a, b, d, e, g, h, we have that it is not independent, it is dependent iff $ae-bd=ah-gb=dh-ge$

5.5.1 Each set is a basis if and only if we can express each vector in the space in a unique way as a linear combination of the given vectors.

a. Yes this is a basis. The relation

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}3\\2\\1\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives

$$\left[\begin{array}{ccc|c}1&3&0&x\\2&2&0&y\\3&1&1&z\end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{ccc|c}1 & 3 & 0 & x\\ 0 & -4 & 0 & -2x+y\\ 0 & 0 & 1 & x-2y+z\end{array}\right]$$

which has the unique solution $c_3 = x - 2y + z$, $c_2 = x/2 - y/4$, and $c_1 = -x/2 + 3y/4$.

b. This is not a basis. Setting it up as in the prior part

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives a linear system whose solution

$$\left[\begin{array}{cc|c}1 & 3 & x\\ 2 & 2 & y\\ 3 & 1 & z\end{array}\right]$$

gives

$$\left[\begin{array}{cc|c}1 & 3 & x\\ 0 & -4 & -2x+y\\ 0 & 0 & x-2y+z\end{array}\right]$$

is possible if and only if the three-tall vector's components x, y, and z satisfy $x - 2y + z = 0$. For instance, we can find the coefficients c_1 and c_2 that work when $x = 1$, $y = 1$, and $z = 1$. However, there are no c's that work for $x = 1$, $y = 1$, and $z = 2$. Thus this is not a basis; it does not span the space.

c. Yes, this is a basis. Setting up the relationship leads to this reduction

$$\left[\begin{array}{ccc|c}0 & 1 & 2 & x\\ 2 & 1 & 5 & y\\ -1 & 1 & 0 & z\end{array}\right]$$

gives

$$\left[\begin{array}{ccc|c}-1 & 1 & 0 & z\\ 0 & 3 & 5 & y+2z\\ 0 & 0 & 1/3 & x-y/3-2z/3\end{array}\right]$$

which has a unique solution for each triple of components x, y, and z.

d. No, this is not a basis. The reduction of

$$\left[\begin{array}{ccc|c}0 & 1 & 1 & x\\ 2 & 1 & 3 & y\\ -1 & 1 & 0 & z\end{array}\right]$$

gives

$$\left[\begin{array}{ccc|c}-1 & 1 & 0 & z\\ 0 & 3 & 3 & y+2z\\ 0 & 0 & 0 & x-y/3-2z/3\end{array}\right]$$

which does not have a solution for each triple x, y, and z. Instead, the span of the given set includes only those vectors where $x = y/3 + 2z/3$.

5.5.2

a. This is a basis for $\mathcal{P}_2$. To show that it spans the space we consider a generic $a_2x^2 + a_1x + a_0 \in \mathcal{P}_2$ and look for scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that $a_2x^2 + a_1x + a_0 = c_1 \cdot (x^2 - x + 1) + c_2 \cdot (2x + 1) + c_3(2x - 1)$. Gauss's Method on the linear system

$$\begin{aligned} c_1 \qquad\qquad &= a_2\\ 2c_2 + 2c_3 &= a_1\\ c_2 - c_3 &= a_0\end{aligned}$$

shows that given the a_i's we can compute the c_j's as $c_1 = a_2$, $c_2 = (1/4)a_1 + (1/2)a_0$, and $c_3 = (1/4)a_1 - (1/2)a_0$. Thus each element of $\mathcal{P}_2$ is a combination of the given three.

To prove that the set of the given three is linearly independent we can set up the equation $0x^2 + 0x + 0 = c_1 \cdot (x^2 - x + 1) + c_2 \cdot (2x + 1) + c_3(2x - 1)$ and solve, and it will give that $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

b. This is not a basis. It does not span the space since no combination of the two $c_1 \cdot (x + x^2) + c_2 \cdot (x - x^2)$ will sum to the polynomial $3 \in \mathcal{P}_2$.

