Modeling and Stabilization of port-Hamiltonian Systems

Analytical and Numerical Studies

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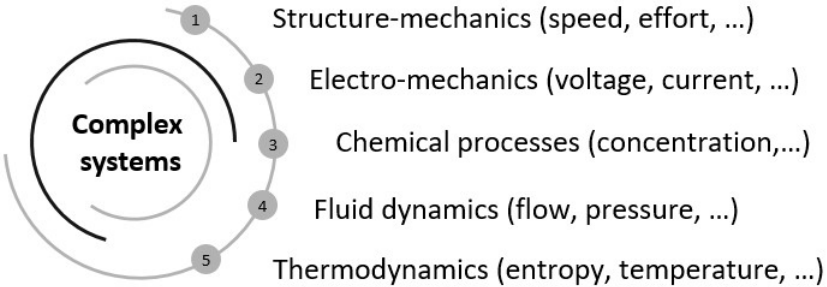
Physical based modeling

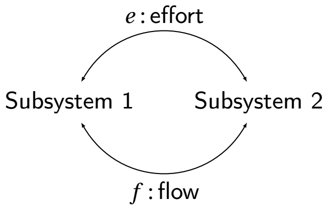


Port-Hamiltonian Systems

Hamiltonian formalism

$$f = J.e, \text{ with } P = \frac{dH}{dt} = e^* f$$

where P is the power, H is the Hamiltonian (i.e the total energy), and $J = J^*$ is the structure operator.



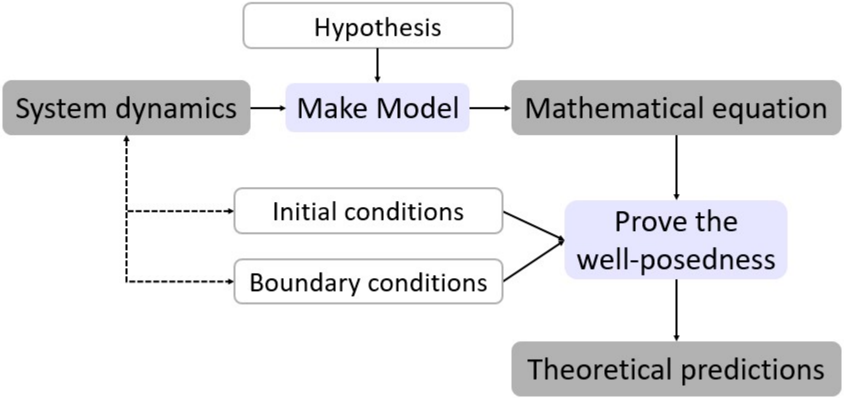
Linear port-Hamiltonian system

- Hamiltonian: $H(t) = \frac{1}{2}\int_a^b x(t,\xi)^* \mathscr{H}(\xi) x(t,\xi) d\xi$, where x is the state energy variable.
- Linear PHS of order N:

$$\dot{x}(t,\xi) = J(x)\frac{\partial H}{\partial x}(x) = \sum_{k=0}^{N} P_k \frac{\partial^k (\mathscr{H} x)}{\partial \xi^k}(t,\xi), \quad t \geq 0, \xi \in [a,b],$$

$$P_k^* = (-1)^{k+1} P_k, \quad N \geq k \geq 0.$$

From a mathematical point of view



Outline

Motivation

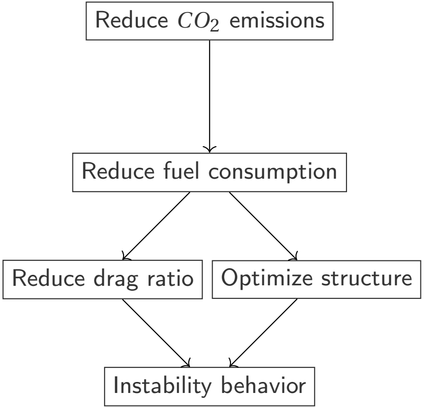


Figure: Boeing TTBW

Figure: Lockheed T-33

Our system

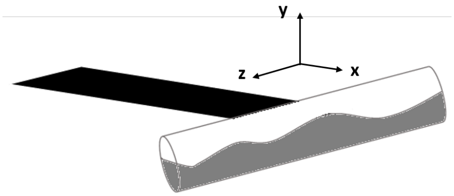


Figure: Schematic representation

1. Beam: a flexible unidirectional structure:
   - Bending: $\mu(x) \rightsquigarrow$ mass density, $E(x) \rightsquigarrow$ Young modulus, $I(x) \rightsquigarrow$ inertia
   - Torsion: $G \rightsquigarrow$ material shear constant, $J \rightsquigarrow$ section torsion constant, $I_p \rightsquigarrow$ section polar moment of inertia per unit length.
2. Rigid tank: $m_R \rightsquigarrow$ tank mass, I_R^B and $I_R^T \rightsquigarrow$ tank rotational inertias
3. Fluid: incompressible, non viscous

