ON THE ORIENTED ALGEBRAIC COBORDISM SPECTRUM

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Abstract. In this paper we study the motivic Thom spectrum MSL^c, which is a universal homotopy commutative ring spectrum (up to phantoms) among those admitting Thom classes for oriented vector bundles in the sense of Fabien Morel, i.e. with a choice of a square root of the determinant bundle. We prove an interpolation between MSL and MSL^c and receive an equivalence of the motivic spectra $\mathrm{MSL}^c \cong \mathrm{MSL} \oplus \Sigma^1_{\mathbb{P}^1}\mathrm{MGL}$. Using this splitting, we compute various invariants of this Thom spectrum over a field (after inverting the exponential characteristic). For example, we determine the geometric diagonal of the homotopy groups in terms of Stong's complex-spin cobordism ring. Also, we compute the slices and use them to establish description of the category of 2-inverted MSL^c-modules.

Contents

1. Introduction

The second half of the twentieth century has seen a spectacular development in the theories of bordism and cobordism. The interest in cobordism theories and Thom spectra, started with the Pontryagin–Thom construction. In modern terms, it states that the cobordism theory Ω^G_* as a generalized homology theory is isomorphic to the generalized homology theory represented by the Thom spectrum $\mathrm{M}G$ [Pon55]. Therefore, the computation of the respective cobordism ring is equivalent to the study of the homotopy groups of a spectrum, which is a problem of stable homotopy theory. One of the most significant achievements was the description of the complex (or unitary) bordism ring Ω^U by Milnor [Mil60], and Novikov [Nov60, Nov62]. Later, Quillen's [Qui69] construction of an explicit isomorphism from Ω^U to the Lazard ring, and identification of the MU-formal group law as the universal one, has changed the course of stable homotopy theory. Other bordisms like special unitary (MSU), and symplectic (MSp) bordism also have sparked the interest of some of the leading topologists of the previous century like Stong, Conner, Floyd, and Novikov [Sto67], [CF66], [Nov62].

In the setting of motivic homotopy theory, the Thom spectrum MGL was introduced by Voevodsky in his ICM address [Voe98]. This spectrum is the universal oriented commutative ring spectrum [PPR08], where "oriented" means that it possesses Thom classes for all vector bundles. There are interesting theories like Milnor-Witt motivic cohomology ($\widetilde{\mathrm{H}\mathbb{Z}}$), Hermitian K-theory (KQ), derived Witt theory (W) etc, that can not be orientable. Nonetheless, all of them satisfy a weaker form of orientation. They have Thom isomorphism for special linear (SL) vector bundles, or bundles with trivialized determinant. Panin-Walter [PW23] constructed special linear algebraic cobordism spectrum MSL and proved its universality among SL-oriented theories.

Quadratically oriented or metalinear vector bundles have a central place in enumerative geometry. The structure group of such bundles is the special group scheme $\mathrm{SL}^c_n := \mathrm{Ker}(\mathrm{GL}_n \times \mathbb{G}_m \xrightarrow{\det\times()^{-2}} \mathbb{G}_m)$. These are the vector bundles whose determinants admit a square root. Theories with metalinear orientation are fundamental in $\mathbb{A}^1$-homotopy theory [BM00], [Fas08], [Mor12], [BW23]. Special linear, and metalinear orientations are closely related to each other. By construction, any SL^c-oriented theory is also SL-oriented. A rather surprising result by Ananyevsky showed that the existence of Thom orientation for SL-bundles implies the existence of Thom orientation for SL^c-bundles [Ana20, Theorem 1.1]. Nonetheless, the precise relation between the Thom spectrum MSL, and the universal Thom spectrum admitting Thom classes for SL^c-vector bundles, MSL^c is not well understood. In this article, we study this spectrum MSL^c in detail.

1.1. **Overview of results.** Now, we formulate the main results of our work. First, we describe MSL^c in the category of MSL-modules, as the free MSL-module associated with $\Sigma^{\infty-1}_{\mathbb{P}^1}\mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(-2))$.

Theorem 1.1 (Theorem 5.7)**.** *Let S be a base scheme. Then the morphism of motivic spectra over S*

$$\mathrm{MSL}\otimes\Sigma^{-1}_{\mathbb{P}^1}\mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(-2)) \xrightarrow{\mathrm{id}\otimes\Sigma^{-1}_{\mathbb{P}^1}\mathrm{th}(\mathcal{O}(-2))} \mathrm{MSL}\otimes\mathrm{MSL}^c \xrightarrow{\mathrm{act}} \mathrm{MSL}^c$$

is an equivalence of MSL*-modules. Here* act *is the action on the* MSL*-module (see Lemma 3.6), and* $\mathrm{th}(\mathcal{O}(-2))$ *is the Thom class of $\mathcal{O}(-2)$ with an obvious orientation in* MSL^c*-cohomology.*

We can use the well-known trick of "reduction to determinant" (see Lemma 5.8) to improve Theorem 1.1, and get the following splitting of motivic spectra.

Theorem 1.2 (Theorem 5.11)**.** *Let S be a base scheme. Then there is an isomorphism of* MSL*-modules*

$$\mathrm{MSL}^c \cong \mathrm{MSL}\oplus\Sigma^1_{\mathbb{P}^1}\mathrm{MGL}.$$

As the first motivic Hopf map η acts trivially on MGL, our splitting result (Theorem 1.2) immediately gives the equivalence of MSL, and MSL^c in the η-periodic category $\mathrm{SH}(S)[1/\eta]$ [Hau23, Proposition 3.2.8] (see also [BW25, Corollary 1.3]).

Corollary 1.3 (Corollary 5.13)**.** *Let S be a base scheme. Then, the map* $\mathrm{MSL}\to\mathrm{MSL}^c$ *induces an equivalence of $\mathbb{E}_\infty$-ring spectra over S*

$$\mathrm{MSL}[\eta^{-1}] \xrightarrow{\simeq} \mathrm{MSL}^c[\eta^{-1}].$$

After proving these structural results, we move on to specific computations. First, we show that the complex realization of MSL^c is Stong's complex-spin bordism $\mathrm{M}\Sigma$ [Sto67].

Proposition 1.4 (Proposition 6.5)**.** *The complex realization of* MSL^c *is* $\mathrm{M}\Sigma$.

We compute the geometric diagonal of MSL^c. Just like the situation for MSL, $\pi_{2*,*}(\mathrm{MSL}^c)$ depends on the base field we work on [Zol24].

Theorem 1.5 (Proposition 6.6, and Theorem 6.7)**.** *Let k be a local Dedekind domain (i.e. a field or a discrete valuation ring) and let $e\neq 2$ be the exponential characteristic of the residue field of k.*

(1) Then there is an isomorphism of rings

$$\pi_{2*,*}(\mathrm{MSL}^c)\Big/{}_\eta\pi_{2*,*}(\mathrm{MSL}^c)\,[1/e] \cong \mathrm{W}(k)[1/e][y_4,y_8,\dots],\ \textit{where } |y_i|=(2i,i).$$

Here ${}_\eta\pi_{2*,*}(\mathrm{MSL})$ denotes the annihilator of η in $\pi_{2*,*}(\mathrm{MSL}^c)$.

(2) Let $\mathrm{I}_{\mathrm{MSL}}(k)$ denotes the graded subgroup of $\pi_{2*,*}(\mathrm{MSL})$ which in degree n is given by

$$\begin{cases}\eta\cdot\pi_{2n-1,n-1}(\mathrm{MSL}), & \textit{if } n\equiv 0\ (\textit{mod } 4),\\ 0, & \textit{otherwise.}\end{cases}$$

Then, there is an isomorphism of rings

$$\pi_{2*,*}(\mathrm{MSL}^c)\Big/\mathrm{I}_{\mathrm{MSL}^c}(k)\,[1/e] \cong \pi_{2*}(\mathrm{M}\Sigma)[1/e].$$

The quotient of $\pi_{2*,*}(\mathrm{MSL}^c)$ by the ideal $\mathrm{I}_{\mathrm{MSL}}(k)$ is the base field independent part, which happens to be isomorphic to its complex topological counterpart $\pi_{2*}(\mathrm{M}\Sigma)$, at least after inverting the exponential characteristic.

Remark 1.6. In particular, the splitting result of Theorem 1.1 allows us to compute the first few homotopy modules of MSL^c using known similar computations of MSL, and MGL. For the definition of the homotopy module $\underline{\pi}_n(-)_*$, please see section 6.1. We have

$$\underline{\pi}_n(\mathrm{MSL}^c)_* \cong \underline{\pi}_n(\mathrm{MSL})_*\oplus\underline{\pi}_{n-1}(\mathrm{MGL})_{*+1}.$$

Spitzweck computed $\underline{\pi}_0(\mathrm{MGL})_*$, and $\underline{\pi}_1(\mathrm{MGL})_*$ [Spi10]. The computations of $\underline{\pi}_0(\mathrm{MSL})_*$, and $\underline{\pi}_1(\mathrm{MSL})_*$ can be found in [BH21], and [Nan23, Theorem 1.4]. in the upcoming work [NRZ], $\underline{\pi}_2(\mathrm{MSL})_*$ will be computed in terms of $\underline{\pi}_2(-)_*$ of the very effective cover of Hermitian K-theory (kq).

We summarize these computations in the form of the following result.

Theorem 1.7 (Proposition 6.1, Proposition 6.2, Proposition 6.3)**.** *Let k be a field of characteristic* 0*.*

(1) The unit map $\mathbb{1} \to \mathrm{MSL}^c$ induces an isomorphism

$$\underline{\pi}_0(\mathbb{1})_* \cong \underline{\pi}_0(\mathrm{MSL}^c)_*,$$

(2) the morphisms $\mathrm{MSL}^c \to \mathrm{kq}$ and $\mathrm{MSL}^c \to \Sigma^1_{\mathbb{P}^1}\mathrm{MGL}$ induce an isomorphism

$$\underline{\pi}_1(\mathrm{MSL}^c)_* \cong \underline{\pi}_1(\mathrm{kq})_* \oplus \underline{\mathrm{K}}^{\mathrm{M}}_{*+1},$$

where $\underline{\mathrm{K}}^{\mathrm{M}}_*$ is the unramified Milnor K-theory.

(3) We have an isomorphism

$$\underline{\pi}_2(\mathrm{MSL}^c)_* \cong \underline{\pi}_2(\mathrm{kq})_* \oplus \underline{\pi}_1(\mathrm{MGL})_{*+1},$$

where $\underline{\pi}_1(\mathrm{MGL})_m$ fits into the following exact sequence of Nisnevich sheaves

$$\underline{\mathrm{K}}^{\mathrm{M}}_{m+1} \to \underline{\pi}_1(\mathrm{MGL})_m \to \underline{\pi}_1(\mathrm{H}\mathbb{Z})_m \to 0.$$

Remark 1.8. In the Theorem 1.7, the statement about $\underline{\pi}_0(\mathrm{MSL}^c)_*$ works over any qcqs base scheme. The other two statements are true over fields of odd prime characteristic, after inverting its characteristic.

We have further studied the stable category of 2-inverted MSL^c-modules over a field. The main tool here is a description of all the slices of MSL^c (see Theorem 7.2). Again, Theorem 1.1 gives a way of describing the slices of MSL^c in terms of the known computation of slices of MSL ([NRZ]), and MGL [Spi10]. We have compared the slice spectral sequence of 2-inverted MSL^c, and MGL to get the following equivalence.

Theorem 1.9 (Theorem 7.4)**.** *Let k be a field of characteristic zero. Then the canonical map of $\mathbb{E}_\infty$-ring spectra over S*

$$(\mathrm{MSL}^c)^\wedge_\eta[1/2] \to \mathrm{MGL}[1/2]$$

is an equivalence. For a field k of characteristic $p > 0$ the same result holds after inverting p.

the category of 2-inverted MSL^c-modules splits (as a symmetric monoidal ∞-category) into the product

$$\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2] \cong \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^+ \times \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^-. \tag{1.10}$$

The projection of the motivic Hopf element in the plus-part is trivial, while it is invertible in minus part. Due to formal reasons, $\mathrm{MSL}^c[1/2]^+$ coincides with the η-completion $(\mathrm{MSL}^c)^\wedge_\eta[1/2]$. Now, Theorem 1.9 gives us the following equivalence.

Theorem 1.11 (Theorem 7.12)**.** *Let k be a field. After inverting the exponential characteristic of k, there is an equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^+ \cong \mathrm{Mod}_{\mathrm{MGL}}(\mathrm{SH}(k))[1/2].$$

Our main tool in understanding the minus part is the symmetric monoidal equivalence between $\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^-$, and $\mathrm{Mod}_{\mathrm{MSL}^c[1/\eta,1/2]}(\mathrm{SH}(k))$. Then, we use the fact that MSL, and MSL^c are equivalent after η-periodization (see Corollary 1.3). We also use Bachmann's work on real étale realization [Bac18], giving the following equivalence of symmetric monoidal ∞-categories

$$\mathrm{SH}(k)[1/2]^- \cong \mathrm{Sp}(\mathrm{Sper}\, k)[1/2].$$

For the specific notations used here, and a quick overview of real étale realization, please see Appendix A. The following result summarizes this discussion.

Theorem 1.12 (Theorem 7.15)**.** *Let k be a base field. Then the real étale realization induces an equivalence of the stable symmetric monoidal ∞-categories*

$$\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^- \cong \mathrm{Mod}_{\underline{\mathrm{MSO}}}(\mathrm{Sp}(\mathrm{Sper}\, k))[1/2].$$

Here $\underline{\mathrm{MSO}} \in \mathrm{Sp}(\mathrm{Sper}\, k)$ is a constant sheaf on the Thom spectrum $\mathrm{MSO} \in \mathrm{Sp}$.

1.2. **Outline.** In Section 2, we recall some constructions about the classifying space of SL_n^c-torsors. Here, we have also written down the construction of the motivic Thom functor following [BH21]. In Section 3, we recall the construction of MSL^c as an $\mathbb{E}_\infty$-ring spectrum. We have also given a geometric model of this using "metalinear" Grassmannians. In Section 4, we show the (weak) universality of MSL^c among theories admitting Thom classes for oriented vector bundles in the sense of Fabien Morel, i.e. with a choice of a square root of the determinant bundle. We have mostly followed the ideas from [PW23]. In Section 5, we prove Theorem 1.1, and Theorem 1.2. Section 6 contains computations of some homotopy modules, and the geometric diagonal of MSL^c. Finally, in section 7, we give a description of the category of MSL^c-modules after inverting 2. We have recalled some needed results about real étale realization in Appendix A. In Appendix B, we have summarized some facts about Stong's complex-spin cobordism spectrum $\mathrm{M}\Sigma$. For the reader's convenience, we have summarized some results about geometric diagonal of MSL from [Zol24] in Appendix C. Since Section 7 heavily relies on the description of the slices of MSL^c, a draft containing only the relevant slice computation of MSL is available [NRZ].

