Topic 4
Numerical solution & quantification

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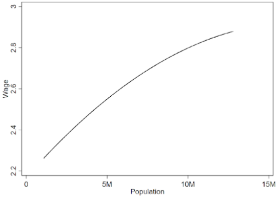
Introduction

Lessons from tutorial

```
Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)       -10.512      1.678  -6.265 7.73e-07 ***
log(avfloor_res)    9.257      0.680  13.614 3.99e-14 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.3606 on 29 degrees of freedom
Multiple R-squared:  0.8647,	Adjusted R-squared:   0.86
F-statistic: 185.3 on 1 and 29 DF,  p-value: 3.985e-14
```

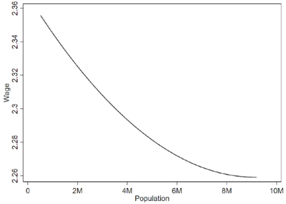
```
Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)       -12.34078    0.19591  -62.99   <2e-16 ***
log(avfloor_comm)   8.80076    0.07161  122.90   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.0454 on 29 degrees of freedom
Multiple R-squared:  0.9981,	Adjusted R-squared:  0.998
F-statistic: 1.51e+04 on 1 and 29 DF,  p-value: < 2.2e-16
```

Lower fundamental productivity



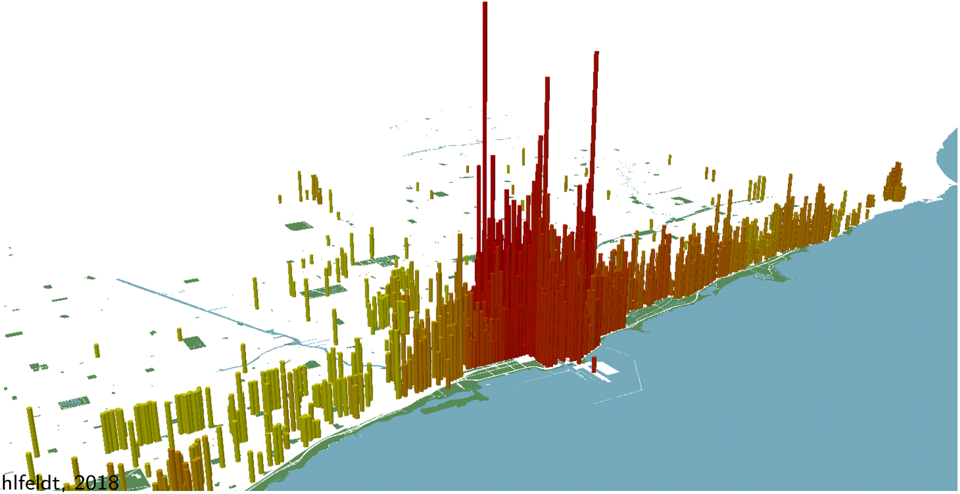
Higher fundamental productivity

Lower fundamental amenity



Higher fundamental amenity

Fuzzy skyline of Chicago



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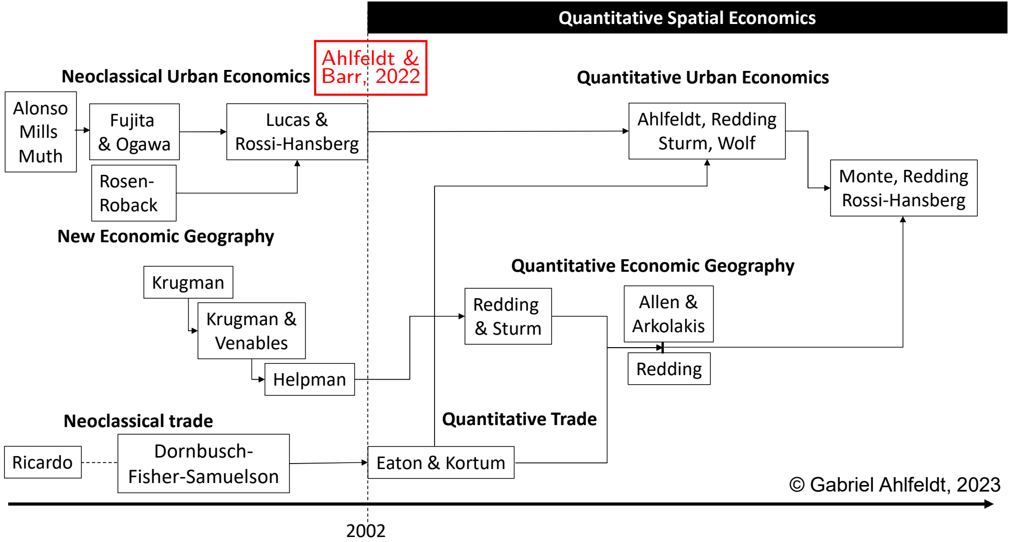
Roadmap

