Modeling and Stabilization of port-Hamiltonian Systems

Analytical and Numerical Studies

A project report submitted by

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Engineering degree

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Abstract. In this graduation project, we studied the modeling and stabilization for a class of linear port-Hamiltonian systems. Firstly, we analyzed these systems with respect to the derivation of the equations from physical laws that govern the model dynamics (e.g. Newton's law). Then we wrote the different energies (kinetic and potential energy) to obtain the port-Hamiltonian framework. In practice, we develop a mathematical model to describe a flexible beam, which consists in manipulating the wave and Euler-Bernoulli equations. This study is motivated by an aeronautical issue: vibrations of flexible wings that lead to airplane instability. Then we reformulate this model into a port-Hamiltonian system. Secondly, we focused on the theoretical side including the second-order port-Hamiltonian system's well-posedness, by defining suitable boundary conditions that guarantee the well-posedness of this partial differential equation. For that, we employed the rich theory of one-parameter C_0-semigroups of linear operators. In addition, to reduce the beam's vibrations, we are interested in stabilization via feedback applied to external ports, because the change of energy of these systems is only possible via the boundary of its spatial domain. Furthermore, exponential stability is also proved under sufficient conditions thanks to Gearhart theorem. Finally, we investigated a discretization of the infinite-dimensional system into a finite-dimensional system that preserves the port-Hamiltonian structure: the staggered grid finite-difference space discretization is proposed for first-order port-Hamiltonian system. In this context, a numerical simulation is performed to compare the continuous and the discrete Hamiltonians (global energies). We have also shown that with this approach the discrete spectrum converges to the spectrum of the infinite-dimensional model for Dirichlet problem.

Keywords: Structure Mechanics, Euler-Bernoulli equation, port-Hamiltonian system, exponential stability, stabilization via feedback, C_0-semigroup, finite difference.

Résumé. Dans ce projet de recherche, nous avons étudié la modélisation et la stabilisation des systèmes hamiltoniens linéaires à port. Tout d'abord, nous avons analysé ces systèmes en ce qui concerne la dérivation des équations de mouvement à partir des lois physiques qui régissent la dynamique du modèle (par exemple, la loi de Newton). Ensuite, nous avons écrit les différentes énergies (énergie cinétique et énergie potentielle) pour obtenir la formulation port-hamiltonien. En pratique, nous avons développé un modèle mathématique pour décrire une poutre flexible (cette étude est motivée par une problématique aéronautique : les vibrations des ailes flexibles qui conduisent à l'instabilité des avions), qui consiste à manipuler les deux équations d'onde et d'Euler-Bernoulli, et avons reformulé ce modèle en un système hamiltonien à port. Ce modèle nous a amenés à nous concentrer sur un aspect théorique : le caractère bien posé du système hamiltonien à port du second ordre, en définissant des conditions aux limites appropriées qui garantissent l'existence, l'unicité et la stabilité des solutions de ce type d'équations aux dérivées partielles. Pour cela, nous avons utilisé la théorie des C_0-semigroupes à un paramètre d'opérateurs linéaires. En outre, pour réduire les vibrations de la poutre, nous nous intéressons à la stabilisation par retour d'état appliquée aux ports externes, car le changement d'énergie de ces systèmes n'est possible que via les frontières de son domaine spatial. De plus, la stabilité exponentielle est également prouvée sous certaines conditions grâce au théorème de Gearhart. Enfin, nous avons étudié une discrétisation du système continu de dimension infinie en un système de dimension finie qui préserve la structure hamiltonienne : il s'agit dans notre étude de la discrétisation spatiale par différences finies à grille décalée pour le système hamiltonien à port du premier ordre. Dans ce contexte, une simulation numérique est réalisée pour comparer les hamiltoniens continus et discrets (énergies globales). Nous avons également montré qu'avec cette approche, le spectre discret converge vers le spectre du modèle continu pour le problème de Dirichlet.

Mots clés : Mécanique de structure, équation d'Euler-Bernoulli , systèmes Hamiltoniens à port, stabilité exponentielle, stabilisation par retour d'état, C_0-semigroupe, différence finie.

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General context

This graduation project was carried out within "Laboratoire de Mathématiques et Dynamique de Populations" (LMDP) at Cadi Ayyad University, whose research fields include control theory and modeling of complex systems, in collaboration with Al-Khwarizmi department at University Mohammad VI Polytechnic. This department has as purpose to provide support to the various departments and research laboratories within the UM6P in the fields of applied computer science, scientific computing, and applied mathematics (differential models, operational research, optimization, financial mathematics, actuarial science, social sciences, artificial intelligence, etc.) in order to develop adequate teaching programs to the industry, services and applied research, both in Morocco and in Africa.

Research in the modeling of complex systems seems to be unlimited in several fields (e.g. biomedical engineering, aerospace industry, building structure) because it is necessary to have a mathematical model to interact with a system. Ranging from physical-based modeling that facilitates the creation of models capable of describing system dynamics, to port-Hamiltonian modeling that provides a unified framework inspired by the port-based modeling approach for modeling interconnected multi-physical systems using its Hamiltonian (i.e. total energy). Modeling of port-Hamiltonian systems has received a significant focus in the last years due to their many advantages in the modeling, analysis, and control of complex systems. In this context, the main goal of this work is to investigate the use of this formalism for modeling a flexible structure from the physical model represented by hypotheses and physical laws. From a mathematical point of view, we also focus on the well-posedness and exponential stability of second order port-Hamiltonian system.

In what follows, we give the motivation of elaborating this research and a short literature review on the port-Hamiltonian framework. We also provide the outline of this document.

Motivation

Greenhouse gas emissions caused by the aviation sector represent 12 % (source: ATAG 2019), but this percentage is relatively small compared to other sectors, the key challenge is that it is particularly hard to decarbonize air travel. Thus the need for improving the performance characteristics of aircraft in order to reduce fuel consumption.

By optimizing the structure (using light-weight materials, e.g. composite materials) or reducing drag (using long wings) we could save the fuel and improve the aerodynamic efficiency. However, these strategies usually contribute to increasing the structural flexibility of new designs. For instance, Boeing TTBW (Fig. 1) has a higher wing aspect ratio and therefore exhibits larger structural flexibility than the previous generation (check This for more details on this ratio).

Figure 1 – Boeing TTBW.

On the other hand, the structure flexibility could possibly lead to changes in the stability behavior (undamped oscillations could be exhibited). Also, the fuel inside tanks (which are wings is several airplanes) can modify the flight dynamics of airplanes (Graham and Rodriguez, 1951 [8]), potentially leading to fatal accidents caused by an unstable coupling between the low frequency modes and the fluid such as crashes of the Lockheed F-80C. Moreover, flight tests verified that a coupling between the fuel motion and airplane dynamics led to unstable motion. The influence of fuel sloshing on the flight mechanics of the aircraft is a difficult subject that is addressed for example in (Pizzoli M. 2020 [17]), a paper that provides an investigation of the effects of linear slosh dynamics on aeroelastic stability and response of flying wing configuration.

Indeed, there are multiple factors that cause aircraft imbalance including the flow of fuel from the tank to the engine, but the fuel quantity in various tanks is carefully monitored so no imbalance occurs. Fuel from any tank can be used by any engine and it is burned sequentially so that no tank is emptied. Those factors such as mass imbalance and damaged propeller impact on the overall balance of the aircraft, which brings about some unwanted vibration. However, with the right vibration meter at hand, it is easier to detect vibrations caused by such factors, some aircraft vibrations are very rough and dangerous. It can cause serious problems if an emergency landing is not deployed, for instance, the fuel sloshing inside the rigid tank (see Fig. 2) can cause a self-excited vibration (Farhat et al., 2013 [7]) of the wing around which air is flowing. Thus it impacts the aeroelastic stability with "flutter", but this concerns mainly military planes, which leads to intense vibrations in the aircraft if no control system is implemented.

Figure 2 – Lockheed T-33.

It is important to know that the two main characteristics that make the sloshing of the fuel can be penalized are the mass of liquid that ballots [4] (a part of the liquid at the bottom of the tank participates little in the sloshing) and the frequency associated with the first "mode" of sloshing. The objective is to decrease the balloting mass, decrease the forces generated on the structure, and increase the balloting frequency (1^{st} resonance frequency of the sloshing liquid) to limit the coupling. To do this, the simplest way is to add internal walls in the tank (anti-slosh baffles), or even to compartmentalize it as shown in Fig. 3. But this requires adding structural mass to the device, so there is a compromise to find.

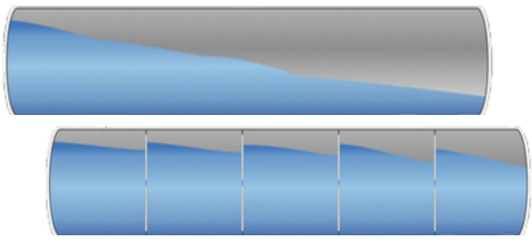


Figure 3 – Difference of quantity balloting between normal tank and tank with baffle.

The subject of flutter control is still being studied a lot for space launchers (especially with the new types of reusable launchers like SpaceX) and satellites. There is also currently a European project SLOWD that aims at improving the simulation of the flutter of the wings of airliners which shows that this field of research is still very active.

Furthermore, to achieve the mission of flying for months without landing, the spacecraft performance requirements become even more critical. Thus, most HALE airplanes have wings with aspect ratios several times bigger than usual, but large structural deflections lead to the loss of aircraft. As it's mentioned in [11], the accident report (Noll et al., 2004 [15]) highlighted that the "lack of adequate analysis methods (...)" related to the "(...) complex interactions among the flexible structure (...)" was among the root causes of the accident.

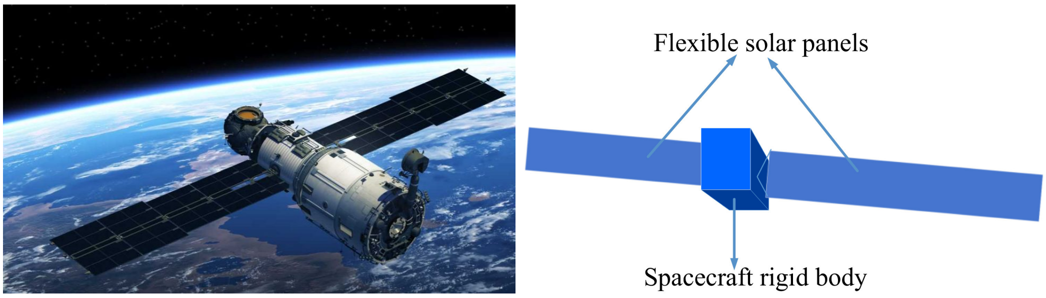


Figure 4 – Spacecraft with two flexible solar panels, and it's schema.

For all these reasons, airplanes with large fuel tanks and increased structural flexibility necessitate new tools to be used for modeling and control of such complex multidisciplinary coupled systems. This will be very useful in the future of the aerospace industry.

In this study, we investigate the use of the port-Hamiltonian formalism for modeling a system consisting of a flexible structure coupled to a tank partially filled with fluid. This device mimics the dynamics of a flexible wing with a tip fuel tank. We also focus on the theoretical side that includes studying the first and second order port-Hamiltonian systems well-posedness and stabilization.

Literature review

Firstly proposed in (B. Maschke, A. van der Schaft 1992 [12]), the port-Hamiltonian approach offers a geometric description for analysis, modeling, control, and simulation of complex physical systems, for lumped and distributed parameter models [13]. It represents an energy-based framework to describe the behavior of interconnected multi-physics systems through the study of their external energy exchanges. It presents a unified modeling structure taking into account the connection between subsystems, since energy and power are the only common parameters between all physics fields (e.g. fields shown in Fig. 5). As a result, any interconnection of port-Hamiltonian systems (PHS) could be written as port-Hamiltonian system. It's also well suited for the modeling and control of non-linear multi-physical systems. The PHS framework has been extended to systems described by boundary-controlled partial differential equations and led to powerful results regarding the analysis of control problems in infinite dimensions.

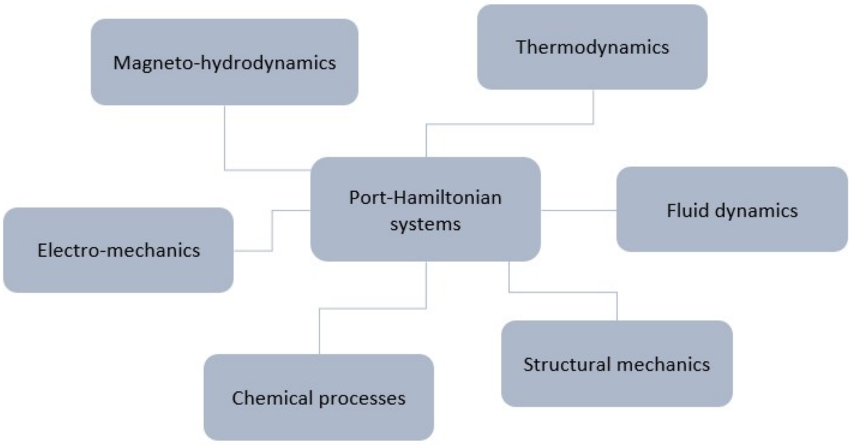


Figure 5 – Different physical domains in which the PH framework was applied to in the literature.

Port-based modelling is originally developed by H. M. Paynter (1961) [16]. It is based on a set of graphical notations to represent the structure of the physical system, as a set of ideal elements connected by edges, and linked by energy flow. In PH approach, linking is done via conjugate vector pairs of flow variables $f \in \mathbb{R}^k$ and effort variables $e \in \mathbb{R}^k$, with product $e^* f$ equal to power. Some examples of such pairs are voltages and currents in the electrical domain, velocities and forces in translational mechanics, flows and pressures in hydraulics, etc.

PHS express a system's fundamental internal interconnection structure through its geometric structure, defined by a set of structure matrices in the finite-dimensional case and differential operators in the infinite-dimensional case, namely,

$$f = Je, \tag{0.0.1}$$

where J is a skew-adjoint operator, i.e. $J^* = -J$.
Most times, the flow variable is the time derivative of state space variables, and the effort variable equals the gradient of the energy.

Outline

This document is divided into four chapters:

Chapter 1

We develop the mathematical model of our system that consists of a flexible beam with two motions (bending and torsion) modeled independently, a rigid tank, and sloshing fluid, and we end this chapter by pointing out the difficulties we found during the modeling of such a system.

Chapter 2

We introduce the finite and infinite linear port-Hamiltonian framework, and we focus on the well-posedness of second order port Hamiltonian systems. After that, we investigate the exponential stability of those systems.

Chapter 3

We are interested in writing our system as a port-Hamiltonian system. We determine suitable boundary conditions that guarantee the well-posedness of this system, then we prove the exponential stability via a feedback law.

Chapter 4

We introduce the general wave equation with a Neumann type boundary condition, and we discretize the continuous equation using the finite difference approximation in space and time. Then we show that the staggered finite difference is structure preserving discretization method for the first order port-Hamiltonian system. Afterwards, we compare the discrete Hamiltonian with the discrete wave energy and show that with this approach the discrete spectrum converges to the spectrum of the infinite-dimensional model for Dirichlet problem.

1 | Physical-based Modeling

Physically-based techniques facilitate the creation of models capable of automatically synthesizing complex shapes and realistic motions of the system. This modeling adds new levels of representation to graphics objects. In addition to geometry, forces, torques, velocities, accelerations, heat, and other physical quantities are used to describe the evolution of models, by applying physical laws that govern model behavior such as Ohm's law, Newton's law, ...

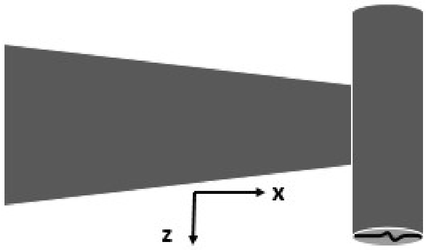


Figure 1.1 – Schematic representation.

