CEGEP Linear Algebra Problems

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CEGEP level linear algebra problems

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Contents

Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

1.1.1 [**GH**] State which of the following equations is a linear equation. If it is not, state why.

a. $x + y + z = 10$
b. $xy + yz + xz = 1$
c. $-3x + 9 = 3y - 5z + x - 7$
d. $\sqrt{5}y + \pi x = -1$
e. $(x - 1)(x + 1) = 0$
f. $\sqrt{x_1^2 + x_2^2} = 25$
g. $x_1 + y + t = 1$
h. $\frac{1}{x} + 9 = 3\cos(y) - 5z$
i. $\cos(15)y + \frac{x}{4} = -1$
j. $2^x + 2^y = 16$

1.1.2 [**JH**] In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy-plane. By geometrical reasoning, show that there are three possibilities: there is a unique solution, there is no solution, and there are infinitely many solutions.

1.1.3 [**JH**] Is there a two-unknowns linear system whose solution set is all of $\mathbb{R}^2$?

1.2 Gaussian and Gauss-Jordan Elimination

1.2.1 [**JH**] Use Gauss's Method to find the unique solution for each system.

a.
$$\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$$

b.
$$\begin{aligned} x \phantom{{}+y} - z &= 0 \\ 3x + y \phantom{{}+z} &= 1 \\ -x + y + z &= 4 \end{aligned}$$

1.2.2 [**GH**] Solve the system of linear equations.

a.
$$\begin{aligned} x + y &= -1 \\ 2x - 3y &= 8 \end{aligned}$$

b.
$$\begin{aligned} 2x - 3y &= 3 \\ 3x + 6y &= 8 \end{aligned}$$

c.
$$\begin{aligned} x - y + z &= 1 \\ 2x + 6y - z &= -4 \\ 4x - 5y + 2z &= 0 \end{aligned}$$

d.
$$\begin{aligned} x + y - z &= 1 \\ 2x + y \phantom{{}-z} &= 2 \\ y + 2z &= 0 \end{aligned}$$

1.2.3 [**YL**] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 + x_5 &= 3 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 \phantom{{}+3x_2} + 17x_3 - 2x_4 - x_5 &= 1 \end{aligned}$$

a. Solve the following system by Gauss-Jordan elimination.
b. Find two particular solution to the above system.
c. Find a solution to the above system when $x_3 = 1$.

1.2.4 [**YL**] Given

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 &= 0 \\ 4x_1 \phantom{{}+3x_2} + 17x_3 - 2x_4 &= 0 \\ 9x_1 + 6x_2 + 27x_3 - 4x_4 &= 0 \end{aligned}$$

a. Solve the system by Gauss-Jordan elimination.
b. Find two particular nontrivial solution to the system.

c. Find a solution to the system when $x_1 = 1$.

1.2.5 [**JH**] Find the coefficients a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.2.6 [**JH**] True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.2.7 [**JH**] For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

1.2.8 [**YL**] Given the augmented matrix of a linear system:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a^2 - 1 & b^2 - a^2 \end{bmatrix}$$

If possible for what values of a and b the system has

a. no solution? Justify.

b. exactly one solution? Justify.

c. infinitely many solutions? Justify.

1.2.9 [**YL**] Given the augmented matrix of a linear system

$$\begin{bmatrix} 1 & 3 & 1 & -4 & b_1 \\ 3 & -2 & 4 & 5 & b_2 \\ 4 & 1 & 5 & 1 & b_3 \\ 7 & -1 & 9 & 6 & b_4 \end{bmatrix}.$$

Determine the restrictions on the b_i's for the system to be consistent.