Mathematical and port-Hamiltonian modeling

Beam bending: Euler-Bernoulli equation

$$\mu(x)\frac{\partial^2}{\partial t^2}\omega(t,x) = -\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2}{\partial x^2}\omega(t,x)\right), \tag{1}$$

where $\omega(t,x)$ is the deflection at point x at time t.

- Energy variables: $x_1^B(t,x) = \frac{\partial^2}{\partial x^2}\omega(t,x)$ and $x_2^B(t,x) = \mu(x)\frac{\partial}{\partial t}\omega(t,x)$
- Hamiltonian: $H^B(t) = \frac{1}{2}\int_0^L EI(x)(x_1^B)^2 + \frac{1}{\mu(x)}(x_2^B)^2 dx$
- Co-energy variables: $\frac{\delta H^B}{\delta x_1^B} = E.I(x)x_1^B$ and $\frac{\delta H^B}{\delta x_2^B} = \frac{1}{\mu(x)}x_2^B$
- PH formalism:

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1^B(x,t) \\ x_2^B(x,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{\partial^2}{\partial x^2}\left(\begin{pmatrix} EI(x) & 0 \\ 0 & \frac{1}{\mu(x)} \end{pmatrix}\begin{pmatrix} x_1^B(x,t) \\ x_2^B(x,t) \end{pmatrix}\right),$$

$$\frac{\partial}{\partial t}X^B = P_2\frac{\partial^2}{\partial x^2}\mathscr{H}^B X^B \quad \Longrightarrow \quad \text{Linear } 2^{nd} \text{ order port-Hamiltonian system.}$$

Port-Hamiltonian modeling

Beam torsion: 1D wave equation

$$I_p(x)\frac{\partial^2}{\partial t^2}\theta(t,x) = \frac{\partial}{\partial x}(GJ(x)\frac{\partial}{\partial x}\theta(t,x)), \tag{2}$$

where $\theta(x,t)$ is the local torsional angle, x is the position along the beam, t is time.

- Energy variables: strain $x_1^T(t,x) = \frac{\partial}{\partial x}\theta(t,x)$, and momentum $x_2^T(t,x) = I_p(x)\frac{\partial}{\partial t}\theta(t,x)$
- Hamiltonian: $H^T(t) = \frac{1}{2}\int_0^L GJ(x_1^T)^2 + \frac{1}{I_p(x)}(x_2^T)^2 dx$
- Co-energy variables: $\frac{\delta H^T}{\delta x_1^T} = GJx_1^T$ and $\frac{\delta H^T}{\delta x_2^T} = \frac{1}{I_p(x)}x_2^T$
- PH formalism:

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1^T(x,t) \\ x_2^T(x,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\left(\begin{pmatrix} GJ(x) & 0 \\ 0 & \frac{1}{I_p(x)} \end{pmatrix}\begin{pmatrix} x_1^T(x,t) \\ x_2^T(x,t) \end{pmatrix}\right).$$

$$\frac{\partial}{\partial t}X^T = P_1\frac{\partial}{\partial x}\mathscr{H}^T X^T \quad \Longrightarrow \quad \text{Linear } 1^{st} \text{ order port-Hamiltonian system.}$$

Mathematical and port-Hamiltonian modeling

Rigid tank

$$
\begin{aligned}
m_R \ddot{\omega}_B(t) &= F_{ext} \\
I_R^B \ddot{\theta}_B(t) &= M_{ext}^B \\
I_R^T \ddot{\theta}_t(t) &= M_{ext}^T,
\end{aligned}
\tag{3}
$$

where F_{ext} is the sum of forces applied to the tank, M_{ext}^B is the sum of moments in bending direction and M_{ext}^T is the sum of moments in torsion direction