- **How do we solve numerically for the equilibrium?**
  - Have discussed how to **reference the equilibrium**
    - Find all endogenous objects that need to be solved simultaneously
  - Need to solve for **target variables** and use recursive structure
  - Now, let's write our programmes...
- **How do we rationalize observed data for real geographies?**
  - Standard MCM imposes **stylized geography** with smooth gradients
  - In reality, rents and densities can **vary remarkably over short distances**
  - Let's discretize space and
    - estimate parameters
    - adjust fundamentals to rationalize data

History of thought



Motivation

- Recall the structure of a **quantitative model**
  - Consists of **exogenous objects** and **endogenous objects** (like any model)
  - Consists of **parameters** and **fundamentals**, inverted to **rationalize observed data**
  - Allows for **quantitative counterfactuals** to evaluate real-world shocks and policies
- **Ahlfeldt & Barr (2022) model bridges worlds**
  - **Intuitive** stylized vs. **tractable** quantitative models
  - Amenable to 'textbook' world' with **smooth gradients** & fully quantifiable
- Last week
  - Cover building blocks and **intuition** with a **stylized city structure**
- **This week**
  - **Quantification** and **computational implementation**

Equilibrium solver

Discrete space

- To solve numerically, we 'discretize' space, **indexing locations by** $i \in J$
  - In Ahlfeldt & Barr (2022) toolkit, we create $J = 10,001$ locations along a line
    - Values range from -50 to 50 in 0.1 steps $\Rightarrow$ can think of unit as km
    - `_PROGS.do`, lines 42-25 ▸ Open AB2022-toolkit
- As an example, bid-rent function becomes $\bar{p}_i^U = \frac{a_i^U}{1+\omega^U}\left(S_i^U\right)^{\omega^U}$,
  - where discrete variation in a_i^U will generate discrete variation in $\{\bar{p}_i^U, S_i^U\}$
- Instead of integrating over continuous space, we sum over discrete locations:

$$\sum_i L_i^J \times \mathbb{I}\left(r_i^C > r_i^R \wedge r_i^C > r^A\right) = \sum_i n_i^J \times \mathbb{I}\left(r_i^R > r_i^C \wedge r_i^R > r^A\right) = N,$$

  - where $\mathbb{I}(.)$ returns 1 when the condition is true and zero otherwise

Recall: Recursive structure in Ahlfeldt & Barr (2022) (Topic 3)

- Assume we have values for wage, y, and total employment, N (and all primitives)
  - We can use $\{y, N\}$ in 1. and 2. to get $\{a^C, a^R\}$
  - We can use $\{a^C, a^R\}$ in 3. and 4. to get $\{\tilde{S}^C, \tilde{S}^R\}$
  - We can use $\{\tilde{S}^C, \tilde{S}^R\}$ in 5. and 6. to get $\{r^C, r^R\}$
  - We can use $\{\tilde{S}^C, \tilde{S}^R\}$ and $\{S^C, S^R\}$ in 7. and 8. to get $\{S^C, S^R\}$
  - We can use $\{a^C, a^R\}$ and $\{S^C, S^R\}$ in 9. and 10. to get $\{\bar{p}^C, \bar{p}^R\}$
  - We can use $\{y, N\}$, $\{\bar{p}^C, \bar{p}^R\}$ and $\{S^C, S^R\}$ in 11. and 12. to get $\{n(x), L(x)\}$
  - We can use $\{n(x), L(x)\}$ in 13. to compute N
- Just one combination of values for $\{y, N\}$ that ensures labour market clearing