1.2.10 [**JH**] Prove that, where $a, b, \ldots, e$ are real numbers and $a \neq 0$, if

$$ax + by = c$$

has the same solution set as

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

1.2.11 [**JH**] Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

1.3 Applications of Linear Systems

1.3.1 Place Holder

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Chapter 2

Matrix Algebra

2.1 Introduction to Matrices and Matrix Operations

2.1.1 [**HE**] Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 4 \\ 5 & -1 \\ 1 & -1 \end{bmatrix}, \; C = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \; E = \begin{bmatrix} 3 & 4 \\ -2 & 3 \\ 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \; G = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Compute each of the following and simplify, whenever possible. If a computation is not possible, state why.

a. $3C - 4D$
b. $A - (D + 2C)$
c. $A - E$
d. AE
e. $3BC - 4BD$
f. $CB + D$
g. GC
h. FG
i. Illustrate the associativity of matrix multiplication by multiplying $(AB)C$ and $A(BC)$ where A, B, and C are matrices above.

2.1.2 [**YL**] A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$.
Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

2.1.3 [**YL**] A square matrix A is called *idempotent* if $A^2 = A$.
Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A.

2.2 Matrix Inverses and Algebraic Properties

2.2.1 [**YL**] Solve of A given that it satisfies

$$(I - A^T)^{-1} = (\mathrm{tr}(B)B^2)^T$$

where

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

2.2.2 [**YL**] Solve of X given that it satisfies

$$DXD^T = \mathrm{tr}(BC)BC$$

where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} \; C = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} \; D = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}.$$

2.2.3 [**YL**] Given

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & \frac{1}{2} \end{bmatrix}.$$

a. Find A^{-1}.
b. Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2.2.4 [**YL**] Prove: If A and B are square matrices satisfying $AB = I$, then $A = B^{-1}$.

2.2.5 [**YL**] Prove: If AB and BA are both invertible then A and B are both invertible.

2.2.6 [**YL**] Prove: If B and C are $n \times n$ matrices such that $A = B^T C + C^T B$ is invertible then A^{-1} is symmetric.

2.3 Elementary Matrices

2.3.1 [**YL**] Write the given matrix as a product of elementary matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

2.3.2 [**YL**] Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

as a product of 4 elementary matrices.

2.3.3 [**YL**] Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3E_2E_1A = B$.

2.4 Linear Systems and Matrices

2.4.1 [**YL**] Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. Find A^{-1}.

b. Using A^{-1} solve $Ax = b$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Chapter 3

Determinants

3.1 The Laplace Expansion

3.1.1 [**YL**] Solve for λ.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

3.2 Determinants and Elementary Operations

3.2.1 [**YL**] Consider

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} \text{ and } B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}.$$

If $\det(B) = 5$ then determine $\det(A)$.

3.3 Properties of Determinants

3.3.1 [**YL**] Let A and B be $n \times n$ matrices such that $AB = -BA$ and n is odd, show that either A or B has no inverse.

3.4 Applications of the Determinant

3.4.1 [**YL**] Solve only for x_1 using Cramer's Rule.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_2 - 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned}$$

Chapter 4

Vector Geometry

4.1 Introduction to Vectors and Lines

4.1.1 Place Holder

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4.2 Dot Product and Projections

4.2.1 Cauchy-Schwartz Inequality [YL] Prove *without assuming that the law of cosine holds in* $\mathbb{R}^n$: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$.

4.3 Cross Product and Planes

4.3.1 Place Holder

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4.4 Areas, Volumes and Distances

4.4.1 Place Holder

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4.5 Geometry of Solutions of Linear Systems

4.5.1 Place Holder

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Chapter 5

Vector Spaces

5.1 Introduction to Vector Spaces

5.1.1 [**JH**] Name the zero vector for each of these vector spaces.

a. The space of degree three polynomials under the natural operations.
b. The space of 2×3 matrices.
c. The space $\{\, f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\,\}$.
d. The space of real-valued functions of one natural number variable.

5.1.2 [**JH**] Find the additive inverse, in the vector space, of the vector.

a. In $\mathcal{P}_3$, the vector $-3-2x+x^2$.
b. In the space $\mathcal{M}_{2\times2}$,
$$\begin{bmatrix}1 & -1\\ 0 & 3\end{bmatrix}.$$
c. In $\{\, ae^x + be^{-x} \mid a,b\in\mathbb{R}\,\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

5.1.3 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of linear polynomials $\mathcal{P}_1 = \{\, a_0 + a_1x \mid a_0, a_1 \in \mathbb{R}\,\}$ under the usual polynomial addition and scalar multiplication operations.
b. The set of linear polynomials $\{\, a_0 + a_1x \mid a_0 - 2a_1 = 0\,\}$, under the usual polynomial addition and scalar multiplication operations.