5.5.3

a. We solve

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

with

$$\left[\begin{array}{cc|c}1 & -1 & 1\\ 1 & 1 & 2\end{array}\right]$$

and Gaussian elimination gives

$$\left[\begin{array}{cc|c}1 & -1 & 1\\ 0 & 2 & 1\end{array}\right]$$

and conclude that $c_2 = 1/2$ and so $c_1 = 3/2$. Thus, the representation is this.

$$\mathrm{Rep}_B(\begin{pmatrix}1\\2\end{pmatrix}) = \begin{pmatrix}3/2\\1/2\end{pmatrix}_B$$

b. The relationship $c_1 \cdot (1) + c_2 \cdot (1 + x) + c_3 \cdot (1 + x + x^2) + c_4 \cdot (1 + x + x^2 + x^3) = x^2 + x^3$ is easily solved by inspection to give that $c_4 = 1$, $c_3 = 0$, $c_2 = -1$, and $c_1 = 0$.

$$\mathrm{Rep}_D(x^2 + x^3) = \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}_D$$

5.5.4 A natural basis is $\langle 1,\ x,\ x^2\rangle$. There are bases for $\mathcal{P}_2$ that do not contain any polynomials of degree one or degree zero. One is $\langle 1 + x + x^2,\ x + x^2,\ x^2\rangle$. (Every basis has at least one polynomial of degree two, though.)

5.5.5 The reduction

$$\left[\begin{array}{cccc|c}1 & -4 & 3 & -1 & 0\\ 2 & -8 & 6 & -2 & 0\end{array}\right]$$

Gaussian elimination gives

$$\left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

gives that the only condition is that $x_1 = 4x_2 - 3x_3 + x_4$. The solution set is

$$\left\{ \begin{pmatrix} 4x_2 - 3x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_2, x_3, x_4 \in \mathbb{R} \right\}$$
$$= \left\{ x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_3, x_4 \in \mathbb{R} \right\}$$

and so the obvious candidate for the basis is this.

$$\langle \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

We've shown that this spans the space, and showing it is also linearly independent is routine.

5.5.6 There are many bases. This is a natural one.

$$\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$$

5.5.7 For each, many answers are possible.

a. One way to proceed is to parametrize by expressing the a_2 as a combination of the other two $a_2 = 2a_1 + a_0$. Then $a_2x^2 + a_1x + a_0$ is $(2a_1 + a_0)x^2 + a_1x + a_0$ and

$$\{ (2a_1 + a_0)x^2 + a_1x + a_0 \;\big|\; a_1, a_0 \in \mathbb{R} \}$$
$$= \{ a_1 \cdot (2x^2 + x) + a_0 \cdot (x^2 + 1) \;\big|\; a_1, a_0 \in \mathbb{R} \}$$

suggests $\langle 2x^2 + x, x^2 + 1 \rangle$. This only shows that it spans, but checking that it is linearly independent is routine.

b. Parametrize $\{ (a\ \ b\ \ c) \mid a + b = 0 \}$ to get $\{ (-b\ \ b\ \ c) \mid b, c \in \mathbb{R} \}$, which suggests using the sequence $\langle (-1\ \ 1\ \ 0), (0\ \ 0\ \ 1) \rangle$. We've shown that it spans, and checking that it is linearly independent is easy.

c. Rewriting

$$\left\{ \begin{bmatrix} a & b \\ 0 & 2b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} = \left\{ a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

suggests this for the basis.

$$\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \rangle$$

5.5.8

a. Asking which $a_0 + a_1x + a_2x^2$ can be expressed as $c_1 \cdot (1 + x) + c_2 \cdot (1 + 2x)$ gives rise to three linear equations, describing the coefficients of x^2, x, and the constants.

$$\begin{aligned} c_1 + c_2 &= a_0 \\ c_1 + 2c_2 &= a_1 \\ 0 &= a_2 \end{aligned}$$