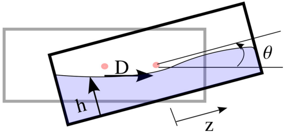
- Energy variables: moments variables: $p = m_R \dot{\omega}_B, p_{\theta B} = I_R^B \dot{\theta}_B$, and $p_{\theta T} = I_R^T \dot{\theta}_T$
- Hamiltonian: $H^R = \frac{1}{2}\left(\frac{p^2}{m_R} + \frac{p_{\theta B}^2}{I_R^B} + \frac{p_{\theta T}^2}{I_R^T}\right)$
- PH formalism:

$$\dot{X}^R(t) = J.\frac{\partial H^R}{\partial X^R}(t) + U^F(t).$$

$\Longrightarrow$ Finite dimensional port-Hamiltonian system.

Mathematical and port-Hamiltonian modeling

Sloshing fluid : Saint-Venant equation



$$\begin{aligned} \rho \frac{\partial w}{\partial t} &= -\frac{\partial P}{\partial z} \\ \frac{\partial w}{\partial t} &= -\dot{D} - \frac{\partial (g\tilde{h} + gz\theta)}{\partial z} \end{aligned} \tag{4}$$

- Energy variables: $p_F = I_F \dot{\theta}$, $x_1^F = b\tilde{h}$, $x_2^F = \rho(\omega + D)$, $x_3^F = \theta$,
- Hamiltonian: $H^F = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{\rho} x_1^F (x_2^F)^2 + \frac{\rho g}{b}(x_1^F)^2 + \rho g z x_1^F x_3^F \, dz + \frac{p_F^2}{I_f}$,
- Co-energy variables: $\frac{\delta H^f}{\delta x_1^F} = \frac{1}{\rho}(x_2^F)^2 + \frac{\rho g}{b} x_1^F + \rho b g z x_3^F$, $\frac{\delta H^F}{\delta x_2^F} = \frac{1}{\rho} x_1^F x_2^F$, $\frac{\delta H^F}{\delta x_3^F} = \rho b g z x_1^F$, and $\frac{\delta H^F}{\delta p_F} = \frac{p_F}{I_F}$

$\Longrightarrow$ Nonlinear port-Hamiltonian formulation.

Mathematical and port-Hamiltonian modeling

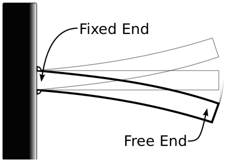
Beam

Beam system can be writing as:

$$\frac{\partial X}{\partial t}(t,\xi) = P_2 \frac{\partial^2}{\partial \xi^2}\mathcal{H}(\xi)X(t,\xi) + P_1 \frac{\partial}{\partial \xi}\mathcal{H}(\xi)X(t,\xi) + P_0\mathcal{H}(\xi)X(t,\xi), \tag{5}$$

where

$$X = \begin{pmatrix} X^B \\ X^T \end{pmatrix},\ \mathcal{H} = \begin{pmatrix} \mathcal{H}_B & 0 \\ 0 & \mathcal{H}_T \end{pmatrix},\ P_2 = \begin{pmatrix} P_2^B & 0 \\ 0 & 0 \end{pmatrix},\ P_1 = \begin{pmatrix} 0 & 0 \\ 0 & P_1^T \end{pmatrix},\ \text{and } P_0 = \mathbf{O}.$$



$$\begin{aligned}
x_1^B(t,L) &= \frac{\partial^2 \omega}{\partial x^2}(t) = 0 && \text{Free side (no strain)} \\
x_2^B(t,0) &= \mu(0)\frac{\partial \omega}{\partial t}(t) = 0 && \text{Fixed side} \\
x_1^T(t,L) &= \frac{\partial \theta}{\partial \xi}(t) = 0 && \text{Free side (no momentum)} \\
x_2^T(t,0) &= I_p(0)\frac{\partial \theta}{\partial t}(t) = 0 && \text{Fixed side.}
\end{aligned}$$

Well-posedeness of linear 2^{nd} order PHS

Boundary conditions

We define

$$\begin{pmatrix} f \\ e \end{pmatrix} = R\Phi(\mathcal{H}X),$$

where

$$R = \begin{pmatrix} P_1 & P_2 & -P_1 & -P_2 \\ -P_2 & 0 & P_2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \Phi(\mathcal{H}X) = \begin{pmatrix} \mathcal{H}X(b) \\ \frac{\partial}{\partial\xi}\mathcal{H}X(b) \\ \mathcal{H}X(a) \\ \frac{\partial}{\partial\xi}\mathcal{H}X(a) \end{pmatrix}.$$