5.1.4 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of 2×2 matrices with real entries under the usual matrix operations.
b. The set of 2×2 matrices with real entries where the $2,1$ entry is zero, under the usual matrix operations.

5.1.5 [**JH**] For each, list three elements and then show it is a vector space.

a. The set of three-component row vectors with their usual operations.
b. The set
$$\{\, (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x+y-z+w = 0\,\}$$
under the operations inherited from $\mathbb{R}^4$.

5.1.6 [**JH**] Show that the following are not vector spaces.

a. Under the operations inherited from $\mathbb{R}^3$, this set
$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x+y+z = 1\,\}$$
b. Under the operations inherited from $\mathbb{R}^3$, this set
$$\{\, (x,\ y,\ z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\,\}$$
c. Under the usual matrix operations,
$$\left\{ \begin{bmatrix}a & 1\\ b & c\end{bmatrix} \;\middle|\; a,b,c\in\mathbb{R} \right\}$$
d. Under the usual polynomial operations,
$$\{\, a_0 + a_1x + a_2x^2 \mid a_0,a_1,a_2 \in \mathbb{R}^+\,\}$$
where $\mathbb{R}^+$ is the set of reals greater than zero
e. Under the inherited operations,
$$\{\, (x,\ y) \in \mathbb{R}^2 \mid x+3y=4,\ 2x-y=3 \text{ and } 6x+4y=10\,\}$$

5.1.7 [**JH**] Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

5.1.8 [**JH**] Prove that the following is not a vector space: the set of two-tall column vectors with real entries subject to these operations.
$$\begin{pmatrix}x_1\\ y_1\end{pmatrix} + \begin{pmatrix}x_2\\ y_2\end{pmatrix} = \begin{pmatrix}x_1-x_2\\ y_1-y_2\end{pmatrix} \qquad r\cdot\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}rx\\ ry\end{pmatrix}$$

5.1.9 [**JH**] Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

a. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

b. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5.1.10 [**JH**] For each, decide if it is a vector space; the intended operations are the natural ones.

a. The set of *diagonal* 2×2 matrices

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

b. The set of 2×2 matrices

$$\left\{ \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

c. $\{ (x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x + y + w = 1 \}$

d. The set of functions $\{ f\colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 0 \}$

e. The set of functions $\{ f\colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 1 \}$

5.1.11 [**YL**] Let $V = \{ A \mid A \in \mathcal{M}_{2\times 2} \text{ and } \det(A) \neq 0 \}$ with the following operations:

$$A + B = AB \text{ and } kA = kA$$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.

a. Does V satisfy closure under vector addition? Justify.

b. Does V contain a zero vector? If so find it. Justify.

c. Does V contains an additive inverse for all of its vectors? Justify.

d. Does V satisfy closure under scalar multiplication? Justify.

5.1.12 [**JH**] Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x + y$' to mean the product of x and y (so that $2 + 3$ is 6), and we interpret '$r \cdot x$' as the r-th power of x.

5.1.13 [**JH**] Prove or disprove that the following is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

5.1.14 [**JH**]
Is $\{ (x,\ y) \mid x, y \in \mathbb{R} \}$ a vector space under these operations?

a. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$

b. $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$ and $r \cdot (x,\ y) = (rx,\ 0)$

5.1.15 [**JH**]
Prove the following:

a. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself.

b. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

5.1.16 [**JH**]
The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

5.1.17 [**JH**]
Prove or disprove that the following is a vector space: the set of all matrices, under the usual operations.

5.1.18 [**JH**]
In a vector space every element has an additive inverse. Is the additive inverse unique *(Can some elements have two or more)*?

