Measuring quality of life under spatial frictions: Toy version of the model

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Abstract

This document outlines a toy version of the model developed by Ahlfeldt, Bald, Roth, Seide (WIP), which represents a special case with one group, no trade cost, and one sector. We solve it analytically in relative differences for the two-regions case. This toy model represents the theoretical foundation of the ABRS toolkit, which illustrates how spatial equilibrium adjusts to changes in fundamentals depending on the dispersion of idiosyncratic tastes (and other parameters).

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1 Preferences and demand

Worker ω living in location i derives utility from the consumption of goods ($C_{i\omega}$) and floor space ($h_{i\omega}$) according to

$$U_{i\omega} = \left(\frac{Q_{i\omega}}{\alpha}\right)^{\alpha} \left(\frac{h_{i\omega}}{1-\alpha}\right)^{1-\alpha} \exp[a_{i\omega}], \tag{1}$$

where $Q_{i\omega}$ summarises the consumption of a final good that is locally assembled at zero cost under perfect competition from tradable intermediate goods, shipped from origin j to destination i, $q_{ji\omega}$, according to the CES-aggregator

$$Q_{i\omega} = \left[\sum_{j \in J} (q_{ji\omega})^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{2}$$

. The idiosyncratic amenity component $\exp[a_{i\omega}]$ is modelled as a stochastic preference shock, with $a_{i\omega}$ being drawn from a type-I-extreme value (Gumbel) distribution:

$$F_i(a) = \exp\left(-\tilde{A}_i \exp\{-[\gamma a + \Gamma]\}\right) \quad \text{with} \quad \gamma > 0, \tag{3}$$

where $\tilde{A}_i \equiv (A_i)^{\gamma}$ represents the mean of the amenity shock and Γ is the Euler-Mascheroni constant. We take the component A_i as a measure of local quality of life (QoL) and consider it as an exogenous utility shifter that captures the effects of (dis)amenities. The parameter γ governs the dispersion of individual amenity shocks.

Immobile landowners earn a rent r_i per unit of land, which they spend locally on goods. Utility maximisation provides optimal aggregate demand at location i for tradable intermediates from origin j, $q_{ji} = (p_{ji})^{-\sigma}(P_i)^{\sigma-1}E_i$, where $E_i = \alpha w_i L_i + r_i \bar{T}_i$ and p_{ji} is the consumer price of a variety produced in j and consumed in location i. Workers receive region-specific wages w_i as compensation, while $P_i = [\sum_j (p_{ji})^{1-\sigma}]^{1/(1-\sigma)}$ describes the price index of the final good dual to Eq. (2). The optimal aggregate demand for housing is given by $H_i^r = (1-\alpha)w_i L_i / p_i^H$.

2 Technology

Developers supply floor space under perfect competition according to a Cobb-Douglas production function that combines a share of the globally available capital stock, K_i (available at unit prices), with location-specific land, $\bar{T}_i$:

$$H_i^S = \eta_i \left(\frac{\bar{T}_i}{\delta}\right)^{\delta} \left(\frac{K_i}{1-\delta}\right)^{1-\delta}. \tag{4}$$

η_i denotes total factor productivity, capturing the role of regulatory (e.g. height restrictions) and physical (e.g. a rugged surface) constraints, and δ controls the relative importance of both input factors.

Each location produces a unique variety of a tradable intermediate good using labour L_i as the only production input under perfect competition according to

$$q_i = \varphi_i L_i. \tag{5}$$

We also allow for endogenous labour productivity $\varphi_i = \bar{\varphi}_i L_i^{\zeta}$, which increases in local employment according to the agglomeration elasticity ζ.

Trade in intermediate goods is free. Perfect competition equates prices to marginal costs, so we get $p_{ji} = w_j/\varphi_j$. The trade structure implies an expenditure share of customers in i on intermediate goods shipped from j as follows:

$$\chi_{ji} = \frac{p_{ji}q_{ji}}{\sum_k p_{ki}q_{ki}} = \frac{\left(w_j/\varphi_j\right)^{1-\sigma}}{\sum_k \left(w_k/\varphi_k\right)^{1-\sigma}} \tag{6}$$

Intuitively, workers in i consume more from good produced in j if the price at j, which directly maps into marginal costs w_j/φ_j, is lower.