Gauss's Method with back-substitution shows, provided that $a_2 = 0$, that $c_2 = -a_0 + a_1$ and $c_1 = 2a_0 - a_1$. Thus, with $a_2 = 0$, we can compute appropriate c_1 and c_2 for any a_0 and a_1. So the span is the entire set of linear polynomials $\{ a_0 + a_1x \mid a_0, a_1 \in \mathbb{R} \}$. Parametrizing that set $\{ a_0 \cdot 1 + a_1 \cdot x \mid a_0, a_1 \in \mathbb{R} \}$ suggests a basis $\langle 1, x \rangle$ (we've shown that it spans; checking linear independence is easy).

b. With

$$a_0 + a_1x + a_2x^2 = c_1 \cdot (2 - 2x) + c_2 \cdot (3 + 4x^2) = (2c_1 + 3c_2) + (-2c_1)x + ($$

we get this system.

$$\begin{aligned} 2c_1 + 3c_2 &= a_0 \\ -2c_1 &= a_1 \\ 4c_2 &= a_2 \end{aligned}$$

and Gaussian elimination gives

$$\begin{aligned} 2c_1 + 3c_2 &= a_0 \\ 3c_2 &= a_0 + a_1 \\ 0 &= (-4/3)a_0 - (4/3)a_1 + a_2 \end{aligned}$$

Thus, the only quadratic polynomials $a_0 + a_1x + a_2x^2$ with associated c's are the ones such that $0 = (-4/3)a_0 - (4/3)a_1 + a_2$. Hence the span is this.

$$\{ (-a_1 + (3/4)a_2) + a_1x + a_2x^2 \;\big|\; a_1, a_2 \in \mathbb{R} \}$$

Parametrizing gives $\{ a_1 \cdot (-1 + x) + a_2 \cdot ((3/4) + x^2) \mid a_1, a_2 \in$ which suggests $\langle -1 + x, (3/4) + x^2 \rangle$ (checking that it is linearly independent is routine).

5.5.9

a. The subspace is this.

$$\{ a_0 + a_1x + a_2x^2 + a_3x^3 \;\big|\; a_0 + 7a_1 + 49a_2 + 343a_3 = 0 \}$$

Rewriting $a_0 = -7a_1 - 49a_2 - 343a_3$ gives this.

$$\{ (-7a_1 - 49a_2 - 343a_3) + a_1x + a_2x^2 + a_3x^3 \;\big|\; a_1, a_2, a_3 \in \mathbb{R} \}$$

On breaking out the parameters, this suggests $\langle -7 + x, -49 + x^2, -343 + x^3 \rangle$ for the basis (it is easily verified).

b. The given subspace is the collection of cubics $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_0 + 7a_1 + 49a_2 + 343a_3 = 0$ and $a_0 + 5a_1 + 25a_2 + 125a_3 = 0$. Gauss's Method

$$\begin{aligned} a_0 + 7a_1 + 49a_2 + 343a_3 &= 0 \\ a_0 + 5a_1 + 25a_2 + 125a_3 &= 0 \end{aligned}$$

gives

$$\begin{aligned} a_0 + \quad 7a_1 + 49a_2 + 343a_3 &= 0 \\ -2a_1 - 24a_2 - 218a_3 &= 0 \end{aligned}$$

gives that $a_1 = -12a_2 - 109a_3$ and that $a_0 = 35a_2 + 420a_3$. Rewriting $(35a_2 + 420a_3) + (-12a_2 - 109a_3)x + a_2x^2 + a_3x^3$ as $a_2 \cdot (35 - 12x + x^2) + a_3 \cdot (420 - 109x + x^3)$ suggests this for a basis $\langle 35 - 12x + x^2, 420 - 109x + x^3\rangle$. The above shows that it spans the space. Checking it is linearly independent is routine. (*Comment.* A worthwhile check is to verify that both polynomials in the basis have both seven and five as roots.)