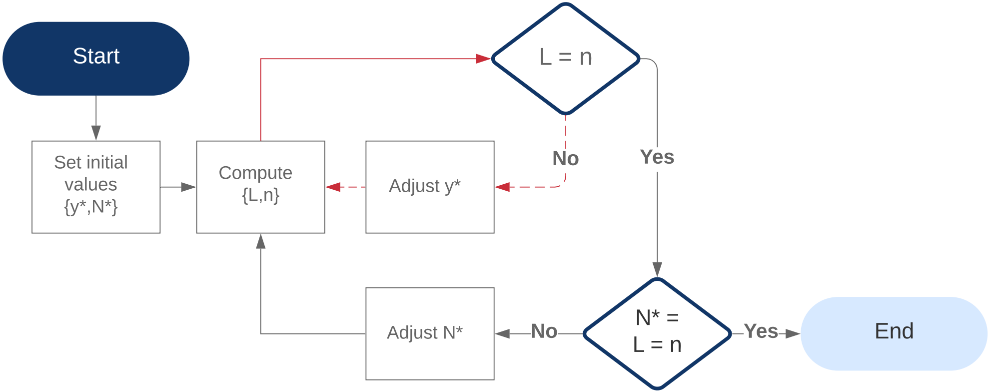
Aim is to get from **primitives to endogenous objects**.
Recursive structure implies that **we need to find** $\{y, N\}$

Conceptualize

- We **need some code** that
  - 'guesses' the value of our target variables $\{y, N\}$
  - goes through the recursive structure to solve for the other referencing objects
  - checks if we clear the labour market
  - If not, update our guesses and repeat until we converge
- **What programmes** do we need to achieve that?
  - Want compact programmes where it is easy to think about inputs and outputs
  - **Nesting various simple programmes** is usually **more accessible and flexible**
    - As opposed to writing one convoluted programme

Think before you code!

Flow chart



Good way of thinking through 'while loops' and 'if conditions'

What programmes do we need to implement the flow chart?

- `SOLVER`: to compute other referencing objects for given values of target variables
  - Takes $\{y, N\}$ as given
- `WAGE`: Finds the market-clearing wage for given guess of N (inner red loop)
  - Takes N as given
  - Calls `SOLVER` as it keeps updating guess of y
- `FINDEQ`: Finds the N that satisfies spatial equilibrium
  - Calls `WAGE` as it keeps updating guess of N
- Of course, all programmes take primitives as given
  - We deal with $\{\alpha^U, \beta, \omega^U, \theta^U, \tau^U, \bar{a}^U, \tilde{c}^U, \bar{S}^U, r^a, \bar{U}\}$ in the **quantification**

What follows is from the *codebook* in the toolkit, check it out!

SOLVER

Algorithm 1: Solver for endogenous variables: `SOLVER`

Data: Given values for primitives $\{\alpha^U, \beta, \omega^U, \theta^U, \tau^U, \bar{a}^U, \tilde{c}^U, \bar{S}^U, r^a, \tilde{U}\}$
Given values of target objects wage and population $\{y, N\}$

Compute $\bar{p}^U(x)$ using bid-rent eqs.
Compute $\tilde{S}^U(x)$ using profit-maximizing building height eq.
Compute $r^U(x)$ using land bid rent eq.
Allocate land to use with the highest land rent
Compute local workplace employment $L(x)$ using MRS eq. within commercial zone
Compute local residence employment $n(x)$ using Marshallian demand eq. within residential zone
Compute labour demand (total L within commercial zone)
Labour supply (total $\hat{N}$) within residential zone