2.1 Location choice

Under the distributional assumptions on the idiosyncratic utility component, we obtain the probability λ_i that a worker lives in location i:

$$\lambda_i = \frac{\left(A_i w_i/\mathcal{P}_i\right)^{\gamma}}{\sum_{j\in J}\left(A_j w_j/\mathcal{P}_j\right)^{\gamma}}, \tag{7}$$

where we have defined the aggregate consumer price index $\mathcal{P}_i \equiv (P_i)^{\alpha}(p_i^H)^{1-\alpha}$.

3 General equilibrium

Goods market clearing.

$$w_i L_i = \sum_{j\in J} \frac{\left(w_i/\varphi_i\right)^{1-\sigma}}{\sum_k \left(w_k/\varphi_k\right)^{1-\sigma}} \left(\alpha w_j L_j + r_j \bar{T}_j\right). \tag{8}$$

This condition states that what firms spend on the wage bill at i (left-hand side) must equate to the revenues they generate across all locations j (right-hand side). This is simply the share of the wage bill at j spent on non-housing goods $\left(\alpha w_j L_j + r_j \bar{T}_j\right)$, plus the landlord income, $r_j\bar{T}_j$, weighted by the share of goods produced at i that workers consume in j, $\sum_{j\in J}\frac{(w_i/\varphi_i)^{1-\sigma}}{\sum_k (w_k/\varphi_k)^{1-\sigma}}$,

Labor market clearing.

$$L_i = \lambda_i \bar{L} \tag{9}$$

This condition simply states that the labour input L_i must equate labour supply, which is product of the share of workers living in i, λ_i, and the national labour endowment $\bar{L}$.

Floor-space market clearing. To derive the floor space market equilibrium $H_i^S = H_i^D$ we start with the demand side. In this toy version of the model, H_i^r, is the only component of housing demand $_i^D$. First-order conditions of the housing production function and zero profits deliver $H_i^S = \eta^{\frac{1}{\delta}} \bar{T}_i (p_i^H)^{\frac{1-\delta}{\delta}} / \delta$. The market clearing price for floor space solves $H_i^S = H_i^D$ and is given by

$$p_i^H = \left(\frac{\tilde{\alpha} \delta w_i L_i}{\eta_i^{\frac{1}{\delta}} \bar{T}_i} \right)^{\delta}, \tag{10}$$

where $\tilde{\alpha} \equiv (1-\alpha) + \alpha(1-\beta)(1-\delta)$ is a constant. Since land rents will be irrelevant to the applications in the toolkit, we abstract from land market clearing.

4 Mapping from primitives to endogenous objects

Under the implications made in this toolkit, we can derive analytical solutions and a direct mapping from primitives to the endogenous objects for relative differences in the two-regions case. This is because we can reduce the system to three equations in three unknowns. Specifically, we need the clearing conditions for tradable goods, labour, and floor space that solve for (relative) wages, employment and floor-space prices.

Goods market clearing.

$$\begin{aligned} \frac{w_i L_i}{w_j L_j} &= \frac{\frac{(w_i/\varphi_i)^{1-\sigma}}{(w_i/\varphi_i)^{1-\sigma} + (w_j/\varphi_j)^{1-\sigma}} \left(\alpha w_i L_i + r_i \bar{T}_i + \alpha w_j L_j + r_j \bar{T}_j \right)}{\frac{(w_j/\varphi_j)^{1-\sigma}}{(w_i/\varphi_i)^{1-\sigma} + (w_j/\varphi_j)^{1-\sigma}} \left(\alpha w_i L_i + r_i \bar{T}_i + \alpha w_j L_j + r_j \bar{T}_j \right)} \\ \Leftrightarrow \quad \hat{L} &= \hat{\varphi}^{\frac{\sigma-1}{1-\zeta(\sigma-1)}} \hat{w}^{-\frac{\sigma}{1-\zeta(\sigma-1)}} = \hat{\varphi}^{\frac{\sigma-1}{\Lambda}} \hat{w}^{-\frac{\sigma}{\Lambda}}, \end{aligned} \tag{A.GMC}$$

where $\hat{\cdot}$ indicates ratios, e.g. $\hat{L} = L_i/L_j$, and we define $\Lambda = 1 - \zeta(\sigma - 1)$.