Well-posedeness of linear 2^{nd} order PHS

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Finite dimensional system

$$\begin{cases} \dot{X}(t) = AX(t), \quad t \geq 0, \\ X(t=0) = X_0 \in \mathbb{R}^n, \end{cases} \implies \quad X(t) = e^{tA}X_0.$$

Well-posedeness of linear 2^{nd} order PHS

Theorem

Let A be the operator defined as

$$AX(\xi) = P_2 \frac{\partial^2}{\partial \xi^2} \mathcal{H}(\xi) X(x) + P_1 \frac{\partial}{\partial \xi} \mathcal{H}(\xi) X(\xi) + P_0 \mathcal{H}(\xi) X(\xi),$$

where

$$D(A) = \left\{ X \in L^2([a,b], \mathbb{R}^n) \,/\, \mathcal{H}X \in H^2([a,b], \mathbb{R}^n),\ W \begin{bmatrix} f \\ e \end{bmatrix} = 0,\ W \in \mathbb{R}^{n \times 2n} \right\},$$

Then, A generates a contraction semigroup if and only if $\operatorname{Re}\langle AX, X \rangle_{\mathcal{H}} \leq 0$.

Application

$$\operatorname{Re}\langle AX, X \rangle_{\mathcal{H}} = \frac{1}{2}(f^* e + e^* f) = 0.$$

Stabilization of linear 2^{nd} order PHS

$$\frac{dH}{dt}(t) = \langle AX, X\rangle_{\mathcal{H}} + \langle X, AX\rangle_{\mathcal{H}} = 0 \quad \Longrightarrow \text{ Conservative system}$$

Acting on the system at the end $x = L$ by a feedback control, we find

$$\frac{dH}{dt}(t) = \frac{1}{2}\left(m(t)\frac{\partial^2\omega}{\partial x\partial t}(t,L) - \frac{\partial\omega}{\partial t}(t,L)f(t) + \tau(t)\frac{\partial\theta}{\partial t}(t,L)\right).$$

Theorem

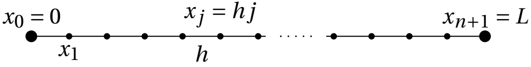
Let A be an operator defined as above. Assume that $\mathcal{H}x \in H^1\left([a,b];\mathbb{R}^{n\times n}\right)$. Case $N = 2$, if

$$\operatorname{Re}\langle Ax, x\rangle_{\mathcal{H}} \leq -\kappa\left[|(\mathcal{H}x)(a)|^2 + \left|(\mathcal{H}x)'(a)\right|^2 + \left\{\begin{array}{c} |(\mathcal{H}x)(b)|^2 \\ \text{or} \\ \left|(\mathcal{H}x)'(b)\right|^2 \end{array}\right\}\right], \quad x \in D(A)$$

holds for some $\kappa > 0$. Then A generates an exponentially stable contractive C_0-semigroup.

In other words, the energy of the system decays exponentially to 0 as t becomes sufficiently large.

Space discretization of 1^{st} order PHS



$$\dot{x_1}(x_j) = c^2 \frac{1}{h} x_2(x_j) - x_2(x_{j-1})$$

$$\dot{x_2}(x_j) = \frac{1}{h} x_1(x_{j+1}) - x_1(x_j)$$

$$\dot{X}(t) = \begin{pmatrix} 0 & D \\ -D^T & 0 \end{pmatrix} \mathscr{H}_d X(t) + \frac{1}{h} \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_2(t,0) \\ c^2 x_1(t,L) \end{pmatrix}$$

General wave equation

$$\begin{cases} \dfrac{\partial^2\theta}{\partial t^2}(t,x) - c^2 \dfrac{\partial^2\theta}{\partial x^2}(t,x) & = 0 \\ \dfrac{\partial\theta}{\partial x}(t,L) & = u(t) \\ \dfrac{\partial\theta}{\partial t}(t,0) & = v(t) \\ \theta(0,x) & = f(x), \quad \dfrac{\partial\theta}{\partial t}(0,x) = w(x) \end{cases} \Longrightarrow \begin{cases} \dfrac{\theta_j^{i+1} - 2\theta_j^i + \theta_j^{i-1}}{dt^2} & = c^2 \dfrac{\theta_{j+1}^i - 2\theta_j^i + \theta_{j-1}^i}{h^2} \\ \theta_{n+1}^i & = \theta_n^i + h u^i \\ \theta_0^{i+1} & = \theta_0^i + dt\, v^i \\ \theta_j^0 & = f_j, \quad \theta_j^1 = \theta_j^0 + dt\, w_j \end{cases}$$