1.3. **Table of notations.** Throughout the paper we employ the following notations and conventions.

k	local Dedekind domain, i.e. a field or a discrete valuation ring
base scheme S	quasi-compact quasi-separated scheme S
Sm_S	category of smooth schemes over S
$\mathrm{PSh}(\mathrm{Sm}_S)$	∞-category of (space-valued) presheaves on Sm_S
$\mathrm{B}G$	sheaf that classifies Nisnevich G-torsors for a group scheme G
$\mathrm{Th}_X(E)$	Thom space of a vector bundle $E \to X$
$\mathrm{SH}(S)$	stable ∞-category of motivic spectra over S
$\Sigma^{i,j}$	(i,j)-suspension endofunctor of $\mathrm{SH}(k)$
Σ^∞, Ω^∞	infinite $\mathbb{P}^1$-suspension and $\mathbb{P}^1$-loop functors
$\mathbb{1}$	motivic sphere spectrum
MGL, MSL, MSL^c	algebraic cobordism, special linear algebraic cobordism, and metalinear algebraic cobordism
kq	very effective cover of the Hermitian K-theory spectrum
$\mathrm{H}A$	Spitzweck's motivic cohomology spectrum with A-coefficients
$\mathcal{E} \in \mathrm{CAlg}(\mathrm{hSH}(k))$	homotopy commutative ring spectrum $\mathcal{E}$
$\mathcal{E} \in \mathrm{CAlg}(\mathrm{SH}(k))$	$\mathbb{E}_\infty$-ring spectrum $\mathcal{E}$
$[-,-]$	homotopy classes of maps in $\mathrm{SH}(k)$
$\underline{\pi}_n(\mathcal{E})_m$	Nisnevich sheaf associated with the presheaf $U \mapsto [\Sigma^n\Sigma^\infty_{\mathbb{P}^1}U_+, \Sigma^m_{\mathbb{G}_m}\mathcal{E}]$.
$\underline{\pi}_n(\mathcal{E})_*$	The homotopy module $\bigoplus_{m\in\mathbb{Z}} \underline{\pi}_n(\mathcal{E})_m$
$\pi_{2*,*}(\mathcal{E})$	$\mathbb{P}^1$-diagonal/geometric diagonal/geometric part of a spectrum $\mathcal{E}$
$\mathcal{E}_{i,j}(-)$, $\mathcal{E}^{i,j}(-)$	homology and cohomology theory represented by a spectrum $\mathcal{E}$
$\eta \in \pi_{1,1}(\mathbb{1})$	The first motivic Hopf element, i.e. stabilization of $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$
${}_{\eta^m}\pi_{2*,*}(\mathrm{MSL})$	annihilator $\mathrm{Ann}_{\pi_{2*,*}(\mathrm{MSL})}(\eta^m) = \{\alpha \in \pi_{2*,*}(\mathrm{MSL}) \mid \alpha\cdot\eta^m = 0\}$
$\mathcal{E}[\eta^{-1}]$	$\mathrm{colim}(\mathcal{E} \xrightarrow{\eta} \Sigma^{-1,-1}\mathcal{E} \xrightarrow{\eta} \Sigma^{-2,-2}\mathcal{E} \xrightarrow{\eta} \dots)$
SH	stable ∞-category of spectra
$\eta_{\mathrm{top}} \in \pi_1(\mathbb{1}_{\mathrm{top}})$	classical Hopf element
$\eta_{\mathrm{top}} \in \pi_{1,0}(\mathbb{1})$	image of η_{top} under the constant functor $\mathrm{SH} \to \mathrm{SH}(k)$
MU, MSU, MSO	complex cobordism, special unitary cobordism, oriented cobordism
$\mathrm{M}\Sigma$	complex-spin bordism, see Appendix B
Sper A	the real spectrum Sper A of a commutative ring A, see Appendix A.
$\mathrm{Re}_{\mathrm{rét}}$	real étale realization functor, see Appendix A.

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2. Preliminaries

Throughout the text S denotes a base scheme which is assumed to be quasi-compact and quasi-separated. By Sm_S we denote the category of quasi-compact quasi-separated smooth S-schemes and by $\mathrm{PSh}(\mathrm{Sm}_S)$ the ∞-category of presheaves (of spaces) on Sm_S.

2.1. **Classifying spaces.** Let $\tau \in \{\mathrm{Zar}, \mathrm{Nis}, \text{ét}\}$ be one of the three Grothendieck topologies on the category of smooth S-schemes. Suppose that G is an algebraic group over S (or, more generally, a

τ-sheaf of groups on Sm_S). Then we denote by $\mathrm{B}_\tau G \in \mathrm{PSh}(\mathrm{Sm}_S)$ the presheaf

$$X \mapsto \{\text{groupoid of } \tau\text{-locally trivial } G\text{-torsors over } X\}.$$

We refer to this presheaf as *τ-classifying space of G*. It is well-known that this presheaf is a τ-sheaf and that $\mathrm{B}_\tau G$ is equivalent to the τ-sheafification of the usual simplicial construction $(\mathcal{B}G)_n = G^{\times n}$ (see e.g. [AHW18, Lemma 2.3.2]). By the Yoneda lemma, to give a morphism of presheaves $X \to \mathrm{B}_\tau G$ one has to choose a τ-locally trivial G-torsor over $X \in \mathrm{Sm}_S$.

Remark 2.1. Recall that the linear algebraic group G over S is called *special* if every fppf-torsor over a (not necessarily smooth) S-scheme is locally trivial in the Zariski topology. Note that if G is a smooth affine group scheme over S, then every fppf-torsor is automatically étale locally trivial.

For the Morel–Voevodsky construction of a geometric model of the étale classifying space of an algebraic group, please see [MV99, Section 4.2, p. 133].

2.2. **Oriented vector bundles and SL^c-torsors.** Consider the following epimorphism of linear algbraic groups over S

$$\nu_n\colon \mathrm{GL}_n \times \mathbb{G}_m \to \mathbb{G}_m,\ (g,t) \mapsto t^{-2} \cdot \det(g). \tag{2.2}$$

Following [PW18, §3] we denote by SL^c_n its kernel $\mathrm{SL}^c_n := \mathrm{Ker}(\nu_n)$. This is a smooth linear algebraic group over S. It appears as a structure group of rank n oriented vector bundles in the sense of F. Morel.

Definition 2.3 ([Mor12, Definition 4.3])**.** A *rank n oriented vector bundle* over a scheme $X \in \mathrm{Sm}_S$ is a triple (E, L, λ) with E being a vector bundle over X, L being a line bundle over X and $\lambda\colon\ \det(E) \xrightarrow{\simeq} L^{\otimes 2}$ being an isomorphism of line bundles. An isomorphism of oriented vector bundles is defined in the expected way.

The direct sum $(E, L, \lambda) \oplus (E', L', \lambda')$ of oriented vector bundles (E, L, λ) and (E', L', λ') is given by the triple $(E \oplus E', L \otimes L', \lambda \otimes \lambda')$. Here by abuse of notation $\lambda \otimes \lambda'$ denotes the following isomorphism

$$\det(E \oplus E') \cong \det(E) \otimes \det(E') \xrightarrow{\lambda\otimes\lambda'} L^{\otimes 2} \otimes (L')^{\otimes 2} \cong (L \otimes L')^{\otimes 2}.$$

The trivial rank n oriented vector bundle over X is a rank n oriented vector bundle over X isomorphic to $(\mathcal{O}_X, \mathcal{O}_X, \theta)^{\oplus n}$, where θ is the canonical isomorphism $\theta : \mathcal{O}_X \xrightarrow{\simeq} \mathcal{O}_X^{\otimes 2}$. According to [Ana20, Lemma 2.7] oriented vector bundles are locally trivial in the Zariski topology.

Notation 2.4. We denote by $\mathrm{Vect}^{\mathrm{or}}_n(X)$ the groupoid of oriented vector bundles over X and by $\mathrm{Vect}^{\mathrm{or}}(X)$ the groupoid of oriented vector bundles of all ranks, i.e. the coproduct

$$\coprod_n \mathrm{Vect}^{\mathrm{or}}_n(X).$$

For a scheme X we have a canonical equivalence of groupoids (see [Ana20, Remark 2.8])

$$\mathrm{Vect}^{\mathrm{or}}_n(X) \cong \mathrm{B}_{\mathrm{Zar}}\mathrm{SL}^c_n(X). \tag{2.5}$$

Remark 2.6. This is the reason why oriented vector bundles are also sometimes called "vector SL^c-bundles" in the literature; see [Ana20, §2], [PW18, §3]. However, we prefer to use the above terminology, since it agrees well with the real Betti realization. Also, we caution the reader that some authors by oriented vector bundles mean vector SL-bundles, but it seems that the latter should be called "strictly oriented vector bundles"; see [MS23, Definition 2.14].

The next lemma seems to be known, but we do not know a reference.

Lemma 2.7. *Let S be a base scheme. Then SL^c_n is a special linear algebraic group over S (i.e. every SL^c_n-torsor over an S-scheme is locally trivial in the Zariski topology).*

Proof. By smoothness of SL^c_n it is sufficient to show that every étale SL^c_n-torsor over a spectrum of a local ring over S is trivial. Let R be such a local ring. The definition of SL^c_n induces an exact sequence of étale cohomology

$$\mathrm{GL}_n(R) \times \mathbb{G}_m(R) \xrightarrow{\nu_n(R)} \mathbb{G}_m(R) \to H^1_{\text{ét}}(R, \mathrm{SL}^c_n) \to H^1_{\text{ét}}(R, \mathrm{GL}_n \times \mathbb{G}_m).$$

The last pointed set is trivial since $\mathrm{GL}_n \times \mathbb{G}_m$ is a special linear group over S (as a product of special groups). Hence, the boundary map $\mathbb{G}_m(R) \to H^1_{\text{ét}}(R, \mathrm{SL}^c_n)$ is surjective. It is easy to show that the morphism $\nu_n(R)$ is also surjective. Now the claim follows from the exactness of the above sequence. $\square$

Remark 2.8. In particular, the natural morphisms of the classifying spaces

$$\mathrm{B}_{\mathrm{Zar}}\mathrm{SL}_n^c \to \mathrm{B}_{\mathrm{Nis}}\mathrm{SL}_n^c \to \mathrm{B}_{\mathrm{\acute{e}t}}\mathrm{SL}_n^c$$

are equivalences of presheaves on Sm_S. It allows us to use the symbol BSL_n^c for any of them without specifying the underlying Grothendieck topology.

2.3. **Thom functor and K-theory spaces.** In this subsection, we recall the motivic Thom functor formalism from [EHK+20, pp. 23]. Let Pic(SH) denotes the presheaf that takes a smooth S-scheme X to the $\mathbb{E}_\infty$-space of $\wedge$-invertible motivic spectra $\mathrm{Pic}(\mathrm{SH}(X))$. To a morphism of presheaves $\beta\colon \mathrm{B} \to \mathrm{Pic}(\mathrm{SH})$ we associate a motivic Thom spectrum $\mathrm{M}\beta \in \mathrm{SH}(k)$ by the colimit construction

$$\mathrm{M}(\beta\colon \mathrm{B} \to \mathrm{Pic}(\mathrm{SH})) := \operatorname*{colim}_{\substack{f\colon X\to S \text{ smooth} \\ b\in \mathrm{B}(X)}} f_\# \beta(b).$$

This defines a symmetric monoidal functor of ∞-categories $\mathrm{M}\colon \mathrm{PSh}(\mathrm{Sm}_k)_{/\,\mathrm{Pic(SH)}} \to \mathrm{SH}(k)$. Moreover, this functor inverts Nisnevich equivalences and even motivic equivalences over a motivic space; see [**?**, Proposition 16.9].

For a scheme X we denote by $\mathrm{Vect}(X)$ the ∞-groupoid of vector bundles over X. Taking the group completion and the Zariski localization of the presheaf $\mathrm{Vect} \in \mathrm{PSh}(\mathrm{Sm}_k)$ we get the Thomason-Trobaugh K-theory presheaf $\mathrm{K} := \mathrm{L}_{\mathrm{Zar}}(\mathrm{Vect}^{\mathrm{gp}})$ (see [TT90, Theorems 7.6 and 8.1]). Using this K-theory space we can construct the motivic J-homomorphism

$$\mathrm{J}\colon \mathrm{K} \to \mathrm{Pic}(\mathrm{SH}),\ \ E \mapsto \Sigma^\infty \mathrm{Th}(E).$$

This is a map of grouplike $\mathbb{E}_\infty$-spaces. Restricting the motivic Thom functor along the J-homomorphism, we obtain a symmetric monoidal functor of ∞-categories

$$\mathrm{M}\colon \mathrm{PSh}(\mathrm{Sm}_k)_{/\mathrm{K}} \to \mathrm{SH}(k).$$

Example 2.9.
(1) Let X be a smooth (ind-)scheme over S, let $E \ominus \mathcal{O}^n$ be a virtual vector bundle over X, and let $[E \ominus \mathcal{O}^n]\colon X \to \mathrm{K}$ be its class in the K-theory space. Then the corresponding Thom spectrum $\mathrm{M}([E \ominus \mathcal{O}^n] : X \to \mathrm{K})$ is given by $\Sigma^{\infty-(2n,n)}\mathrm{Th}_X(E)$. We denote this spectrum by $\mathrm{Th}_X(E \ominus \mathcal{O}^n) \in \mathrm{SH}(k)$ or $\mathrm{Th}(E \ominus \mathcal{O}^n)$ if X is clear from the context.
(2) The motivic Thom spectrum of the rank zero summand in K-theory $\iota\colon \mathrm{K}_{\mathrm{rk}=0} = \mathrm{K} \times_{\underline{\mathbb{Z}}} \{0\} \to \mathrm{K}$ is the Voevodsky algebraic cobordism spectrum

$$\mathrm{M}(\mathrm{K}_{\mathrm{rk}=0} \xrightarrow{\iota} \mathrm{K}) \simeq \mathrm{MGL}.$$

This can be shown using a motivic description of $\mathrm{K}_{\mathrm{rk}=0}$ in terms of Grassmanians together with some basic properties of the motivic Thom functor; see [BH20, Lemma 4.6].
(3) For a scheme X consider the ∞-groupoid of SL-oriented vector bundles $\mathrm{Vect}^{\mathrm{SL}}(X)$ (see e.g. [Ana20, §2]). Taking the group completion and the Zariski localization of the presheaf $\mathrm{Vect}^{\mathrm{SL}} \in \mathrm{PSh}(\mathrm{Sm}_k)$ we get the special linear K-theory presheaf K^{SL}. Applying the Thom functor to $\mathrm{K}^{\mathrm{SL}}_{\mathrm{rk}=0} = \mathrm{K} \times_{\underline{\mathbb{Z}}} \{0\}$ through the natural map $\mathrm{K}^{\mathrm{SL}}_{\mathrm{rk}=0} \to \mathrm{K}_{\mathrm{rk}=0}$, we obtain the special linear algebraic cobordism spectrum of Panin and Walter [PW23]

$$\mathrm{M}(\mathrm{K}^{\mathrm{SL}}_{\mathrm{rk}=0} \to \mathrm{K}) \simeq \mathrm{MSL}.$$

Remark 2.10. The presheaf K^{SL} from the last example is equivalent to the fiber of the determinant map $\det\colon \mathrm{K} \to \mathrm{Pic}$; see [EHK+20, Example 3.3.4]. We stress that K^{SL} is a presheaf of $\mathbb{E}_1$-spaces, while $\mathrm{K}^{\mathrm{SL}}_{\mathrm{rk}=0}$ is a presheaf of $\mathbb{E}_\infty$-spaces; see e.g. [EHK+20, Example A.0.6].

3. Motivic Thom spectrum MSL^c

In this section, we define the motivic Thom spectrum MSL^c with a natural $\mathbb{E}_\infty$-ring structure applying the motivic Thom functor to the K-theory space of oriented vector bundles. Then we deduce an explicit description of MSL^c in terms of appropriate ind-Grassmanians $\mathrm{Gr}^c(n,\infty)$, which give motivic models for the classyfying spaces BSL^c_n.

3.1. **Construction via the Thom functor.** Denote by $\mathrm{Vect}^{\mathrm{or}} \in \mathrm{PSh}(\mathrm{Sm}_S)$ the stack (see Notation 2.4)

$$X \mapsto \mathrm{Vect}^{\mathrm{or}}(X).$$

The direct sum of oriented vector bundles induces a natural $\mathbb{E}_1$-ring structure on the presheaf $\mathrm{Vect}^{\mathrm{or}}$. It admits a natural forgetful ring map to the stack of vector bundles $\mathrm{Vect}^{\mathrm{or}} \to \mathrm{Vect}$. On the other hand, since each special linear vector bundle is oriented in a natural way, we get the ring morphism from the stack of special linear vector bundles $\mathrm{Vect}^{\mathrm{SL}} \to \mathrm{Vect}^{\mathrm{or}}$.