c. Here there are three conditions on the cubics, that $a_0 + 7a_1 + 49a_2 + 343a_3 = 0$, that $a_0 + 5a_1 + 25a_2 + 125a_3 = 0$, and that $a_0 + 3a_1 + 9a_2 + 27a_3 = 0$. Gauss's Method

$$\begin{aligned} a_0 + 7a_1 + 49a_2 + 343a_3 &= 0 \\ a_0 + 5a_1 + 25a_2 + 125a_3 &= 0 \\ a_0 + 3a_1 + \quad 9a_2 + \quad 27a_3 &= 0 \end{aligned}$$

gives

$$\begin{aligned} a_0 + \quad 7a_1 + 49a_2 + 343a_3 &= 0 \\ -2a_1 - 24a_2 - 218a_3 &= 0 \\ 8a_2 + 120a_3 &= 0 \end{aligned}$$

yields the single free variable a_3, with $a_2 = -15a_3$, $a_1 = 71a_3$, and $a_0 = -105a_3$. The parametrization is this.

$$\begin{aligned} &\{(-105a_3) + (71a_3)x + (-15a_3)x^2 + (a_3)x^3 \mid a_3 \in \mathbb{R}\} \\ &= \{a_3 \cdot (-105 + 71x - 15x^2 + x^3) \mid a_3 \in \mathbb{R}\} \end{aligned}$$

Therefore, a natural candidate for the basis is $\langle -105 + 71x - 15x^2 + x^3\rangle$. It spans the space by the work above. It is clearly linearly independent because it is a one-element set (with that single element not the zero object of the space). Thus, any cubic through the three points $(7, 0)$, $(5, 0)$, and $(3, 0)$ is a multiple of this one. (*Comment.* As in the prior question, a worthwhile check is to verify that plugging seven, five, and three into this polynomial yields zero each time.)

d. This is the trivial subspace of $\mathcal{P}_3$. Thus, the basis is empty $\langle\rangle$.

5.5.10

a. $B = \{1 + x^3,\ x^2 + x^3\}$

b. $(p(x))_B = (-2, 2)$

5.5.11 No linearly independent set contains a zero vector.

5.5.12

a. To show that it is linearly independent, note that if $d_1(c_1\vec{\beta}_1) + d_2(c_2\vec{\beta}_2) + d_3(c_3\vec{\beta}_3) = \vec{0}$ then $(d_1c_1)\vec{\beta}_1 + (d_2c_2)\vec{\beta}_2 + (d_3c_3)\vec{\beta}_3 = \vec{0}$, which in turn implies that each d_ic_i is zero. But with $c_i \neq 0$ that means that each d_i is zero. Showing that it spans the space is much the same; because $\langle\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ is a basis, and so spans the space, we can for any $\vec{v}$ write $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$, and then $\vec{v} = (d_1/c_1)(c_1\vec{\beta}_1) + (d_2/c_2)(c_2\vec{\beta}_2) + (d_3/c_3)(c_3\vec{\beta}_3)$. If any of the scalars are zero then the result is not a basis, because it is not linearly independent.

b. Showing that $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ is linearly independent is easy. To show that it spans the space, assume that $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$. Then, we can represent the same $\vec{v}$ with respect to $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ in this way $\vec{v} = (1/2)(d_1 - d_2 - d_3)(2\vec{\beta}_1) + d_2(\vec{\beta}_1 + \vec{\beta}_2) + d_3(\vec{\beta}_1 + \vec{\beta}_3)$.