Labour market clearing. Relating the labour-market clearing conditions, $L_i = \lambda_i \bar{L}$, for both locations delivers

$$\hat{L} = \left(\hat{A} \hat{w} (\hat{p}^H)^{\alpha-1} \right)^{\gamma}. \tag{A.LMC}$$

Floor-space market clearing. We obtain

$$\hat{p}^H = \hat{k} \left(\hat{w} \hat{L} \right)^{\delta}, \tag{A.FMC}$$

where $k_i \equiv \frac{1}{\eta_i} \left(\frac{\tilde{\alpha}\delta}{\bar{T}_i} \right)^{\delta}$ collects model primitives.

Combining Eqs. (A.GMC), (A.LMC) and (A.FMC) allows us to solve the spatial equilibrium in closed form. Defining $\Delta \equiv \Lambda[1-(1-\alpha)\delta]+\sigma[1/\gamma+(1-\alpha)\delta]$, we get

$$\hat{L} = \hat{k}^{-\frac{(1-\alpha)\sigma}{\Delta}} \hat{\varphi}^{\frac{(\sigma-1)(1-(1-\alpha)\delta)}{\Delta}} \hat{A}^{\frac{\sigma}{\Delta}}$$

$$\hat{w} = \hat{k}^{\frac{(1-\alpha)\Lambda}{\Delta}} \hat{\varphi}^{\frac{(\sigma-1)(1/\gamma+(1-\alpha)\delta)}{\Delta}} \hat{A}^{-\frac{\Lambda}{\Delta}}$$

$$\hat{p}^{H} = \hat{k}^{\frac{\Lambda+\sigma/\gamma}{\Delta}} \hat{\varphi}^{\frac{(\sigma-1)(1+1/\gamma)\delta}{\Delta}} \hat{A}^{\frac{\delta(\sigma-\Lambda)}{\Delta}}$$

Hence, we have a direct mapping from the structural parameters and the fundamentals in relative differences $\{\hat{A}, \hat{\varphi}, \hat{\eta}\}$ to the endogenous outcomes in relative differences $\{\hat{L}, \hat{w}, \hat{p}^H\}$. We exploit this mapping in the Stata and HTML implementations of the `ABRS toolkit`.

5 Spatial general equilibrium in four quadrants

For didactic purposes, it is useful to rearrange the equilibrium conditions to give equilibrium relationships between the key endogenous outcomes in relative differences. First, it is instructive to realize that the goods market clearing condition (A.GMC) is the labour demand equation. We can equate it to (A.LMC), which essentially gives labour supply, to clear the labour market. Then, we can use (A.FMC) to derive the combination of prices and quantities on the labour market that satisfy all equilibrium conditions.

$$\hat{L} = \hat{k}^{-\frac{1-\alpha}{\frac{1}{\gamma}+(1-\alpha)\delta}} \cdot \hat{A}^{\frac{1}{\frac{1}{\gamma}+\delta(1-\alpha)}} \cdot \hat{w}^{\frac{1-\delta(1-\alpha)}{\frac{1}{\gamma}+\delta(1-\alpha)}}$$

This is the first quadrant generated by the Sata ado file `ABRS` that is part of the `ABRS toolkit`. This curve is upward-sloping since higher wages attract greater labour supply. Quality of life shifts the curve upwards since it increases labour supply. Housing rent enters indirectly via $\hat{k}$, which depends on $\hat{\eta}$, as greater housing productivity implies lower rents and greater housing supply.