Definition 3.1. The K-theory presheaf of oriented vector bundles $\mathrm{K}^{\mathrm{or}} \in \mathrm{PSh}(\mathrm{Sm}_S)$ is the Zariski localization of the group completion of the stack of oriented vector bundles $\mathrm{Vect}^{\mathrm{or}}$:

$$\mathrm{K}^{\mathrm{or}} := \mathrm{L}_{\mathrm{Zar}}(\mathrm{Vect}^{\mathrm{or},\mathrm{gp}}).$$

This presheaf admits a natural $\mathbb{E}_1$-ring structure. We also denote by $\mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} = \mathrm{K}^{\mathrm{or}} \times_{\underline{\mathbb{Z}}} \{0\}$ the respective rank zero presheaf, where $\mathrm{K}^{\mathrm{or}} \to \underline{\mathbb{Z}}$ is induced by the rank of underlying vector bundles.

Remark 3.2. The presheaf $\mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0}$ is a presheaf of $\mathbb{E}_\infty$-ring spaces. Indeed, already an even rank summand $\mathrm{K}^{\mathrm{or}}_{\mathrm{ev}} = \mathrm{K}^{\mathrm{or}} \times_{\underline{\mathbb{Z}}} \underline{2\mathbb{Z}}$ is an $\mathbb{E}_\infty$-ring, since the direct sum of oriented vector bundles is homotopy commutative on the subgroupoid of oriented vector bundles of even rank $\mathrm{Vect}^{\mathrm{or}} \times_{\mathbb{Z}} 2\mathbb{Z}$; see [BH21, Example 16.22] and [EHK+20, Example A.0.6] for the case of vector SL-bundles.

By construction, we have the following morphisms of presheaves of $\mathbb{E}_1$-spaces (resp. $\mathbb{E}_\infty$-spaces)

$$\mathrm{K}^{\mathrm{SL}} \to \mathrm{K}^{\mathrm{or}} \to \mathrm{K} \text{ (resp. } \mathrm{K}^{\mathrm{SL}}_{\mathrm{rk}=0} \to \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} \to \mathrm{K}_{\mathrm{rk}=0}). \tag{3.3}$$

We have the following motivic description of the rank zero summand of the K-theory presheaf of oriented vector bundles.

Lemma 3.4. *The canonical morphism* $\mathrm{BSL}^c \to \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0}$ *is a motivic equivalence of presheaves on* Sm_S*.*

Definition 3.5. The *motivic Thom spectrum of oriented vector bundles* is the motivic Thom spectrum associated with the composition $\mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} \to \mathrm{K}_{\mathrm{rk}=0} \xrightarrow{\iota} \mathrm{K}$:

$$\mathrm{MSL}^c_S := \mathrm{M}(\mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} \to \mathrm{K}) \in \mathrm{SH}(S).$$

We usually omit S from the notation and denote this spectrum simply by MSL^c.

Lemma 3.6. *The spectrum* MSL^c *admits a canonical* $\mathbb{E}_\infty$*-ring structure such that the following forgetful maps are morphisms of* $\mathbb{E}_\infty$*-motivic spectra:*

$$\mathrm{MSL} \to \mathrm{MSL}^c \to \mathrm{MGL}.$$

Proof. Applying $\mathrm{M}(-)$ to the sequence of maps 3.3 we obtain the result, since the motivic Thom functor is symmetric monoidal. □

Lemma 3.7. *The motivic spectrum* MSL^c *is stable under base change.*

Proof. This follows from [BW23, Proposition 4.10], since SL^c_n is an affine and finitely presented group scheme. □

3.2. **Description via Grassmanians.** We denote the Grassmannian of n-dimensional vector subbundles in $\mathcal{O}^m_S$ by $\mathrm{Gr}(n, m)$, and the tautological n-bundle over it by $\gamma_{n,m}$. Taking colimit along the closed immersions $\mathrm{Gr}(n, m) \hookrightarrow \mathrm{Gr}(n, m + 1)$, we obtain the ind scheme $\mathrm{Gr}(n, \infty)$ in $\mathrm{PSh}(\mathrm{Sm}_S)$. we denote the corresponding colimit of the tautological bundles by $\gamma_{n,\infty}$.

We denote the finite special linear Grassmannian $\mathrm{Gr}^c(n, m; N)$ as the complement of the zero section of the line bundle $\det(\gamma_{n,m}) \boxtimes \mathcal{O}(-2)$ over $\mathrm{Gr}(n, m) \times \mathbb{P}^N$. As shown in [BW25, Corollary 3.7.], BSL^c_n can be written as the following ind scheme

$$\mathrm{BSL}^c_n \simeq \operatorname*{colim}_{m,N} \mathrm{Gr}^c(n, m, N).$$

The pullback of $\gamma_{n,m}$ along the map $\mathrm{Gr}^c(n, m, N) \to \mathrm{Gr}(n, m)$ is denoted by $\gamma^c_{n,m,N}$. We denote the corresponding colimit of the oriented tautological bundles as $\gamma^c_{n,\infty}$ over BSL^c_n. Then, we can construct MSL^c as follows:

$$\mathrm{MSL}^c_n \cong \mathrm{Th}_{\mathrm{BSL}^c_n}(\gamma^c_{n,\infty}).$$
$$\mathrm{MSL}^c \cong \operatorname{colim} \Sigma^{\infty-(2n,n)} \mathrm{MSL}^c_n.$$

Consider the following commutative diagram

$$\begin{tikzcd}
 & \pi^*(\det(\gamma_{n,np}) \boxtimes \mathcal{O}(-2)) \arrow[r] \arrow[d] & \det(\gamma_{n,np}) \boxtimes \mathcal{O}(-2) \arrow[r] \arrow[d] & \mathcal{O}(-1) \boxtimes \mathcal{O}(-1) \arrow[d] \\
X \arrow[r, "f"'] & (\det(\gamma_{n,np}) \boxtimes \mathcal{O}(-2))^\circ \arrow[r, "\pi"'] & \mathrm{Gr}(n, np) \times \mathbb{P}^N \arrow[r, "(\mathrm{pl}, V_2)"'] & \mathbb{P}^{\tilde{N}} \times \mathbb{P}^{N'}.
\end{tikzcd}$$

Here pl denotes the Plücker embedding. As $\det(\gamma_{n,np})$ is the pullback of $\mathcal{O}(-1)$ on $\mathbb{P}^{\binom{np}{p}-1}$ along the Plücker embedding of $\mathrm{Gr}_{n,np}$ [Ful98, B.5.7], this map works as a model for the determinant map

$\det : \mathrm{BGL}_n \to \mathrm{B}\mathbb{G}_m$, as shown in [BW25, Proposition 3.6].
The map V_2 is the degree two Veronese embedding $\mathbb{P}^N \to \mathbb{P}^{\binom{N+2}{2}-1}$. An argument similar to the proof of [BW25, Proposition 3.6] gives that the degree two Veronese embedding can be a model for the squaring map $\mathrm{B}\mathbb{G}_m \to \mathrm{B}\mathbb{G}_m$. For any $X \in \mathrm{Sm}_S$, a map $f : X \to (\det(\gamma_{n,np}) \boxtimes \mathcal{O}(-2))^\circ =: \mathrm{Gr}^c(n, np, N)$ is equivalent to the following data:

- uniquely determined maps $X \to \mathrm{Gr}(n, np)$, and $X \to \mathbb{P}^N$.
- an isomorphism λ between the pullback of $\gamma_{n,np} \boxtimes \mathcal{O}(-2)$, and $\mathcal{L}^{\otimes 2}$, where $\mathcal{L}$ is the line bundle on X, uniquely (up to isomorphism) determined by the map $X \to \mathbb{P}^N$.

Using this we get the following maps of representable presheaves

$$\mathrm{Hom}_{Sm_S}(X, \mathrm{Gr}^c(n, np, N)) \times \mathrm{Hom}_{Sm_S}(X, \mathrm{Gr}^c(m, mp, N')) \to \mathrm{Hom}_{Sm_S}(X, \mathrm{Gr}^c(m+n, (m+n)p, NN' + N + N')$$
$$(f_1, \lambda_1, \mathcal{L}_1) \times (f_2, \lambda_2, \mathcal{L}_2) \to (f_1 \oplus f_2, \lambda_1 \otimes \lambda_2, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

In the colimit it induces the folowing multiplication map:

$$\mathrm{BSL}^c_n \times \mathrm{BSL}^c_m \to \mathrm{BSL}^c_{n+m}$$
$$\mathrm{MSL}^c_n \wedge \mathrm{MSL}^c_m \to \mathrm{MSL}^c_{n+m}$$

4. Thom classes and the universal property

Definition 4.1. A homotopy commutative motivic ring spectrum E is said to have (normalized) SL^c-*orientation* if for every rank n oriented vector bundle $p : V \to X$, there exists an element $\mathrm{th} \in E^{2n,n}(\mathrm{Th}(V))$, that satisfies the following properties:

- For an oriented vector bundle isomorphism $\phi : V \to V'$, we get
$$\mathrm{th}(V) = \phi^E(\mathrm{th}(V')).$$
- For a morphism $f : Y \to X$ of smooth varieties
$$f^E(\mathrm{th}(V)) = \mathrm{th}(f^*(V)).$$
- We have **Thom isomorphism**. In other words, the map $E^{*,*}(X) \to E^{*+2n,*+n}(\mathrm{Th}(V))$, taking $a \to p^*(a) \cup \mathrm{th}(V)$ is an $E^{*,*}(X)$-module isomorphism.
- Let $q_i : V_1 \oplus V_2 \to V_i$ denote the projection map for $i \in \{1, 2\}$. We have the following multiplicative property of Thom classes
$$q_1{}^*\big(\mathrm{th}(V_1)\big) \cup q_2{}^*\big(\mathrm{th}(V_2)\big) = \mathrm{th}(V_1 \oplus V_2) \in E^{*,*}\big(\mathrm{Th}(V_1 \oplus V_2)\big).$$
Here we have actually considered the maps induced by q_i's on Thom spaces.
- The unit of E gives rise to an element $1 \in E^{0,0}(\mathrm{Spec}(S)_+)$. Applying T-suspension on this, we get an element $\Sigma_T(1) \in E^{2,1}(T)$. For the trivial line bundle $\mathcal{O}_X$, we have $\mathrm{th}(\mathcal{O}_X) = \Sigma_T(1)$.

We have the following weak universal property of MSL^c, following Panin-Walter's treatment of MSL from [PW23, Section 5].

Lemma 4.2. *A monoid morphism* $\mathrm{MSL}^c \to E$ *in* $\mathrm{SH}(S)$ *determines a natural* SL^c*-orientation on* E*.*

Proof. For a monoid morphism $\mathrm{MSL}^c \xrightarrow{\phi} E$ in $\mathrm{SH}(S)$, and a given SL^c-bundle (V, L, λ) over $X \in \mathrm{Sm}_S$, we construct a thom class $\mathrm{th}^E(V, L, \lambda)$ as follows.
There is a canonical choice of thom class in the MSL^c-cohomology of BSL^c_n. Here BSL^c_n is considered as the presheaf sending a smooth S-scheme X to the groupoid of SL^c_n-torsors over X. Applying the motivic Thom functor along the inclusion $\mathrm{BSL}^c_n \to \mathrm{BSL}^c \to \mathrm{K}$, we get a map $\Sigma^{-2n,-n}\mathrm{M}(\mathrm{BSL}^c_n) \to \mathrm{MSL}^c$. Pulling back id along this morphism gives the canonical thom element. For an SL^c-bundle (V, L, λ) over $X \in \mathrm{Sm}_S$, by Yoneda lemma, we have a map $X \to \mathrm{BSL}^c_n$. Pulling back the previously constructed thom element along $\mathrm{M}(X \to \mathrm{BSL}^c_n)$ gives a thom element in $(\mathrm{MSL}^c)^{2n,n}(\mathrm{Th}(V))$. Post composing with the map $\mathrm{MSL}^c \to E$ gives a thom element in $E^{2n,n}(\mathrm{Th}(V))$. The first two, and the last properties of Eq. (4.1) are satisfied by construction, and the third property about mulplicativity of thom classes work as the map $\mathrm{MSL}^c \to E$ is a monoid map. □

Remark 4.3. In the proof of [PW23, Lemma 5.4], for a given special linear bundle over a scheme $X \in \mathrm{Sm}_S$, the authors construct a morphism $X \to \mathrm{BSL}_n$, via constructing a map to a specific model of finite special linear Grassmannians. This requires the use of Jouanolou's device. That is why they impose the condition of admitting an ample family of line bundles for all schemes in Sm_S. This condition seems to be redundant.

Following [PW23, Theorem 5.6, 5.7], we are going to use the following two special cases of Milnor exact sequence.

Theorem 4.4. [PW23, Theorem 5.8] *Let $M = \mathrm{colim}_n M_n$ be a sequential colimit of homotopy commutative $\mathbb{P}^1$-ring spectra. Then for any $\mathbb{P}^1$-spectrum E, we have the following short exact sequences of abelian groups.*

$$0 \to \lim{}^1 E^{2n-1,n}(M_n) \to \mathrm{Hom}_{\mathrm{SH}(S)}(M, E) \to \lim E^{2n,n}(M_n) \to 0$$

$$0 \to \lim{}^1 E^{4n-1,2n}(M_n \wedge M_n) \to \mathrm{Hom}_{\mathrm{SH}(S)}(M \wedge M, E) \to \lim E^{4n,2n}(M_n \wedge M_n) \to 0$$

Construction 4.5. We write

$${}^{<p>}\mathrm{MSL}^c_n := \mathrm{Th}(\gamma^c_{n,np,np})$$

This gives a sequence of $\mathbb{P}^1$-monoids ${}^{<1>}\mathrm{MSL}^c \hookrightarrow {}^{<2>}\mathrm{MSL}^c \hookrightarrow ....$, whose colimit is MSL^c. We get the monoidal structured by restricting the multiplicative structure described in Section 3.2 on specific finite oriented Grassmannians.

Let's define the monoid ${}^{\mathrm{fin}}\mathrm{MSL}^c$ as follows

$${}^{\mathrm{fin}}\mathrm{MSL}^c_n := {}^{\mathrm{n}}\mathrm{MSL}^c_n.$$

The monoidal inclusion ${}^{\mathrm{fin}}\mathrm{MSL}^c \hookrightarrow \mathrm{MSL}^c$ is an equivalence in $\mathrm{SH}(S)$ using a cofinality argument.

Lemma 4.6. *Every SL^c-orientation on E determines a morphism $\mathrm{MSL}^c \to E$ that preserves the Thom classes for every special linear bundle.*

More precisely, let E be given a (normalized) SL^c-orientation with Thom classes $\mathrm{th}^E(V, L, \lambda)$, for every SL^c-bundle (V, L, λ) over some smooth scheme X. There exists a morphism $\phi : \mathrm{MSL}^c \to E$, such that

$$\phi(\mathrm{th}^{\mathrm{MSL}^c}(V, L, \lambda)) = \mathrm{th}^E(V, L, \lambda)$$

However, the morphism ϕ might not be unique, and might not be monoidal.

Proof. For each p and n, we get a Thom class for the oriented bundle $\gamma^c_{n,np,np}$ over $\mathrm{Gr}^c(n, np, np)$. Let's denote it by $\mathrm{th}_{n,p} \in E^{2n,n}({}^{<p>}\mathrm{MSL}^c_n)$. Using compatibility of $\mathrm{th}_{n,p}$ along the inclusion ${}^{<p>}\mathrm{MSL}^c_n \hookrightarrow {}^{<p+1>}\mathrm{MSL}^c_n$, and the monoidal multiplication of ${}^{<p>}\mathrm{MSL}^c_n$, we get an element

$$\mathrm{th} \in \lim_{p,n} E^{2n,n}({}^{<p>}\mathrm{MSL}^c_n) \cong \lim_n E^{2n,n}({}^{n}\mathrm{MSL}^c_n).$$

From the first exact sequence of Eq. (4.4), there exists an element $\phi \in \mathrm{Hom}_{\mathrm{SH}(S)}(\mathrm{MSL}^c, E)$, that maps to th. However, obstructions to uniqueness of ϕ lie in the group $\lim^1 E^{2n-1,n}({}^{n}\mathrm{MSL}^c_n)$.