5.5.13 Each forms a linearly independent set if we omit $\vec{v}$. To preserve linear independence, we must expand the span of each. That is, we must determine the span of each (leaving $\vec{v}$ out), and then pick a $\vec{v}$ lying outside of that span. Then to finish, we must check that the result spans the entire given space. Those checks are routine.

a. Any vector that is not a multiple of the given one, that is, any vector that is not on the line $y = x$ will do here. One is $\vec{v} = \vec{e}_1$.

b. By inspection, we notice that the vector $\vec{e}_3$ is not in the span of the set of the two given vectors. The check that the resulting set is a basis for $\mathbb{R}^3$ is routine.

c. For any member of the span $\{c_1 \cdot (x) + c_2 \cdot (1 + x^2) \mid c_1, c_2 \in \mathbb{R}\}$, the coefficient of x^2 equals the constant term. So we expand the span if we add a quadratic without this property, say, $\vec{v} = 1 - x^2$. The check that the result is a basis for $\mathcal{P}_2$ is easy.

5.5.14 To show that each scalar is zero, simply subtract $c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k - c_{k+1}\vec{\beta}_{k+1} - \cdots - c_n\vec{\beta}_n = \vec{0}$. The obvious generalization is that in any equation involving only the $\vec{\beta}$'s, and in which each $\vec{\beta}$ appears only once, each scalar is zero. For instance, an equation with a combination of the even-indexed basis vectors (i.e., $\vec{\beta}_2$, $\vec{\beta}_4$, etc.) on the right and the odd-indexed basis vectors on the left also gives the conclusion that all of the coefficients are zero.

5.5.15 Here is a subset of $\mathbb{R}^2$ that is not a basis, and two different linear combinations of its elements that sum to the same vector.

$$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}\right\} \qquad 2 \cdot \begin{pmatrix}1\\2\end{pmatrix} + 0 \cdot \begin{pmatrix}2\\4\end{pmatrix} = 0 \cdot \begin{pmatrix}1\\2\end{pmatrix} + 1 \cdot \begin{pmatrix}2\\4\end{pmatrix}$$

Thus, when a subset is not a basis, it can be the case that its linear combinations are not unique.
But just because a subset is not a basis does not imply that its combinations must be not unique. For instance, this set

$$\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$$

does have the property that

$$c_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} = c_2 \cdot \begin{pmatrix}1\\2\end{pmatrix}$$

implies that $c_1 = c_2$. The idea here is that this subset fails to be a basis because it fails to span the space.

5.5.16

a. Describing the vector space as

$$\left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

suggests this for a basis.

$$\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rangle$$

Verification is easy.

b. This is one possible basis.

$$\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rangle$$

c. As in the prior two questions, we can form a basis from two kinds of matrices. First are the matrices with a single one on the diagonal and all other entries zero (there are n of those matrices). Second are the matrices with two opposed off-diagonal entries are ones and all other entries are zeros. (That is, all entries in M are zero except that $m_{i,j}$ and $m_{j,i}$ are one.)

5.5.17

a. Any four vectors from $\mathbb{R}^3$ are linearly related because the vector equation

$$c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + c_3 \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} + c_4 \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives rise to a linear system

$$\begin{aligned} x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 &= 0 \\ y_1c_1 + y_2c_2 + y_3c_3 + y_4c_4 &= 0 \\ z_1c_1 + z_2c_2 + z_3c_3 + z_4c_4 &= 0 \end{aligned}$$

that is homogeneous (and so has a solution) and has four unknowns but only three equations, and therefore has nontrivial solutions. (Of course, this argument applies to any subset of $\mathbb{R}^3$ with four or more vectors.)

b. We shall do just the two-vector case. Given $x_1, \ldots, z_2$,

$$S = \left\{ \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\}$$

to decide which vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

are in the span of S, set up

$$c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and row reduce the resulting system.

$$\begin{aligned} x_1c_1 + x_2c_2 &= x \\ y_1c_1 + y_2c_2 &= y \\ z_1c_1 + z_2c_2 &= z \end{aligned}$$

There are two variables c_1 and c_2 but three equations, so when Gauss's Method finishes, on the bottom row there will be some relationship of the form $0 = m_1x + m_2y + m_3z$. Hence, vectors in the span of the two-element set S must satisfy some restriction. Hence the span is not all of $\mathbb{R}^3$.