We assimilate the airplane wing to a flexible beam and the fuel tank to a cylinder partially filled with an incompressible fluid. So our system consists of an aluminum beam with a fluid tank near the free tip. A schematic representation of the system is shown in Fig. 1.1. Since by the port-Hamiltonian formulation, it is possible to describe each element of the system separately, physically relevant variables appear as interconnected ports and the various subsystems can be coupled, ensuring that the overall system is also a port-Hamiltonian system. We will study each subsystem independently.

1.1 Beam

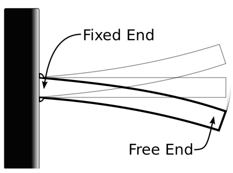


Figure 1.2 – Vibrating beam.

The beam is simplified as a 1D-beam with bending and torsion modeled independently. The linear Euler-Bernoulli equation is used for beam bending and the wave equation for beam torsion.

1.1.1 Euler-Bernoulli equation

Since we considered a long beam relatively to its depth and width (stresses developed perpendicular to beam length are ignored, otherwise we talk about the model of the Timoshenko beam) with an isotropic material that obeys Hooke's Law (i.e. it is linear and elastic), a small deformations, and plane sections along the time (this is true when a beam is subject to pure bending, and experiences zero shear deformation), we could apply the beam theory to get the equation

$$\mu(\xi)\frac{\partial^2}{\partial t^2}\omega(t,\xi) = -\frac{\partial^2}{\partial \xi^2}(E.I(\xi)\frac{\partial^2}{\partial \xi^2}\omega(t,\xi)), \tag{1.1.1}$$

where $\mu(\xi)$ is the mass density of the beam, $\omega(t,\xi)$ is the deflection at point ξ at time t, E is the Young modulus, and I the inertia.

Proof. Considering that the deflection is in y direction and the length of the beam is along x axis. Using Hooke's law: A rod of any elastic material may be viewed as a linear spring. Therefore for a rod with length L and cross-sectional area A, its tensile stress σ is linearly proportional to its fractional extension or strain ε by the modulus of elasticity E: $\sigma = E\varepsilon$ with $\varepsilon = \frac{\Delta L}{L}$.

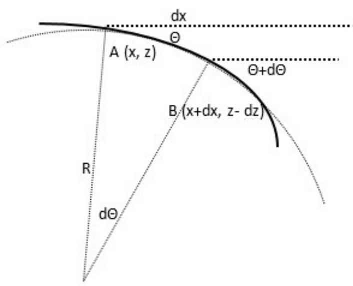


Figure 1.3 – 1D beam element in bending.

We have $\tan\theta = -\frac{dy}{ds} = -\frac{dy}{dx} \sim \theta$ (because θ is very small, the same for $d\theta$), and $\tan d\theta = \frac{dx}{R} \sim d\theta$. Thus $\frac{1}{R} = \frac{d\theta}{dx} = \frac{d}{dx}\left(-\frac{dy}{dx}\right) = -\frac{d^2y}{dx^2}$.

Bending moment is defined by $M = \frac{E.I}{R}$ with $I = \int y^2 dA$ is the second moment of area (see [14] for more details), and the force per element is given by f(x) $= \frac{d^2M}{dx^2} = -\frac{d^2}{dx^2}\left(E \cdot I \cdot \frac{d^2y}{dx^2}\right)$.

By applying second Newton's laws of motion, we get

$$\mu(x)\frac{d^2y}{dt^2}(x,t) = -\frac{d^2}{dx^2}\left(E \cdot I \cdot \frac{d^2y}{dx^2}\right).$$

□

1.1.2 Wave equation

Torsion in beams generally arises when the action of shear loads has points of application that do not coincide with the shear center of the beam section. It is described by the following wave equation:

$$I_p(\xi)\frac{\partial^2}{\partial t^2}\theta(t,\xi) = \frac{\partial}{\partial \xi}(GJ(\xi)\frac{\partial}{\partial \xi}\theta(t,\xi)), \tag{1.1.2}$$

where $\theta(t,\xi)$ is the local torsional angle, ξ is the position along the beam, t is time, G is the material shear constant (the transverse modulus of elasticity), J is the section torsion constant and I_p is the section polar moment of inertia per unit length.

Proof. The torsion angle per unit length being $\frac{\partial\theta}{\partial x}$, the torsion torque C in the section of abscissa x has the value

$$C = -G.J(x)\frac{\partial\theta}{\partial x}.$$

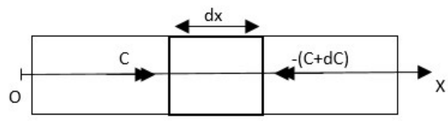


Figure 1.4 – Torque density applied to a beam element.

Let us write the fundamental equation of dynamics (motion about the Ox axis of the beam) for the beam element of moment of inertia $I_p dx$ lying between the x and $x+dx$ abscissa sections. Suppose that no torque density is applied to the beam; the torques applied to the element are C, and $-C-dC$ as shown in Fig. 1.4. Thus,

$$I_p \ \mathrm{d}x\frac{\partial^2\theta}{\partial t^2} = -\mathrm{d}C.$$

We thus obtain, given the value of C, the equation

$$I_p\frac{\partial^2\theta}{\partial t^2} = \frac{\partial}{\partial x}\left(G.J(x)\frac{\partial\theta}{\partial x}\right).$$

In the following, we consider that $c = \sqrt{G.J}$ is a constant that has the dimensions of a velocity. □

1.2 Rigid body

Let us consider a rigid tank with tree degrees of freedom, two are due to the beam bending (translation speed $\omega_B(t)$ and rotation speed $\theta_B(t)$) and one degree of freedom to rotation related torsion.

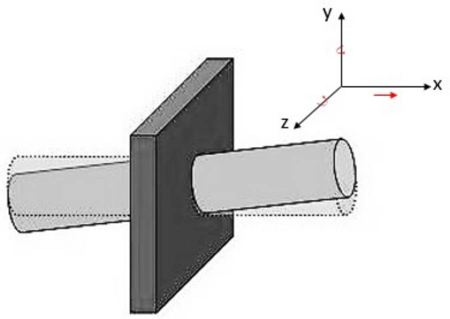


Figure 1.5 – Tank's degrees of freedom.

We simply apply second Newton's law to get the equations of motions

$$
\begin{aligned}
m_R \ddot{\omega_B}(t) &= F_{ext} \\
I_R^B \ddot{\theta_B}(t) &= M_{ext}^B \\
I_R^T \ddot{\theta}_t(t) &= M_{ext}^T,
\end{aligned}
\tag{1.2.1}
$$

where m_R is the tank mass, I_R^B and I_R^T are the tank rotational inertias. F_{ext} is the sum of forces applied to the tank, M_{ext}^B is the sum of moments in bending direction and M_{ext}^T is the sum of moments in torsion direction.

1.3 Saint-Venant equation

We consider a non-viscous and incompressible fluid within the rigid tank. The simplest infinite-dimensional approach for modeling sloshing fluid dynamics is the 1D Saint-Venant equations (the shallow-water equations in unidirectional form). There are other applications of the 1-D Saint-Venant equations including flood routing along rivers, dam break analysis, storm pulses in an open channel, as well as storm runoff in overland flow. These equations are obtained assuming that the fluid depth is very small compared to the tank length:

$$
\begin{aligned}
\frac{\partial \tilde{h}}{\partial t} &= -\frac{\partial \tilde{h} w}{\partial z} \\
\frac{\partial w}{\partial t} &= -\frac{\partial (g\tilde{h} + gz\theta)}{\partial z},
\end{aligned}
\tag{1.3.1}
$$

where w is the fluid speed along tank length, θ is the fluid rotation, $\tilde{h}$ is the difference between h, the fluid height at point z at time t, and the equilibrium height $\bar{h}$, with g the acceleration due to gravity.

Proof. For the first step of the derivation of the shallow-water equations, we consider the global mass conservation:

$$
\frac{\partial \rho}{\partial t} + \nabla(\rho V) = 0, \text{ then } \nabla(V) = 0, \tag{1.3.2}
$$

where $\rho = \text{cste}$ is the density and $V = (u, v, w)$ the three dimensional velocity.

We apply the momentum conservation equation to the fluid height:

$$v = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + V \cdot \nabla(h) = \frac{\partial h}{\partial t} + u \cdot \frac{\partial h}{\partial x} + v \cdot \frac{\partial h}{\partial y} + w \cdot \frac{\partial h}{\partial z} = \frac{\partial h}{\partial t} + u \cdot \frac{\partial h}{\partial x} + w \cdot \frac{\partial h}{\partial z}.$$

We integrate the continuity equation (1.3.2) vertically as follows:

$$\begin{aligned}\int_{\bar{h}}^{h} \nabla(V) \cdot dy &= \int_{\bar{h}}^{h} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \cdot dy \\ &= \frac{\partial}{\partial x} \int_{\bar{h}}^{h} u \cdot dy + \frac{\partial}{\partial z} \int_{\bar{h}}^{h} \omega \cdot dy - u(h) \cdot \frac{\partial h}{\partial x} - w(h) \cdot \frac{\partial h}{\partial z} + v(h) + u(\bar{h}) \cdot \frac{\partial \bar{h}}{\partial x} + w(\bar{h}) \cdot \frac{\partial \bar{h}}{\partial z} - v(\bar{h}) \\ &= \frac{\partial \widetilde{hu}}{\partial x} + \frac{\partial \widetilde{hw}}{\partial z} + \frac{\partial h}{\partial t} = 0.\end{aligned}$$

In the case where the horizontal length scale is greater than the height scale, the velocity in the x direction of the fluid is small compared to the velocity in the z direction, this implies that no displacement along x axis is shown:

$$\frac{\partial \widetilde{hw}}{\partial z} + \frac{\partial \tilde{h}}{\partial t} = 0.$$

Therefore the vertical pressure gradients are nearly hydrostatic, that is, $p = \rho g H + p_0$ with $H = h + z\theta$. By applying Euler's equation to this non viscous fluid, assuming all the conditions already mentioned, we get

$$\begin{aligned}\rho \frac{\partial w}{\partial t} &= -\frac{\partial P}{\partial z} \\ \frac{\partial w}{\partial t} &= -\frac{\partial (gh + gz\theta)}{\partial z}.\end{aligned}$$

Indeed, this second equation is true only when the frame of reference is considered fix relatively to the beam (in addition to assumptions already cited) but for modeling the fluid in same frame as the beam, we have to add tank translation speed to fluid speed D:

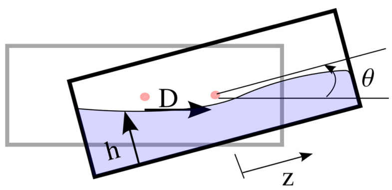


Figure 1.6 – Moving tank [11].

In this case, we have

$$\frac{\partial w}{\partial t} = -\dot{D}\cos\theta - \frac{\partial (g\tilde{h}\cos\theta + gz\sin\theta)}{\partial z}$$

Assuming that the angular rotation is small ($\theta \ll 0$), then

$$\begin{aligned}\rho \frac{\partial w}{\partial t} &= -\frac{\partial P}{\partial z} \\ \frac{\partial w}{\partial t} &= -\dot{D} - \frac{\partial (g\tilde{h} + gz\theta)}{\partial z}.\end{aligned} \tag{1.3.3}$$

□

Remark 1.3.1. The structural dynamics model used here is quite simple, we neglect the bending rotation inertia and shear stresses. Additionally, instead of using two models, one for the bending and other for the torsion, the use of a coupled system could provide more accurate results. For the fluid sloshing, it is unclear how it could be represented without the considered hypothesis, especially the non-viscous property, since the fuel is a viscous fluid.

2 | Port-Hamiltonian systems

2.1 Finite dimensional port-Hamiltonian systems

The general formulation of a finite dimensional Hamiltonian system without dissipation is given by:

$$\dot{x} = J.\frac{\partial H}{\partial x}(x), \tag{2.1.1}$$

where $x \in \mathbb{R}^d$ denotes the state variables, also named as energy variables, $J \in \mathbb{R}^{d\times d}$ is the interconnection matrix and is skew-symmetric representing the energy exchange in the system. H represents the Hamiltonian of the system. The time derivative of the Hamiltonian writes:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial^T H}{\partial x}\dot{x}. \tag{2.1.2}$$

According to (0.0.1), we can define a pair of power conjugated variables, flow variable f and effort variable e as

$$f = \dot{x}, \quad e = \frac{\partial H}{\partial x}(x).$$

Because of the skew-symmetry of the matrix $J(x)$, (2.1.2) equals:

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\partial^T H}{\partial x}(x)\dot{x} \\ &= \frac{\partial^T H}{\partial x}(x)J\frac{\partial H}{\partial x}(x) \\ &= 0.\end{aligned}$$

2.2 Infinite dimensional port-Hamiltonian systems

In the sequel, we focus on one-dimensional port-Hamiltonian systems, (i.e. systems with unidirectional motions), and we restrict ourselves to the linear case, in which the Hamiltonian is quadratic, i.e.,

$$H(t) = \frac{1}{2}\int_0^1 x(t,\xi)^* \mathcal{H}(\xi) x(t,\xi) d\xi.$$

This expression defines a new norm which is associated to the inner product

$$\langle f, g\rangle_{\mathcal{H}} := \frac{1}{2}\int_0^1 f(\xi)^* \mathcal{H}(\xi) g(\xi) d\xi, \quad f, g \in X,$$

where $X = L_2\left([0,1]\,;\mathbb{C}^d\right)$. Since we equip the Hilbert space X with this inner product, it is called the energy state space because the squared norm $\|x\|^2_{\mathcal{H}}$ equals the energy $H(t)$ associated to the linear port-Hamiltonian system.

In this case, the variational derivative of the energy functional H is given by

$$\frac{\partial H}{\partial x}(t,\xi) = \mathcal{H}(\xi)x(t,\xi),$$

where x is called the state space variable. Here we talk about linear port-Hamiltonian systems. This class covers in particular the wave equation, the transport equation, the Timoshenko beam equation, but also the Schrödinger equation and the Euler-Bernoulli beam equation. The general form is given by:

$$\frac{\partial x}{\partial t}(t,\xi) = \sum_{k=0}^{N} P_k \frac{\partial^k(\mathcal{H}x)}{\partial \xi^k}(t,\xi), \quad t \geq 0, \xi \in (0,1). \tag{2.2.1}$$

Here $P_k \in \mathbb{C}^{d\times d}, k = 0,1,\ldots,N$, always denote some complex matrices satisfying the condition

$$P_k^* = (-1)^{k+1}P_k, \quad k \geq 0.$$

The system (2.2.1) is called a N order linear port-Hamiltonian system associated to $\mathcal{H}$ the Hamiltonian density. In this work, we require P_0 to be skew-adjoint for simplicity. Moreover, we always assume that P_N is invertible. The Hamiltonian density matrix function $\mathcal{H} : (0,1) \to \mathbb{C}^{d\times d}$ is a measurable function such that it is uniformly positive, i.e. there exist $0 < m \leq M$ such that for almost every $\xi \in (0,1)$ the matrix $\mathcal{H}(\xi)$ is self-adjoint and

$$m|\xi|^2 \leq \xi^*\mathcal{H}(\xi)\xi \leq M|\xi|^2, \quad \xi \in \mathbb{C}^d.$$

Note that $\|\cdot\|_{\mathcal{H}}$ is equivalent to the standard L_2-norm $\|\cdot\|_{L_2}$.

Remark 2.2.1. In all definitions, we consider $[0,1]$ as the space interval. We could pass to any other interval $[a,b]$ by a change of variable using the following bijective function

$$\begin{aligned} f\colon [0,1] &\to [a,b] \\ x &\mapsto (b-a)x + a. \end{aligned}$$

The operator $A_0 : D\left(A_0\right) \subseteq X \to X$ corresponding to equation (2.2.1) is given by

$$\begin{aligned} A_0 x &= \sum_{k=0}^{N} P_k \frac{d^k}{d\xi^k}(\mathcal{H}x) \\ D\left(A_0\right) &= \left\{x \in X : \mathcal{H}x \in H^N\left(0,1;\mathbb{C}^d\right)\right\}. \end{aligned}$$

Thanks to the invertibility of P_N, the operator A_0 is closed, and $D(A_0)$ is the maximal domain.