Consider the following diagram

$$\begin{tikzcd}
\Sigma^{\infty-(2n,n)}_{\mathbb{P}^1}({}^{<n>}\mathrm{MSL}^c_n) \wedge \Sigma^{\infty-(2n,n)}_{\mathbb{P}^1}({}^{<n>}\mathrm{MSL}^c_n) \arrow[r, "\mu^{\mathrm{fin}}_{nn}"] \arrow[d, "u_n \wedge u_n"'] \arrow[ddd, bend right=60, "\mathrm{th}_{n,n} \wedge \mathrm{th}_{n,n}"'] & \Sigma^{\infty-(4n,2n)}_{\mathbb{P}^1}({}^{<2n>}\mathrm{MSL}^c_{2n}) \arrow[d, "u_{2n}"] \arrow[ddd, bend left=60, "\mathrm{th}_{2n,2n}"] \\
{}^{\mathrm{fin}}\mathrm{MSL}^c \wedge {}^{\mathrm{fin}}\mathrm{MSL}^c \arrow[r, "\mu^{\mathrm{fin}}"'] \arrow[d, "\mathrm{inclusion}"] & {}^{\mathrm{fin}}\mathrm{MSL}^c \arrow[d, "\mathrm{inclusion}"'] \\
\mathrm{MSL}^c \wedge \mathrm{MSL}^c \arrow[r, "\mu_{\mathrm{MSL}^c}"'] \arrow[d, "\phi \wedge \phi"] & \mathrm{MSL}^c \arrow[d, "\phi"'] \\
E \wedge E \arrow[r, "\mu_E"] & E
\end{tikzcd}$$

Using multiplicativity, and functoriality of Thom classes, the outer perimeter, the top two squares, and the half circles of the previous diagram commute. We get that, for all n,

$$(\mu_E \circ (\phi \wedge \phi) - \phi \circ \mu_{\mathrm{MSL}^c}) \circ \mathrm{inclusion} \circ (u_n \wedge u_n) = 0$$

So, the image of $(\mu_E \circ (\phi \wedge \phi) - \phi \circ \mu_{\mathrm{MSL}^c})$ under the surjective map $\mathrm{Hom}_{\mathrm{SH}(S)}(\mathrm{MSL}^c \wedge \mathrm{MSL}^c, E) \to \lim E^{4n,2n}({}^{<n>}\mathrm{MSL}^c_n \wedge {}^{<n>}\mathrm{MSL}^c_n)$ goes to 0. From the second short exact sequence of Eq. (4.4), the obstruction to monoidality of ϕ, $(\mu_E \circ (\phi \wedge \phi) - \phi \circ \mu_{\mathrm{MSL}^c})$ comes from the group $\lim^1 E^{4n-1,2n}({}^{<n>}\mathrm{MSL}^c_n \wedge {}^{<n>}\mathrm{MSL}^c_n)$. □

5. Interpolation and splitting

In this section we prove several models for MSL^c in the category of MSL-modules. To be more precise, we show that this Thom spectrum is equivalent to the free MSL-module associated with $\Sigma^{\infty-1}_{\mathbb{P}^1}\mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(-2))$ and then split it into a direct sum $\mathrm{MSL} \oplus \Sigma^1_{\mathbb{P}^1}\mathrm{MGL}$. Along the way, we prove several important properties of the K-theory space K^{or}.

5.1. **Free MSL-module model.** First we recall basics about Quillen's plus construction in ∞-topoi (see e.g. [Hoy19]). Let $\mathcal{C}$ be an ∞-topos and let $f\colon X \to Y$ be a map in $\mathcal{C}$. Then f is called *acyclic* if it is an epimorphism in the categorical sense. The class of such maps is closed under composition, (co)base change, colimits, and finite products; see [BEH+21, Lemma 3.1]. Moreover, if $g \circ f$ and f are acyclic, then g is acyclic. We denote by $X \to X^+$ the final object in the ∞-category of acyclic maps out of X, and call the respective functor $X \mapsto X^+$ the *plus construction*. We need the following fact.

Lemma 5.1. *Suppose that $\mathcal{C}$ is an ∞-topos where π_1 preserves products, and that we have a diagram $X \to Z \leftarrow Y$ in $\mathcal{C}$ such that the objects Y and Z are equivalent to their plus constructions (e.g. grouplike $\mathbb{E}_1$-objects). Then the canonical morphism $(X \times_Z Y)^+ \to X^+ \times_Z Y$ is ∞-connective. In particular, if $\mathcal{C}$ is moreover hypercomplete, then the above map is an equivalence.*

Proof. We have the following commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} & X \times_Z Y \arrow[dl] \arrow[dr] & \\ (X \times_Z Y)^+ \arrow[rr] & & X^+ \times_Z Y. \end{tikzcd}$$

The left diagonal arrow is acyclic by definition of the plus construction, and the right diagonal map is acyclic being the base change of $X \to X^+$. It follows that the desired morphism is acyclic as well. By [Hoy19, Corollary 5], it remains to show that $\pi_1((X \times_Z Y)^+)$ is hypoabelian (i.e. it has no nontrivial perfect subgroups). This holds by the assumption on π_1, since the map $X \times_Z Y \to (X \times_Z Y)^+$ kills the entire perfect core of $\pi_1(X \times_Z Y)$; see Lemma 3 and Theorem 8 of *loc.cit.*. $\square$

Now we return to the setup $\mathcal{C} = \mathrm{PSh}(\mathrm{Sm}_S)$. Consider a presheaf of stable oriented vector bundles $\mathrm{sVect}^{\mathrm{or}} \in \mathrm{PSh}(\mathrm{Sm}_S)$ that is defined as the colimit of the sequence

$$\mathrm{Vect}^{\mathrm{or}} \xrightarrow{\oplus \mathcal{O}} \mathrm{Vect}^{\mathrm{or}} \xrightarrow{\oplus \mathcal{O}} \mathrm{Vect}^{\mathrm{or}} \xrightarrow{\oplus \mathcal{O}} \cdots .$$

By definition, the canonical map $\mathrm{Vect}^{\mathrm{or}} \to \mathrm{Vect}^{\mathrm{or,gp}}$ factor through $\mathrm{sVect}^{\mathrm{or}}$. We also denote by sVect the presheaf of stable vector bundles defined by a similar colimit. The forgetful morphism $\mathrm{Vect}^{\mathrm{or}} \to \mathrm{Vect}$ induces $\mathrm{sVect}^{\mathrm{or}} \to \mathrm{sVect}$.

Lemma 5.2. *The morphism of presheaves of $\mathbb{E}_1$-spaces $\mathrm{sVect}^{\mathrm{or}} \to \mathrm{Vect}^{\mathrm{or,gp}}$ is Quillen's plus construction.*

Proof. Consider $\mathrm{Vect}^{\mathrm{or}}$ as a module over the substack of oriented vector bundles of even rank $\mathrm{Vect}^{\mathrm{or}}_{\mathrm{ev}}$. Then the statement is a content of a slightly modified version (see [BEH+21, Proposition 3.2]) of the McDuff–Segal group completion theorem [MS76]. $\square$

Consider the morphism $\sqrt{\det}\colon \mathrm{Vect}^{\mathrm{or}} \to \mathrm{Pic}$ that maps an oriented vector bundle (E, L, λ) to the line bundle L. This is a morphism of the presheaves of $\mathbb{E}_1$-spaces, which induces arrows $\mathrm{sVect}^{\mathrm{or}} \to \mathrm{Pic}$, $\mathrm{Vect}^{\mathrm{gp}} \to \mathrm{Pic}$, and $\mathrm{K}^{\mathrm{or}} \to \mathrm{Pic}$. Slightly abusing notation, we denote all these maps also by $\sqrt{\det}$.

Proposition 5.3. *Let S be a base scheme. Then there is a cartesian square in* $\mathrm{PSh}(\mathrm{Sm}_S)$

$$\begin{tikzcd} \mathrm{K}^{\mathrm{or}} \arrow[r, "\sqrt{\det}"] \arrow[d] & \mathrm{Pic} \arrow[d, "\cdot 2"] \\ \mathrm{K} \arrow[r, "\det"] & \mathrm{Pic}\,. \end{tikzcd}$$

Proof. First, note that by definition, the stack $\mathrm{Vect}^{\mathrm{or}}$ is the pullback of the diagram $\mathrm{Vect} \xrightarrow{\det} \mathrm{Pic} \xleftarrow{\cdot 2} \mathrm{Pic}$ with the arrows $\mathrm{Pic} \leftarrow \mathrm{Vect}^{\mathrm{or}} \to \mathrm{Vect}$ being $\sqrt{\det}$ and forgetful map, respectively. Then by universality of colimits we see that the following square is cartesian

$$\begin{tikzcd} \mathrm{sVect}^{\mathrm{or}} \arrow[r, "\sqrt{\det}"] \arrow[d] & \mathrm{Pic} \arrow[d, "\cdot 2"] \\ \mathrm{sVect} \arrow[r, "\det"] & \mathrm{Pic}\,. \end{tikzcd}$$

Now using Lemma 5.1 ($\mathrm{PSh}(\mathrm{Sm}_S)$ is a hypercomplete ∞-topos where π_1 preserves products) and Lemma 5.2 (together with a similar equivalence $\mathrm{sVect}^+ \xrightarrow{\simeq} \mathrm{Vect}^{\mathrm{gp}}$) we obtain that the limit

$$\lim(\mathrm{Vect}^{\mathrm{gp}} \xrightarrow{\det} \mathrm{Pic} \xleftarrow{\cdot 2} \mathrm{Pic})$$

is given by the group completion $\mathrm{Vect}^{\mathrm{or,gp}}$. Moreover, morphisms out of $\mathrm{Vect}^{\mathrm{or,gp}}$ in this pullback square are $\sqrt{\det}$ and forgetful one, as always. In order to proceed to K-theory spaces, it remains to notice that the Zariski localization commutes with finite limits. □

Remark 5.4. In papers [HJNY22, Remark 7.11] and [BW25, §6.3] the authors define the Thom spectrum MSL^c as the Thom spectrum associated with the rank zero part of

$$\lim(\mathrm{K} \xrightarrow{\det} \mathrm{Pic} \xleftarrow{\cdot 2} \mathrm{Pic}).$$

This definition is equivalent to ours according to the previous proposition.

Corollary 5.5. *There is a fiber sequence $\mathrm{K}^{\mathrm{SL}} \to \mathrm{K}^{\mathrm{or}} \xrightarrow{\sqrt{\det}} \mathrm{Pic}$ in $\mathrm{PSh}(\mathrm{Sm}_S)$, where the first map is a forgetful morphism. A similar fiber sequence exists on the level of rank zero presheaves.*

Proof. This follows from the previous proposition and the fiber sequence $\mathrm{K}^{\mathrm{SL}} \to \mathrm{K} \xrightarrow{\det} \mathrm{Pic}$ (see [EHK+20, Example 3.3.4]). □

Consider the morphism of groupoids $s : \mathrm{Pic}(X) \to \mathrm{Vect}^{\mathrm{or}}(X)$, taking a line bundle L over $X \in \mathrm{Sm}_S$ to $(L^{\otimes 2}, L, \mathrm{id})$. This gives a section to the map $\sqrt{\det} : \mathrm{Vect}^{\mathrm{or}} \to \mathrm{Pic}$. After group completion, and Zariski localization, this gives a section of $\sqrt{\det}\colon \mathrm{K}^{\mathrm{or}} \to \mathrm{Pic}$. Restricting on the rank zero summand, we get a section $s_{\mathrm{rk}=0} : \mathrm{Pic} \to \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0}$ of $\sqrt{\det}_{\mathrm{rk}=0} : \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} \to \mathrm{Pic}$.

Proposition 5.6. *For any base scheme S, the following morphism of presheaves over S is an equivalence*

$$\mathrm{K}^{\mathrm{SL}} \times \mathrm{Pic} \xrightarrow{\mathrm{id} \times s} \mathrm{K}^{\mathrm{SL}} \times \mathrm{K}^{\mathrm{or}} \xrightarrow{\mathrm{act}} \mathrm{K}^{\mathrm{or}}.$$

Here act *denotes the action as K^{SL}-modules. For the rank zero parts of the previous presheaves, this leads to the following equivalence $\mathrm{K}^{\mathrm{SL}}_{\mathrm{rk}=0} \times \mathrm{Pic} \cong \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0}$.*

Proof. This follows from the previous corollary by the usual topological argument (see [**?**, Lemma 3.12] for a discussion in general ∞-categories with finite limits). □

Theorem 5.7. *Let S be a base scheme. Then the morphism if motivic spectra over S*

$$\mathrm{MSL} \otimes \Sigma^{-1}_{\mathbb{P}^1}\mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(-2)) \xrightarrow{\mathrm{id} \otimes \Sigma^{-1}_{\mathbb{P}^1} \mathrm{th}(\mathcal{O}(-2))} \mathrm{MSL} \otimes \mathrm{MSL}^c \xrightarrow{\mathrm{act}} \mathrm{MSL}^c$$

is an equivalence of MSL*-modules. Here* act *is the action on the* MSL*-module (see Lemma 3.6), and* $\mathrm{th}(\mathcal{O}(-2))$ *is the Thom class of $\mathcal{O}(-2)$ with an obvious orientation in MSL^c-cohomology.*

Proof. It is easy to see that the above morphism is a map of MSL-modules. By the previous proposition the following composition

$$\mathrm{K}^{\mathrm{SL}}_{\mathrm{rk}=0} \times \mathrm{Pic} \xrightarrow{\mathrm{id} \times s_{\mathrm{rk}=0}} \mathrm{K}^{\mathrm{SL}}_{\mathrm{rk}=0} \times \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} \xrightarrow{\mathrm{act}} \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0}$$

is an equivalence. Moreover, a straightforward verification shows that this map is an equivalence in $\mathrm{PSh}(\mathrm{Sm}_S)_{/K}$ (where Pic viewed as an object of the slice ∞-category via $\mathrm{Pic} \xrightarrow{s_{\mathrm{rk}=0}} \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} \to \mathrm{K}$). Hence, applying the motivic Thom functor we obtain an isomorphism in $\mathrm{SH}(S)$

$$\mathrm{MSL} \otimes \mathrm{M}(\mathrm{Pic} \xrightarrow{s_{\mathrm{rk}=0}} \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} \to \mathrm{K}) \xrightarrow{\mathrm{id} \otimes \mathrm{M}(s_{\mathrm{rk}=0})} \mathrm{MSL} \otimes \mathrm{MSL}^c \xrightarrow{\mathrm{act}} \mathrm{MSL}^c.$$

It remains to identify the Thom spectrum of $\mathrm{Pic} \to \mathrm{K}$. For this purpose, consider the isomorphism of Thom spectra induced by the usual motivic equivalence $\mathcal{O}(-1)\colon \mathbb{P}^\infty \to \mathrm{Pic}$ (Thom functor inverts motivic equivalences over K by [BH21, Proposition 16.9.(2) and Remark 16.11]). The composition

$$\Sigma^{-1}_{\mathbb{P}^1}\mathrm{Th}(\mathcal{O}(-2)) \cong \mathrm{M}(\mathbb{P}^\infty \xrightarrow{\mathcal{O}(-2)\ominus\mathcal{O}} \mathrm{K}) \xrightarrow{\mathrm{M}(\mathcal{O}(-1))} \mathrm{M}(\mathrm{Pic} \xrightarrow{s_{\mathrm{rk}=0}} \mathrm{K}^{\mathrm{or}}_{\mathrm{rk}=0} \to \mathrm{K}) \xrightarrow{\mathrm{M}(s_{\mathrm{rk}=0})} \mathrm{MSL}^c$$

coincides with the $\mathbb{P}^1$-desuspension of the Thom class of $\mathcal{O}(-2)$ in MSL^c-cohomology by the construction of the Thom classes. This concludes the proof. □

5.2. **Decomposition into the direct sum.** In order to split the obtained free MSL-module into a direct sum, we need to recall a few facts about the stable ∞-category of MSL-modules. Here we follow exposition of [DFJK21, §7].