5.5.18 We have (using these odball operations with care)

$$\left\{ \begin{pmatrix} 1-y-z \\ y \\ z \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -y+1 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z+1 \\ 0 \\ z \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\}$$
$$= \left\{ y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\}$$

and so a natural candidate for a basis is this.

$$\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

To check linear independence we set up

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(the vector on the right is the zero object in this space). That yields the linear system

$$\begin{aligned} (-c_1 + 1) + (-c_2 + 1) - 1 &= 1 \\ c_1 \qquad\qquad &= 0 \\ c_2 \quad &= 0 \end{aligned}$$

with only the solution $c_1 = 0$ and $c_2 = 0$. Checking the span is similar.

5.6.1 One basis is $\langle 1, x, x^2 \rangle$, and so the dimension is three.

5.6.2 For this space

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$
$$= \left\{ a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \cdots + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

this is a natural basis.

$$\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$$

The dimension is four.

5.6.3 The solution set is

$$\left\{ \begin{pmatrix} 4x_2 - 3x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_2, x_3, x_4 \in \mathbb{R} \right\}$$

so a natural basis is this

$$\langle \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

(checking linear independence is easy). Thus the dimension is three.

5.6.4

a. As in the prior exercise, the space $\mathcal{M}_{2\times 2}$ of matrices without restriction has this basis

$$\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$$

and so the dimension is four.

b. For this space

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a = b - 2c \text{ and } d \in \mathbb{R} \right\}$$
$$= \left\{ b \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; b, c, d \in \mathbb{R} \right\}$$

this is a natural basis.

$$\langle \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$$

The dimension is three.

c. Gauss's Method applied to the two-equation linear system gives that $c = 0$ and that $a = -b$. Thus, we have this description

$$\left\{ \begin{bmatrix} -b & b \\ 0 & d \end{bmatrix} \;\middle|\; b, d \in \mathbb{R} \right\} = \left\{ b \cdot \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; b, d \in \mathbb{R} \right\}$$

and so this is a natural basis.

$$\langle \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$$

The dimension is two.

5.6.5 $\left\{ -1 + x^2,\ -x + x^3 \right\}$ is a basis of W, therefore W is of dimension 2.

5.6.6 The bases for these spaces are developed in the answer set of the prior subsection.

a. One basis is $\langle -7+x,\ -49+x^2,\ -343+x^3 \rangle$. The dimension is three.

b. One basis is $\langle 35 - 12x + x^2,\ 420 - 109x + x^3 \rangle$ so the dimension is two.

c. A basis is $\left\{ -105 + 71x - 15x^2 + x^3 \right\}$. The dimension is one.

d. This is the trivial subspace of $\mathcal{P}_3$ and so the basis is empty. The dimension is zero.

5.6.7 First recall that $\cos 2\theta = \cos^2\theta - \sin^2\theta$, and so deletion of $\cos 2\theta$ from this set leaves the span unchanged. What's left, the set $\left\{ \cos^2\theta, \sin^2\theta, \sin 2\theta \right\}$, is linearly independent (consider the relationship $c_1 \cos^2\theta + c_2 \sin^2\theta + c_3 \sin 2\theta = Z(\theta)$ where Z is the zero function, and then take $\theta = 0$, $\theta = \pi/4$, and $\theta = \pi/2$ to conclude that each c is zero). It is therefore a basis for its span. That shows that the span is a dimension three vector space.

5.6.8 A basis is

$$\langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rangle$$

and thus the dimension is $3 \cdot 5 = 15$.

5.6.9 In a four-dimensional space a set of four vectors is linearly independent if and only if it spans the space. The form of these vectors makes linear independence easy to show (look at the equation of fourth components, then at the equation of third components, etc.).

5.6.10

a. One

b. Two

c. n

5.6.11 $\dim(\mathbb{R}^2) = 2$, hence a set of four vectors is linearly independent.