Using (A.FMC) to replace wages, we can derive all combinations of total employment and rents that satisfy all equilibrium conditions.

$$\hat{p}^{H} = \hat{k}^{\frac{1}{1-(1-\alpha)\delta}} \cdot \hat{A}^{-\frac{\delta}{1-\delta(1-\alpha)}} \cdot \hat{L}^{\frac{\delta(\frac{1}{\gamma}+1)}{1-(1-\alpha)\delta}}$$

This curve, which enters the second quadrant, is upward-sloping since larger total employment means implies greater housing demand, which results in higher equilibrium rents given imperfectly elastic floor space supply (originating from fixed land supply). Housing productivity (enters via $\hat{k}$) shifts the curve to the right since higher housing productivity increases housing supply, reducing equilibrium rent. Quality of life also shifts the curve to the right since greater quality of life leads to a negative compensating differential in wages, which reduces housing demand for any given employment level (notice that equilibrium rents increase in total employment since the labour supply effect dominates).

Finally, can use Eq. (A.FMC) in Eq. (A.GMC) to derive all combinations of between wages and rents that satisfy our equilibrium conditions.

$$\hat{w} = \left(\frac{\hat{p}^H}{\hat{k}}\right)^{\frac{\zeta(\sigma-1)-1}{\delta(\sigma-1)(\zeta+1)}} \cdot \hat{\bar{\varphi}}^{\frac{1}{\zeta+1}}$$

This curve enters the third quadrant and is downward-sloping. For given primitives, a higher rent implies more total employment. Goods market clearing, which essentially establishes a downward-sloping labour demand function, requires a lower wage.

The fourth quadrant simply consists of a 45-degree line that maps the firm wages from the third quadrant onto worker wages in the first quadrant. The only combination of wages, employment, and rents, that satisfies all equilibrium conditions is defined by the rectangle whose corners intersect with all curves in all four quadrants. Intuitively, the worker wage determines employment via labour supply in the first quadrant. Employment maps into rents via housing demand in the second quadrant. Employment also maps into firm the firm wage via labour demand in the third quadrant. Market clearing requires the firm wage to correspond to the worker wage, which enforced via the mirroring along the 45-degree line in the fourth quadrant. The Stata implementation of the `ABRS toolkit` visually conveys this intuition. To see the four quadrants under the baseline parameterization, simply type:

```
ssc install ABRS
ABRS
```

This will return figure 1 showing the four-quadrant figure under the baseline parameterization described in Table 1.