The motivic ∞-category $\mathrm{Mod}_{\mathrm{MSL}}(\mathrm{SH}(S))$ admits a canonical SL-orientation in the strong sense (see Definition 7.3 and Remark 7.6 of *loc.cit.*). Essentially, this means that we have coherent data of multiplicative Thom isomorphisms for virtual vector SL-bundles. Consider a virtual vector bundle ξ over S. Then its class in the K-theory space can be expressed as a sum of $\xi - \det(\xi)$ and $\det(\xi)$. The first one admits a canonical structure of a virtual vector SL-bundle. Then we have

$$\mathrm{MSL} \otimes \mathrm{Th}(\xi) \cong \mathrm{MSL} \otimes \mathrm{Th}(\xi - \det(\xi)) \otimes \mathrm{Th}(\det(\xi)) \cong \mathrm{MSL} \otimes \Sigma^{\mathrm{rk}(\xi)}_{\mathbb{P}^1}\mathrm{Th}(\det(\xi)),$$

where the last identification is given by the Thom isomorphism. We call this operation *reduction to the determinant* (the induced isomorphism in cohomology was first treated in [Ana20, Corollary 1]). This is most interesting in the case of ξ being a sum of two line bundles $L \oplus M$. Then we obtain the following equivalences of MSL-modules

$$\mathrm{MSL} \otimes \mathrm{Th}(L) \otimes \mathrm{Th}(M) \cong \mathrm{MSL} \otimes \Sigma^2_{\mathbb{P}^1}\mathrm{Th}(\det(L \oplus M)) \cong \mathrm{MSL} \otimes \Sigma^2_{\mathbb{P}^1}\mathrm{Th}(L \otimes M).$$

Lemma 5.8 (c.f. [DFJK21, 7.13])**.** *Let X be a smooth ind-scheme over S and let L be a line bundle over X. Then there is an isomorphism of* MSL*-modules*

$$\mathrm{MSL} \otimes \mathrm{Th}_X(L^{\otimes 2}) \cong \mathrm{MSL} \otimes \Sigma^{\infty+1}_{\mathbb{P}^1} X_+.$$

Proof. Suppose first that X is a smooth S-scheme and put $M = L^\vee$ in the previous equation. Applying Ananyevskiy's (unstable) motivic equivalence $\mathrm{Th}(L) \cong \mathrm{Th}(L^\vee)$ (see [Ana20, Lemma 4.1]) and taking the reduction to the determinant, we obtain an equivalence over X. We then use the smooth projection formula to transfer the result along the structure morphism $X \to S$. The general case follows by passing to the colimit. □

For the sake of completeness we also recover the following more general fact.

Proposition 5.9. *Let X be a smooth ind-scheme over S and let (E, L, λ) be an oriented vector bundle over X. Then there is an isomorphism of* MSL*-modules*

$$\mathrm{MSL} \otimes \mathrm{Th}_X(E) \cong \mathrm{MSL} \otimes \Sigma^{\infty+\mathrm{rk}(E)}_{\mathbb{P}^1} X_+.$$

Proof. Taking the reduction to the determinant we can replace E by $\det(E) \cong L^{\otimes 2}$. Then the result follows from the previous lemma. □

Remark 5.10. Note, however, that these Thom isomorphisms are not multiplicative in the following precise sense. Assume that E_1 and E_2 are oriented vector bundles over S and denote by $\mathrm{th}^{\mathrm{tw}}(E_i) \in \mathrm{MSL}^{2\mathrm{rk}(E_i),\mathrm{rk}(E_i)}(\mathrm{Th}(E_i))$ the 'Thom class' that induced by the constructed Thom isomorphism (through composition with the unit map $\mathbb{1} \to \mathrm{MSL}$). Then in general we have

$$\mathrm{th}^{\mathrm{tw}}(E_1 \oplus E_2) \neq \mathrm{th}^{\mathrm{tw}}(E_1) \times \mathrm{th}^{\mathrm{tw}}(E_2).$$

Here by $\neq$ we mean that these elements do not correspond to each other under the identification $\mathrm{Th}(E_1 \oplus E_2) \cong \mathrm{Th}(E_1) \otimes \mathrm{Th}(E_2)$. Indeed, if the above equality holds, then replacing MSL with MGL via $\mathrm{MSL} \to \mathrm{MGL}$, we obtain a similar formula in $\mathrm{MGL}^{2*,*}(\mathrm{Th}(E_1 \oplus E_2))$. But one can check that the following is valid for a vector bundle V

$$\mathrm{th}^{\mathrm{tw}}(V) = \frac{[-1](c_1(\det(V)))}{c_1(\det(V))} \cdot \mathrm{th}(V) \in \mathrm{MGL}^{2\mathrm{rk}(V),\mathrm{rk}(V)}(\mathrm{Th}(V)),$$

where $[-1](-)$ is the formal inverse with respect to the formal group law of MGL. This additional correction appear because of the equivalence $\mathrm{Th}(\det(V)) \simeq \mathrm{Th}(\det(V)^\vee)$; see [Zol24, Claim (3) in the proof of Theorem 3.16]. Now it is easy to see that this additional multiplier is not multiplicative. This provides a negative answer to Ananyevskiy's question [Ana20, Remark 4.4] (the isomorphism he constructed is slightly different from ours, but the same argument works in his context as well). We also remark that by [Hau23, Proposition 3.2.8] this problem disappears after η-periodization.

Now we are ready to formulate the main result of this subsection.

Theorem 5.11. *Let S be a base scheme. Then there is an isomorphism of* MSL*-modules*

$$\mathrm{MSL}^c \cong \mathrm{MSL} \oplus \Sigma^1_{\mathbb{P}^1}\mathrm{MGL}.$$

Proof. Combining Theorem 5.7 with Lemma 5.8 we get the following equivalence
$$\mathrm{MSL}^c \cong \mathrm{MSL} \otimes \Sigma^\infty_{\mathbb{P}^1} \mathbb{P}^\infty_+.$$
The resulting MSL-module splits into a direct sum of MSL and $\mathrm{MSL} \otimes \Sigma^\infty_{\mathbb{P}^1} \mathbb{P}^\infty$. In turn, the latter is isomorphic to the $\mathbb{P}^1$-suspension of MGL via the free MSL-module model of the algebraic cobordism spectrum [Nan23, Theorem 1.1] (see also [Zol24, Corollary 3.3]). ☐

Remark 5.12. Let us comment the arrows that appear in the above equivalence. A direct (but tedious) check shows that the inclusion of the direct summand MSL coincide with the canonical morphism $\mathrm{MSL} \to \mathrm{MSL}^c$ from Lemma 3.6.

We can rephrase above facts saying that there is a cofiber sequence of MSL-modules
$$\mathrm{MSL} \to \mathrm{MSL}^c \xrightarrow{c_1(\det \gamma)} \Sigma^1_{\mathbb{P}^1} \mathrm{MGL} \xrightarrow{0} \Sigma \mathrm{MSL}.$$
We do not know how to prove this cofiber sequence directly. Furthermore, we see no a priori reason why the boundary morphism is zero.

As a corollary we also re-prove Haution's result [Hau23, Proposition 3.2.8] (see also [BW25, Corollary 1.3]).

Corollary 5.13. *Let S be a base scheme. Then the map* $\mathrm{MSL} \to \mathrm{MSL}^c$ *induces an equivalence of $\mathbb{E}_\infty$-ring spectra over S*
$$\mathrm{MSL}[\eta^{-1}] \xrightarrow{\simeq} \mathrm{MSL}^c[\eta^{-1}].$$

Proof. Indeed, it is well-known that η acts trivially on GL-oriented spectra. Hence, $\mathrm{MGL}[\eta^{-1}] = 0$ and we have $\mathrm{MSL}^c[\eta^{-1}] \cong \mathrm{MSL}[\eta^{-1}] \oplus \Sigma^1_{\mathbb{P}^1} \mathrm{MGL}[\eta^{-1}] = \mathrm{MSL}[\eta^{-1}]$. ☐

6. Computation of the homotopy groups

In this section we apply the obtained splitting to the computation of the homotopy groups (and sheaves) of MSL^c over a field. We deduce formulas for the first few homotopy modules of MSL^c and compute its geometric diagonal (away from the exponential characteristic).

6.1. **Homotopy modules.** In this subsection we work over a field k. Recall that for a motivic spectrum $\mathcal{E} \in \mathrm{SH}(k)$ symbol $\underline{\pi}_n(\mathcal{E})_m$ denotes the Nisnevich sheaf associated with the presheaf
$$U \mapsto [\Sigma^n \Sigma^\infty_{\mathbb{P}^1} U_+, \Sigma^m_{\mathbb{G}_m} \mathcal{E}].$$
We also put $\underline{\pi}_n(\mathcal{E})_* := \bigoplus_{m \in \mathbb{Z}} \underline{\pi}_n(\mathcal{E})_m$. Such an object admits a canonical structure of a homotopy module in the sense of Morel [Mor03, Definition 5.2.4]. The category of homotopy modules is equivalent to the heart of the homotopy t-structure on $\mathrm{SH}(k)$. We assume that the reader is familiar with these notions; see [Hoy15, §2.1] for an overview.

Proposition 6.1. *We have $\underline{\pi}_n(\mathrm{MSL}^c)_* = 0$ for negative n. The unit map $\mathbb{1} \to \mathrm{MSL}^c$ induces an isomorphism*
$$\underline{\pi}_0(\mathbb{1})_* \cong \underline{\pi}_0(\mathrm{MSL}^c)_*,$$
where the left side can be identified with the unramified Milnor–Witt K-theory $\underline{\mathrm{K}}^{\mathrm{MW}}_*$ by Morel's computation; see [Mor12, Theorem 1.23].

Proof. The connectivity of MSL^c with respect to the homotopy t-structure is rather standard; see [Hoy15, Lemma 3.1]. The isomorphism for the zeroth homotopy module follows from our splitting (see previous section), connectivity of MGL, and analogous result about MSL (see [BH21, Example 16.35]). ☐

Now we turn to the first and second homotopy modules. Since for that we use computations with the slices of the algebraic cobordism spectrum, we need to invert the exponential characteristic. Following [?] we denote by kq the very effective cover of the hermitian K-theory spectrum. It admits a canonical morphism $\mathrm{MSL}^c \to \mathrm{kq}$ of $\mathbb{E}_\infty$-ring spectra; see [HJNY22, Remark 7.11].

Proposition 6.2. *Assume that the field k has characteristic zero. Then the morphisms* $\mathrm{MSL}^c \to \mathrm{kq}$ *and* $\mathrm{MSL}^c \to \Sigma^1_{\mathbb{P}^1} \mathrm{MGL}$ *induce an isomorphism*
$$\underline{\pi}_1(\mathrm{MSL}^c)_* \cong \underline{\pi}_1(\mathrm{kq})_* \oplus \underline{\mathrm{K}}^{\mathrm{M}}_{*+1},$$
where $\underline{\mathrm{K}}^{\mathrm{M}}_*$ is the unramified Milnor K-theory. If k is a field of characteristic $p > 0$, the same result holds after inverting p.

Proof. Consider first the case of $\mathrm{char}(k) = 0$. By our splitting we have

$$\underline{\pi}_1(\mathrm{MSL}^c)_* \cong \underline{\pi}_1(\mathrm{MSL})_* \oplus \underline{\pi}_0(\mathrm{MGL})_{*+1} \cong \underline{\pi}_1(\mathrm{kq})_* \oplus \underline{\mathrm{K}}^{\mathrm{M}}_{*+1},$$

where the last identifications follows from [Nan23, Theorem 1.4] and [Hoy15, Remark 3.10]. The projection onto the direct summand $\underline{\pi}_1(\mathrm{kq})_*$ induced by the desired map of spectra since the isomorphism $\underline{\pi}_1(\mathrm{MSL})_* \cong \underline{\pi}_1(\mathrm{kq})_*$ is obtained via $\mathrm{MSL} \to \mathrm{MSL}^c \to \mathrm{kq}$. The same proof works in characteristic $p > 0$ away from p by [Nan23, Remark 4.8]. □

We end this subsection with a similar description of the second homotopy module.

Proposition 6.3. *Assume that the field k has characteristic zero. Then the morphisms $\mathrm{MSL}^c \to \mathrm{kq}$ and $\mathrm{MSL}^c \to \Sigma^1_{\mathbb{P}^1}\mathrm{MGL}$ induce an isomorphism*

$$\underline{\pi}_2(\mathrm{MSL}^c)_* \cong \underline{\pi}_2(\mathrm{kq})_* \oplus \underline{\pi}_1(\mathrm{MGL})_{*+1},$$

where $\underline{\pi}_1(\mathrm{MGL})_m$ fits into the following exact sequence of Nisnevich sheaves

$$\underline{\mathrm{K}}^{\mathrm{M}}_{m+1} \to \underline{\pi}_1(\mathrm{MGL})_m \to \underline{\pi}_1(\mathrm{H}\mathbb{Z})_m \to 0.$$

If k is a field of characteristic $p > 0$, the same result holds after inverting p.

Proof. The proof is similar to the proof of the previous proposition. The isomorphism

$$\underline{\pi}_2(\mathrm{MSL})_*[1/e] \cong \underline{\pi}_2(\mathrm{kq})_*[1/e]$$

will be given in [NRZ] (here e is the exponential characteristic of k). The outlined description of $\underline{\pi}_1(\mathrm{MGL})_m[1/e]$ is presented in [Spi10, Corollary 7.6]. □

Remark 6.4. The above answers are given in terms of the first homotopy modules of the very effective hermitian K-theory. They can be computed via the slice spectral sequence; see [ARØ20, Proposition 4.4 and below] for a discussion of $\underline{\pi}_1(\mathrm{kq})_*$.

6.2. **Geometric diagonal.** We show that the complex realization of MSL^c is Stong's complex-spin bordism [Sto67]. Using Stong's computation of $\mathrm{M}\Sigma$, we get the following description of the geometric diagonal of MSL^c.

Proposition 6.5. *The complex realization of MSL^c is $\mathrm{M}\Sigma$.*

Proof. We have the following pullback (stably also pushout) diagram

$$\begin{tikzcd}
\mathrm{BSL}^c_n \arrow[r] \arrow[d] & \mathrm{BGL}_n \arrow[d] \\
\mathrm{B}\mathbb{G}_m \arrow[r, "(-)^2"] & \mathrm{B}\mathbb{G}_m
\end{tikzcd}$$

Looking at the complex points, we have the following commutative diagram, where the rows are fiber sequences.

$$\begin{tikzcd}
\mathrm{K}(\mathbb{Z}_2,1) \arrow[r] \arrow[d, "="] & \mathrm{BSL}^c_n(\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{BGL}_n(\mathbb{C}) \cong \mathrm{BU}_n \arrow[d] \\
\mathrm{K}(\mathbb{Z}_2,1) \arrow[r] & S^1 \cong \mathrm{B}\mathbb{G}_m \arrow[r, "(-)^2"] & \mathrm{B}\mathbb{G}_m(\mathbb{C}) \cong S^1
\end{tikzcd}$$

Now, we get morphism of fiber sequences

$$\begin{tikzcd}
\mathrm{K}(\mathbb{Z}_2,1) \arrow[r] \arrow[d] & \mathrm{B}\widetilde{\mathrm{U}}_n \arrow[r] \arrow[d] & \mathrm{BU}_n \arrow[d, "="] \\
\mathrm{K}(\mathbb{Z}_2,1) \arrow[r] \arrow[d, "="] & \mathrm{BSL}^c_n(\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{BGL}_n(\mathbb{C}) \cong \mathrm{BU}_n \arrow[d] \\
\mathrm{K}(\mathbb{Z}_2,1) \arrow[r] & S^1 \cong \mathrm{B}\mathbb{G}_m(\mathbb{C}) \arrow[r, "(-)^2"] & \mathrm{B}\mathbb{G}_m(\mathbb{C}) \cong S^1.
\end{tikzcd}$$

As defined in Eq. (B.5), $\widetilde{\mathrm{U}}_n$ is the kernel of the following map

$$\mathrm{U}(n) \times \mathrm{U}(1) \to \mathrm{U}(1),\ (u,t) \mapsto t^{-2} \cdot \det(u).$$

Following a similar argument as the proof of Theorem B.6, the map $\mathrm{B}\widetilde{\mathrm{U}}_n \to \mathrm{BSL}^c_n(\mathbb{C})$ is an equivalence. The required equivalence $\mathrm{MSL}^c(\mathbb{C}) \xleftarrow{\cong} \mathrm{M}\widetilde{\mathrm{U}} \xrightarrow{\cong} \mathrm{M}\Sigma$ follows from construction.

