Analysis 2 (Assignment / My Notes)

We measured $y_1, y_2 \dots y_N$ at known $x_1, x_2 \dots x_N$.

we believe the true relationship to be $y = ax^2$
but our measurement is noisy so we measure $y = ax^2 + \text{noise}$

Task: 1) Find the best value of a
2) Quantify the uncertainty in value, given noise.

Assumptions: * Each measurement error is random (not systematic)
*) Errors are Gaussian (normal) with standard deviation (σ)
*) Errors on different point are independent

Mathematically $\boxed{\epsilon_k = y_k - ax_k^2 \sim N(0, \sigma^2)}$

Likelihood is, if the true coefficient was a, how propobable is that we would measure the data we saw.

For One data point: $\text{Prob}(y_k | a) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{\left[-\frac{(y_k - ax_k^2)^2}{2\sigma^2}\right]}$

Because measurements are independent,

$$\boxed{\mathcal{L}(a) = \underbrace{\prod_{k=1}^{N}}_{\text{Product}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\left[-\frac{(y_k - ax_k^2)^2}{2\sigma^2}\right]}}$$

Likely hood function

Because product of many small / Large numbers is numerically messy.
* Taking log turns products into sums
* does not change where maximum is.

So we, $\ln\left(\mathcal{L}(a)\right) = -N\ln\left(\sqrt{2\pi}\,\sigma\right) - \frac{1}{2\sigma^2}\sum_{K=1}^{N}\left(y_K - a x_K^2\right)^2$

or, $-\ln\left(\mathcal{L}(a)\right) = +N\ln\left(\sqrt{2\pi}\,\sigma\right) + \frac{1}{2\sigma^2}\sum_{K=1}^{N}\left(y_K - a x_K^2\right)^2$

Since first term does not depend on a, maximizing $\ln(\mathcal{L})$ is same as minimizing $\sum_{K=1}^{N}(y_K - a x_K^2)^2$

$\therefore$ Gaussian maximum likelihood is least square fitting" not approx. exact

$\Rightarrow$ Now, to find the best $\hat{a}$, we differentiate & set to zero

$$\frac{d}{da}\left(-\ln\mathcal{L}\right) = \frac{1}{\sigma^2}\sum_{K=1}^{N}\left(a x_K^4 - y_K x_K^2\right) = 0$$

$$\hookrightarrow \frac{d}{da}\sum_{K=1}^{N}\left(y_K - a x_K^2\right)^2 = 0 \longrightarrow \sum_{K=1}^{N} 2\left(y_K - a x_K^2\right)\times\left(-x_K^2\right) = 0$$

$$= \sum_{K=1}^{N} -y_K x_K^2 + a x_K^4 = 0$$

$$\Rightarrow \sum_{K=1}^{N} a x_K^4 = \sum_{K=1}^{N} y_K x_K^2$$

$$\therefore \hat{a} = \frac{\sum_{K=1}^{N} y_K x_K^2}{\sum_{K=1}^{N} x_K^4}$$

* Width of parabola is uncertainty
* σ, second derivative:

$$\frac{d^2}{da^2}\left(-\ln(\mathcal{L})\right) = \frac{1}{\sigma^2}\sum_{K=1}^{N} x_K^4$$

For 1 parameter MLE Variance is inverse of Curvature, $\therefore$ $\sigma_a^2 = \left(\frac{d^2}{da^2} - \ln\mathcal{L}\right)^{-1} = \frac{\sigma^2}{\sum_{K=1}^{N} x_K^4}$

$$\therefore \sigma_a = \sqrt{\sigma_a^2} = \frac{\sigma}{\sqrt{\sum_{K=1}^{N} x_K^4}}$$

Remember to upload the code!