Figure 1: Spatial equilibrium in four quadrants

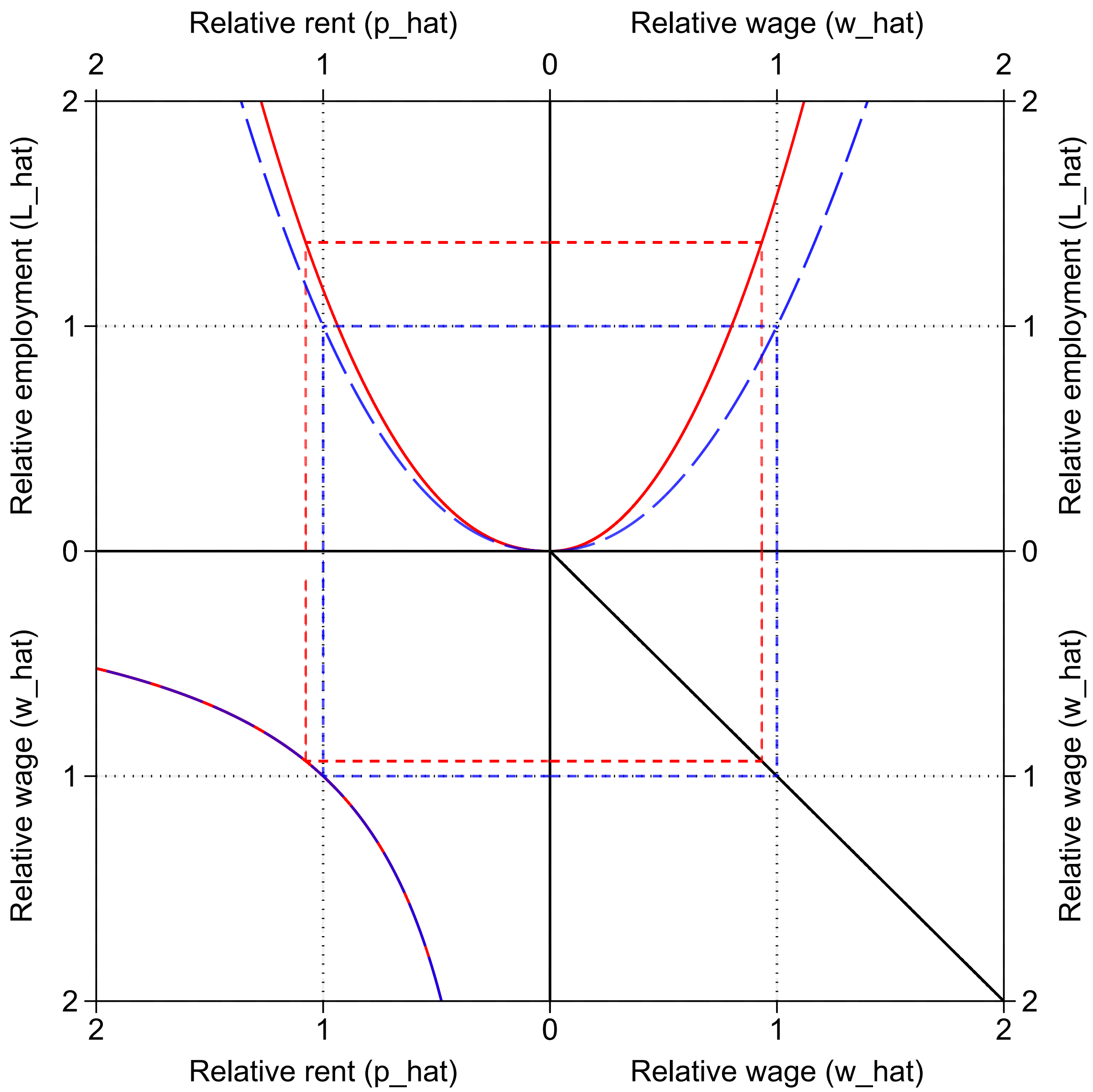

Table 1: Baseline parameterization

Symbol	Description	Value
Parameters		
α	Expenditure share on non-housing goods	0.66
γ	Preference heterogeneity (inversely related to dispersion)	3
δ	Share of land in housing (inversely related to labour supply elasticity)	0.3
σ	Elasticity of substitution between varieties	4
ζ	Agglomeration elasticity	0.04
Red solid lines		
$\hat{A}_1$	Relative quality of life advantage of region 1	0.2
$\hat{\eta}_1$	Relative housing productivity advantage of region 1	0
$\hat{\bar{\varphi}}_1$	Relative labor productivity advantage of region 1	0
Blue dashed lines		
$\hat{A}_2$	Relative quality of life advantage of region 2	0
$\hat{\eta}_2$	Relative housing productivity advantage of region 2	0
$\hat{\bar{\varphi}}_2$	Relative labor productivity advantage of region 2	0
Technical parameters governing the size of the panels		
$\hat{w}_{\text{min}}$	Minimum x-value: Wage	0
$\hat{w}_{\text{step}}$	x-steps: wage	1
$\hat{w}_{\text{max}}$	Maximum x-value: Wage	2
$\hat{L}_{\text{min}}$	Minimum x-value: Employment	0
$\hat{L}_{\text{step}}$	x-steps: Employment	1
$\hat{L}_{\text{max}}$	Maximum x-value: Employment	3
$\hat{p}_{\text{min}}$	Minimum x-value: Rent	0
$\hat{p}_{\text{step}}$	x-steps: Rent	1
$\hat{p}_{\text{max}}$	Maximum x-value: Rent	2