Example 2.2.2 (Timoshenko beam). *This model takes into account shear deformation and rotational bending effects. Its motions equations are given by*

$$\rho(\xi)\frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial}{\partial \xi}\left(K(\xi)\left(\frac{\partial w}{\partial \xi}(t,\xi) - \phi(t,\xi)\right)\right), \quad \xi \in (0,1),\ t \geq 0,$$

$$I_\rho(\xi)\frac{\partial^2 \phi}{\partial t^2}(t,\xi) = \frac{\partial}{\partial \xi}\left(EI(\xi)\frac{\partial \phi}{\partial \xi}(t,\xi)\right) + K(\xi)\left(\frac{\partial w}{\partial \xi}(t,\xi) - \phi(t,\xi)\right),$$

where $w(t,\xi)$ is the transverse displacement of the beam and $\phi(t,\xi)$ is the rotation angle of a filament of the beam. The coefficients $\rho(\xi), I_\rho(\xi), EI(\xi)$, and $K(\xi)$ are the mass per unit length, the rotary moment of inertia of a cross section, the product of Young's modulus of elasticity and the moment of inertia of a cross section, and the shear modulus, respectively.

The energy/Hamiltonian for this system is given by

$$E(t) = \frac{1}{2}\int_0^1 (K(\xi)(\frac{\partial w}{\partial \xi}(t,\xi) - \phi(t,\xi))^2 + \rho(\xi)(\frac{\partial w}{\partial t}(t,\xi))^2 + EI(\xi)(\frac{\partial \phi}{\partial \xi}(t,\xi))^2 + I_\rho(\xi)(\frac{\partial \phi}{\partial t}(t,\xi))^2)d\xi.$$

We introduce the following energy variables.

$$\begin{aligned}
x_1(t,\xi) &= \tfrac{\partial w}{\partial \xi}(t,\xi) - \phi(t,\xi) && \textit{shear displacement}\\
x_2(t,\xi) &= \rho(\xi)\tfrac{\partial w}{\partial t}(t,\xi) && \textit{momentum}\\
x_3(t,\xi) &= \tfrac{\partial \phi}{\partial \xi}(t,\xi) && \textit{angular displacement}\\
x_4(t,\xi) &= I_\rho(\xi)\tfrac{\partial \phi}{\partial t}(t,\xi) && \textit{angular momentum.}
\end{aligned}$$

Formulating the Hamiltonian in those variables $x_1, \ldots, x_4$:

$$E(t) = \frac{1}{2}\int_0^1 K(\xi)x_1(t,\xi)^2 + \frac{1}{\rho(\xi)}x_2(t,\xi)^2 + EI(\xi)x_3(t,\xi)^2 + \frac{1}{I_\rho(\xi)}x_4(t,\xi)^2 d\xi.$$

Calculating the time derivative of the variables $x_1, \ldots, x_4$, we find

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1(t,\xi) \\ x_2(t,\xi) \\ x_3(t,\xi) \\ x_4(t,\xi) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \xi}\left(\frac{x_2(t,\xi)}{\rho(\xi)}\right) - \frac{x_4(t,\xi)}{I_\rho(\xi)} \\ \frac{\partial}{\partial \xi}\left(K(\xi)x_1(t,\xi)\right) \\ \frac{\partial}{\partial \xi}\left(\frac{x_4(t,\xi)}{I_\rho(\xi)}\right) \\ \frac{\partial}{\partial \xi}\left(EI(\xi)x_3(t,\xi)\right) + K(\xi)x_1(t,\xi) \end{pmatrix}.$$

Thus, we obtain

$$\dot{x} = P_1\frac{\partial}{\partial \xi}\mathcal{H}x + P_0\mathcal{H}x,$$

where $P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $P_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, *and* $\mathcal{H} = \begin{pmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & \frac{1}{T_\rho} \end{pmatrix}$.

We verify easily that

$$E(t) = \frac{1}{2}\int_0^1 x^*(t,\xi)\mathcal{H}(\xi)x(t,\xi)d\xi.$$

This shows that the Timoshenko beam can be written as a port-Hamiltonian system on $X = L^2\left([0,1];\mathbb{R}^4\right)$*. It is called a first order port-Hamiltonian system [9] (Always when* $N=1$*).*

Next we calculate the power using the above formulation, we find that

$$\frac{dE}{dt}(t) = \left[K(\xi)x_1(t,\xi)\frac{x_2(t,\xi)}{\rho(\xi)} + EI(\xi)x_3(t,\xi)\frac{x_4(t,\xi)}{I_\rho(\xi)}\right]_0^1.$$

Remark 2.2.3. We notice that the power goes via the boundary of the spatial domain. In other words, the change of energy of these systems is only possible via the boundary of its spatial domain (the ports).

Therefore, the best way to define boundary conditions based on the physical model is to reformulate them in the boundary effort and boundary flow. Let

$$\Phi : H^N\left(0,1;\mathbb{C}^d\right) \to \mathbb{C}^{2Nd}, \Phi(x) = \left(x(1),\ldots,x^{(N-1)}(1), x(0)\ldots,x^{(N-1)}(0)\right)$$

be the boundary trace operator and introduce the boundary port variables $\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$ defined via

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} Q & -Q \\ I & I \end{pmatrix}\Phi(\mathcal{H}x)$$

$$Q_{ij} = \begin{cases} (-1)^{j-1}P_{i+j-1}, & i+j \leq N+1 \\ 0, & \text{else.} \end{cases}$$

Note that the boundary port variables do not depend on the matrix P_0, so all finite dimensional port-Hamiltonian systems with $P_0^* = -P_0$ are conservatives.

In the following, we are interested in the second order port-Hamiltonian systems ($N=2$), and we follow the same approach of [9] that contains a complete study of first order port-Hamiltonian system. More specifically, the well-posedness and the stability theory or stabilization via feedback.

2.3 Well-posedness of port-Hamiltonian systems

For PDEs, which is a field extensively used for physical modelling (heat equation, transport equation, wave equation, etc), the question of existence, uniqueness and stability of solutions is more difficult than for ODEs. In real life, physicians see that the solution exists and is unique, because in the real world there is only one solution but after many hypothesis that we have taken to develop the mathematical model, we have to prove the well-posedness of this obtained PDE from a mathematical point of view, otherwise we would not know which solution would model what happens, for example if we look for controlling a system or predicting it's future state, if the PDE‘s solution is unique, then we are done, but if the solution is not unique, then we could not prove that the solution we found is what will actually happen in the real world, this means that the model we have proposed does not replicate necessary the reality of what is happening. So we must pose a new model with a unique solution as we only expect one result in testing, from a Mathematical perspective, we then have to specify additional boundary conditions that lead to our problem's well-posedness (i.e. existence of a solution, and only one solution.). To do this, we employ the rich theory of one-parameter C_0-semigroups of linear operators and only consider homogeneous PDE.

Definition 2.3.1. *Let X be a Hilbert space. $(T(t))_{t\geq 0}$ is called a strongly continuous semigroup if the following holds:*

1. *For all $t \geq 0, T(t)$ is a bounded linear operator on X, i.e., $T(t) \in \mathcal{L}(X)$;*
2. *$T(0) = I$;*
3. *$T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$;*
4. *For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$, i.e., $t \mapsto T(t)$ is strongly continuous at zero.*

In addition, if $\|T(t)\| \leq 1$ for every $t \geq 0$, then $(T(t))_{t\geq 0}$ is called a contraction semigroup.

We call X the state space, and its elements states. The easiest example of a strongly continuous semigroup is the exponential of a matrix, the matrix-valued function $T(t) = e^{At}$ defines a C_0-semigroup on the Hilbert space $\mathbb{R}^n$ with A an $n \times n$ matrix.

Definition 2.3.2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X. If the following limit exists*

$$\lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t},$$

then we say that x_0 is an element of the domain of A, shortly $x_0 \in D(A)$, and we define Ax_0 as

$$Ax_0 = \lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t}.$$

We call A the infinitesimal generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$

In [9], it's proven that for every $x_0 \in D(A)$ the function $t \mapsto T(t)x_0$ is differentiable. This enables us to link a strongly continuous semigroup (uniquely) to an abstract differential equation, thus if A the associated operator of a PDE generates a strongly continuous semigroup, then our problem is well posed (possess a solution, and it's unique).

Next, we follow the unified approach of port-Hamiltonian systems, and we especially consider the following PDE

$$\frac{\partial X}{\partial t}(t,\xi) = J\mathcal{H}(\xi)X(t,\xi),$$

and it could be writing as

$$\frac{\partial X}{\partial t}(t,\xi) = P_2\frac{\partial^2}{\partial \xi^2}\mathcal{H}(\xi)X(t,\xi) + P_1\frac{\partial}{\partial \xi}\mathcal{H}(\xi)X(t,\xi) + P_0\mathcal{H}(\xi)X(t,\xi), \tag{2.3.1}$$

where

$$J = P_2 \frac{\partial^2}{\partial \xi^2} + P_1 \frac{\partial}{\partial \xi} + P_0.$$

In practical, it's easy to verify that J is a skew symmetric operator using a specific boundary conditions. This differential equation is called a linear second order port-Hamiltonian system. The associated Hamiltonian (most times, it equals the energy of the global system) is given by

$$H(X) = \frac{1}{2} \int X^*(t,\xi)(\mathcal{H}(\xi)X(t,\xi)d\xi.$$

Let's consider the operator A_0 defined on

$$D(A_0) = \{X \in L^2([0,1], \mathbb{R}^n) /\ \mathcal{H}X \in H^2([0,1], \mathbb{R}^n)\}$$

such that

$$A_0 X(\xi) = P_2 \frac{\partial^2}{\partial \xi^2} \mathcal{H}(\xi)X(\xi) + P_1 \frac{\partial}{\partial \xi} \mathcal{H}(\xi)X(\xi) + P_0 \mathcal{H}(\xi)X(\xi)$$

for all $X \in L^2([0,1], \mathbb{R}^n)$.

$D(A_0)$ is the maximal domain, in order to guarantee that this equation possesses a unique solution we have to add boundary conditions. As we already see, it is better to formulate them in the boundary effort and boundary flow, for N=2, they are defined as

$$\begin{pmatrix} f \\ e \end{pmatrix} = R\,\Phi(\mathcal{H}X),$$

where

$$R = \begin{pmatrix} P_1 & P_2 & -P_1 & -P_2 \\ -P_2 & 0 & P_2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \Phi(\mathcal{H}X) = \begin{pmatrix} \mathcal{H}X(1) \\ \frac{\partial}{\partial \xi}\mathcal{H}X(1) \\ \mathcal{H}X(0) \\ \frac{\partial}{\partial \xi}\mathcal{H}X(0) \end{pmatrix}.$$

Let $W \in \mathbb{R}^{n \times 2n}$ with full rank such that $W \begin{pmatrix} f \\ e \end{pmatrix} = 0$,

we define the operator $AX = A_0 X$ such that $D(A) = \left\{ D(A_0) :\ W \begin{pmatrix} f \\ e \end{pmatrix} = 0 \right\}$.

Now, the question is when this closed densely defined operator is an infinitesimal generator of a C_0-semigroup, so we aim to characterize homogeneous boundary conditions such that (2.3.1) possesses a unique solution.

Lemma 2.3.3. *Let us consider the operator A defined as above, then we have*

$$\operatorname{Re} \langle AX, X \rangle = \frac{1}{4}(f^* e + e^* f).$$

Proof.

$$\langle AX, X\rangle + \langle X, AX\rangle = \frac{1}{2}\int_0^1 X^*\mathcal{H}AXd\xi + \frac{1}{2}\int_0^1 (AX)^*\mathcal{H}Xd\xi$$

$$= \frac{1}{2}\int_0^1 X^*\mathcal{H}(P_2\frac{\partial^2}{\partial\xi^2}\mathcal{H}X + P_1\frac{\partial}{\partial\xi}\mathcal{H}X + P_0\mathcal{H}X) + (P_2\frac{\partial^2}{\partial\xi^2}\mathcal{H}X + P_1\frac{\partial}{\partial\xi}\mathcal{H}X + P_0\mathcal{H}X)^*\mathcal{H}Xd\xi.$$

Since P_1 is self-adjoint, and P_0 and P_2 are skew-adjoint, we find

$$= \frac{1}{2}\int_0^1 (\mathcal{H}X)^*P_2\frac{\partial^2}{\partial\xi^2}\mathcal{H}X - (\frac{\partial^2}{\partial\xi^2}\mathcal{H}X)^*P_2\mathcal{H}Xd\xi$$

$$+\frac{1}{2}\int_0^1 (\mathcal{H}X)^*P_1\frac{\partial}{\partial\xi}\mathcal{H}X + (\frac{\partial}{\partial\xi}\mathcal{H}X)^*P_1\mathcal{H}Xd\xi$$

$$+\frac{1}{2}\int_0^1 (\mathcal{H}X)^*P_0\mathcal{H}X - (\mathcal{H}X)^*P_0\mathcal{H}Xd\xi$$

$$= \frac{1}{2}\int_0^1 (\mathcal{H}X)^*P_2\frac{\partial^2}{\partial\xi^2}\mathcal{H}X - (\frac{\partial^2}{\partial\xi^2}\mathcal{H}X)^*P_2\mathcal{H}Xd\xi + \frac{1}{2}\int_0^1 \frac{\partial}{\partial\xi}((\mathcal{H}X)^*P_1\mathcal{H}X)\xi.$$

By partial integration of 1st integral, we find

$$\mathrm{Re}\,(\langle AX, X\rangle) = \frac{1}{2}((\mathcal{H}X)^*P_2\frac{\partial}{\partial\xi}\mathcal{H}X(1) - (\mathcal{H}X)^*P_2\frac{\partial}{\partial\xi}\mathcal{H}X(0) + \frac{1}{2}((\mathcal{H}X)^*P_1\mathcal{H}X(1) - (\mathcal{H}X)^*P_1\mathcal{H}X(0))).$$

Then

$$\mathrm{Re}\,\langle AX, X\rangle = \frac{1}{4}(f^*e + e^*f).$$

□

Lemma 2.3.4.

$$\forall \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} \in K^{4n} \quad , \quad \exists x_0 \in D(A_0) tel\ que \begin{pmatrix} \mathcal{H}x_0(1) \\ \frac{\partial}{\partial\xi}\mathcal{H}x_0(1) \\ \mathcal{H}x_0(0) \\ \frac{\partial}{\partial\xi}\mathcal{H}x_0(0) \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix}.$$

Proof. For all $\begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} \in K^{4n}$, we could define x_0 as follow

$$x_0(\xi) = \mathcal{H}^{-1}(a\xi^3 + b\xi^2 + c\xi + d).$$

We write a, b,c, and d in terms of u,v,w,y, we have $x_0 \in D(A_0)$ and satisfies boundary conditions, by solving the following linear system

$$\begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

thanks to the invertibility of this matrix. □

Lemma 2.3.5. *Let W be a $n \times 2n$ matrix and let $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Then W has rank n and $W\Sigma W^* \geq 0$ if and only if there exist a matrix $V \in K^{n\times n}$ and an invertible matrix $S \in K^{n\times n}$ such that*

$$W = S \begin{pmatrix} I+V & I-V \end{pmatrix}$$

with $VV^* \leq I$, or equivalently $V^*V \leq I$.

Lemma 2.3.6. *Suppose that the $n \times 2n$ matrix W can be written in the format defined in Lemma 2.3.5, i.e. $W = S \begin{pmatrix} I+V & I-V \end{pmatrix}$ with S and V square matrices, and S invertible. Then the kernel of W equals the range of $\begin{pmatrix} I-V \\ -I-V \end{pmatrix}$.*

Lemma 2.3.7. *Let Z be a Hilbert space with inner product $\langle .,. \rangle$, and let $\mathcal{P} \in L(Z)$ be a coercive operator on Z, i.e., P is self-adjoint and $\mathcal{P} \geq \epsilon I$, for some $\epsilon \geq 0$. We define $Z_{\mathcal{P}}$ as the Hilbert space Z with the inner product $\langle .,. \rangle_{\mathcal{P}} = \langle ., \mathcal{P}. \rangle$. Then the operator A with domain D(A) generates a contraction semigroup on Z if and only if $A\mathcal{P}$ with domain $D(A\mathcal{P}) = \{z \in Z | \mathcal{P}_z \in D(A)\}$ generates a contraction semigroup on $Z_{\mathcal{P}}$.*

Definition 2.3.8. *Let A be linear operator $A : D(A) \subset X \to X$, it is called dissipative, if*

$$\langle AX, X \rangle \leq 0 \ , \ x \in D(A)$$

Theorem 2.3.9 (Lumer-Phillips Theorem)**.** *Let A be a linear operator with domain $D(A)$ on a Hilbert space X. Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ on X if and only if A is dissipative and* $\mathrm{ran}(I - A) = X$.