Result: $L(x), n(x), \bar{p}^U(x), r^U(x), S^U(x), L, \hat{N}$

- `SOLVER` essentially implements the **recursive structure** we have developed

  ▸ Open AB2022-toolkit

WAGE

Algorithm 2: Finding equilibrium wage: `WAGE`

Data: Given values for primitives $\{\alpha^U, \beta, \omega^U, \theta^U, \tau^U, \bar{a}^U, \tilde{c}^U, \bar{S}^U, r^a, \tilde{U}\}$
Given values for labour supply and labour demand $\{L, \hat{N}\}$
Guess of wage y
Algorithm 1

while $L \neq \hat{N}$ **do**
- Update y to $\hat{y} = y \times \left(\frac{\hat{N}}{L}\right)^{\rho>0}$
- Use Algorithm 1 to local workplace and residence employment
- Update labour demand (L) within commercial zone
- Update labour supply ($\hat{N}$) within residential zone

Result: Updated values $\{\hat{y}, L, \hat{N}\}$

- `WAGE` updates y until labour market clears ▸ Open AB2022-toolkit
  - increases guess of y if labour supply exceeds demand, i.e. $\frac{\hat{N}}{L} > 1$
  - reduces guess of y if labour demand exceeds supply, i.e. $\frac{\hat{N}}{L} < 1$
  - $0 < \rho < 1$ ensures smoother convergence (avoids overshooting)

FINDEQ

Algorithm 3: Finding the equilibrium: `FINDEQ`

Data: Given values for primitives $\{\alpha^U, \beta, \omega^U, \theta^U, \tau^U, \bar{a}^U, \tilde{c}^U, \bar{S}^U, r^a, \tilde{U}\}$
Guesses of wage and employment $\{y, N\}$
Algorithm 1
Algorithm 2

1 Solve endogenous variables for guesses of $\{y, N\}$ using Algorithm 1
2 **while** *N changes* **do**
3 Use Algorithm 2 to clear labour market and optain new $\{\hat{y}, N\}$
4 Update guesses to weighted combination of old guess of N and new value N

Result: Equilibrium values of $L(x), n(x), \bar{p}^U(x), r^U(x), S^U(x), y, N$

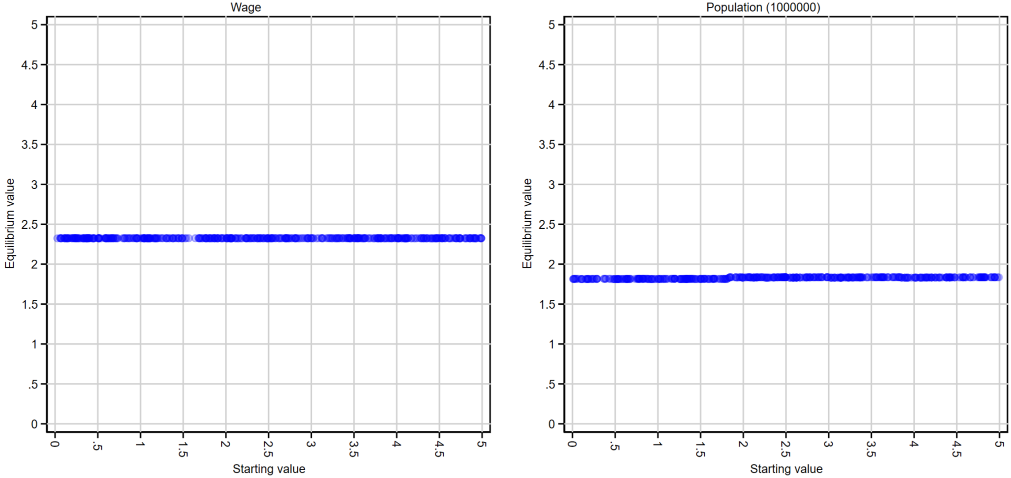
- `FINDEQ` finds N that leads to matching reservation utility $\bar{U}$
  - updates N until recursively solved value of N equates to guess
  - Main programme that calls the other programmes and contains the syntax