5.6.12 A plane has the form $\{\vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 \mid t_1, t_2 \in \mathbb{R}\}$. When the plane passes through the origin we can take the particular vector $\vec{p}$ to be $\vec{0}$. Thus, in the language we have developed in this chapter, a plane through the origin is the span of a set of two vectors.

Now for the statement. Asserting that the three are not coplanar is the same as asserting that no vector lies in the span of the other two no vector is a linear combination of the other two. That's simply an assertion that the three-element set is linearly independent. Since $\dim(\mathbb{R}^3) = 3$ then the set spans $\mathbb{R}^3$. The set is a basis for $\mathbb{R}^3$

5.6.13 Let the space V be finite dimensional. Let S be a subspace of V.

a. The empty set is a linearly independent subset of S. It can be expanded to a basis for the vector space S.

b. Any basis for the subspace S is a linearly independent set in the superspace V. Hence it can be expanded to a basis for the superspace, which is finite dimensional. Therefore it has only finitely many members.

5.6.14 Let B_U be a basis for U and let B_W be a basis for W. Consider the concatenation of the two basis sequences.

If there is a repeated element then the intersection $U \cap W$ is nontrivial. Otherwise, the set $B_U \cup B_W$ is linearly dependent as it is a six member subset of the five-dimensional space $\mathbb{R}^5$. In either case some member of B_W is in the span of B_U, and thus $U \cap W$ is more than just the trivial space $\left\{\vec{0}\right\}$.

Generalization: if U, W are subspaces of a vector space of dimension n and if $\dim(U) + \dim(W) > n$ then they have a nontrivial intersection.

5.6.15 First, note that a set is a basis for some space if and only if it is linearly independent, because in that case it is a basis for its own span.

a. The answer to the question in the second paragraph is "yes" (implying "yes" answers for both questions in the first paragraph). If B_U is a basis for U then B_U is a linearly independent subset of W. It is possible to expand it to a basis for W. That is the desired B_W.

The answer to the question in the third paragraph is "no", which implies a "no" answer to the question of the fourth paragraph. Here is an example of a basis for a superspace with no sub-basis forming a basis for a subspace: in $W = \mathbb{R}^2$, consider the standard basis $\mathcal{E}_2$. No sub-basis of $\mathcal{E}_2$ forms a basis for the subspace U of $\mathbb{R}^2$ that is the line $y = x$.

b. It is a basis (for its span) because the intersection of linearly independent sets is linearly independent (the intersection is a subset of each of the linearly independent sets).

It is not, however, a basis for the intersection of the spaces. For instance, these are bases for $\mathbb{R}^2$:

$$B_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle \quad \text{and} \quad B_2 = \langle [\rangle r] \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

and $\mathbb{R}^2 \cap \mathbb{R}^2 = \mathbb{R}^2$, but $B_1 \cap B_2$ is empty. All we can say is that the $\cap$ of the bases is a basis for a subset of the intersection of the spaces.

c. The $\cup$ of bases need not be a basis: in $\mathbb{R}^2$

$$B_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \quad \text{and} \quad B_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

$B_1 \cup B_2$ is not linearly independent. A necessary and sufficient condition for a $\cup$ of two bases to be a basis

$$B_1 \cup B_2 \text{ is linearly independent} \quad \Longleftrightarrow \quad \text{span}\,(B_1 \cap B_2) = \text{span}\,(B_1) \cap \text{span}\,(B_2)$$

it is easy enough to prove (but perhaps hard to apply).

d. The complement of a basis cannot be a basis because it contains the zero vector.

5.6.16

a. A basis for U is a linearly independent set in W and so can be expanded to a basis for W. The second basis has at least as many members as the first.

b. One direction is clear: if $V = W$ then they have the same dimension. For the converse, let B_U be a basis for U. It is a linearly independent subset of W and so can be expanded to a basis for W. If $\dim(U) = \dim(W)$ then this basis for W has no more members than does B_U and so equals B_U. Since U and W have the same bases, they are equal.

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