Theorem 2.3.10. *The following are equivalent:*

1. *A is the infinitesimal generator of a contraction semigroup*
2. $\mathrm{Re}\,\langle Ax, x \rangle \leq 0$
3. $W_B \Sigma W_B^* \geq 0$

Proof. $1 \implies 2$, by Lumer-Phillips theorem.

$2 \implies 3$, we assume that $\mathrm{Re}\,\langle AX, X \rangle \leq 0$ for every $x \in D(A)$.

Lemma 2.3.3 implies $f^*e + e^*f \leq 0$ for every $x \in D(A)$. Furthermore, by Lemma 2.3.4 for every pair For all $\begin{pmatrix} f \\ e \end{pmatrix} \in KerW_B$, there exists a function $x \in D(A)$ with boundary effort and flow $\begin{pmatrix} f \\ e \end{pmatrix}$, Thus we have $f^*e + e^*f \leq 0$ for every pair $\begin{pmatrix} f \\ e \end{pmatrix} \in Ker\,W_B$.

We write W_B as $W_B = W_1 + W_2$. Let y be in $ker(W_1 + W_2)$, then $W_B \begin{pmatrix} y \\ y \end{pmatrix} = 0$, means $y^*y + yy^* = 0$ which implies $y = 0$. This shows that the matrix $W_B = W_1 + W_2$ is injective, and hence invertible.

Let's define $V = (W_1 + W_2)^{-1}(W_1 - W_2)$, we have $\begin{pmatrix} W_1 & W_2 \end{pmatrix} = \frac{1}{2}(W_1 + W_2) \begin{pmatrix} I+V \\ I-V \end{pmatrix}$. Let $\begin{pmatrix} f \\ e \end{pmatrix} \in KerW_B$ be arbitrary. By Lemma 2.3.6 there exists a vector l such that $\begin{pmatrix} f \\ e \end{pmatrix} = \begin{pmatrix} I-V \\ -I-V \end{pmatrix} l$. This implies $f^*e + e^*f = l^*(-2I + 2V^*V)l \leq 0$. This inequality holds for any $\begin{pmatrix} f \\ e \end{pmatrix} \in KerW_B$. Since the $n \times 2n$ matrix W_B has rank n, $dim(kerW_B) = n$, and so the set of vectors l satisfying $\begin{pmatrix} f \\ e \end{pmatrix} = \begin{pmatrix} I-V \\ -I-V \end{pmatrix} l$ equals the whole space K^n. Hence $V^*V \leq I$,and by Lemma 2.3.5 we obtain $W_B \Sigma W_B^* \geq 0$

$3 \implies 1$ By lemma 2.3.7, it is sufficient to prove this implication for $\mathcal{H} = I$.

Dissipativity of A:

Let $x \in D(A)$. Lemma 2.3.3 implies

$$\langle AX, X\rangle + \langle X, AX\rangle = \frac{1}{2}(f^*e + e^*f)$$

By assumption, the vector $\begin{pmatrix} f \\ e \end{pmatrix}$ in $kerW_B$, Using lemma 2.3.6, $\begin{pmatrix} f \\ e \end{pmatrix} = \begin{pmatrix} I-V \\ -I-V \end{pmatrix} l$ for some l. Thus we get

$$\langle AX, X\rangle + \langle X, AX\rangle = \frac{1}{2}(f^*e + e^*f) = l^*(-2I + 2V^*V)l \leq 0$$

where we used again Lemma 2.3.3. This implies the dissipativity of A.

By the Lumer-Phillips Theorem 2.3.9 it remains to show that the range of $I - A$ equals X. The equation $(I - A)x = y$ is equivalent to the differential equation

$$x(\xi) - P_2\ddot{x}(\xi) - P_1\dot{x}(\xi) - P_0x(\xi) = y(\xi), \quad \xi \in [0, 1] \tag{2.3.2}$$

Defining Y such that $Y = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$, the equation (2.3.2) becomes

$$\dot{Y} = \begin{pmatrix} 0 & I \\ P_2^{-1}(I - P_0) & P_2^{-1}(P_1) \end{pmatrix} Y + \begin{pmatrix} 0 \\ P_2^{-1} \end{pmatrix} \tag{2.3.3}$$

Thanks to the invertibility of the matrix P_2, the solution of (2.3.3)is given by

$$Y = e^{\begin{pmatrix} 0 & I \\ P_2^{-1}(I-P_0) & P_2^{-1}(P_1) \end{pmatrix}(\xi-0)} Y(0) + \int_0^{\xi} e^{\begin{pmatrix} 0 & I \\ P_2^{-1}(I-P_0) & P_2^{-1}(P_1) \end{pmatrix}(\xi-t)} \begin{pmatrix} 0 \\ P_2^{-1} \end{pmatrix} y(t)dt \tag{2.3.4}$$

$x \in D(A)$ if and only if

$$W_B R_0 \begin{pmatrix} Y(1) \\ Y(0) \end{pmatrix} = W_B R_0 \begin{pmatrix} EY(0)+q \\ Y(0) \end{pmatrix} = 0, \quad (III)$$

where

$$E = e^{\begin{pmatrix} 0 & I \\ P_2^{-1}(I-P_0) & P_2^{-1}(P_1) \end{pmatrix}(1-0)}, \quad q = \int_0^{1} e^{\begin{pmatrix} 0 & I \\ P_2^{-1}(I-P_0) & P_2^{-1}(P_1) \end{pmatrix}(\xi-t)} \begin{pmatrix} 0 \\ P_2^{-1} \end{pmatrix} y(t)dt$$

This Equation can be equivalently written as

$$W_B R_0 \begin{pmatrix} E \\ I \end{pmatrix} Y(0) = -W_B R_0 \begin{pmatrix} q \\ 0 \end{pmatrix}$$

Next we prove that the square matrix $W_B R_0 \begin{pmatrix} E \\ I \end{pmatrix}$ is invertible, it's sufficient to prove it is injective.

Therefore, we assume that there exists a vector $r \neq 0$ such that $W_B R_0 \begin{pmatrix} E \\ I \end{pmatrix} r = 0.$

For Y being the solution of (2.3.3) with $Y(0) = r$ and $y = 0$, is an eigenfunction of A with eigenvalue one. However A possesses no eigenvalues positive, which leads to a contradiction, implying that $W_B R_0 \begin{pmatrix} E \\ I \end{pmatrix}$ is invertible, and so (2.3.2) has a unique solution in $D(A)$. The function y was arbitrary, so we have proved that $\operatorname{ran}(I - A) = X$.

Using the Lumer-Phillips Theorem 2.3.9, we conclude that A generates a contraction semigroup. □

Proposition 2.3.11. *The operator A has compact resolvent, i.e., its resolvent $R(\lambda, A)$ is compact for $\lambda \in \rho(A)$.*

Proof. First, let us show that the embedding $D(A_0) \hookrightarrow X$ is compact. The operators $\mathcal{H} : L^2(0,1;\mathbb{C}^n) \to X$ and $\mathcal{H}^{-1} : D(A_0) \to H^N(0,1;\mathbb{C}^n)$ are continuous, and by Rellich-Kondrachov theorem, the embedding $i_{H^N} : H^N(0,1;\mathbb{C}^n) \hookrightarrow L^2(0,1;\mathbb{C}^n)$ is compact. Therefore, the embedding $i_{D(A_0)} : D(A_0) \hookrightarrow X$ is compact since $i_{D(A_0)} = \mathcal{H} \circ i_{H^N} \circ \mathcal{H}^{-1}$ is the composition of a compact operator with bounded operators. Now, let $\lambda \in \rho(A)$. Since the operator $R(\lambda, A) : X \to D(A) \subseteq D(A_0)$ is bounded and $D(A_0) \hookrightarrow X$ is compact, then $R(\lambda, A) : X \to X$ is compact. □

Example 2.3.12 (Vibrating string). *Let's consider a vibrating string, it's fixed in the right end side and free at the left end side, using law's Hooke and second law's Newton, it's equation of motion is given by :*

$$\frac{\partial^2 \omega}{\partial t^2}(t,\xi) = \frac{1}{\rho(\xi)} \cdot \frac{\partial}{\partial t}\left(T(\xi) \cdot \frac{\partial \omega}{\partial \xi}(t,\xi)\right) \tag{2.3.5}$$

Derivation of motion equation motion:

Let's consider $\vec{F}$ the effort applied to en element of the string at position ξ, where the vertical is z axis:

$$\begin{aligned}\vec{F} \cdot \vec{z} =& T(\xi + d\xi, t) \cdot \sin(\theta(\xi + d\xi, t)) - T(t,\xi) \cdot \sin(\theta(t,\xi)) \\ =& \frac{\partial}{\partial \xi}(T(t,\xi) \cdot \sin(\theta(t,\xi)))\end{aligned}$$

In the other hand, we have : $\sin(\theta(t,\xi)) = \frac{\partial \omega}{\partial \xi}(t,\xi)$ *By applying second law's Newton with projection on vertical axis, we obtain :*

$$\rho(\xi) \cdot \frac{\partial^2 \omega}{\partial t^2}(t,\xi) = \frac{\partial}{\partial \xi}\left(T(\xi) \cdot \frac{\partial \omega}{\partial \xi}(t,\xi)\right)$$

Defining energy variables as the following : $x_1 = \rho(\xi) \cdot \frac{\partial \omega}{\partial t}(t,\xi)$ *et* $x_2 = \frac{\partial \omega}{\partial \xi}(t,\xi)$*, with boundary conditions: no stress in the right side end is applied :* $x_2(L,t) = 0$*, and since it's fixed in the other end then* $x_1(0,t) = 0$

The Hamiltonian associated is the sum of kinetic energy : $dE_c = \frac{1}{2}\rho(\xi) \cdot \left(\frac{\partial \omega}{\partial t}(t,\xi)\right)^2$ *and potential elastic energy :* $dE_p = \frac{1}{2}T(\xi) \cdot \left(\frac{\partial \omega}{\partial \xi}(t,\xi)\right)^2$*, then*

$$H = \frac{1}{2}\int_a^b T(\xi) \cdot \left(\frac{\partial \omega}{\partial \xi}(t,\xi)\right)^2 + \rho(\xi) \cdot \left(\frac{\partial \omega}{\partial t}(t,\xi)\right)^2 d\xi$$

so (2.3.5) *became:*

$$\frac{\partial x_1}{\partial t} = \frac{\partial}{\partial \xi}\left(T(\xi) \cdot x_2\right)$$

Therefore :

$$\dot{\mathrm{X}} = \frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{\partial}{\partial \xi}\begin{pmatrix} \frac{1}{\rho(\xi)} & 0 \\ 0 & T(\xi) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We have $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $P_0 = 0$*, and* $\mathcal{H}(\xi) = \begin{pmatrix} \frac{1}{\rho(\xi)} & 0 \\ 0 & T(\xi) \end{pmatrix}$*. The boundary variables are given by*

$$f_\partial = \frac{1}{\sqrt{2}}\begin{pmatrix} T(b)\frac{\partial w}{\partial \xi}(b) - T(a)\frac{\partial w}{\partial \xi}(a) \\ \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \end{pmatrix}, e_\partial = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ T(b)\frac{\partial w}{\partial \xi}(b) + T(a)\frac{\partial w}{\partial \xi}(a) \end{pmatrix}.$$

The boundary condition becomes in these variables

$$\begin{aligned}\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} T(b)\frac{\partial w}{\partial \xi}(b,t) \\ \frac{\partial w}{\partial t}(a,t) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = W_B\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}\end{aligned}$$

with $W_B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$. *Since* W_B *is a* 2×4 *malrix with rank 2 , and since* $W_B \Sigma W_B^T = 0$, *we conclude from Theorem* (2.3.10)*that the operator associated to the p.d.e. generates a contraction semigroup.*

2.4 Exponential stability

In this section, we investigate the exponential stability of port-Hamiltonian systems, we refer to [2, 3] for more details.

Definition 2.4.1. *The* C_0*-semigroup* $(T(t))_{t \geq 0}$ *on the Hilbert space* X *is exponentially stable if there exist positive constants* M *and* ω *such that*

$$\|T(t)\| \leq M e^{-\omega t} \quad \textit{for } t \geq 0$$

Theorem 2.4.2 (Exponential Stability: Gearhart, Greiner, Prüss, Huang)**.** *Let* B *generate a bounded* C_0*-semigroup* $(T(t))_{t \geq 0}$ *on a Hilbert space* Y*. Then* $(T(t))_{t \geq 0}$ *is exponentially stable if and only if*

$$\sigma(B) \subseteq \mathbb{C}_0^-, \quad \sup_{\omega \in \mathbb{R}} \|R(i\omega, B)\| < +\infty$$

Remark 2.4.3. The uniform boundedness of the resolvent on $i\mathbb{R}$ in Theorem 2.2 is equivalent to the condition

$$\left.\begin{array}{l} (x_n, \beta_n) \subseteq D(B) \times \mathbb{R} \\ \sup_{n \in \mathbb{N}} \|x_n\| < +\infty, |\beta_n| \to \infty \\ Bx_n - i\beta_n x_n \to 0 \end{array}\right\} \Rightarrow x_n \to 0$$

Proof. First assume that

$$\sup_{\omega \in \mathbb{R}} \|R(i\omega, B)\| < +\infty$$

and take any sequence $(x_n, \beta_n) \subseteq D(B) \times \mathbb{R}$ where $\sup_{n \in \mathbb{N}} \|x_n\|_X < +\infty$ and $|\beta_n| \to \infty$ and such that $Bx_n - i\beta_n x_n \to 0$ as $n \to \infty$. Then we obtain

$$\|x_n\|_X \leq \sup_{\omega \in \mathbb{R}} \|R(i\omega, B)\| \, \|(B - i\beta_n) \, x_n\|_X \overset{n\to\infty}{\longrightarrow} 0$$

Conversely, if $\sup_{\omega \in \mathbb{R}} \|R(i\omega, B)\| = +\infty$, then there is a sequence $(z_n, \beta_n)_{n \geq 1} \subseteq D(B) \times \mathbb{R}$ such that $\|z_n\|_X = 1 (n \in \mathbb{N})$ and $|\beta_n| \overset{n\to\infty}{\longrightarrow} \infty$ with

$$\|R\left(i\beta_n, B\right) z_n\| \overset{n\to\infty}{\longrightarrow} +\infty.$$

Then we set $x_n := \frac{R(i\beta_n, B) z_n}{\|R(i\beta_n, B) z_n\|} \in D(B)$ and observe that $\|x_n\|_X = 1 (n \in \mathbb{N})$, but still

$$\|Bx_n - i\beta_n x_n\| = \frac{\|z_n\|}{\|R\left(i\beta_n, B\right) z_n\|} \overset{n\to\infty}{\longrightarrow} 0,$$

so that the sequence criterion cannot hold. $\square$

Lemma 2.4.4. *Let $0 \leq k < N \in \mathbb{N}_0$ and $\theta \in (0,1)$ such that $\eta := \theta N \in \left(k + \frac{1}{2}, k+1\right)$ Then there exist a constant $c_\theta > 0$ such that for all $f \in H^N\left(0,1;\mathbb{C}^d\right)$*

$$\|f\|_{C^k} \leq c_\theta \|f\|_{L_2}^{1-\theta} \|f\|_{H^N}^{\theta}.$$

Further for $\sigma := \frac{k}{N}$ there exists a constant $c_\sigma > 0$ such that for all $f \in H^N\left(0,1;\mathbb{C}^d\right)$

$$\|f\|_{H^k} \leq c_\sigma \|f\|_{L_2}^{1-\sigma} \|f\|_{H^N}^{\sigma}$$

Proof. It's detailed in [3] □

Theorem 2.4.5. *Let A be an operator defined as above, assume that $\mathcal{H}x \in H^1\left([a,b];\mathbb{C}^{d\times d}\right)$.*

1. *Case $N = 1$, if*

$$\operatorname{Re}\langle Ax, x\rangle_{\mathcal{H}} \leq -\kappa |(\mathcal{H}x)(a)|^2, \quad x \in D(A)$$

   holds for some $\kappa > 0$, then A generates an exponentially stable C_0 semigroup.