5.1.19 [**JH**]
Assume that $\vec{v} \in V$ is not $\vec{0}$.

a. Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.

b. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

c. Prove that any nontrivial vector space is infinite.

5.2 Subspaces

5.2.1 [**JH**]

a. Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ under the inherited operations.

b. What if it doesn't contain the origin?

5.2.2 [**JH**] Is the following a subspace under the inherited natural operations: the real-valued functions of one real variable that are differentiable?

5.3 Spanning Sets

5.3.1 [**YL**] Given the following two subspace of $\mathbb{R}^3$: $W_1 = \{ x \mid A_1 x = 0 \}$ and $W_2 = \{ x \mid A_2 x = 0 \}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix},\ A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}.$$

Determine whether the two subspaces are equal or whether one of the subspaces is contained in the other.

5.4 Linear Independence

5.4.1 [**YL**] Let $\vec{u} = (1,\ \lambda,\ -\lambda)$, $\vec{v} = (-2\lambda\ \ -2\ \ 2\lambda)$ and $\vec{w} = (\lambda - 2,\ -5\lambda - 2,\ -2)$.

a. For what value(s) of λ will $\{\vec{u},\ \vec{v}\}$ be linearly dependent.

b. For what value(s) of λ will $\{\vec{u},\ \vec{v},\ \vec{w}\}$ be linearly independent.

5.5 Basis

5.5.1 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_2x^2 + a_3x^3 \ \middle|\ p(-1) = 0 \right\}$$

a subspace of $\mathcal{P}_3$.

a. Find a basis B for $\mathcal{W}$.

b. Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis B.

5.6 Dimension

5.6.1 [**YL**] Given

$$W = \left\{ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \ \middle|\ p(1) = 0 \text{ and } p(-1) = 0 \right\}$$

a subspace of P_3. Determine the dimension of W.

Answers to Exercises

1.1.1

a. Yes
b. No
c. Yes
d. Yes
e. No
f. No
g. Yes
h. No
i. Yes
j. No

1.1.2 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line.
Thus the lines can share one point (giving a unique solution), share no points (giving no solutions), or share at least two points (which makes them the same line).

1.1.3 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

1.2.1

a. $x = 2$, $y = 3$
b. $x = -1$, $y = 4$, and $z = -1$.

1.2.2

a. $x = 1$, $y = -2$
b. $x = 2$, $y = \frac{1}{3}$
c. $x = -1$, $y = 0$, and $z = 2$.
d. $x = 1$, $y = 0$, and $z = 0$.

1.2.3

a. $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (60s - 55t + 30,\ -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3},\ -14s + 13t - 7,\ s,\ t)$ where $s,\ t \in \mathbb{R}$.
b. If $s = t = 0$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (30,\ -\frac{38}{3},\ -7,\ 0,\ 0)$.
If $s = 0$ and $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-25,\ \frac{35}{3},\ 6,\ 0,\ 1)$.
c. If $t = 0$ then $s = -\frac{4}{7}$ and $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-\frac{30}{7},\ \frac{316}{21},\ 1,\ \frac{4}{7},\ 0)$.

1.2.4

a. $(x_1,\ x_2,\ x_3,\ x_4) = (60t,\ -\frac{79}{3}t,\ -14t,\ t)$ where $t \in \mathbb{R}$.
b. If $t = 1$ then $(x_1,\ x_2,\ x_3,\ x_4) = (60,\ -\frac{79}{3},\ -14,\ 1)$.
If $t = 3$ then $(x_1,\ x_2,\ x_3,\ x_4) = (180,\ -79,\ 42,\ 3)$.
c. If $t = \frac{1}{60}$ then $(x_1,\ x_2,\ x_3,\ x_4) = (1,\ -\frac{79}{180},\ -\frac{14}{60},\ \frac{1}{60})$.

