School of Engineering and Applied Science (SEAS), Ahmedabad University

CSE 400: Fundamentals of Probability in Computing

Lecture 5: Bayes' Theorem, Random Variables, and PMF

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1. **Bayes' Theorem: Weighted Average of Conditional Probabilities**

   Let A and B be events. We may express A as

   $$A = AB \cup AB^c$$

   for, in order for an outcome to be in A, it must either be in both A and B or be in A but not in B.

   As AB and AB^c are mutually exclusive, by Axiom 3,

   $$\Pr(A) = \Pr(AB) + \Pr(AB^c)$$

   $$= \Pr(A \mid B)\Pr(B) + \Pr(A \mid B^c)[1 - \Pr(B)]$$

   The probability of event A is a weighted average of the conditional probabilities with weights given as the probability of the event on which it is conditioned has of occurring.

2. **Bayes' Theorem: Learning by Example**

   Example 3.1 (Part 1)

   An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident prone. If 30% of the population is accident prone, find the probability that a new policyholder will have an accident within a year.

   Solution:

   Let A_1 denote the event that the policyholder has an accident within a year, and let A denote the event that the policyholder is accident prone.

   $$\Pr(A_1) = \Pr(A_1 \mid A)\Pr(A) + \Pr(A_1 \mid A^c)\Pr(A^c)$$
   $$= (0.4)(0.3) + (0.2)(0.7) = 0.26$$

3. **Bayes Formula: Learning by Example**

   Example 3.1 (Part 2)

   Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that the policyholder is accident prone?

   Solution:

$$\begin{aligned}\Pr(A \mid A_1) &= \frac{\Pr(A \cap A_1)}{\Pr(A_1)} \\ &= \frac{\Pr(A)\Pr(A_1 \mid A)}{\Pr(A_1)} \\ &= \frac{(0.3)(0.4)}{0.26} = \frac{6}{13}\end{aligned}$$

4. **Law of Total Probability**

   If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events, then

   $$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap B_i)$$

   This is known as the **Law of Total Probability** (Formula 3.4).

5. **Bayes Formula (Proposition 3.1)**

   Using

   $$\Pr(A \cap B_i) = \Pr(B_i \mid A)\Pr(A)$$

   we obtain the Bayes Formula:

   $$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_j \Pr(A \mid B_j)\Pr(B_j)}$$

   Here,

   - $\Pr(B_i)$ is the *a priori probability*
   - $\Pr(B_i \mid A)$ is the *a posteriori probability*

6. **Bayes Formula: Card Example**

   Three cards are mixed: one red-red, one black-black, and one red-black. A card is selected randomly and placed face up. Given that the upper side is red, find the probability that the other side is black.

   Let RR, BB, and RB denote the three cards. Let R denote the event that the upper side is red.

   $$\Pr(RB \mid R) = \frac{\Pr(R \mid RB)\Pr(RB)}{\Pr(R \mid RR)\Pr(RR) + \Pr(R \mid RB)\Pr(RB) + \Pr(R \mid BB)\Pr(BB)}$$

   $$= \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)} = \frac{1}{3}$$

7. **Random Variables: Motivation and Concept**

   A random variable is a real-valued function defined on the sample space.

   Values are determined by the outcomes of an experiment.

   Probabilities are assigned to possible values of random variables.

8. **Random Variable Example**

   Suppose an experiment consists of tossing 3 fair coins. Let Y denote the number of heads.

   $$\Pr(Y=0)=\frac{1}{8},\quad \Pr(Y=1)=\frac{3}{8},\quad \Pr(Y=2)=\frac{3}{8},\quad \Pr(Y=3)=\frac{1}{8}$$

   Since Y must take one of the values $0, 1, 2, 3$,

   $$\sum_y \Pr(Y=y)=1$$

9. **Probability Mass Function**

   A random variable that can take on at most a countable number of possible values is said to be **discrete**.

   Let X be a discrete random variable with range

   $$R_X=\{x_1, x_2, x_3, \dots\}$$

   The function

   $$p(x_k)=\Pr(X=x_k)$$

   is called the **Probability Mass Function (PMF)** of X.

   Since X must take one of the values x_k,

   $$\sum_k p(x_k)=1$$

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