2. *Case $N = 2$, if*

$$\operatorname{Re}\langle Ax, x\rangle_{\mathcal{H}} \leq -\kappa \left[|(\mathcal{H}x)(a)|^2 + \left|(\mathcal{H}x)'(a)\right|^2 + \left\{ \begin{array}{c} |(\mathcal{H}x)(b)|^2 \\ or \\ |(\mathcal{H}x)'(b)|^2 \end{array} \right\} \right], \quad x \in D(A)$$

   holds for some $\kappa > 0$. Then $(T(t))_{t\geq 0}$ is an exponentially stable and contractive C_0-semigroup.

Proof. 1. it's obvious that A generates a contractive C_0 semigroup $(T(t))_{t\geq 0}$, since

$$\operatorname{Re}\langle Ax, x\rangle_{\mathcal{H}} \leq -\kappa |(\mathcal{H}x)(a)|^2 \leq 0$$

where $\kappa > 0$, for all $x \in D(A)$, with a compact resolvent $(\sigma(A) = \sigma_p(A))$.

Let $1 \in \mathbb{R}$ and $x \in D(A)$ and $x \neq 0$ with

$$i1x = Ax$$

then

$$|(\mathcal{H}x)(a)|^2 \leq -\frac{1}{\kappa}\operatorname{Re}\langle Ax, x\rangle = -\frac{1}{\kappa}\operatorname{Re}\langle i1x, x\rangle = 0$$

we have a system of ordinary differential equations

$$i1x(\xi) = P_1(\mathcal{H}x)^{'}(\xi) + P_0(\mathcal{H}x)(\xi), \quad \xi \in (a,b)$$

Since P_1 is invertible the unique solution of this initial value problem is $x = 0$, so $i\mathbb{R}\cap\sigma_p(A) = \emptyset$.

Since $(T(t))_{t\geq 0}$ is a bounded C_0 semigroup, then $\sigma(A) = \sigma_p(A) \subseteq \mathbb{C}_0^-$.

Let $((x_n,\beta_n))_{n\geq 1} \subseteq D(A)\times\mathbb{R}$ be any sequence with $\|x_n\| \leq c$ and $|\beta_n| \to \infty$ such that $Ax_n - i\beta_n x_n \overset{n\to\infty}{\longrightarrow} 0$ it follows that

$$0 \leftarrow \frac{-1}{\kappa}\operatorname{Re}\langle i\beta_n x_n - Ax_n, x_n\rangle \geq ||(\mathcal{H}x_n)(a)||^2 \geq 0$$

i.e. $\quad |(\mathcal{H}x_n)(a)|^2 \overset{n\to\infty}{\longrightarrow} 0.$

Then $\frac{x_n}{\beta_n}$ is bounded in the graph norm $\|\cdot\|_A = \|\cdot\|_{L_2} + \|A\cdot\|_{L_2}$, and since this norm is equivalent to the norm $\|\mathcal{H}\cdot\|_{H^1}$ where $\|\mathcal{H}x\|_{H^T} = \|\mathcal{H}x\|_{L_2} + \|(\mathcal{H}x)'\|_{L_2}$, hence

$$\left\|\frac{(\mathcal{H}x_n)'}{\beta_n}\right\|_{L_2} \leq c, \qquad \text{for all } n\in\mathbb{N}$$

By integration by parts we get

$$\operatorname{Re}\langle x', \mathcal{H}x\rangle = -\frac{1}{2}\operatorname{Re}\langle x, \mathcal{H}'x\rangle + \frac{1}{2}\left[x(\xi)^*\mathcal{H}(\xi)x(\xi)\right]_a^b.$$

Let $q\in C^1([0,1];\mathbb{R})$ with $q(1)=0$ and we find

$$\begin{aligned}
&0 \leftarrow \frac{1}{\beta_n}\operatorname{Re}\langle Ax_n - i\beta_n x_n, iq(\mathcal{H}x_n)'\rangle_{L_2}\\
=&\frac{1}{\beta_n}\operatorname{Re}\langle P_1(\mathcal{H}x_n)', iq(\mathcal{H}x_n)'\rangle_{L_2} + \frac{1}{\beta_n}\operatorname{Re}\langle P_0(\mathcal{H}x_n), iq(\mathcal{H}x_n)'\rangle_{L_2} - \operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2}\\
=&\frac{1}{2\beta_n}\left(\langle \mathcal{H}x_n, iq'P_0(\mathcal{H}x_n)\rangle_{L_2} - [(\mathcal{H}x_n)(\xi)^* iq(\xi)P_0(\mathcal{H}x_n)(\xi)]_0^1\right) - \operatorname{Re}\langle x_n, q\mathcal{H}x_n'\rangle_{L_2} - \langle x_n, q\mathcal{H}'x_n\rangle_{L_2}\\
=&\frac{1}{2}\langle x_n, (q\mathcal{H})'x_n\rangle_{L_2} - \frac{1}{2}[x_n(\xi)^*q(\xi)\mathcal{H}(\xi)x_n(\xi)]_0^1 - \langle x_n, q\mathcal{H}'x_n\rangle_{L_2} + o(1)\\
=&-\frac{1}{2}\langle x_n, (q\mathcal{H}' - q'\mathcal{H})x_n\rangle_{L_2} + o(1)
\end{aligned}$$

In particular we may choose $q\leq 0$ with $q'>0$ such that

$$\lambda q + mq' > 0, \quad \xi\in[0,1].$$

where $\mathcal{H}(\xi)\geq mI$ and $\pm\mathcal{H}'(\xi)\leq \lambda I$ for a.e. $\xi\in[0,1]$, so

$$\langle (q'\mathcal{H} - q\mathcal{H}\prime)x, \mathcal{H}x\rangle = q\prime\langle \mathcal{H}x, x\rangle - q\langle \mathcal{H}'x, x\rangle \geq q\prime m\|x\|^2 + q\lambda\|x\|^2$$

then $q'\mathcal{H} - q\mathcal{H}'$ is uniformly positive. This implies

$$\|x_n\|_{L_2} \simeq \|x_n\|_{L_{2,q'\mathcal{H}-q\mathcal{H}'}} \overset{n\to\infty}{\longrightarrow} 0$$

this leads to $x_n\to 0$

So thanks to Theorem 2.4.2, we show that $\|T(t)_{t\geq 1}\|$ is exponentially stable.

2. Let $\beta\in\mathbb{R}$ and $x\in D(A)$ and $x\neq 0$ with

$$i\beta x = Ax$$

then

$$|(\mathcal{H}x)(a)|^2 + \left|(\mathcal{H}x)'(a)\right|^2 + \left\{ \begin{array}{c} |(\mathcal{H}x)(b)|^2 \\ \text{or} \\ \left|(\mathcal{H}x)'(b)\right|^2 \end{array} \right\} \leq -\frac{1}{\kappa}\operatorname{Re}\langle Ax, x\rangle = -\frac{1}{\kappa}\operatorname{Re}\langle i1x, x\rangle = 0$$

Hence

$$|(\mathcal{H}x)(a)|^2 = \left|(\mathcal{H}x)'(a)\right|^2 = 0$$

we have a system of ordinary differential equations

$$i1x(\xi) = P_2(\mathcal{H}x)''(\xi) + P_1(\mathcal{H}x)'(\xi) + P_0(\mathcal{H}x)(\xi), \quad \xi \in (a, b)$$

Since P_2 is invertible the unique solution of this initial value problem is $x = 0$ (see (2.3.4)), so $i\mathbb{R} \cap \sigma_p(A) = \emptyset$. Since $(T(t))_{t\geq 0}$ is a contraction C_0 semigroup, then $\sigma(A) = \sigma_p(A) \subseteq \mathbb{C}_0^-$.
Let $(x_n, \beta_n)_{n\geq 1} \subseteq D(A) \times \mathbb{R}$ be a sequence with $\sup_{n\geq 1} \|x_n\|_{L_2} < \infty$ for all $n \in \mathbb{N}$ and $|\beta_n| \to +\infty$ as $n \to +\infty$ such that

$$i\beta_n x_n - Ax_n \overset{n\to\infty}{\longrightarrow} 0.$$

The sequence $\left(\frac{\mathcal{H}x_n}{\beta_n}\right)_{n\geq 1} \subseteq H^2\left(0, 1; \mathbb{C}^d\right)$ is bounded and by Lemma 2.4.4 $\frac{\mathcal{H}x_n}{\beta_n}$ converges to zero in $C^1\left([0, 1]; \mathbb{C}^d\right)$ (since $|\beta_n| \to \infty$). Let $q \in C^2([0, 1]; \mathbb{R})$ be some real function. Integrating by parts and employing the assumptions on the matrices P_1 and P_2 we conclude

$$\begin{aligned}
0 \longleftarrow & \operatorname{Re}\left\langle Ax_n - i\beta_n x_n, \frac{iq}{\beta_n}\left(\mathcal{H}x_n\right)'\right\rangle_{L_2} \\
=& \operatorname{Re}\frac{1}{\beta_n}\left\langle P_2\left(\mathcal{H}x_n\right)'', iq\left(\mathcal{H}x_n\right)'\right\rangle_{L_2} + \frac{1}{\beta_n}\operatorname{Re}\left\langle P_1\left(\mathcal{H}x_n\right)', iq\left(\mathcal{H}x_n\right)'\right\rangle_{L_2} \\
& - \operatorname{Re}\left\langle x_n, q\left(\mathcal{H}x_n\right)'\right\rangle_{L_2} + o(1) \\
=& -\frac{1}{2\beta_n}\left\langle P_2\left(\mathcal{H}x_n\right)', iq'\left(\mathcal{H}x_n\right)'\right\rangle_{L_2} + \frac{1}{2}\left\langle x_n, \left(q'\mathcal{H} - q\mathcal{H}'\right)x_n\right\rangle_{L_2} \\
& + \frac{1}{2\beta_n}\left[\left(\mathcal{H}x_n\right)'(\xi)^* P_2^* iq(\xi)\left(\mathcal{H}x_n\right)'(\xi)\right]_0^1 - \frac{1}{2}\left[x_n(\xi)^* q(\xi)\mathcal{H}(\xi)x_n(\xi)\right]_0^1 + o(1) \quad (I)
\end{aligned}$$

and

$$\begin{aligned}
0 \longleftarrow & \operatorname{Re}\left\langle Ax_n - i\beta_n x_n, \frac{iq'}{\beta_n}\left(\mathcal{H}x_n\right)\right\rangle_{L_2} \\
=& \frac{1}{\beta_n}\operatorname{Re}\left\langle P_2\left(\mathcal{H}x_n\right)'', iq'\left(\mathcal{H}x_n\right)\right\rangle_{L_2} - \left\langle x_n, q'\mathcal{H}x_n\right\rangle_{L_2} + o(1) \\
=& -\frac{1}{\beta_n}\left\langle P_2\left(\mathcal{H}x_n\right)', iq'\left(\mathcal{H}x_n\right)'\right\rangle_{L_2} - \left\langle x_n, q'\mathcal{H}x_n\right\rangle_{L_2} \\
& + \frac{1}{\beta_n}\operatorname{Re}\left[\left(\mathcal{H}x_n\right)'(\xi)^* P_2^* iq'(\xi)\left(\mathcal{H}x_n\right)(\xi)\right]_0^1 + o(1) \quad (II)
\end{aligned}$$

Subtracting (II) from two times (I) this implies

$$\begin{aligned}
0 \longleftarrow & \left\langle x_n, \left(2q'\mathcal{H} - q\mathcal{H}'\right)x_n\right\rangle_{L_2} + \frac{1}{\beta_n}\left[\left(\mathcal{H}x_n\right)'(\xi)^* P_2^* iq(\xi)\left(\mathcal{H}x_n\right)'(\xi)\right]_0^1 \\
& + \frac{1}{\beta_n}\operatorname{Re}\left[\left(\mathcal{H}x_n\right)'(\xi)^* P_2^* iq'(\xi)\left(\mathcal{H}x_n\right)(\xi)\right]_0^1 - \left[x_n(\xi)^* q(\xi)\mathcal{H}(\xi)x_n(\xi)\right]_0^1
\end{aligned}$$

Choosing $q \in C^2([0,1];\mathbb{R})$ such that $q(1)=0$ and $2q'\mathcal{H} - q\mathcal{H}'$ is uniformly positive this leads in the case that also $f(x_n) \to 0$ to

$$\|x_n\|_{L_2} \simeq \|x_n\|_{q\mathcal{H}'-2q'\mathcal{H}} \overset{n\to\infty}{\longrightarrow} 0$$

and thus by Theorem 2.4.2, we show that $(T(t))_{t\geq 0}$ is exponentially stable..

□

Example 2.4.6 (Transport Equation)**.** *This equation describes how a quantity is transported in a space (e.g. temperature or chemical concentration) inside an incompressible flow. It is also called the convection-diffusion equation, which is a first-order PDE. Let consider the nonuniform transport equation*

$$\frac{\partial}{\partial t}x(t,\xi) = \frac{\partial}{\partial \xi}(c(\xi)x(t,\xi)) =: (\mathfrak{A}x(t))(\xi)$$

so that we have for all $x \in D(\mathfrak{A}) = \left\{x \in L_2(0,1) : cx \in H^1(0,1)\right\}$,

$$\mathrm{Re}\langle \mathfrak{A}x, x\rangle_c = \mathrm{Re}\left\langle cx, (cx)'\right\rangle_{L_2} = \frac{1}{2}\left[|(cx)(1)|^2 - |(cx)(0)|^2\right].$$

Then it is clear that a dissipative boundary condition has the form

$$(cx)(1) = \lambda(cx)(0),$$

where $\lambda \in \mathbb{K}$ *with* $|\lambda| \leq 1$. *Moreover, we observe that for the corresponding operators* $A_\lambda = \mathfrak{A}|_{D(A_\lambda)}$ *with* $D(A_\lambda) = \{x \in D(\mathfrak{A}) : (cx)(1) = \lambda(cx)(0)\}$, *we have*

$$\mathrm{Re}\langle A_\lambda x, x\rangle \begin{cases} = 0, & |\lambda| = 1 \\ \leq -\sigma_\lambda\left(|(cx)(1)|^2 + |(cx)(0)|^2\right), & |\lambda| \in (0,1), \\ \leq -|(cx)(1)|^2, & \lambda = 0, \end{cases}$$

where $\sigma_\lambda > 0$ *for every* $\lambda \in \mathbb{K}$ *with* $|\lambda| < 1$. *As a result, the* C_0*-semigroups* $(T_\lambda(t))_{t\geq 0}$ *generated by* A_λ *are uniformly exponentially stable for* $|\lambda| < 1$.

Proposition 2.4.7. *Let's* A *be an operator defined as above,*
then A *generates an exponentially stable semigroup if and only if there exists a* $k \geq 0$ *such that*

$$\mathrm{Re}\langle AX, X\rangle \leq -k(|\mathcal{H}X(0)|^2 + |\mathcal{H}X(0)|'^2 + |\mathcal{H}X(L)|^2 + |\mathcal{H}X(L)|'^2).$$

3 Our system

3.1 PHS modeling

3.1.1 Beam bending

We reintroduce Euler-Bernoulli equation:

$$\mu(\xi)\frac{\partial^2}{\partial t^2}\omega(t,\xi) = -\frac{\partial^2}{\partial \xi^2}(E.I(\xi)\frac{\partial^2}{\partial \xi^2}\omega(t,\xi)), \tag{3.1.1}$$

with $\xi \in [0, L]$.