▸ Open AB2022-toolkit

Uniqueness I

- Now, we have a procedure to emulate the mapping
  - **From primitives to endogenous objects**
- But is the equilibrium **unique**?
  - Is there only one solution for $\{y, N\}$?
- Can use **Monte Carlo Simulations** to answer the question
  - **Randomly draw guesses** of $\{y, N\}$ from distributions (e.g. uniform or normal)
    - Random, hence, Monte Carlo element
  - Use `FINDEQ` to solve for $\{y, N\}$
  - Repeat many times
  - Check if we **always obtain the same values**

Uniqueness II



Estimation

101 of QSE

- **Estimation**
  - **Use endogenous variables to estimate structural parameters**
  - **Structural fundamentals are the residuals**
- Inversion
  - Use structural parameters and endogenous variables to recover fundamentals
- Simulation
  - Use structural parameters and fundamentals to solve for endogenous variables

Estimation of structural parameters

- For **estimation**, we can derive **'reduced-form'** equations from the model
  - Can then infer structural parameter values from reduced-form estimates
  - Recall Glaeser-Gottlieb example, Topic 1
- Alternatively, we can employ what is often called **'structural estimation'**
  - Derive moment conditions from the model
  - Estimate structural parameters directly within the structure of the model
    - GMM, ML, NLS, SMM, II, etc.

Can only proceed to inversion once we have parameter values

Reduced-from approach

- Say, we want to estimate τ^R, decay in amenity
  - Look for model equations that contain τ^U and variables observed in data
    - $\bar{p}_i^R = \frac{a_i^R}{1+\omega^R}\left(S_i^R\right)^{\omega^R}$; $S_i^R = \left(\frac{a_i^R}{c^R(1+\theta^R)}\right)^{\frac{1}{\theta^R-\omega^R}}$; $a_i^R = \left(\bar{a}_i^R \exp(-\tau^R x_i)\right)^{\frac{1}{1-\alpha^R}}(y)^{\frac{1}{1-\alpha^R}}$
  - Substituting and taking logs delivers
    - $\ln \bar{p}_i^R = b_0 - \frac{\tau^R\theta^R}{(1-\alpha^R)(\theta^R-\omega^R)}\ln x_i + \frac{\theta^R}{\theta^R-\omega^R}\ln \bar{a}_i^R \Rightarrow \ln \bar{p}_i^R = b_0 + b_1 \ln x_i + \epsilon_i$
    - Gradient depends on amenity decay, plus expenditure share, vertical cost and benefits
  - Straightforward reduced-form specification using **within-city** variation
    - From the estimate $\hat{b}_1$, we can recover $\tau^R = -\hat{b}_1\frac{(1-\alpha^R)(\theta^R-\omega^R)}{\theta^R}$
    - if we have values for $\{\alpha^R, \theta^R, \omega^R\}$

In structural work, we often **borrow canonical parameter** values from literature and focus on **estimating** parameters that are **central** to the research question

Stuctural estimation I

- Let's begin with the commercial bid rent curve (and add m to index cities)
    - $\bar{p}_{im}^{C} = \frac{a_{im}^{C}}{1+\omega^{C}} \left(S_{im}^{C}\right)^{\omega^{C}}$; $a_{im}^{C} = \left(\bar{a}_{im}^{C} N_m^{\beta} \exp(-\tau^{C} x_i)\right)^{\frac{1}{1-\alpha^{C}}} (y_m)^{\frac{\alpha^{C}}{\alpha^{C}-1}}$
  - 1. Substitute and rearrange for the **structural residual** (usually a transformed fundamental)
    - $\ln(p_{im}^{R}) + \frac{\tau^{C}}{1-\alpha^{C}} x_i - \omega^{C} \ln(S_{im}^{R}) - z_m = \varepsilon_i,$
    - z_m is a city fixed effect that collects variation in y_m, N_m, and
    - city-level variation in fundamental productivity $\bar{a}_i^{C}$;
    - $\varepsilon_i = \frac{1}{1-\alpha^{C}} \ln \bar{a}_i^{C}$
  - 2. Our **identifying assumption** can be expressed as a **moment condition**:
    - The structural residual is uncorrelated with distance x_i: $\mathbb{E}\left(\varepsilon_i x_i\right) = 0$
  - 3. Get moment condition by substituting structural residual into moment condition
    - $\mathbb{E}\left(\left(\ln(p_{im}^{R}) + \frac{\tau^{C}}{1-\alpha^{C}} x_i - \omega^{C} \ln(S_{im}^{R}) - z_m\right) x\right) = 0$