□

Our approach of getting a description of the geometric diagonal of MSL^c closely follows the second named author's work on the same for MSL (see [Zol24]). For the reader's convenience, we have summarized the results we use in Section C.

Proposition 6.6. *Let k be a local Dedekind domain (i.e. a field or a discrete valuation ring) and let $e \neq 2$ be the exponential characteristic of the residue field of k. Then there is an isomorphism of rings*

$$\pi_{2*,*}(\mathrm{MSL}^c)\Big/{}_{\eta}\pi_{2*,*}(\mathrm{MSL}^c)\,[1/e] \cong \mathrm{W}(k)[1/e][y_4, y_8, \dots], \text{ where } |y_i| = (2i, i).$$

Here ${}_{\eta}\pi_{2*,*}(\mathrm{MSL})$ denotes the annihilator of η in $\pi_{2*,*}(\mathrm{MSL}^c)$.

Proof. The map $\eta : \pi_{*,*}(\mathrm{MGL}) \to \pi_{*,*}(\mathrm{MGL})$ is 0, see [Ana15, Corollary 1]. Therefore, from our splitting (Eq. (5.11)), we have that

$$\pi_{2*,*}(\mathrm{MSL}^c)\Big/{}_{\eta}\pi_{2*,*}(\mathrm{MSL}^c) \cong \pi_{2*,*}(\mathrm{MSL})\Big/{}_{\eta}\pi_{2*,*}(\mathrm{MSL})$$

The rest follows from Eq. (C.2). □

Theorem 6.7. *Suppose that k is a local Dedekind domain and $e \neq 2$ is the exponential characteristic of k.*

(1) The subgroup $\mathrm{I}_{\mathrm{MSL}^c}[1/e]$ is a graded ideal and there is an isomorphism of rings

$$\pi_{2*,*}(\mathrm{MSL}^c)\Big/\mathrm{I}_{\mathrm{MSL}^c}(k)\,[1/e] \cong \pi_{2*}(\mathrm{M}\Sigma)[1/e].$$

(2) The following diagram is a pullback of graded rings

$$\begin{tikzcd}
\pi_{2*,*}(\mathrm{MSL}^c)[1/e] \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \pi_{2*}(\mathrm{M}\Sigma)[1/e] \arrow[d, two heads] \\
\mathrm{W}(k)[1/e][y_4, y_8, \dots] \arrow[r, two heads, "\overline{\mathrm{rk}}"] & \mathbb{Z}/2[y_4, y_8, \dots],
\end{tikzcd}$$

where the left vertical arrow is a map from the previous proposition, and the right one is the quotient by the annihilator of $\eta_{\mathrm{top}} \in \pi_{2*}(\mathrm{M}\Sigma)$.

7. Slices and the category of modules

In this section we study the stable category of 2-inverted MSL^c-modules over a field (away from the characteristic). For that we compute the slices of MSL^c and use them to compare $(\mathrm{MSL}^c)^{\wedge}_{\eta}[1/2]$ with $\mathrm{MGL}[1/2]$. Then we categorify this comparison getting the description of the plus-part. After that we obtain a characterization of the minus-part via the real étale realization.

7.1. **Comparison with MGL.** In this section we use Voevodsky's slice filtration; see [RØ16, §2] for an overview. We denote by $\mathrm{s}_q(-)$ the q-th slice functor. Giving a spectrum $\mathcal{E} \in \mathrm{SH}(S)$, the slice filtration of $\mathcal{E}$ induces a spectral sequence

$$E^1_{p,q,n}(\mathcal{E}) = \pi_{p,n}(\mathrm{s}_q(\mathcal{E})).$$

Simply by construction, this spectral sequence conditionally converges to the homotopy groups of a motivic spectrum $\mathrm{sc}(\mathcal{E})$, which is called *slice completion of $\mathcal{E}$* (see [RSØ19, Definition 3.1]). Now we consider the case of MSL^c.

Proposition 7.1. *Let S be a base scheme. We have $\mathrm{sc}(\mathrm{MSL}^c) \cong (\mathrm{MSL}^c)^{\wedge}_{\eta}$, i.e. there is a conditionally convergent slice spectral sequence*

$$E^1_{p,q,n}(\mathrm{MSL}^c) = \pi_{p,n}(\mathrm{s}_q(\mathrm{MSL}^c)) \Longrightarrow \pi_{p,n}((\mathrm{MSL}^c)^{\wedge}_{\eta}).$$

Proof. By effectivity of MSL^c and [RSØ19, Lemma 3.13] it is enough to show that MSL^c/η is η-complete. By our splitting, we need to prove that MSL/η and $\Sigma^{2,1}\mathrm{MGL}/\eta$ are η-complete. The second statement is well-known (note that $\mathrm{MGL}/\eta \cong \mathrm{MGL} \oplus \Sigma^{2,1}\mathrm{MGL}$) and the first one is proven in [NRZ]. □

Now we compute the slice filtration of MSL^c over a field away from the characteristic.

Theorem 7.2. *Suppose that k is a field of characteristic zero. Then the slices of* MSL^c *are given by*

$$\mathrm{s}_q(\mathrm{MSL}^c) \cong \Sigma^q_{\mathbb{P}^1}\mathrm{H}\mathbb{Z}^{p(q)} \oplus \bigoplus_{n=1}^{q} \Sigma^q_{\mathbb{P}^1}\mathrm{H}(\mathbb{Z}/2)^{f(n)}[-n],$$

where $p(-)$ is the partition function. In positive characteristic the same result holds after inverting the characteristic.

Proof. We implicitly invert the exponential characteristic in the proof. Since slices commute with finite direct sums, we have

$$\mathrm{s}_q(\mathrm{MSL}^c) \cong \mathrm{s}_q(\mathrm{MSL}) \oplus \mathrm{s}_q(\Sigma^1_{\mathbb{P}^1}\mathrm{MGL}) \cong \mathrm{s}_q(\mathrm{MSL}) \oplus \Sigma^1_{\mathbb{P}^1}\mathrm{s}_{q-1}(\mathrm{MGL}).$$

Here the last equivalence holds by [RØ16, Lemma 2.1]. Combining this expression with the computations of $\mathrm{s}_q(\mathrm{MSL})$ (see [NRZ]) and $\mathrm{s}_{q-1}(\mathrm{MGL})$ (see [Spi10, Theorem 3.1]) we obtain the result. □

Remark 7.3. The conceptual explanation of the above formula is (here we omit inversion of the exponential characteristic)

$$\mathrm{s}_q(\mathrm{MSL}^c) \cong \bigoplus_{p=0}^{q} \Sigma^q_{\mathbb{P}^1}\mathrm{HExt}^{p,2q}_{\mathrm{MU}^*\mathrm{MU}}(\mathrm{MU}^*(\mathrm{M}\Sigma), \mathrm{MU}^*)[-p].$$

This can be seen from our splitting, the analogous splitting in topology $\mathrm{M}\Sigma \cong \mathrm{MSU} \oplus \Sigma^2\mathrm{MU}$ and the similar view on the slices of MSL; see [NRZ].

Since Voevodsky's algebraic cobordism spectrum MGL is η-complete, the $\mathbb{E}_\infty$-ring map $\mathrm{MSL}^c \to \mathrm{MGL}$ factors through the η-completion of MSL^c. Inverting 2, we obtain the following morphism

$$(\mathrm{MSL}^c)^\wedge_\eta[1/2] \to \mathrm{MGL}[1/2] \in \mathrm{CAlg}(\mathrm{SH}(k)).$$

Theorem 7.4. *Let S be a scheme over a field of characteristic zero. Then the canonical map of $\mathbb{E}_\infty$-ring spectra over S*

$$(\mathrm{MSL}^c)^\wedge_\eta[1/2] \to \mathrm{MGL}[1/2]$$

is an equivalence. In positive characteristic the same result holds away from the characteristic.

Proof. We implicitly invert the exponential characteristic of the base field k below. Since all ingredients are stable under base change, we can assume that S is the spectrum of k. Furthermore, passing to the perfection, we can assume that k is perfect; see [EK20, Corollary 2.1.7].

We claim that the morphism $\mathrm{MSL}^c[1/2] \to \mathrm{MGL}[1/2]$ induces an isomorphism on all slices. From the shape of the slices, we see that the map $\mathrm{s}_q(\mathrm{MSL}^c)[1/2] \to \mathrm{s}_q(\mathrm{MGL})[1/2]$ is an equivalence if and only if it induces an isomorphism after applying $\pi_{2q,q}(-)$. There are conditionally convergent slice spectral sequences (see [Hoy15, 8.14] and Proposition 7.1 above)

$$E^1_{p,q,n}(\mathrm{MSL}^c)[1/2] = \pi_{p,n}(\mathrm{s}_q(\mathrm{MSL}^c))[1/2] \Longrightarrow \pi_{p,n}((\mathrm{MSL}^c)^\wedge_\eta)[1/2]$$

and

$$E^1_{p,q,n}(\mathrm{MGL})[1/2] = \pi_{p,n}(\mathrm{s}_q(\mathrm{MGL}))[1/2] \Longrightarrow \pi_{p,n}(\mathrm{MGL})[1/2]$$

A direct argument with the standard vanishing areas of motivic cohomology (c.f. [LYZ21, Theorem 2.1]) shows that the groups $E^1_{2q,q,q}(\mathrm{MSL}^c)[1/2]$ and $E^1_{2q,q,q}(\mathrm{MGL})[1/2]$ survive to the E^∞-pages. Moreover, they are the only groups that contribute to $\pi_{2q,q}(-)$'s. Thus we have the following commutative diagram

$$\begin{tikzcd}
\pi_{2q,q}(\mathrm{s}_q(\mathrm{MSL}^c))[1/2] \arrow[r, equal] \arrow[d] & E^1_{2q,q,q}(\mathrm{MSL}^c)[1/2] \arrow[r, equal] \arrow[d] & E^\infty_{2q,q,q}(\mathrm{MSL}^c)[1/2] \arrow[d] & \pi_{2q,q}((\mathrm{MSL}^c)^\wedge_\eta)[1/2] \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
\pi_{2q,q}(\mathrm{s}_q(\mathrm{MGL}))[1/2] \arrow[r, equal] & E^1_{2q,q,q}(\mathrm{MGL})[1/2] \arrow[r, equal] & E^\infty_{2q,q,q}(\mathrm{MGL})[1/2] & \pi_{2q,q}(\mathrm{MGL})[1/2]. \arrow[l, "\simeq"']
\end{tikzcd}$$

The right vertical morphism is an isomorphism by Corollary. The claim follows.

An isomorphism on the slices gives us an isomorphism between the E^1-pages

$$E^1_{p,q,n}(\mathrm{MSL}^c)[1/2] \xrightarrow{\simeq} E^1_{p,q,n}(\mathrm{MGL})[1/2] \text{ for all } p,q,n \in \mathbb{Z}.$$

Consequently, the map $\mathrm{MSL}^c[1/2] \to \mathrm{MGL}[1/2]$ induces an isomorphism between the E^∞-pages of the above spectral sequences. Then applying [Boa99, Theorem 7.2], we see that the desired morphism induces an isomorphism on all homotopy groups. It remains to use Morel's theorem about strictly $\mathbb{A}^1$-invariant sheaves (see [Mor03, Theorem 2.11]) to conclude that isomorphism is valid at the level of homotopy sheaves as well. □

Remark 7.5. Suppose that S is a scheme over characteristic zero field and $\mathcal{E}$ is an η-complete SL^c-oriented homotopy commutative ring spectrum over S. Then it follows from the previous theorem, that $\mathcal{E}[1/2]$ admits a GL-orientation (i.e. it is oriented in the usual sense). Indeed, lets choose an orientation morphism $\mathrm{MSL}^c \to \mathcal{E}$ and consider the following diagram

$$\begin{tikzcd}
\mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(2)) \arrow[rr, "\mathrm{Th}(\nu_2)"] \arrow[d, "\mathrm{th}(\mathcal{O}(2))"'] & & \mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(1)) \arrow[d, "\mathrm{th}(\mathcal{O}(1))"] \\
\Sigma^1_{\mathbb{P}^1}(\mathrm{MSL}^c)^\wedge_\eta[1/2] \arrow[rr, "\simeq"] \arrow[dr] & & \Sigma^1_{\mathbb{P}^1}\mathrm{MGL}[1/2] \arrow[dl, dashed] \\
& \Sigma^1_{\mathbb{P}^1}\mathcal{E}[1/2], &
\end{tikzcd}$$

where $\mathrm{Th}(\nu_2)$ is the map on the Thom spectra that induced by the degree 2 Veronese embedding $\nu_2\colon \mathbb{P}^\infty \to \mathbb{P}^\infty$ and $\mathrm{th}(\mathcal{O}(2))$ (resp. $\mathrm{th}(\mathcal{O}(1))$) is the Thom class of $\mathcal{O}(2)$ (resp. $\mathcal{O}(1)$). Then composing the dashed arrow with $\mathrm{th}(\mathcal{O}(1))$, we obtain an element that defines the GL-orientation of $\mathcal{E}[1/2]$. Note that apriori we can't claim that the dashed arrow give us a ring morphism after $\mathbb{P}^1$-desuspension (even up to homotopy), since $\mathrm{MSL}^c \to \mathcal{E}$ may be not multiplicative. However, the above information is enough. In positive characteristic $p > 0$ the same conclusion holds if $\mathcal{E} \in \mathrm{SH}(S)[1/p]$.

7.2. **Category of modules.** In order to apply the above result to the category of modules, we need several categorical definitions and lemmas. Since we use them several times for different ∞-categories, we state it in a general form.

Definition 7.6 ([MNN17, Definition 5.27])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be presentable, symmetric monoidal stable ∞-categories where tensor products commute with colimits in each variable. Suppose that

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$$

is an adjunction with L being symmetric monoidal (it follows that R is lax symmetric monoidal). We say that (L, R) *satisfies the projection formula* if for $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ the canonical morphism

$$R(Y) \otimes X \to R(Y \otimes L(X)) \tag{7.7}$$

adjoint to the map $L(R(Y) \otimes X) \cong L(R(Y)) \otimes L(X) \xrightarrow{\epsilon \otimes \mathrm{id}} Y \otimes L(X)$ is an equivalence.

Lemma 7.8. *Suppose that we have two adjunctions $L_1\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R_1$ and $L_2\colon \mathcal{D} \rightleftarrows \mathcal{E}\colon R_2$ that satisfy all assumptions of the above definition. If they satisfy the projection formula, then the composition adjunction $(L_2 \circ L_1)\colon \mathcal{C} \rightleftarrows \mathcal{E}\colon (R_1 \circ R_2)$ does as well.*

Example 7.9. We will use the following examples of adjunctions that satisfy the projection formula.