Defining energy variables $x_1^B(t,\xi) = \frac{\partial^2}{\partial \xi^2}\omega(t,\xi)$ and $x_2^B(t,\xi) = \mu(\xi)\frac{\partial}{\partial t}\omega(t,\xi)$, then we get :

$$\frac{\partial}{\partial t}x_2^B(t,\xi) = -\frac{\partial^2}{\partial \xi^2}(E.I(\xi)x_1^B(t,\xi)).$$

The energy/ Hamiltonian, the sum of the elastic potential energy and the kinetic energy, is given using those variables by :

$$H^B(t) = \frac{1}{2}\int_0^L EI(\xi)(x_1^B)^2 + \frac{1}{\mu(\xi)}(x_2^B)^2 d\xi.$$

We calculate the variational derivative of H, with respect to x_1^B and x_2^B :

$$e_1^B(t,\xi) = \frac{\delta H^B}{\delta x_1^B} = EIx_1^B$$
$$e_2^B(t,\xi) = \frac{\delta H^B}{\delta x_2^B} = \frac{x_2^B}{\mu}.$$

e_1^B and e_2^B are called the effort (or co-energy) variables: e_1^B is the local bending moment $EI\frac{\partial^2 w}{\partial \xi^2}(t,\xi)$ and e_2^B is the local vertical speed $\frac{\partial w}{\partial t}(t,\xi)$.

We can now compute the state variable rate of change

$$\frac{\partial x_1^B}{\partial t} = \frac{\partial}{\partial t}\frac{\partial^2 \omega}{\partial \xi^2} = \frac{\partial^2}{\partial \xi^2}\frac{1}{\mu(\xi)}x_2^B = \frac{\partial^2}{\partial \xi^2}(\frac{\delta H}{\delta x_2^B})$$
$$\frac{\partial x_2^B}{\partial t} = -\frac{\partial^2}{\partial \xi^2}(E.I(\xi)x_1^B) = -\frac{\partial^2}{\partial \xi^2}(\frac{\delta H}{\delta x_1^B}).$$

We rewrite the Euler-Bernoulli beam equation using these new variables as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1^B \\ x_2^B \end{pmatrix} = \begin{pmatrix} 0 & \partial^2_{\xi^2} \\ -\partial^2_{\xi^2} & 0 \end{pmatrix}\begin{pmatrix} e_1^B \\ e_2^B \end{pmatrix}.$$

Therefore

$$\dot{X}^B(t,\xi) = P_2^B \frac{\partial^2}{\partial \xi^2} \mathcal{H}^B X^B(t,\xi), \tag{3.1.2}$$

with $P_2^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\mathcal{H}^B = \begin{pmatrix} EI(\xi) & 0 \\ 0 & \frac{1}{\mu(\xi)} \end{pmatrix}$.

Then the beam under bending motion is a linear second order port-Hamiltonian system.

3.1.2 Beam torsion

Let's consider the wave equation that described torsion motion of the flexible beam :

$$I_p(\xi) \frac{\partial^2}{\partial t^2} \theta(t,\xi) = \frac{\partial}{\partial \xi}(GJ(\xi) \frac{\partial}{\partial \xi} \theta(t,\xi)). \tag{3.1.3}$$

We define the strain $x_1^T(t,\xi) = \frac{\partial}{\partial \xi}\theta(t,\xi)$ and the momentum $x_2^T(t,\xi) = I_p(\xi)\frac{\partial}{\partial t}\theta(t,\xi)$.

Then the equation (3.1.3) can equivalently be written as

$$\frac{\partial}{\partial t} x_2^T(t,\xi) = \frac{\partial}{\partial x} GJ x_1^T(t,\xi).$$

It is easy to verify that Eq. (3.1.3) can now be rewritten using the port-Hamiltonian formulation by calculating the time derivative of the variable x_1^T. Then we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} x_1^T(t,\xi) \\ x_2^T(t,\xi) \end{pmatrix} &= \begin{pmatrix} 0 & \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \xi} & 0 \end{pmatrix} \begin{pmatrix} GJx_1^T \\ \frac{x_2^T}{I_p(\xi)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \left(\begin{pmatrix} GJ(\xi) & 0 \\ 0 & \frac{1}{I_p(\xi)} \end{pmatrix} \begin{pmatrix} x_1^T(t,\xi) \\ x_2^T(t,\xi) \end{pmatrix} \right) \\ &= P_1 \frac{\partial}{\partial \xi} \left(\mathcal{H}(\xi) \begin{pmatrix} x_1^T(t,\xi) \\ x_2^T(t,\xi) \end{pmatrix} \right) \end{aligned}$$

We then reformulate the Hamiltonian using those energy variables :

$$\begin{aligned} H^T(t) &= \frac{1}{2} \int_0^L GJ(x_1^T)^2 + \frac{(x_2^T)^2}{I_p(\xi)} d\xi \\ &= \frac{1}{2} \int_0^L \begin{pmatrix} x_1^T(t,\xi) & x_2^T(t,\xi) \end{pmatrix} \mathcal{H}(\xi) \begin{pmatrix} x_1^T(t,\xi) \\ x_2^T(t,\xi) \end{pmatrix} d\xi. \end{aligned}$$

We conclude that the wave equation leads to a linear first order port-Hamiltonian system.

3.1.3 Rigid tank

In order to show that the system (1.2.1) is a port-Hamiltonian system, we introduce the following moments variables : $p = m_R \dot{\omega}_B, p_{\theta B} = I_R^B \dot{\theta}_B$, and $p_{\theta T} = I_R^T \dot{\theta}_T$, then we reformulate the system with these variables :

$$\frac{\partial}{\partial t} \begin{pmatrix} p \\ \omega_B \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{p}{m_R} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} F_{ext}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} p_{\theta B} \\ \theta_B \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{P_{\theta B}}{I_R^B} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} M_{ext}^B$$

$$\frac{\partial}{\partial t}\begin{pmatrix} p_{\theta T} \\ \theta_T \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{P_{\theta T}}{I_R^T} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} M_{ext}^T$$

where the Hamiltonian is equal to the kinetic energy:

$$H^R = \frac{1}{2}\left(\frac{p^2}{m_R} + \frac{p_{\theta B}^2}{I_R^B} + \frac{p_{\theta T}^2}{I_R^T}\right)$$

then we define the following notations:

$$X^F = \begin{pmatrix} p \\ \omega_B \\ p_{\theta B} \\ \theta_B \\ p_{\theta T} \\ \theta_T \end{pmatrix}, \quad J^R = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = -J^{R*}, \quad \text{and } U^F = \begin{pmatrix} F_{ext} \\ 0 \\ M_{ext}^B \\ 0 \\ M_{ext}^T \\ 0 \end{pmatrix}.$$

The obtained system

$$\dot{X}^R(t) = J\frac{\partial H}{\partial X^R}(t) + U^R(t)$$

is called a finite dimensional port-Hamiltonian system as presented (2.1.1) .

3.1.4 Fluid sloshing

Here we are interested in modeling the fluid within the tank taking into account both translation and planar rotation. The fluid Kinetic energy and gravitational potential energy are given by:

$$E_c = \frac{\rho b}{2}\int_{-\frac{a}{2}}^{\frac{a}{2}} \tilde{h}((w+D)^2 + (z\dot{\theta})^2)dz$$
$$E_p = \rho b g\int_{-\frac{a}{2}}^{\frac{a}{2}} z\tilde{h}\theta + \frac{\tilde{h}^2}{2}dz$$

where ρ fluid density, b is the tank width, a is the tank length and $\dot{\theta}_F(t)$ is the tank rotation velocity. Let's define the energy variables as follow:

$$p_F = I_F\dot{\theta}\ ,\ x_1^F = b\tilde{h}\ ,\ x_2^F = \rho(\omega + D),\ x_3^F = \theta,$$

where x_1^F is the fluid section area and x_2^F is the fluid momentum by area. We write the Hamiltonian as function of these variables

$$H^F = E_c + E_p = \int_{-\frac{a}{2}}^{\frac{a}{2}} x_1^F\frac{(x_2^F)^2}{\rho} + \rho g\frac{(x_1^F)^2}{b} + \rho g z x_1^F x_3^F dz + \frac{p_F^2}{I_f},$$

where $I_f = \rho b \int_{-\frac{a}{2}}^{\frac{a}{2}} z^2 dz$ is the inertia of the fluid (constant).
By computing the partial derivatives with respect to energy variables:

$$\begin{aligned}
\frac{\delta H^f}{\delta x_1^F} &= \frac{1}{\rho}(x_2^F)^2 + \frac{\rho g}{b}x_1^F + \rho bgzx_3^F \\
\frac{\delta H^F}{\delta x_2^F} &= \frac{1}{\rho}x_1^F x_2^F \\
\frac{\delta H^F}{\delta x_3^F} &= \rho bgzx_1^F \\
\frac{\delta H^F}{\delta p_F} &= \frac{p_F}{I_F},
\end{aligned}$$

we could notice that it is not possible to recover the dynamic Eqs. (1.3.3) using the Hamiltonian framework:

$$\begin{aligned}
\frac{\partial x_1^F}{\partial t} &= -\frac{\partial}{\partial z}(bhu) \\
\frac{\partial x_2^F}{\partial t} &= -\frac{\partial}{\partial z}\left(gh + gz\theta\right).
\end{aligned}$$

To deal with this problem, we often add an energy term in the Hamiltonian formulation. But in this case, the Hamiltonian is not quadratic which means that the sloshing fluid represent a non linear port-Hamiltonian system.
Since we are interested in this study, to linear port Hamiltonian systems, we restricted our system to the beam with both bending and torsion for next part.

Remark 3.1.1. Modeling and stabilisation of the beam only with both bending and torsion motion is useful for studying airplane with long wings as the example of Boing TTBW Fig. 1, if we neglect the fluid motion inside the wings, or for investigating the stability of spacecraft Fig. 4.

3.2 Well-posedness of the system

Let's consider a beam simulated to 1D plate, fixed in the right end side and no strain or momentum is applied in the left end side. Therefore we could define the boundary conditions as follows:

$$\begin{aligned}
x_1^B(t,L) &= \frac{\partial^2 \omega}{\partial x^2}(t) = 0 && \text{Free side (no strain)} \\
x_2^B(t,0) &= \mu(0)\frac{\partial \omega}{\partial t}(t) = 0 && \text{Fixed side} \\
x_1^T(t,L) &= \frac{\partial \theta}{\partial \xi}(t) = 0 && \text{Free side(no momentum)} \\
x_2^T(t,0) &= I_p(0)\frac{\partial \theta}{\partial t}(t) = 0 && \text{Fixed side.}
\end{aligned}$$

The system in the port-Hamiltonian formalism is given by:

$$\frac{\partial X}{\partial t}(t,\xi) = P_2 \frac{\partial^2}{\partial \xi^2}\mathcal{H}(\xi)X(t,\xi) + P_1 \frac{\partial}{\partial \xi}\mathcal{H}(\xi)X(t,\xi) + P_0\mathcal{H}(\xi)X(t,\xi), \tag{3.2.1}$$

where

$$P_2 = \begin{pmatrix} P_2^B & 0 \\ 0 & 0 \end{pmatrix}, \; P_1 = \begin{pmatrix} 0 & 0 \\ 0 & P_1^T \end{pmatrix}, \text{ and } P_0 = 0.$$

The associated Hamiltonian (the global system energy) is given by the sum of each subsystem energy:

$$H(X) = H(t) = H^F(t) + H^T(t) = \frac{1}{2}\int X^*(t,\xi)(\mathcal{H}(\xi)X(t,\xi)d\xi,$$

where

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_{\mathcal{B}} & 0 \\ 0 & \mathcal{H}_{\mathcal{T}} \end{pmatrix}$$

Let's consider the operator A associated to the system (3.2.1) such that

$$D(A) = \left\{X \in L^2([0,L],\mathbb{R}^4),\ /\ \ \mathcal{H}X \in H^2([0,L],\mathbb{R}^4))\right\}.$$

3.2.1 Generation of semigroup

To guarantee that the system (3.2.1) possess a unique solution, we have to define the matrix W as defined above, first we define boundary variables as follow $f = \begin{pmatrix} \frac{\partial}{\partial\xi}(\mathcal{H}X)^B \\ -\mathcal{H}X^T \\ \mathcal{H}X^B \\ 0 \end{pmatrix}$, $e = \begin{pmatrix} (\mathcal{H}X)^B \\ (\mathcal{H}X)^T \\ \frac{\partial}{\partial\xi}(\mathcal{H}X)^B \\ \frac{\partial}{\partial\xi}(\mathcal{H}X)^T. \end{pmatrix}$.

Let $W \in \mathbb{R}^{8\times16}$ such that $W\begin{pmatrix} f \\ e \end{pmatrix} = 0$, therefore

$$W = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

It's easy to verify that $\text{rank}(W) = 8$.

Then we define the operator A on

$$D(A) = \left\{X \in L^2([0,L],\mathbb{R}^4)\ /\ \ \mathcal{H}X \in H^2([0,L],\mathbb{R}^4), W\begin{pmatrix} f \\ e \end{pmatrix} = 0\right\}$$

such that

$$AX(t,\xi) = P_2\frac{\partial^2}{\partial\xi^2}\mathcal{H}(\xi)X(t,\xi) + P_1\frac{\partial}{\partial\xi}\mathcal{H}(\xi)X(t,\xi)$$

for all $X \in L^2([0, L]$,

$$\begin{aligned}
\mathrm{Re}\left(\langle AX, X\rangle\right) &= \frac{1}{2}(f^*e + e * f) \\
&= \frac{1}{2}((\mathcal{H}X)^* P_2 \frac{\partial}{\partial \xi}\mathcal{H}X(L) - (\mathcal{H}X)^* P_2 \frac{\partial}{\partial \xi}\mathcal{H}X(0) + \frac{1}{2}((\mathcal{H}X)^* P_1 \mathcal{H}X(L) - (\mathcal{H}X)^* P_1 \mathcal{H}X(0))) \\
&= 0.
\end{aligned}$$

A is the infinitesimal generator of contraction semigroup.

3.2.2 Stabilization

Since no damping has been taken into account in the modeling of the different components, the global system is power conserving, the Hamiltonian associated is given by: $H(t) = \|X(t, \xi)\|^2 \geq 0$

$$\dot{H} = \langle AX, X\rangle + \langle X, AX\rangle = 0$$

In our case, the flexible beam is connected to a rigid tank partially filled with fluid, so that the free-end boundary conditions will not be zero. Instead of using a free-end boundary, we can specify as boundary conditions at $\xi = L$ the values of force, bending moment and the torsion moment applied by the connected structure. The conjugate port-variables are given by the point speed, angular velocity and the angular velocity at the same point, respectively

$$\begin{aligned}
e_1^B(t, L) &= m(t) \\
\frac{\partial}{\partial x} e_1^B(t, L) &= f(t) \\
e_1^T(t, L) &= \tau(t)
\end{aligned}$$

Therefore

$$\begin{aligned}
\dot{H} &= \langle AX, X\rangle + \langle X, AX\rangle \\
&= \frac{1}{2}((\mathcal{H}X)^* P_2 \frac{\partial}{\partial \xi}\mathcal{H}X(L) + \frac{1}{2}((\mathcal{H}X)^* P_1 \mathcal{H}X(L))) \\
&= \frac{1}{2}(e_1^B(t, L)\frac{\partial}{\partial x}e_2^B(t, L) - e_2^B(t, L)\frac{\partial}{\partial x}e_1^B(t, L) + e_1^T(t, L)e_2^T(t, L)) \\
&= \frac{1}{2}(m(t)\frac{\partial^2 \omega}{\partial x \partial t}(t, L) - \frac{\partial \omega}{\partial t}(t, L)f(t) + \tau(t)\frac{\partial \theta}{\partial t}(t, L))
\end{aligned} \tag{3.2.2}$$

To guarantee that $\frac{dH}{dt} \leq 0$ we could propose feedback laws such as an external actuator applying forces and moments at the coupling point.

$$\begin{aligned}
m(t) &= -k_1 \frac{\partial^2 \omega}{\partial x \partial t}(t, L) \\
f(t) &= k_2 \frac{\partial \omega}{\partial t}(t, L) \\
\tau(t) &= -k_3 \frac{\partial \theta}{\partial t}(t, L))
\end{aligned}$$

with $k_1,\ k_2,\ k_3 \geq 0$.