Structural estimation II

- Let's now look at height
  - an endogenous component in our first moment condition
    - in structural estimation, endogenous variables 'on the right' are fine
    - all depends on credible identifying assumptions
    - $S_{im}^C = \left(\frac{\left(\bar{a}_{im}^C N_m^{\beta} \exp(-\tau^C x_i)\right)^{\frac{1}{1-\alpha^C}} (y_m)^{\frac{\alpha^C}{\alpha^C-1}}}{c^C(1+\theta^C)} \right)^{\frac{1}{\theta^C-\omega^C}}$
  - Solve for the **structural residual**
    - $\ln(S_{im}^C) - \frac{1}{1-\alpha^C}\frac{\beta}{\theta^C-\omega^C}\ln(N_m) + \frac{1}{1-\alpha^C}\frac{\tau^C}{\theta^C-\omega^C}x_{im} - \frac{1}{\theta^C-\omega^C}\frac{\alpha^C}{\alpha^C-1}\ln(y_m) + b_0 = \varepsilon_{im}$
    - where b_0 collects constant terms and $\epsilon_i = \frac{1}{1-\alpha^C}\frac{1}{\theta^C-\omega^C}\ln \bar{a}_{im}^C$

Structural estimation III

- **Identifying assumption**
  - Structural residual is uncorrelated with $\mathbf{Z}$: $\mathbb{E}\left(\epsilon_{im}\mathbf{Z}\right) = 0$
  - where $\mathbf{Z}$ is a $n \times 1$ vector of excludable instruments
  - e.g. one for each variable $\{y_m, N_m, x_i\} \Rightarrow n = 3$
- This gives us $n \times 1$ moment conditions

$$E\left[\left(\ln(S_{im}^C) - \frac{1}{1-\alpha^C}\frac{\beta}{\theta^C - \omega^C}\ln(N_m)\frac{1}{1-\alpha^C}\frac{\tau^C}{\theta^C - \omega^C}x_{im} - \frac{1}{\theta^C - \omega^C}\frac{\alpha^C}{\alpha^C - 1}\ln(y_m) + b_0\right)Z\right] = 0$$

- If we are **exactly identified**, we get three moment conditions
  - Can use x_i again as an included instrument since this is a different equation
  - If we can find IVs for $\{y_m, N_m\}$ that are uncorrelated with fundament productivity
  - That's a big if, of course...

Stuctural estimation IV

- With $1 + 3 = 4$ moment conditions, we can estimate up to 4 parameters
  - For example, $\{\tau^C, \omega^C, \theta^C, \beta\}$
  - Can also estimate fewer, then we are overidentified
  - Can use **GMM for estimation** (`help gmm` in Stata)
- When we are **exactly identified**, there is often a **sequential reduced-form strategy** to get the **same estimates as with structural approach**
  - 2SLS is a special case of GMM
    - Moment condition assumes that excluded IV is uncorrelated with error
- **Structural approach** is useful
  - when **systems of equations are non-linear** ('market potential terms')
  - Can use more information for the estimation of parameters when we are **overidentified**

Inversion

101 of QSE

- Estimation
  - Use endogenous variables to estimate structural parameters
  - Structural fundamentals are the residuals
- **Inversion**
  - **Use structural parameters and endogenous variables to recover fundamentals**
- Simulation
  - Use structural parameters and fundamentals to solve for endogenous variables