1.2.5 Because $f(1) = 2$, $f(-1) = 6$, and $f(2) = 3$ we get a linear system.

$$\begin{aligned} 1a + 1b + c &= 2 \\ 1a - 1b + c &= 6 \\ 4a + 2b + c &= 3 \end{aligned}$$

After performing Gaussian elimination we obtain

$$\begin{aligned} a + \quad b + \quad c &= 2 \\ -2b \qquad &= 4 \\ -3c &= -9 \end{aligned}$$

which shows that the solution is $f(x) = 1x^2 - 2x + 3$.

1.2.6 The following system with more unknowns than equations

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

has no solution.

1.2.7 After performing Gaussian elimination the system becomes

$$\begin{aligned} x - y &= 1 \\ 0 &= -3 + k \end{aligned}$$

This system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

1.2.8

a. Possible if $a = \pm 1$ and $a \neq \pm b$.
b. Not possible.
c. Possible if $a \neq \pm 1$ or $a = \pm b$.

1.2.9 Consistent if $b_3 - b_2 - b_1 = 0$ and $b_4 - 2b_2 - b_1 = 0$.

1.2.10 If $a \neq 0$ then the solution set of the first equation is $\{(x, y) \mid x = (c - by)/a\}$. Taking $y = 0$ gives the solution $(c/a, 0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a) + d \cdot 0 = e$, so $c = e$. Then taking $y = 1$ in $x = (c - by)/a$ gives that

$a((c-b)/a)+d\cdot 1=e$, which gives that $b=d$. Hence they are the same equation.
When $a=0$ the equations can be different and still have the same solution set: e.g., $0x+3y=6$ and $0x+6y=12$.

1.2.11 We take three cases: that $a\neq 0$, that $a=0$ and $c\neq 0$, and that both $a=0$ and $c=0$.
For the first, we assume that $a\neq 0$. Then Gaussian elimination

$$\begin{aligned} ax+ \qquad\qquad by&=j\\ (-(cb/a)+d)y&=-(cj/a)+k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a)+d\neq 0$; remember that $a\neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a)+d=(ad-bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad-bc\neq 0$.
In the second case, if $a=0$ but $c\neq 0$, then we swap

$$\begin{aligned} cx+dy&=k\\ by&=j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b\neq 0$ (we use the case assumption that $c\neq 0$ to get a unique x in back substitution). But where $a=0$ and $c\neq 0$ the condition "$b\neq 0$" is equivalent to the condition "$ad-bc\neq 0$". That finishes the second case.
Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x+by&=j\\ 0x+dy&=k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x,y) will do), but it never has a unique solution. Note that $a=0$ and $c=0$ gives that $ad-bc=0$.

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2.1.1

a. $\begin{bmatrix}16 & -3 & 2\\ -3 & 7 & -1\end{bmatrix}$

b. $\begin{bmatrix}-2 & 0 & -2\\ 3 & -13 & -3\end{bmatrix}$

c. Not possible, since dimension of A and E are not the same.

d. $\begin{bmatrix}7 & 1\\ 5 & 1\end{bmatrix}$

e. $\begin{bmatrix}36 & 19 & 2\\ 83 & -22 & 11\\ 19 & -10 & 3\end{bmatrix}$

f. Not possible, since the dimension of CD is 2×2 and is not equal to the dimension of D.

g. $\begin{bmatrix}9 & -7 & 3\end{bmatrix}$

h. $\begin{bmatrix}4 & -2\\ -6 & 3\end{bmatrix}$

2.1.2 Disprove: Show that it is impossible to obtain a nonzero matrix.

2.1.3 Hint: Apply the definition of an idempotent matrix.

2.2.1 $A=\begin{bmatrix}-\frac{3}{4} & 3\\ 1 & -\frac{3}{4}\end{bmatrix}$

2.2.2 $A=\begin{bmatrix}0 & -1\\ -11 & -\frac{17}{2}\end{bmatrix}$

2.2.3

a. $A=\begin{bmatrix}-\frac{3}{2} & 1 & 0\\ 2 & -1 & 0\\ 1 & -2 & 2\end{bmatrix}$

b. $X=\begin{bmatrix}-\frac{3}{2} & 1 & -\frac{3}{4} & 2 & -1\\ 2 & -1 & 1 & -2 & 1\\ -7 & 2 & \frac{3}{2} & -4 & 2\end{bmatrix}$

2.2.4 Hint: Show that the homogeneous system $Ax=0$ has only the trivial solution.

2.2.5 Hint: Use the definition of the inverse of a matrix.

2.2.6 Hint: Apply the definition of symmetric matrices.

2.3.1

$$\begin{bmatrix}0&0&1\\1&3&0\\2&4&0\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$$

Note: The answer is not unique.