Then the system (3.2.2):

$$\dot{H} = -\frac{1}{2}(k_1 m(t)^2 + k_2 f(t)^2 + k_3 \tau(t)^2) \leq 0$$

Now, we verify if the stabilization is achieved

$$\begin{aligned}
\operatorname{Re}(\langle AX, X\rangle) &= -\frac{1}{4}(\frac{1}{k_1}|e_1^B(L)|^2 + k_2|\frac{\partial}{\partial x}e_1^B(L)|^2 + k_3|e_2^T(L)|^2) \\
&= -\frac{1}{4}(\frac{1}{k_1}|e_1^B(L)|^2 + k_2|e_2^B(L)|^2 + k_3|e_2^T(L)|^2) \\
&= -\frac{1}{4}(\frac{1}{k_1}|e_1^B(L)|^2 + k_2|e_2^B(L)|^2 + \frac{1}{2k_3}|e_2^T(L)|^2) + \frac{k_3}{2}|e_1^T(L)|^2) \\
&\leq -k|(\mathcal{H}X)(L)|^2
\end{aligned}$$

avec $k = \frac{1}{4}min(\frac{1}{k_1}, k_2, \frac{1}{2k_3}, \frac{k_3}{2})$

Hence A generates an exponentially stable contraction semigroup.

Remark 3.2.1. For the wave equation, the boundary conditions that leads to exponential stability

$$\frac{\partial\theta}{\partial\xi}(t, L) = k_3\frac{\partial\theta}{\partial t}(t, L)$$

are not realistic, since it must be applicable at same moment. Because, there's a time delay between the sensors and the actuator who receives the signal and perform the action. It may be interesting if we could determine other boundary conditions that contribute to the system stabilization.

4 Space discretization of port-Hamiltonian systems

In order to perform numerical simulations or implement control schemes for systems governed by partial differential equations (PDEs) which describe a subsystem of a multi-physical model, it is necessary to approximate them by finite-dimensional representations. In this context, preserving the geometric structure of the infinite dimensional system is relevant to preserve the physical properties of the model, such as the conservation of energy, the dissipation profiles and the physical meaning of the inputs and outputs (boundary variables), it is even more relevant to preserve the physical properties of the interconnection variables. Different structure preserving discretization schemes have been proposed in recent years, as for instance the finite-difference method which presents numerous schemes with diverse properties and advantages. Particularly, schemes presenting staggered grids permit to define different state variables on different grids and thus account for their different geometric nature. Here we show how the staggered grids finite difference can be used to discretize infinite-dimensional PHS on 1D spatial domains while preserving its intrinsic PH structure. It is also shown that a centered finite difference can be advantageously used to derive an approximate and efficient simulator for second order port-Hamiltonian systems.

4.1 1D wave equation

Let's introduce the general wave system with a Neumann boundary conditions

$$\begin{cases} \dfrac{\partial^2\theta}{\partial t^2} - c\dfrac{\partial^2\theta}{\partial x^2} &= 0 & & \text{in } (0,T)\times(0,L) \\ \dfrac{\partial\theta}{\partial x}(t,L) &= u(t) & & \text{in } (0,T) \\ \dfrac{\partial\theta}{\partial t}(t,0) &= v(t) & & \text{in } (0,T) \\ \theta(0,x) &= f(x), & \dfrac{\partial\theta}{\partial t}(0,x) = w(x) & \text{in } (0,L). \end{cases}$$

We propose to approach the solution of this system by an explicit scheme of finite difference. For this purpose we approximate the second-order derivative by

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\theta_{j+1} - 2\theta_j + \theta_{j-1}}{h^2},$$

where we have considered a uniform mesh defined by:

$$t_0 = 0 < t_1 = dt < \cdots < t_i = idt < \cdots < t_{NT+1} = T,$$

and

$$x_0 = 0 < x_1 = h < \cdots < x_j = jh < \cdots < x_{n+1} = L,$$

with $dt = \frac{T}{NT+1}$ is the time step and $h = \frac{L}{n+1}$ is the space step. Let θ_j^i be the approximation of $\theta(t_i, x_j)$. Thus for $(i,j) \in (1, NT) \times (1, n)$, we obtain

$$\frac{\theta_j^{i+1} - 2\theta_j^i + \theta_j^{i-1}}{dt^2} = c\frac{\theta_{j+1}^i - 2\theta_j^i + \theta_{j-1}^i}{h^2}$$

For initial and boundary conditions, we used also finite difference for both space and time discretization, then we get:

— Initial conditions

$$\theta_j^0 = f_j$$
$$\theta_j^1 = \theta_j^0 + dtw_j,$$

— Boundary conditions

$$\theta_{n+1}^i = \theta_n^i + hu^i$$
$$\theta_0^{i+1} = \theta_0^i + dtv^i.$$

For all $(i,j) \in (0, NT+1) \times (0, n+1)$

4.2 First order PH system

Let's consider the wave equation as defined in beam torsion part:

$$I_p(\xi)\frac{\partial^2}{\partial t^2}\theta(t,\xi) = \frac{\partial}{\partial \xi}(GJ(\xi)\frac{\partial}{\partial \xi}\theta(t,\xi)) \tag{4.2.1}$$

To simplify the problem, we assume that $GJ(\xi)$ and $I_p(\xi)$ are space-independent variables, then we have:

$$\frac{\partial^2}{\partial t^2}\theta(t,\xi) = c^2\frac{\partial^2}{\partial \xi^2}\theta(t,\xi)) \tag{4.2.2}$$

where $c^2 = GJ$ and $I_p = 1$.

The equation (4.2.2) leads to

$$\dot{X}^T = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} X^T \tag{4.2.3}$$

where

$$X^T = \begin{pmatrix} x_1^T \\ x_2^T \end{pmatrix}, \; x_1^T = \frac{\partial \theta}{\partial \xi} \text{ and } x_2^T = \frac{\partial \theta}{\partial t}$$

$$\dot{X}^T(t,\xi) = J\mathcal{H}^T X^T(t,\xi)$$

for $\xi \in [0, L]$, (in numerical tests, we consider $L = 1$).

4.2.1 Space discretization

In this subsection, we perform the spatial discretization of the wave equation (4.2.2) from its port-Hamiltonian formulation (4.2.3). The first idea is to use right-sided finite difference for space derivative of both energy variables:

$$\dot{x_1}(\xi_j) = \frac{x_2(\xi_{j+1}) - x_2(\xi_j)}{h}$$
$$\dot{x_2}(\xi_j) = c^2\frac{x_1(\xi_{j+1}) - x_1(\xi_j)}{h},$$

where $1 \leq j \leq n$, $\xi_0 = 0$, $\xi_{n+1} = L$, h is space the discretization step, and $n+1$ is the number of space steps. Then we have

$$\dot{x_1} = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_2(\xi_1) \\ x_2(\xi_2) \\ x_2(\xi_3) \\ \vdots \\ x_2(\xi_n) \end{pmatrix} + \frac{1}{h}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_2(L) \end{pmatrix}$$

$$\dot{x_2} = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} c^2x_1(\xi_1) \\ c^2x_1(\xi_2) \\ c^2x_1(\xi_3) \\ \vdots \\ c^2x_1(\xi_n) \end{pmatrix} + \frac{1}{h}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ c^2x_1(L) \end{pmatrix}$$

The matrix $\mathcal{H}_d = \text{diag}(c^2, \cdots, c^2, 1 \cdots, 1)$ is positive definite. Let D be a matrix defined as

$$D = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}$$

Therefore

$$\dot{X}(t) = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}\mathcal{H}_d X(t) + \frac{1}{h}u(t).$$

We remark that $J = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \neq -J^*$, thus the right-sided finite difference for space derivative of both energy variables doesn't preserve the port-Hamiltonian structure.

Now, we follow a method inspired by staggered grid finite difference method introduced in [18]. We use the right-sided finite difference for space derivative of the first energy variable and the left-sided finite difference for space derivative of the second, which means:

$$\dot{x_1}(\xi_i) = \frac{x_2(\xi_j) - x_2(\xi_{j-1})}{h}$$

$$\dot{x_2}(\xi_j) = c^2 \frac{x_1(\xi_{j+1}) - x_1(\xi_j)}{h},$$

then, we get

$$\dot{x_1} = \frac{1}{h} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_2(\xi_1) \\ x_2(\xi_2) \\ x_2(\xi_3) \\ \vdots \\ x_2(\xi_n) \end{pmatrix} - \frac{1}{h} \begin{pmatrix} x_2(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\dot{x_2} = \frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c^2 x_1(\xi_1) \\ c^2 x_1(\xi_2) \\ c^2 x_1(\xi_3) \\ \vdots \\ c^2 x_1(\xi_n) \end{pmatrix} + \frac{1}{h} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ c^2 x_1(L) \end{pmatrix}.$$

In this case, $J = \begin{pmatrix} 0 & -D^T \\ D & 0 \end{pmatrix} = -J^*$, then this spatial discretization called staggered grids finite difference is structure preserving. Moreover, the use of staggered grids permits to directly impose boundary conditions over the effort variables.

4.2.2 Model reduction

Since the resolvent operator of A is a compact, the spectrum of A is composed only of discrete eigenvalues:

$$\lambda X(\xi) = \begin{pmatrix} 0 & \frac{d}{d\xi} \\ \frac{d}{d\xi} & 0 \end{pmatrix} \mathcal{H} X(\xi), \tag{4.2.4}$$

with boundary conditions defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} (\mathcal{H}x)(L) \\ (\mathcal{H}x)(0) \end{pmatrix}$$

and satisfy

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \ker W.$$

Let's consider

$$X(\xi) = \begin{pmatrix} X_1(\xi) \\ X_2(\xi) \end{pmatrix}.$$

To simplify we consider $\mathcal{H} = I$, in this case the system (4.2.4) implies:

$$\frac{d^2 X_2(\xi)}{d\xi^2} = \lambda^2 X_2(\xi),$$

where the solution is given by

$$X_2(\xi) = Ae^{\lambda\xi} - Be^{\lambda\xi}.$$

Therefore

$$X_1(\xi) = Ae^{\lambda\xi} + Be^{-\lambda\xi},$$

with A and B are constants. We consider same Dirichlet boundary conditions as defined above, then we have:

$$\begin{pmatrix} 1 & -1 \\ e^{\lambda L} & e^{-\lambda L} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

If this matrix is invertible, we get $A = B = 0$. So the set of the eigenvalues λ verify:

$$\begin{vmatrix} 1 & -1 \\ e^{\lambda L} & e^{-\lambda L} \end{vmatrix} = e^{-\lambda L} + e^{\lambda L} = 0.$$

So that for $j \in \mathbb{Z}$, we have

$$\lambda_j = \frac{2j+1}{2L}\pi i \quad \text{where } i^2 = 1.$$

For a finite dimensional operator, the spectrum consists of its eigenvalues: the set of values λ such that $Jx = \lambda x$ for some $x \neq 0$.

Let's determine the eigenvalues of the operator J defined as:

$$J = \begin{pmatrix} 0 & -D^T \\ D & 0 \end{pmatrix}$$

where $D \in \mathbb{R}^{N\times N}$ such that

$$D = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}$$

The eigenvalues λ of the operator J are defined as characteristic polynomial roots, i.e., roots λ so that $\det(\lambda I - J) = 0$.

Thanks to the commutativity of λI and D, the eigenvalues λ satisfy:

$$\det\left(\lambda^2 I - D^T D\right) = 0.$$

Let μ be an eigenvalue of $A_k = D_k^T D_k$ where D_k consists of the first k lines and the first k columns of D, which means:

$$A_k = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

Its characteristic polynomial $P_k(\mu) = \det(\mu I - A_k)$ satisfies the following recurrence

$$P_k = (\mu + 2)P_{k-1} - P_{k-2}.$$

After resolution (see the details in [10]) we get:

$$\mu_k = \lambda_k^2 = 2\cos\left(\frac{2k+1}{2n+1}\pi\right) - 2$$

Let's consider $(k \ll N)$. Using a Taylor development of the cosine function we get:

$$\mu_k = \lambda_k^2 = -\left(\frac{2k+1}{2n+1}\pi\right)^2 + o\left(\frac{1}{N^3}\right).$$

We remark that the eigenvalues λ_k are as function of space step $\frac{L}{n+1}$ and space length $[0, L]$.

4.2.3 Numerical results

For numerical tests, we consider $c = 4$ and the following initial conditions

$$\begin{cases} \theta^0(\xi) = \sin(\pi\xi), \\ \theta^1(\xi) = 0. \end{cases}$$

The boundary conditions are

$$\begin{cases} \dfrac{\partial\theta}{\partial\xi}(t, L) = 0, \\ \dfrac{\partial\theta}{\partial t}(t, 0) = 0. \end{cases} \tag{4.2.5}$$

Since the discretization employed does not require a numerical resolution, we develop a python algorithm by two loops, then under these conditions we get the following results (Fig. 4.2).

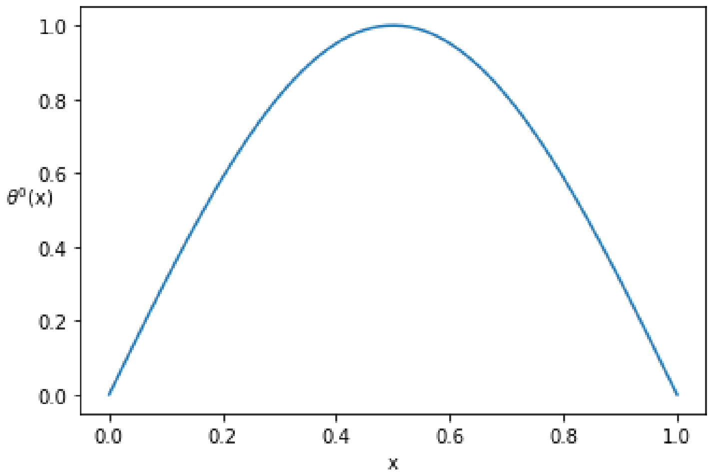
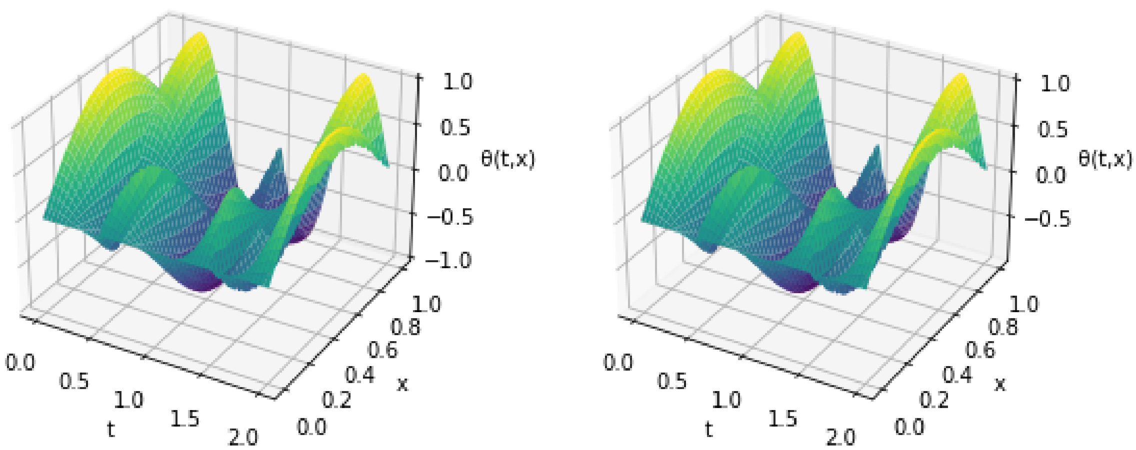


Figure 4.1 – Initial position.



(a) discrete wave

(b) discrete 1st order PHS

Figure 4.2 – Solution of wave equation.

Note that the only difference between the damped system is that the dissipative boundary condition at the endpoint $x = L$ has been replaced by the homogeneous Neumann boundary condition.

