AG.1) Yes, $X(e^{j\omega})$ is periodic, with period 2π, as $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$.

AG.2) (a) We know that $e^{j\omega n}$ is the eigenfunction of the system, with eigenvalue $H(e^{j\omega})$

$\therefore$ When input is $x[n] = e^{j\omega_0 n}$,

$$y[n] = H(e^{j\omega_0}) e^{j\omega_0 n}$$

$$\therefore \boxed{y[n] = H(e^{j\omega_0}) e^{j\omega_0 n}}$$

Proof :-

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

$$\therefore \boxed{y[n] = e^{j\omega_0 n} H(e^{j\omega_0})}$$

(b) Property :-

$$y[n] = x[n] * h[n],$$

then $Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

Consider the DTFT of $y[n]$,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (x[n] * h[n]) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k) h(n-k) e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k) e^{-j\omega n}$$

$$= \left(\sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k}\right) H(e^{j\omega})$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

$$\therefore \boxed{Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})}$$

AG·3b) (b) $y[n] = \sum_{m=0}^{M} b_m x[n-m] - \sum_{\ell=1}^{L} a_\ell y[n-\ell]$

Taking DTFT,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\sum_{m=0}^{M} b_m x[n-m] e^{-j\omega n} - \sum_{n=-\infty}^{\infty}\sum_{\ell=1}^{L} a_\ell y[n-\ell] e^{-j\omega n}$$

$$Y(e^{j\omega}) = \left(\sum_{m=0}^{M} b_m e^{-j\omega m}\right) X(e^{j\omega}) - \left(\sum_{\ell=1}^{L} a_\ell e^{-j\omega L}\right) Y(e^{j\omega})$$

$$\therefore Y(e^{j\omega})\left[1 + \sum_{\ell=1}^{L} a_\ell e^{-j\omega L}\right] = X(e^{j\omega})\left[\sum_{m=0}^{M} b_m e^{-j\omega m}\right]$$

$$\therefore \boxed{\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{1 + \sum_{\ell=1}^{L} a_\ell e^{-j\omega L}}}$$

(c) $y[n] = x[n] + 0.9y[n-1]$

$$Y(e^{j\omega}) = X(e^{j\omega}) + 0.9e^{-j\omega} Y(e^{j\omega})$$

$$Y(e^{j\omega}) = \frac{X(e^{j\omega})}{1 - 0.9e^{-j\omega}}$$

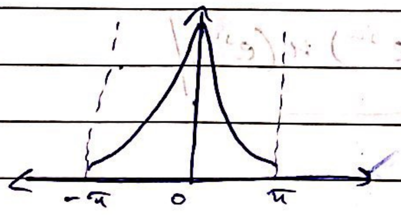
$$\therefore \boxed{\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-0.9e^{-j\omega}}}$$

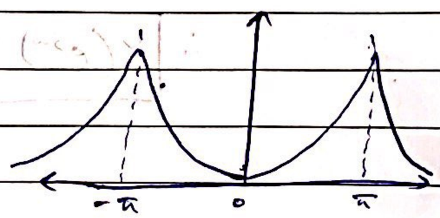
(d) $Y(e^{j\omega}) = X(e^{j\omega}) - 0.9e^{-j\omega} Y(e^{j\omega})$

$$Y(e^{j\omega})\left[1 + 0.9e^{-j\omega}\right] = X(e^{j\omega})$$

$$\boxed{\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1+0.9e^{-j\omega}}}$$

(e)





(c) => Low pass filter

(d) => high pass filter

(f) For impulse response :-

$X(e^{j\omega}) = 1$

(c)

Impulse response, $H(e^{j\omega}) = \dfrac{1}{1 - 0.9e^{-j\omega}}$

$\therefore$ Using property (in class), $a^n u[n] \longleftrightarrow \dfrac{1}{1-ae^{-j\omega}}$, $|a|<1$

$h[n] = (0.9)^n u[n]$

(d) Impulse response, $H(e^{j\omega}) = \dfrac{1}{1+0.9e^{-j\omega}}$

Using property (in class), $a^n u[n] \longleftrightarrow \dfrac{1}{1-ae^{-j\omega}}$, $|a|<1$

$h[n] = (-0.9)^n u[n]$