2.3.2

$$A=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

Note: The answer is not unique.

2.3.3 $E_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$ $E_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$ $E_3=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}$
Note: The answer is not unique.

2.4.1

a. $A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

b. $x = \begin{bmatrix} \frac{16}{3} \\ -\frac{8}{3} \\ \frac{1}{3} \end{bmatrix}$

3.1.1 $\lambda = \frac{3\pm\sqrt{33}}{4}$

3.2.1 $\det(A) = -\frac{5}{12}$

3.3.1 Hint: Apply the determinant to both sides $AB = -BA$.

3.4.1 $x_1 = 4$

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4.2.1 Analyse the squared norm of $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ and $\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$.

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5.1.1

a. $0 + 0x + 0x^2 + 0x^3$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. The constant function $f(x) = 0$

d. The constant function $f(n) = 0$

5.1.2

a. $3 + 2x - x^2$

b. $\begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix}$

c. $-3e^x + 2e^{-x}$

5.1.3

a. $1 + 2x$, $2 - 1x$, and x.

b. $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

5.1.4

a.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.1.5

a. $(1,\ 2,\ 3)$, $(2,\ 1,\ 3)$, and $(0,\ 0,\ 0)$.

b. $(1,\ 1,\ 1,\ -1)$, $(1,\ 0,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ 0)$.

5.1.6

For each part the set is called Q. For some parts, there are more than one correct way to show that Q is not a vector space.

a. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

b. It is not closed under addition.
$$(1,\ 0,\ 0), (0,\ 1,\ 0) \in Q \qquad (1,\ 1,\ 0) \notin Q$$

c. It is not closed under addition.
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in Q \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \notin Q$$

d. It is not closed under scalar multiplication.
$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

e. The set is empty, violating the existance of the zero vector.

5.1.7 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

5.1.8 The '+' operation is not commutative; producing two members of the set witnessing this assertion is easy.

5.1.9

a. It is not a vector space.

$$(1+1)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\neq\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}$$

b. It is not a vector space.

$$1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\neq\begin{pmatrix}1\\0\\0\end{pmatrix}$$

5.1.10 For each “yes” answer, you must give a check of all the conditions given in the definition of a vector space. For each “no” answer, give a specific example of the failure of one of the conditions.

a. Yes.

b. Yes.

c. No, this set is not closed under the natural addition operation. The vector of all 1/4’s is an element of this set but when added to itself the result, the vector of all 1/2’s, is not an element of the set.

d. Yes.

e. No, $f(x)=e^{-2x}+(1/2)$ is in the set but $2\cdot f$ is not (that is, closure under scalar multiplication fails).

5.1.11

a. Closed under vector addition. Hint: Apply determinant properties.

b. $\vec{0}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\in V$

c. Every $A\in V$ has an additive inverse A^{-1}.

d. Yes.

e. Not closed under scalar multiplication. Since $0\vec{0}=0\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\notin V$

5.1.12 Check all 10 conditions of the definition of a vector space.

5.1.13 It is not a vector space since it is not closed under addition, as $(x^2)+(1+x-x^2)$ is not in the set.

5.1.14

a. No since $1\cdot(0,\ 1)+1\cdot(0,\ 1)\neq(1+1)\cdot(0,\ 1)$.

b. No since the same calculation as the prior part shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1\cdot(0,\ 1)\neq(0,\ 1)$.