To compare between the two solutions, we employed total L_2 norm error defined by

$$e = \left(\int \|\theta_w - \theta_{ph}\|_{L^2}^2 dt \right)^{\frac{1}{2}}.$$

In this case, we consider $C = c^2 \frac{dt}{h} = 10^{-5}$, but the number of iterations is big, so we used Numba to solve the problem of CPU time.

The following table gives values of total L_2 norm and code execution time using Numba for different values of n:

Number of space steps	50	100	150	200
Total L_2 norm error	0.064	0.033	0.021	0.016
CPU time using Numba	92.2	264.1	639.8	1363.7

The more we increase number of space steps, the more we obtain small error values.

In the discretized setting, the infinite-dimensional state variables are replaced by the finite-dimensional vector

$$x_d = \begin{pmatrix} x_d^1 & x_d^2 \end{pmatrix}^T \in \mathbb{R}^{2n},$$

with $x_d^1 = \begin{pmatrix} x_1^1 & \dots & x_n^1 \end{pmatrix}^T, x_d^2 = \begin{pmatrix} x_1^2 & \cdots & x_n^2 \end{pmatrix}^T$, where the $x_j^{\{1,2\}}(j \in \{1 \dots n\})$ are the approximation of the state $x^{\{1,2\}}$ respectively evaluated at $\xi = jh$.

For the approximation of the integrals we used the rectangles method, then the discrete Hamiltonian which approximates the original energy such that $hH_d \approx H$ can be defined as

$$H_d = \frac{1}{2} x_d^T \mathcal{H}_d x_d = \frac{1}{2} \sum_{i=1}^{n} \begin{pmatrix} x_i^1 & x_i^2 \end{pmatrix} \mathcal{H}_{\xi_i} \begin{pmatrix} x_i^1 \\ x_i^2 \end{pmatrix},$$

where $\mathcal{H}_d \in \mathbb{R}^{2n \times 2n}$ is a diagonal matrix, composed of $\mathcal{H}_\xi$, evaluated at the corresponding grid points. In the continuous case, we prove that this system is conservative taking into account the precedent boundary conditions (4.2.5):

$$\frac{dH}{dt} = \frac{1}{2} \left[\mathcal{H}(\xi) x(t,\xi) P_1 \mathcal{H}(\xi) x(t,\xi) \right]_0^1 = 0$$

For the discrete energy of the 1D wave defined as follow

$$E(t) = \frac{1}{2} \int_0^1 \frac{\partial \theta}{\partial t}(t,\xi) + c^2 \frac{\partial \theta}{\partial x}(t,\xi) d\xi,$$

we use the right-sided finite difference for the time and space derivative, and we approximate the integral by the sum of the discrete derivatives then we multiply by $d\xi = h$ the space step, then we plot both Discrete wave energy and discrete Hamiltonian of 1st order PHS :

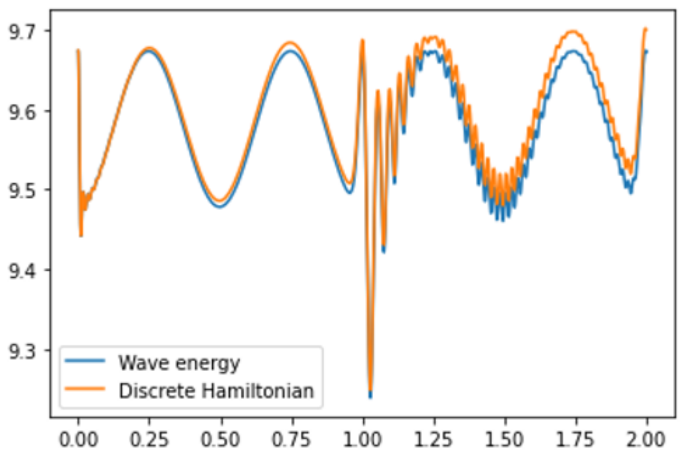


Figure 4.3 – Discrete wave energy and discrete Hamiltonian of 1st order PHS.

In this case, the error equals to 0.8, .

Now we use the left-sided finite difference to approximate the space derivative i.e.

$$\frac{\partial \theta}{\partial x}(t_i, \xi_j) = \frac{\theta^i_j - \theta^i_{j-1}}{h},$$

in this case, the energy looks more constant, therefore the numerical results shows that this discretization method is not only a structure preserving but it preserves also the physical parameters of the systems (the energy).

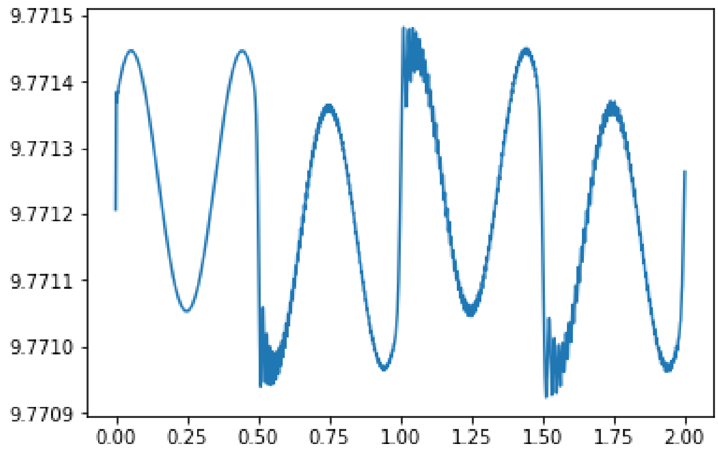


Figure 4.4 – Discrete wave energy.

To be able to compare with the eigenvalues of the infinite dimensional structure as defined above, we define the normalized eigenvalues as follows:

$$\begin{aligned}\hat{\lambda}_k &= \frac{N}{L}\lambda_k = \frac{N}{L}\frac{2k+1}{2N+1}\pi i + o\left(\frac{1}{N^2}\right) \\ &= \frac{2k+1}{2L}\pi i\left(1 - \frac{1}{2N} + o\left(\frac{1}{N}\right)\right), \ \forall k \in \mathbb{Z}.\end{aligned}$$

For $n = 50$:

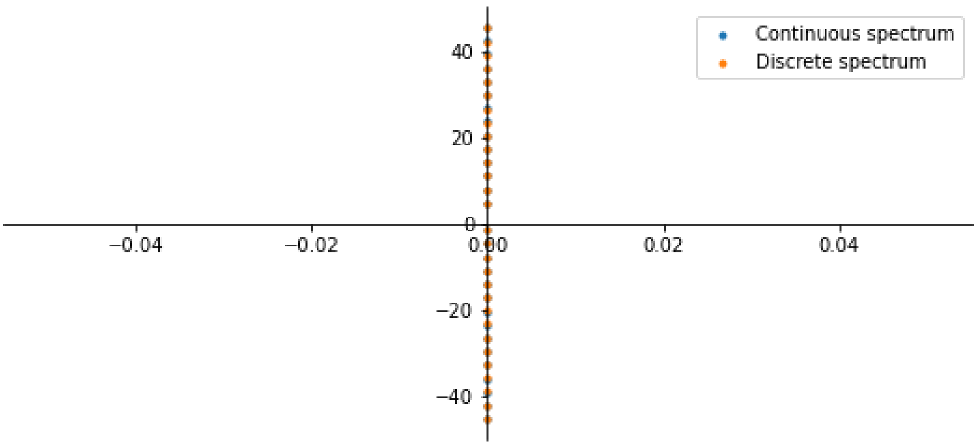


Figure 4.5 – Discrete spectrum and infinite dimensional operator spectrum.

We can deduce that with the chosen boundary conditions, the discrete spectrum associated to staggered grid finite difference converges to the spectrum of the infinite dimensional operator.

Remark 4.2.1. As mentioned, for this part we considered $\mathcal{H} = I$, it means $c = 1$, all these results about eigenvalues, are obtained in this case, but we have just to multiply by c^2.

4.3 Second order PH system

For the first equation (1.1.1) which is a second linear port-Hamiltonian system based on Euler-Bernoulli equation that represents bending motion, we use Taylor expansions to approximate the equation for numerical scheme. This formula is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}.$$

By applying this discretization on each equation of the following system

$$\dot{X}^B(t,\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial^2}{\partial x^2} \mathcal{H} X^B(t,\xi) \quad \text{with } X^B = \begin{pmatrix} X_1^B \\ X_2^B \end{pmatrix}.$$

We get

$$\dot{X}_1^B(\xi_j) = \frac{1}{\mu} \frac{X_2^B(\xi_{j+1}) - 2X_2^B(\xi_j) + X_2^B(\xi_{j-1})}{h^2}$$
$$\dot{X}_2^B(\xi_j) = -EI \frac{X_1^B(\xi_{j+1}) - 2X_1^B(\xi_j) + X_1^B(\xi_{j-1})}{h^2}.$$

For $1 \leq j \leq n$, $\xi_0 = 0$ and $\xi_{n+1} = L$, where h is space discretization step, and $n+1$ is amount of space steps

$$\dot{X_1^B} = \frac{1}{\mu h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} X_2^B(\xi_1) \\ X_2^B(\xi_2) \\ X_2^B(\xi_3) \\ \vdots \\ X_2^B(\xi_n) \end{pmatrix} + \frac{1}{\mu h^2} \begin{pmatrix} X_2^B(0) \\ 0 \\ \vdots \\ 0 \\ X_2^B(L) \end{pmatrix}$$

$$\dot{X_2^B} = -\frac{EI}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} X_1^B(\xi_1) \\ X_1^B(\xi_2) \\ X_1^B(\xi_3) \\ \vdots \\ X_1^B(\xi_n) \end{pmatrix} - \frac{EI}{h^2} \begin{pmatrix} X_1^B(0) \\ 0 \\ \vdots \\ 0 \\ X_1^B(L) \end{pmatrix}$$

$\mathcal{H}_d = \operatorname{diag}(EI, \cdots, EI, \frac{1}{\mu} \cdots, \frac{1}{\mu})$ is positive definite.

$$D = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

$$\dot{X}(t) = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \mathcal{H}_d X(t) + \frac{1}{h} u(t).$$

Hence $J = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} = -J^*$, because $D^* = D$, so the staggered finite difference method preserve the port-Hamiltonian structure for the second order linear port-Hamiltonian systems.

Let's consider the boundary conditions as defined above (based on the physical model):

$$e_2^B(t,0) = e_1^B(t,0) = 0$$

$$e_1^B(t,L) = -k_1 \frac{\partial}{\partial x} e_2^B(t,L)$$

$$e_2^B(t,L) = k_2 \frac{\partial}{\partial x} e_1^B(t,L).$$

Hence

$$X_2^B(0) = X_1^B(0) = 0$$

$$EIX_1^B(L) = -\frac{k_1}{\mu}\frac{\partial}{\partial x} X_2^B(L)$$

$$\frac{1}{\mu} X_2^B(L) = k_2 EI \frac{\partial}{\partial x} X_1^B(L).$$

Now we use the left-sided finite difference for X_1^B and X_2^B space derivative:

$$EIX_1^B(L) = -\frac{k_1}{\mu}\frac{X_2^B(L) - X_2^B(\xi_n)}{h}$$

$$\frac{1}{\mu} X_2^B(L) = k_2 EI \frac{X_1^B(L) - X_1^B(\xi_n)}{h}.$$

We get

$$EIX_1^B(L) = \frac{h^2}{h^2 + k_1 k_2}\left(\frac{k_1}{\mu h} X_2^B(\xi_n) + \frac{k_1 k_2}{h^2} EIX_1^B(\xi_n)\right)$$

$$\frac{1}{\mu} X_2^B(L) = \frac{h^2}{h^2 + k_1 k_2}\left(-\frac{k_2}{h} EIX_1^B(\xi_n) + \frac{k_1 k_2}{\mu h^2} X_2^B(\xi_n)\right).$$

So, $u(t) = KX^B$.

Conclusion and perspectives

In this project, we reviewed the topic of port-Hamiltonian systems that started about more than 20 years ago. The latter is a powerful and unified approach to model physical systems based on energy exchanges between different energy domains of the studied system. More precisely, in this project, we exploited the port-Hamiltonian formulation for the modeling and stabilization of a flexible structure. In this respect, the project consisted firstly of establishing the port-Hamiltonian formalism for this structure from the derivation of the equations of motions. Secondly, we are interested to the well-posedness and the stabilization of port-Hamiltonian systems to reduce the vibrations by a feedback. Finally, to verify the theoretical results, a numerical analysis is essential. This last step consists of a structure preserving discretization of the infinite dimensional system into a finite dimensional system. We are interested in a staggered grid finite difference method given its simplicity. This manuscript highlights the fact that the port-Hamiltonian framework provides a good understanding of different systems dynamics by explicitly separating energetic and interconnection properties of the system. One of the key benefits of the PH framework is the unified language and conceptual insight that can be applied for the synthesis of a distributed parameter system, namely the modeling, discretization, analysis and control. In addition, it is also shown that the PH framework can incorporate both finite and infinite dimensional systems in a similar manner conceptually.

Further work is also needed to include the fluids sloshing, since the previous studies treated 1D Saint-Venant equation which is not a quite realistic case in practice. In this case, a study of nonlinear port-Hamiltonian systems from a mathematical point of view is needed.

Moreover, we could easily remark that the mathematical model does not provide a 100% understanding of the physical system, we take into consideration many hypotheses such as $\sin\theta = \theta$ when θ is too small, but in the real word $\sin\theta = \theta + \epsilon$, $\epsilon > 0$. For this reason, we have to investigate the use of data driven control to elaborate a port Hamiltonian system. Data-driven control is a method in which the identification of the process model and/or the design of the controller are based entirely on experimental data collected from the experiments. As it is shown in this figure, we could use test results to improve the mathematical model. The question is to investigate the reformulation of data-based model into the port-Hamiltonian framework (a paper is published is this context [20]).

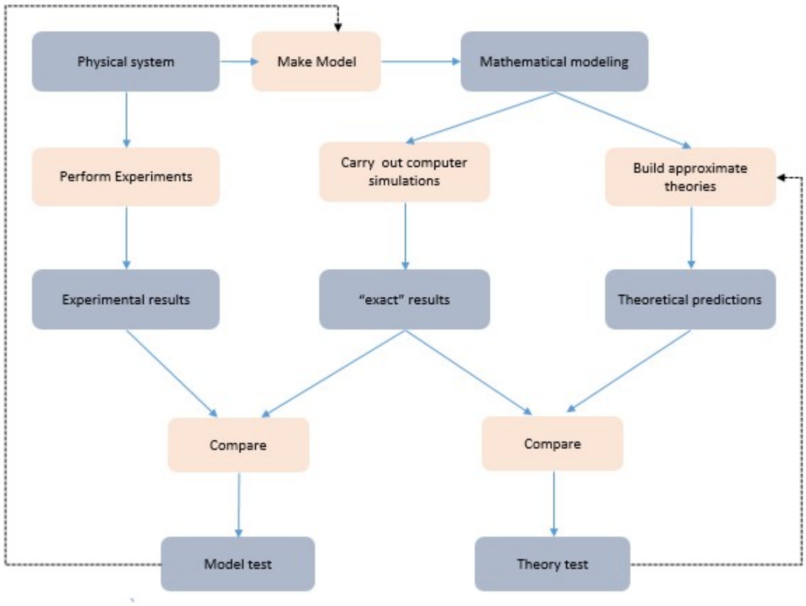


Figure 4.6 – The connection between experiment, theory, and computer simulation.

The dual role of simulation, as a bridge between models and theoretical predictions, and between models and experimental results, is illustrated in Fig. 4.6 inspired by M. P. Allen, D. J. Tildesley [21]. However, to keep similarity between the two systems (physical model represented by hypotheses, and physical systems dynamics), we have to apply the conditions defined in the theoretical studies either to guarantee the existence or the uniqueness or of the solution to the physical system.

The goal of all this modeling work is to create a numerical model that is as close as possible to the real one in order to design airplanes and spacecrafts with as few models and tests as possible. Digital twin are very useful especially when we want to optimize an existing concept or explore new concepts. However, because of the many nonlinear phenomena that are still difficult to model (sloshing fluid, etc.), they are still tests that in the end are used to certify the aircraft before building them ..., it still a lot of work for researchers and engineers in this field. In brief, this work is like a presentation of different points of research that need to be investigated.

«The problem is to find a problem ...».

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