Recovering fundamentals

- **QSMs rationalize observed data**
  - Variation in data explained by **endogenous mechanisms**
  - Residual variation captured by **exogenous components** (fundamentals)
- Fundamentals a_i^U were the regression's residuals in the estimation
  - made identifying assumptions about them to estimate parameters
- Now we want to **invert the model** and **recover fundamentals** a_i^U
  - We take the structural parameter values as given and used observed data

Need the fundamentals to simulate model and run counterfactuals

Recovering fundamentals

- **We want to fit the height profile of Chicago** (approx. linear city)
  - Go to the building blocks of the model and rearrange to **solve for fundamentals**
    $S_i^R = \left(\frac{a_i^R}{c^R(1+\theta^R)}\right)^{\frac{1}{\theta^R-\omega^R}}$; $a_i^R = \left(\bar{a}_i^R \exp(-\tau^R x_i)\right)^{\frac{1}{1-\alpha^R}} (y)^{\frac{1}{1-\alpha^R}}$
  - $\Rightarrow \bar{a}_i^R = \left(\left(S_i^R\right)^{\theta^R-\omega^R} c^R(1+\theta^R)(y)^{-(1-\alpha^R)}\right)^{(1-\alpha^R)} \exp(\tau^R x_i)$
    - (if we are willing to trust ChatGPT)
- Can also approach the problem **numerically**

Add/remove height from stylized gradient until we match the height profile

CONV

Algorithm 4: Updating amenities to match heights CONV

Data: Given values for primitives $\{\alpha^U, \beta, \omega^U, \theta^U, \tau^U, \bar{a}^U, \tilde{c}^U, \bar{S}^U, r^a, \tilde{U}\}$
Given values of wage and population $\{y, N\}$
Given values of model-generated heights, S_i^U
Observed heights, H_i^U
Algorithm 3

1 Adjust $\bar{a}_i^U$ using a function of the adjustment factor $\frac{H_i^U}{S_i^U}$

2 Solve for endogenous objects using Algorithm 3

Result: Updated values of amenities a_i^U and equilibrium values of $L_i, n_i, \bar{p}_i^U, r_i^U, S_i^U, y, N$

- CONV increases/decreases fundamental amenity $\bar{a}_i^U$ and re-solves the model until correlation with observed heights is perfect ▸ Open AB2022-toolkit

EMP

Algorithm 5: Updating amenities to match total employment EMP

Data: Given values for primitives $\{\alpha^U, \beta, \omega^U, \theta^U, \tau^U, \bar{a}^U, \tilde{c}^U, \bar{S}^U, r^a, \tilde{U}\}$
Given values of wage and population $\{y, N\}$
User-specified target population, $\bar{N}$
Algorithm 3

1 Adjust $\bar{a}_i{}^R$ using a function of the adjustment factor $\frac{\bar{N}}{N}$
2 Solve for endogenous objects using Algorithm 3

Result: Updated values of amenities a_i^U and equilibrium values of $L_i, n_i, \bar{p}_i^U, r_i^U, S_i^U, y, N$

- EMP increases/decreases fundamental amenity $\bar{a}_i^U$ and re-solves the model until total employment $\bar{N}$ is matched ▸ Open AB2022-toolkit

INVERT

Algorithm 6: Inverting amenities a_i^U: `INVERT`

Data: Given values for primitives $\{\alpha^U, \beta, \omega^U, \theta^U, \tau^U, \bar{a}^U, \tilde{c}^U, \bar{S}^U, r^a, \tilde{U}\}$
Given values of wage and population $\{y, N\}$
Given values of model-generated heights, S_i^U
Observed heights, H_i^U
User-specified target employment, $\bar{N}$
Algorithm 4
Algorithm 5