5.1.15

a. Let V be a vector space, let $\vec{v}\in V$, and assume that $\vec{w}\in V$ is an additive inverse of $\vec{v}$ so that $\vec{w}+\vec{v}=\vec{0}$. Because addition is commutative, $\vec{0}=\vec{w}+\vec{v}=\vec{v}+\vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

b. Let V be a vector space and suppose $\vec{v},\vec{s},\vec{t}\in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v}+\vec{s}=\vec{v}+\vec{t}$ gives that $-\vec{v}+\vec{v}+\vec{s}=-\vec{v}+\vec{v}+\vec{t}$, which implies that $\vec{0}+\vec{s}=\vec{0}+\vec{t}$ and so $\vec{s}=\vec{t}$.

5.1.16
Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v}=\vec{v}+\vec{0}=\vec{0}+\vec{v}$.

5.1.17
It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

5.1.18
Each element of a vector space has one and only one additive inverse.
For, let V be a vector space and suppose that $\vec{v}\in V$. If $\vec{w}_1,\vec{w}_2\in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1+\vec{v}+\vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1+(\vec{v}+\vec{w}_2)=\vec{w}_1+\vec{0}=\vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1+\vec{v})+\vec{w}_2=\vec{0}+\vec{w}_2=\vec{w}_2$. Therefore, $\vec{w}_1=\vec{w}_2$.

5.1.19
Assume that $\vec{v}\in V$ is not $\vec{0}$.

a. One direction of the if and only if is clear: if $r=0$ then $r\cdot\vec{v}=\vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v}=\vec{0}$ then $(1/r)\cdot r\vec{v}=(1/r)\cdot\vec{0}$ shows that $\vec{v}=\vec{0}$, contrary to the assumption.

b. Where r_1,r_2 are scalars, $r_1\vec{v}=r_2\vec{v}$ holds if and only if $(r_1-r_2)\vec{v}=\vec{0}$. By the prior item, then $r_1-r_2=0$.

c. A nontrivial space has a vector $\vec{v}\neq\vec{0}$. Consider the set $\{k\cdot\vec{v}\ \mid\ k\in\mathbb{R}\}$. By the prior item this set is infinite.

5.2.1

a. Every such set has the form $\{r\cdot\vec{v}+s\cdot\vec{w}\ \mid\ r,s\in\mathbb{R}\}$ where either or both of $\vec{v},\vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v}+s_1\vec{w})+(r_2\vec{v}+s_2\vec{w})=(r_1+r_2)\vec{v}+(s_1+s_2)\vec{w}$ and scalar multiplication $c(r\vec{v}+s\vec{w})=(cr)\vec{v}+(cs)\vec{w}$ is clear.

b. No such set can be a vector space under the inherited operations because it does not have a zero element.

5.2.2 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f+g)'=f'+g'$, and that a multiple of a differentiable function is differentiable and that $(r\cdot f)'=r\,f'$.

5.3.1 Hint: For each subspace determine a set of vectors that spans it.
$W_1\subsetneq W_2$

5.4.1

a. $\lambda=1$

b. $\lambda\neq-1,\ -\frac{1}{2},\ 1$

5.5.1

a. $B = \left\{ 1 + x^3,\ x^2 + x^3 \right\}$
b. $(p(x))_B = (-2, 2)$

5.6.1 $\left\{ -1 + x^2,\ -x + x^3 \right\}$ is a basis of W, therefore W is of dimension 2.

References

[GH] Gregory Hartman, *Fundamentals of Matrix Algebra*, https://github.com/APEXCalculus/Fundamentals-of-Matrix-Algebra Licensed under the Creative Commons Attribution-Noncommercial 3.0 license.

[HE] Harold W. Ellingsen Jr., *Matrix Arithmetic*, Licensed under the Creative Commons Attribution-ShareAlike 2.5 License.

[JH] Jim Hefferon, *Linear Algebra*, http://joshua.smcvt.edu/linearalgebra, Licensed under the GNU Free Documentation License or the Creative Commons Attribution-ShareAlike 2.5 License, 2014.

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