(1) Let $\mathcal{C}$ be as in the previous definition and $A \in \mathrm{CAlg}(\mathcal{C})$. Then the free-forgetful adjunction $\mathcal{C} \rightleftarrows \mathrm{Mod}_A(\mathcal{C})$ satisfies the projection formula. To see this consider the following equivalences

$$Y \otimes_A (A \otimes X) \cong (Y \otimes_A A) \otimes X \cong Y \otimes X.$$

A straightforward verification shows that this composition gives an inverse to the morphism 7.7.

(2) Let $\mathcal{C}$ be as above and assume that it is the product symmetric monoidal ∞-category, i.e. there is a symmetric monoidal equivalence $\mathcal{C} \cong \mathcal{C}_1 \times \mathcal{C}_2$. Then the projection-inclusion adjunction $pr_1\colon \mathcal{C} \rightleftarrows \mathcal{C}_1\colon in_1$ satisfies the projection formula.

Construction 7.10 ([MNN17, Construction 5.23])**.** If we have an adjunction $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ with L being symmetric monoidal, then $R(Y)$ automatically is an $R(\mathbb{1}_\mathcal{D})$-module. In other words, the functor R factors through $\bar{R}\colon \mathcal{D} \to \mathrm{Mod}_{R(\mathbb{1}_\mathcal{D})}(\mathcal{C})$. The functor $\bar{R}$ commutes with limits and its left adjoint is given by the following composition

$$\mathrm{Mod}_{R(\mathbb{1}_\mathcal{D})}(\mathcal{C}) \xrightarrow{L} \mathrm{Mod}_{LR(\mathbb{1}_\mathcal{D})}(\mathcal{D}) \xrightarrow{\otimes_{LR(\mathbb{1}_\mathcal{D})}\mathbb{1}_\mathcal{D}} \mathrm{Mod}_{\mathbb{1}_\mathcal{D}}(\mathcal{D}) = \mathcal{D},$$

where $\otimes_{LR(\mathbb{1}_\mathcal{D})}\mathbb{1}_\mathcal{D}$ is the base change along $\epsilon(\mathbb{1}_\mathcal{D})\colon LR(\mathbb{1}_\mathcal{D}) \to \mathbb{1}_\mathcal{D}$. We denote this composition by $\bar{L}$. Summarizing this discussion, we get a new adjunction

$$\bar{L}\colon \mathrm{Mod}_{R(\mathbb{1}_\mathcal{D})}(\mathcal{C}) \rightleftarrows \mathcal{D}\colon \bar{R},$$

in which $\bar{L}$ is symmetric monoidal functor and $\bar{R}$ is lax symmetric monoidal.

Now we are ready to consider the category of MSL^c-modules. First of all, we note that, as usual, the category of 2-inverted MSL^c-modules splits (as a symmetric monoidal ∞-category) into the product

$$\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2] \cong \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^+ \times \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^-. \tag{7.11}$$

The projection of the motivic Hopf element in the plus-part is trivial $(\mathrm{MSL}^c[1/2] \otimes \eta)^+ = 0$, while its minus part $(\mathrm{MSL}^c[1/2] \otimes \eta)^-$ is invertible. It follows that $\mathrm{MSL}^c[1/2]^+$ coincide with the η-completion $(\mathrm{MSL}^c)^\wedge_\eta[1/2]$, which is already known to be equivalent to $\mathrm{MGL}[1/2]$.

Theorem 7.12. *Let k be a field of characteristic zero. Then the morphism $\mathrm{MSL}^c \to \mathrm{MGL}$ induces an equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^+ \cong \mathrm{Mod}_{\mathrm{MGL}}(\mathrm{SH}(k))[1/2].$$

For a field k of characteristic $p > 0$ the same result holds after inverting p.

Proof. We implicitly invert the characteristic below. Consider the following adjunction

$$\mathrm{SH}(k) \rightleftarrows \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^+.$$

We claim that it satisfies the projection formula. This adjunction is obtained by composing the following ones

$$\mathrm{SH}(k) \rightleftarrows \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k)) \rightleftarrows \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2] \rightleftarrows \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^+.$$

Hence, the claim follows from Lemma 7.8 and Example 7.9. In addition, the right adjoint

$$\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^+ \to \mathrm{SH}(k)$$

is conservative and commute with arbitrary colimits, being the composition of those. Now applying Construction 7.10 we get an adjunction

$$\mathrm{Mod}_{(\mathrm{MSL}^c)^\wedge_\eta[1/2]}(\mathrm{SH}(k)) \rightleftarrows \mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^+,$$

which is an inverse equivalence of symmetric monoidal ∞-categories by [MNN17, Proposition 5.29]. In turn, the left-hand side is equivalent to $\mathrm{Mod}_{\mathrm{MGL}[1/2]}(\mathrm{SH}(k))$ by Theorem 7.4. It remains to identify $\mathrm{MGL}[1/2]$-modules with 2-inverted MGL-modules. This can be done using the same argument for the adjunction $\mathrm{SH}(k) \rightleftarrows \mathrm{Mod}_{\mathrm{MGL}}(\mathrm{SH}(k))[1/2]$. □

Now we describe the minus-part. Of course, for that we use Bachmann's equivalence [Bac18] (below ρ denotes the map $\mathbb{1} \to \mathbb{G}_m$ corresponding to -1)

$$\mathrm{SH}(k)[\rho^{-1}] \cong \mathrm{Sp}(\mathrm{Sper}\, k),$$

which is induced by the real étale realization (see Appendix A for an overview). It follows from [Bac18, Lemma 39] that away from 2 inversion of ρ is the same as inversion of η. In particular, the real étale realization away from two factors through a symmetric monoidal equivalence

$$\mathrm{SH}(k)[1/2]^- \cong \mathrm{Sp}(\mathrm{Sper}\, k)[1/2]. \tag{7.13}$$

Theorem 7.14. *Let k be a base field. Then the real étale realization induces an equivalence of the stable symmetric monoidal ∞-categories*

$$\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^- \cong \mathrm{Mod}_{\underline{\mathrm{MSO}}}(\mathrm{Sp}(\mathrm{Sper}\, k))[1/2].$$

Here $\underline{\mathrm{MSO}} \in \mathrm{Sp}(\mathrm{Sper}\, k)$ is a constant sheaf on the Thom spectrum $\mathrm{MSO} \in \mathrm{Sp}$.

Proof. Repeating the same argument as in the proof of Theorem 7.12 one can get the following equivalences

$$\mathrm{Mod}_{\mathrm{MSL}^c}(\mathrm{SH}(k))[1/2]^- \cong \mathrm{Mod}_{\mathrm{MSL}^c[1/\eta,1/2]}(\mathrm{SH}(k)) \cong \mathrm{Mod}_{\mathrm{MSL}^c[1/\eta,1/2]}(\mathrm{SH}(k)[1/2]^-).$$

We can replace MSL^c with MSL since they become equivalent (as $\mathbb{E}_\infty$-algebras) after inverting η; see Corollary 5.13. Now applying real étale realization and using Bachmann's equivalence stated above we get

$$\mathrm{Mod}_{\mathrm{MSL}[1/\eta,1/2]}(\mathrm{SH}(k)[1/2]^-) \cong \mathrm{Mod}_{\mathrm{Re}_{\text{rét}}(\mathrm{MSL}[1/\eta,1/2])}(\mathrm{Sp}(\mathrm{Sper}\, k)[1/2]) \cong \mathrm{Mod}_{\mathrm{Re}_{\text{rét}}(\mathrm{MSL})}(\mathrm{Sp}(\mathrm{Sper}\, k))[1/2].$$

Thus we need to compute $\mathrm{Re}_{\text{rét}}(\mathrm{MSL})$ as an $\mathbb{E}_\infty$-algebra. It is given by $\underline{\mathrm{MSO}}$ with an obvious ring structure (see Example A.6). Combining everything stated we obtain the desired equivalence. □

Remark 7.15. In fact, the previous theorem is valid over an arbitrary base scheme S, since the equivalence 7.13 and the computation of the real étale realization (see Example A.6) hold over S. We decide to stay over a field for simplicity.

Appendix A. Real étale realization of absolute ring spectra

In this appendix, we recall basics about real étale topology and overview the construction of the real étale realization [Bac18]. Then we compute its for motivic $\mathbb{E}_n$-ring spectra that are defined over $\mathbb{Z}$. As a conclusion, we obtain formulae for the real étale realizations of algebraic cobordism spectra.

Recall, that the real spectrum $\operatorname{Sper} A$ of a commutative ring A is the set consisting of pairs $(\mathfrak{p}, \alpha)$, where $\mathfrak{p} \in \operatorname{Spec} A$ and α is an ordering of $\kappa(\mathfrak{p})$. This set can be endowed with the Harrison topology. More generally, for a scheme S there is a topological space RS called *real space of* S, which is obtained by gluing real spectra and satisfies $R\operatorname{Spec} A = \operatorname{Sper} A$. Giving a scheme S a family of morphisms $\{U_i \to S\}_{i\in I}$ is the *real étale covering* if each morphism $U_i \to S$ is étale and $RS = \bigcup_{i\in I} RU_i$.

Notation A.1. Let S be a base scheme. We denote by $S_{\text{rét}}$ the *small real étale site* consisting of the category Ét_S of étale S-schemes with the real étale topology on it and by $\mathrm{Sm}_{S,\text{rét}}$ the big real étale site (i.e. Sm_S with the real étale topology).

Notation A.2. We write $\mathrm{SH}_{\text{rét}}(S)$ for the localization of $\mathrm{SH}(S)$ at the real étale coverings, $\mathrm{SH}^{S^1}_{\text{rét}}(S)$ for the localization of $\mathrm{SH}^{S^1}(S)$ at the real étale coverings, $\mathrm{Sp}(S_{\text{rét}})$ for the stable ∞-category of sheaves of spectra on $S_{\text{rét}}$, and $\mathrm{Sp}(RS)$ for the stable ∞-category of sheaves of spectra on the topological space RS. All these categories are symmetric monoidal.

Bachmann's theorem (see [Bac18, Theorem 35] and [Bac21, Theorem 4.2]) states that the following natural symmetric monoidal functors are equivalences

$$\mathrm{Sp}(S_{\text{rét}}) \xrightarrow{i^*} \mathrm{SH}^{S^1}_{\text{rét}}(S) \xrightarrow{\sigma^\infty_{\text{rét}}} \mathrm{SH}_{\text{rét}}(S). \tag{A.3}$$

Here the first functor is induced by change of sites $i\colon S_{\text{rét}} \hookrightarrow \mathrm{Sm}_{S,\text{rét}}$ (see [Bac18, Theorem 8]) and the second one is the $\mathbb{G}_m$-stabilization.

Definition A.4. The *real étale realization functor* is the symmetric monoidal functor given by the following composition

$$\mathrm{Re}_{\text{rét}}\colon \mathrm{SH}(S) \xrightarrow{\mathrm{L}_{\text{rét}}} \mathrm{SH}_{\text{rét}}(S) \xrightarrow{\simeq} \mathrm{Sp}(S_{\text{rét}}),$$

where the last functor is the inverse to Bachmann's equivalence stated above. Consider the canonical equivalence $\mathrm{Sp}(S_{\text{rét}}) \cong \mathrm{Sp}(RS)$, which is the stabilization of the Elmanto–Shah–Scheiderer equivalence of the underlying ∞-topoi; see [ES21, Theorem B.10] and [Sch94, Theorem 1.3]. Composing this functor with the above realization functor, we obtain the following symmetric monoidal real étale realization functor (we use the same name and notation for it)

$$\mathrm{Re}_{\text{rét}}\colon \mathrm{SH}(S) \to \mathrm{Sp}(RS).$$

For a scheme S we denote by $\mathrm{Op}(RS)$ the category of open subsets of the topological space RS and by L_R the shefification functor $\mathrm{Fun}(\mathrm{Op}(RS)^{op}, \mathrm{Sp}) \to \mathrm{Sp}(RS)$. This localization functor is symmetric monoidal and left adjoint to the forgetful one.

Theorem A.5. *Suppose that A is a subring of $\mathbb{R}$ such that $\operatorname{Sper} A = *$ (e.g. $\mathbb{Z}$, $\mathbb{Z}[1/2]$, $\mathbb{Q}$), $\mathcal{E}$ is a motivic spectrum over A, and $S \in \mathrm{Sch}_A$. Then the real étale realization of $\mathcal{E}_S$ is given by the constant sheaf on the real Betti realization of $\mathcal{E}$*

$$\mathrm{Re}_{\text{rét}}(\mathcal{E}_S) \cong \mathrm{L}_R(\underline{\mathrm{Re}_{\mathrm{B}\mathbb{R}}(\mathcal{E}_{\mathbb{R}})}) \in \mathrm{Sp}(RS).$$

Moreover, if $\mathcal{E} \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{SH}(A))$ for some $1 \leq n \leq \infty$, then the stated isomorphism is an equivalence of the $\mathbb{E}_n$-algebras.

Proof. First consider the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{SH}(A) & \xrightarrow[\mathrm{Re}_{\text{rét}}]{} & \mathrm{Sp}(\operatorname{Sper} A) & & \\
\downarrow & & \downarrow{=} & & \\
\mathrm{SH}(\mathbb{R}) & \xrightarrow[\mathrm{Re}_{\text{rét}} \cong \mathrm{Re}_{\mathrm{B}\mathbb{R}}]{} & \mathrm{Sp}(\operatorname{Sper} \mathbb{R}) & = & \mathrm{Sp}
\end{array}$$

Here the vertical arrows are base changes along $R \hookrightarrow \mathbb{R}$ and $\operatorname{Sper} \mathbb{R} \xrightarrow{=} \operatorname{Sper} A$ (the equality holds by the assumption on $\operatorname{Sper} A$) and the horizontal morphisms are given by the real étale realizations. The real étale realization over $\mathbb{R}$ is canonically equivalent to the real Betti realization by [Bac18, Proposition 36]. Hence, we have $\mathrm{Re}_{\text{rét}}(\mathcal{E}) \cong \mathrm{Re}_{\mathrm{B}\mathbb{R}}(\mathcal{E}_{\mathbb{R}}) \in \mathrm{Sp}$. Now denote by $\pi_S\colon S \to \operatorname{Spec} A$ the structure morphism of S and lets look at the following diagram

$$\begin{tikzcd}
\mathrm{SH}(A) \arrow[r, "\mathrm{Re}_{\text{rét}}"'] \arrow[d, "\pi_S^*"] & \mathrm{Sp}(\mathrm{Sper}\,A) \arrow[r, "="', "\text{forgetful}"'] \arrow[d, "\pi_{RS}^*"] & \mathrm{Fun}(\mathrm{Op}(\mathrm{Sper}\,A)^{op}, \mathrm{Sp}) = \mathrm{Sp} \arrow[d, "\pi_{RS}^*"] \\
\mathrm{SH}(S) \arrow[r, "\mathrm{Re}_{\text{rét}}"'] & \mathrm{Sp}(RS) & \mathrm{Fun}(\mathrm{Op}(RS)^{op}, \mathrm{Sp}). \arrow[l, "\mathrm{L}_R"]
\end{tikzcd}$$

The left square obviously commutes and the right square commutes by the assumption on $\mathrm{Sper}\,A$. The right vertical base change functor maps a spectrum to the associated constant presheaf. The desired equivalence follows immediately. It automatically preserve $\mathbb{E}_n$-algebra structure, since all functors in the both diagrams are symmetric monoidal. □

Example A.6. Let S be a base scheme, then the real étale realizations of the standard algebraic cobordism spectra are given the following constant sheaves of spectra on RS:

$$\mathrm{Re}_{\text{rét}}(\mathrm{MGL}_S) \cong \underline{\mathrm{MO}}, \qquad \mathrm{Re}_{\text{rét}}(\mathrm{MSL}_S) \cong \underline{\mathrm{MSO}}, \qquad \mathrm{Re}_{\text{rét}}(\mathrm{MSp}_S) \cong \underline{\mathrm{MU}}.$$

Here we omit the localization functor L_R for simplicity. This follows from the previous theorem and the computations of the respective real Betti realizations (see e.g. [BH20, Corollary 4.7]), since these algebraic cobordism spectra are defined over $\mathrm{Spec}(\mathbb{Z})$. Moreover, we claim that the above equivalences are valid in $\mathrm{CAlg}(\mathrm{Sp}(RS))$ given the natural $\mathbb{E}_\infty$-algebra structure on each spectrum. For that it is enough to see, that the real Betti realizations are given by the desired ring spectra. This can be shown by modeling the algebraic cobordism spectra as $\mathbb{E}_\infty$-algebras via symmetric spectra on the appropriate Grassmanians (see [PW23, §4 and §6] for a construction of the symmetric spectra and [AHI24, Construction 7.1 and Remark 7.3.(ii)] for a comparison of $\mathbb{E}_\infty$-structures in the case of MGL, the other two cases are similar).