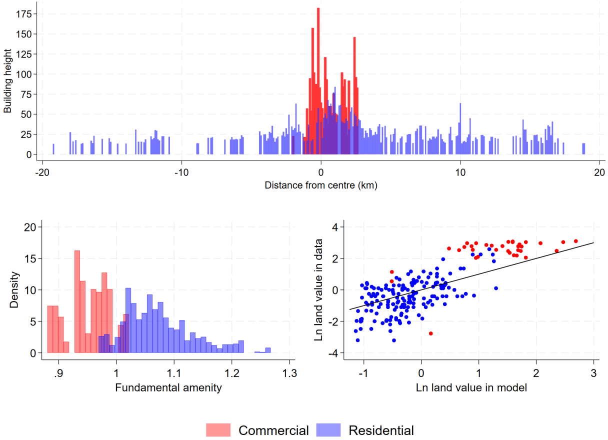
1 **while** $Corr(S_i^U, H_i) \neq 1$ **do**
2   Use Algorithm 4 to update a_i^U and obtain new S_i^U
3 **while** $N \neq \bar{N}$ **do**
4   Use Algorithm 5 to update a_i^U and obtain new N

Result: Updated values of amenities a_i^U and equilibrium values of $L(x), n(x), \bar{p}^U(x), r^U(x), S^U(x), y, N$

- `INVERT` iterates over `CONV` and `EMP` until heights and total employment in model match data ▸ Open AB2022-toolkit

Inverted fundamentals

- Small variation in fundamentals ✓
  - Don't expect huge variation within cities
  - Endogenous forces generate much of the variation in heights
- Model solutions for land rents closely correlated with data ✓
  - Did not use land values in quantification
  - Successful '**overidentification**' test
- **'Sniff tests'** are important!



Counterfactuals

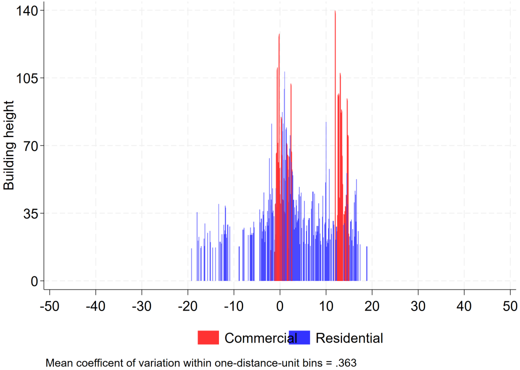
Counterfactuals

- **QSMs are great policy tools**
  - **Rationalize observed data** in initial equilibrium
  - **Change a primitive** to model a shock or policy
  - **Re-solve the model** to derive counterfactual outcomes
  - **Comparison of actual and observed outcomes** delivers GE effect of policy
- **Example**
  - **Increase fundamental production amenity** between +12 and +15 km from the CBD
  - E.g. effect of a subsidy, infrastructure improvement, etc.
  - Let's see the effects on heights, land use, total employment and wage

Predictions of machine learning are highly dependent on observed changes in data $\Rightarrow$ Hard to predict 'unkonwn events'

Counterfactual skyline

- We have created a second CBD
  - Heights increase in **targeted area**
  - Land use changes to commercial
- **Historic CBD shrinks** somewhat
  - Peak height falls from >175 to <140
  - **Relocation** $\Rightarrow$ Strength of GE modelling
- But city has gained overall
  - Wage from 2.128 to 2.223 $\Rightarrow$ +4.5%
  - Total emp. from 1M to 1.38M $\Rightarrow$ +38%



Summary

- Model consist of **exogenous objects and endogenous objects**
  - With analytical solutions, there is a direct **mapping** from the former to the latter
  - Numerical solutions are often required due to due to non-linear systems of equations
- To quantify a QSM, we
  - **Estimate** the structural parameters using data
  - **Invert** the fundamentals using data and structural parameters
- Then we can conduct **counterfactuals**
  - Solve for endogenous objects for any alternative values of fundamentals

Next week: **Idiosyncratic preferences**

Literature I

Core readings

- **Ahlfeldt, G., J. Barr (2022): The economics of skyscrapers: A synthesis. Journal of Urban Economics. 129.**
- **Codebook in AB2022-toolkit**

Other readings

n/a