Solution of wave equation

n: number of space steps	50	100	150	200
e : total error in L_2-norm	0.064	0.033	0.021	0.016
CPU time using Numba	92.2	264.1	639.8	1363.7

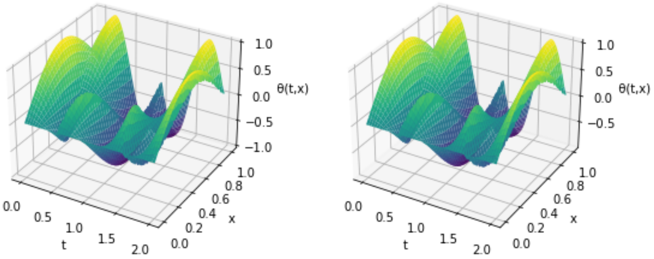


Figure: Discrete wave and discrete 1^{st} PHS, n=200

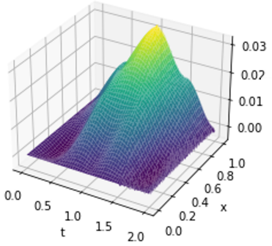
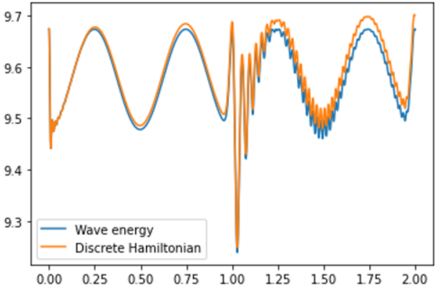


Figure: Difference between two solutions

Energy: conservative system

- Discrete Hamiltonian: $H_d = \frac{h}{2} x_d^T \mathscr{H}_d x_d = \frac{h}{2} \sum_{i=1}^{n} \begin{pmatrix} x_i^1 & x_i^2 \end{pmatrix} \mathscr{H}_{x_i} \begin{pmatrix} x_i^1 \\ x_i^2 \end{pmatrix}$
- Wave energy: $E(t) = \frac{1}{2} \int_0^L (\frac{\partial \theta}{\partial t}(t,x))^2 + c^2 (\frac{\partial \theta}{\partial x}(t,x))^2 dx$



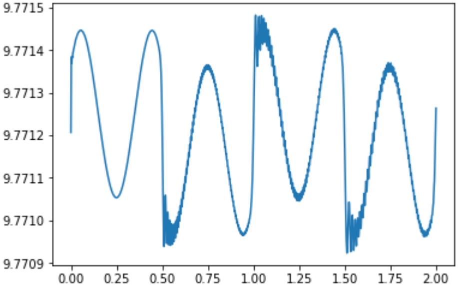


Figure: Discrete wave energy and discrete Hamiltonian of 1^{st} order PHS.

Eigenvalues

- Continuous eigenvalues: $\lambda_k = \frac{2k+1}{2}\pi i, \quad k \in \mathbb{Z};$
- Discrete eigenvalues: $\lambda_k^n \sim \frac{2k+1}{2}\left(1 - \frac{1}{2n}\right)\pi i, \quad k \in \mathbb{Z};$
- For $n = 100$:

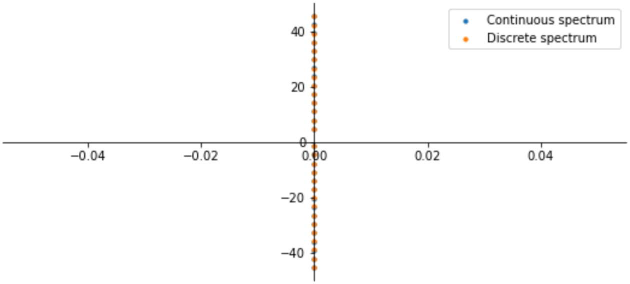


Figure: Discrete spectrum and infinite dimensional operator spectrum

Conclusion and further work

- Structure modeling: Bending and Torsion
- Fluid sloshing modeling
- Stabilization and control

Thank you!

«The problem is to find a problem ...»

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