Remark A.7. Note, that the above situation is radically different from the one with the l-adic étale realization; see [BBX25, §4] for the case of MGL. The reason for this is due to the non-triviality of the small étale site of $\mathrm{Spec}(\mathbb{Z}[1/l])$ (as opposed to the small real étale site of $\mathrm{Spec}(\mathbb{Z})$).

Appendix B. Stong's complex-spin cobordism spectrum

In the paper [Sto67] Stong introduces the complex-spin Thom spectrum in topology and computes the coefficient ring of the respective cobordism theory. In this appendix, we overview his results and prove that it is equivalent to the Thom spectrum built out of the 2-fold coverings of the unitary groups. The respective comparison is inspired by the work of Atiyah [?].

Notation B.1. Consider a space $\mathrm{B}\Sigma(n)$ given by the pullback of the following diagram

$$\mathrm{BU}(n) \to \mathrm{BSO}(2n) \leftarrow \mathrm{BSpin}(2n),$$

where the left map is induced by the inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n)$ and the right map is induced by the 2-fold covering $\mathrm{Spin}(2n) \twoheadrightarrow \mathrm{SO}(2n)$. Note that the desired pullback in the ∞-category of spaces can be computed in a naive way, since the right map is a Kan fibration. We also denote by $\mathrm{B}\Sigma$ the colimit

$$\operatorname*{colim}_n \mathrm{B}\Sigma(n),$$

which is equivalent to the pullback of the following diagram $\mathrm{BU} \to \mathrm{BSO} \leftarrow \mathrm{BSpin}$. This is an $\mathbb{E}_\infty$-ring spectrum.

Definition B.2 ([Sto67])**.** The *complex-spin Thom spectrum* $\mathrm{M}\Sigma \in \mathrm{CAlg}(\mathrm{Sp})$ is the Thom spectrum associated with the following composition $\mathrm{B}\Sigma \to \mathrm{BU} \to \mathrm{BO}$. Note that by construction there are forgetful morphisms $\mathrm{MSU} \to \mathrm{M}\Sigma \to \mathrm{MU}$.

Remark B.3. The above Thom spectrum corresponds under the Pontryagin–Thom construction to the cobordism theory of manifolds with compatible stable unitary and spinor structures.

Theorem B.4. *(1) The coefficient ring $\pi_*(\mathrm{M}\Sigma)$ does not contain odd torsion.*
(2) The map $\pi_*(\mathrm{MSU}) \to \pi_*(\mathrm{M}\Sigma)$ induces an isomorphism on the 2-primary torsion subgroups.
(3) The ring homomorphism $\pi_*(\mathrm{M}\Sigma) \to \pi_*(\mathrm{MU})$ becomes an isomorphism after inverting 2.

Proof. [Sto67, Theorem 1 and Theorem 2] □

Notation B.5. We denote by $\widetilde{\mathrm{U}}(n)$ the kernel of the following homomorphism of Lie groups

$$\mathrm{U}(n) \times \mathrm{U}(1) \to \mathrm{U}(1),\ (u,t) \mapsto t^{-2} \cdot \det(u).$$

This group is a 2-fold covering of the unitary group, i.e. there is an exact sequence of Lie groups

$$1 \to \mathbb{Z}/2 \to \widetilde{\mathrm{U}}(n) \to \mathrm{U}(n) \to 1.$$

The special unitary group $\mathrm{SU}(n)$ can be regarded as a subgroup of $\widetilde{\mathrm{U}}(n)$ via $u \mapsto (u, 1)$. We also write $\mathrm{M}\widetilde{\mathrm{U}}$ for the Thom spectrum associated with the colimit $\mathrm{B}\widetilde{\mathrm{U}} := \mathrm{colim}_n\, \mathrm{B}\widetilde{\mathrm{U}}(n)$ along $\mathrm{B}\widetilde{\mathrm{U}} \to \mathrm{BU} \to \mathrm{BO}$. This is an $\mathbb{E}_\infty$-ring spectrum.

The inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n)$ lifts to the inclusion of the 2-fold covering spaces $\widetilde{\mathrm{U}}(n) \hookrightarrow \mathrm{Spin}(2n)$. Hence, we have the following commutative (up to homotopy) diagram

$$\begin{tikzcd}
\mathrm{B}\widetilde{\mathrm{U}}(n) \arrow[r] \arrow[d] & \mathrm{BSpin}(2n) \arrow[d] \\
\mathrm{BU}(n) \arrow[r] & \mathrm{BSO}(2n).
\end{tikzcd}$$

It induces the morphism between the upper right corner and the pullback of the above diagram $\mathrm{B}\widetilde{\mathrm{U}}(n) \to \mathrm{B}\Sigma(n)$.

Theorem B.6. *The morphism* $\mathrm{B}\widetilde{\mathrm{U}}(n) \to \mathrm{B}\Sigma(n)$ *constructed above is an equivalence of spaces.*

Proof. By definition of $\widetilde{\mathrm{U}}(n)$ the fiber of the map $\mathrm{B}\widetilde{\mathrm{U}}(n) \to \mathrm{BU}(n)$ is $\mathrm{B}\mathbb{Z}/2 = \mathrm{K}(\mathbb{Z}/2, 1)$. From the other hand side, the fiber of $\mathrm{B}\Sigma(n) \to \mathrm{BU}(n)$ coincides with the fiber of $\mathrm{BSpin}(2n) \to \mathrm{BSO}(2n)$, which is also given by $\mathrm{K}(\mathbb{Z}/2, 1)$. Hence, we have morphisms of the fiber sequences

$$\begin{tikzcd}
\mathrm{K}(\mathbb{Z}/2,1) \arrow[r] \arrow[d] & \mathrm{B}\widetilde{\mathrm{U}}(n) \arrow[r] \arrow[d] & \mathrm{BU}(n) \arrow[d, equal] \\
\mathrm{K}(\mathbb{Z}/2,1) \arrow[r] \arrow[d, equal] & \mathrm{B}\Sigma(n) \arrow[r] \arrow[d] & \mathrm{BU}(n) \arrow[d] \\
\mathrm{K}(\mathbb{Z}/2,1) \arrow[r] & \mathrm{BSpin}(2n) \arrow[r] & \mathrm{BSO}(2n).
\end{tikzcd}$$

We need to check that the middle upper vertical arrow is an equivalence. For that it is enough to prove that the left vertical composition induces an isomorphism on π_1. Consider the corresponding map of the long exact sequences of the homotopy groups

$$\begin{tikzcd}
0 \arrow[r] & \pi_2(\mathrm{B}\widetilde{\mathrm{U}}(n)) \arrow[r] \arrow[d] & \pi_2(\mathrm{BU}(n)) \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \pi_1(\mathrm{B}\widetilde{\mathrm{U}}(n)) \arrow[d] \\
0 \arrow[r] & \pi_2(\mathrm{BSpin}(2n)) \arrow[r] & \pi_2(\mathrm{BSO}(2n)) \arrow[r] & \mathbb{Z}/2 \arrow[r] & \pi_1(\mathrm{BSpin}(2n))
\end{tikzcd}$$

Comparing the homotopy groups of the classifying spaces with the homotopy groups of the respective Lie groups (via $\Omega BG \simeq G$) we see that the fundamental groups from the right are trivial and the middle arrow $\pi_2(\mathrm{BU}(n)) \to \pi_2(\mathrm{BSO}(2n))$ is surjective. Thus the homomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$ is not zero and gives a desired equivalence $\mathrm{K}(\mathbb{Z}/2, 1) \to \mathrm{K}(\mathbb{Z}/2, 1)$. □

Corollary B.7. *The map* $\mathrm{B}\widetilde{\mathrm{U}} \to \mathrm{B}\Sigma$ *induces an equivalence of the* $\mathbb{E}_\infty$*-ring Thom spectra* $\mathrm{M}\widetilde{\mathrm{U}} \to \mathrm{M}\Sigma$.

Proof. This follows directly from the previous theorem. □

Appendix C. Recollection on the geometric diagonal of MSL

In this section, k is a local Dedekind domain, and e is its exponential characteristic.

C.1. Let $\eta : \pi_{2n+m,n+m}(\mathrm{MSL}) \to \pi_{2n+m+1,n+m+1}(\mathrm{MSL})$ be the multiplication by the motivic Hopf element $\eta \in \pi_{1,1}(\mathrm{MSL})$. Zolotarev showed the following result.

Theorem C.1. [Zol24, Theorem 5.2] *For* $n \in \mathbb{Z}$*, the map* $\eta : \pi_{2n+m,n+m}(\mathrm{MSL}) \to \pi_{2n+m+1,n+m+1}(\mathrm{MSL})$ *is epimorphism for* $m = 0$*, and isomorphism for* $m \geq 1$*.*

As a corollary of the previous theorem, and computations by Bachmann-Hopkins on the homotopy groups of $\mathrm{MSL}[\eta^{-1}]$, we get the following.

Corollary C.2. [Zol24, Corollary 5.4] *Let $e \neq 2$. Then, we have the following ring isomorphism.*

$$\pi_{2*,*}(\mathrm{MSL})\Big/{}_{\eta}\pi_{2*,*}(\mathrm{MSL})\,[1/e] \cong \mathrm{W}(k)[1/e][y_4, y_8, \dots], \text{ where } |y_i| = (2i, i).$$

Here ${}_{\eta}\pi_{2*,*}(\mathrm{MSL})$ denotes the annihilator of η in $\pi_{2*,*}(\mathrm{MSL})$.

Let MWL denote the motivic c_1-speherical cobordism as defined by Zolotarev in [Zol24, Definition 2.5]. It is the algebraic analog of Conner-Floyd's c_1-spherical cobordism W, see [CF66]. Then, we have the following cofiber sequence connecting MWL, and MSL.

Theorem C.3. [Zol24, Theorem A, Proposition B.4] *There is a cofiber sequence*

$$\Sigma^{1,1}\mathrm{MSL} \xrightarrow{\eta} \mathrm{MSL} \xrightarrow{c} \mathrm{MWL}$$

If we work over complex numbers, taking complex realization of this sequence, we get back the following classical cofiber sequence

$$\Sigma^{1}\mathrm{MSU} \xrightarrow{\eta_{\mathrm{top}}} \mathrm{MSU} \to \mathrm{W}$$

From Eq. (C.3), we have the following long exact sequence

$$\dots \to \pi_{i-1,j-1}(\mathrm{MSL}) \xrightarrow{\eta} \pi_{i,j}(\mathrm{MSL}) \xrightarrow{c_*} \pi_{i,j}(\mathrm{MWL}) \xrightarrow{d_*} \pi_{i-2,j-1}(\mathrm{MSL}) \to \dots.$$

This induces the following short exact sequence.

$$0 \to \eta.\pi_{2n-1,n-1}(\mathrm{MSL}) \to \pi_{2n,n}(\mathrm{MSL}) \to \mathrm{Ker}(\pi_{2n,n}(d)) \to 0$$

Here $\eta.\pi_{2n-1,n-1}(\mathrm{MSL})$ denotes the image of the multiplication by η. Zolotarev has given explicit description of $\eta.\pi_{2n-1,n-1}(\mathrm{MSL})$, and $\mathrm{Ker}(\pi_{2n,n}(d))$. In particular, we have the following result.

Theorem C.4. *(1) Suppose that k is a quadratically closed field of exponential characteristic $e \neq 2$. Then we have*

$$\eta.\pi_{2n-1,n-1}(\mathrm{MSL}_k)[1/e] \simeq \begin{cases} 0, & n \equiv 0,2,3 \ (mod\ 4) \\ (\mathbb{Z}/2)^{p(\frac{n-1}{4})}, & n \equiv 1 \ (mod\ 4) \end{cases}$$

Here p denotes the partition function.

(2) Let k be a local Dedekind domain and $e \neq 2$ is the exponential characteristic of its residue field. Then we have

$$\eta.\pi_{2n-1,n-1}(\mathrm{MSL}_k)[1/e] \simeq \begin{cases} \mathrm{I}(k)[1/e], & n \equiv 0 \ (mod\ 4) \\ (\mathbb{Z}/2)^{p(\frac{n-1}{4})}, & n \equiv 1 \ (mod\ 4) \\ 0, & n \equiv 2,3 \ (mod\ 4) \end{cases}$$

In particular, for morphisms of local Dedekind domains $f : k \to k'$, with exponential character other than 2, the induced base change morphism $\eta.\pi_{2n-1,n-1}(\mathrm{MSL}_k)[1/e] \to \eta.\pi_{2n-1,n-1}(\mathrm{MSL}_{k'})[1/e']$ is an isomorphism for $n \not\equiv 0 \ (mod\ 4)$.

(3) Let's define $(\mathrm{I}_{\mathrm{MSL}}(k))_n := \eta.\pi_{2n-1,n-1}(\mathrm{MSL}_k)$ for $n \equiv 0 \ (mod\ 4)$, and $(\mathrm{I}_{\mathrm{MSL}}(k))_n := 0$, otherwise. We have that $\mathrm{I}_{\mathrm{MSL}}(k)$ is a graded subgroup of $\pi_{2*,*}(\mathrm{MSL}_k)$. In fact, $\mathrm{I}_{\mathrm{MSL}}(k)$ is a graded ideal of $\pi_{2*,*}(\mathrm{MSL}_k)$.

(4) Let k, and $e \neq 2$ be as mentioned above. For a morphism of local Dedekind domains $f : k \to k'$, we have the following ring isomorphism.

$$\pi_{2*,*}(\mathrm{MSL}_k)/\mathrm{I}_{\mathrm{MSL}}(k)[1/e] \simeq \pi_{2*,*}(\mathrm{MSL}_{k'})/\mathrm{I}_{\mathrm{MSL}}(k')[1/e]$$

Let $k = \mathbb{C}$. Then complex realization gives the following commutative diagram.

(C.5)
$$\begin{tikzcd} 0 \arrow[r] & \eta.\pi_{2n-1,n-1}(\mathrm{MSL}_k) \arrow[r] \arrow[d, "\mathrm{Re}_{\mathbb{BC}}"] & \pi_{2n,n}(\mathrm{MSL}_k) \arrow[r] \arrow[d, "\mathrm{Re}_{\mathbb{BC}}"] & \mathrm{Ker}(\pi_{2n,n}(d_k)) \arrow[r] \arrow[d, "\mathrm{Re}_{\mathbb{BC}}"] & 0 \\ 0 \arrow[r] & \eta_{\mathrm{top}}.\pi_{2n-1}(\mathrm{MSU}) \arrow[r] & \pi_{2n}(\mathrm{MSU}) \arrow[r] & \mathrm{Ker}(\pi_{2n}(d)) \arrow[r] & 0 \end{tikzcd}$$

Here the rightmost vertical arrow is an isomorphism [Zol24, Proposition 6.9].

Theorem C.6. *Let k, and e be as mentioned previously. From Eq. (C.5), and various base change, we have the following isomorphism.*

$$\pi_{2*,*}(\mathrm{MSL})\Big/\mathrm{I}_{\mathrm{MSL}}(k)\,[1/e] \cong \pi_{2*}(\mathrm{MSU})[1/e]$$

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