Codebook for:
Toolkit for Quantitative Spatial Models *

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Abstract

This codebook is part of the toolkit for quantitative spatial models. The toolkit covers a class of models established by Monte, Redding, Rossi-Hansberg (2018): Commuting, Migration, and Local Employment Elasticities, *American Economic Review*, 108(12), pp. 3855-90. The toolkit builds on data and code used by Seidel, Wieckerth (2020): Rush hours and urbanization, *Regional Science and Urban Economics*, 85, who apply a variant of the Monte, Redding, Rossi-Hansberg (2018) model to Germany. The toolkit introduces a subset of codes that are crucial for the quantification and simulation of the model, with applications that serve didactic purposes and are unrelated to the substantive analyses in both papers. This codebook summarizes the primitives and endogenous objects of the models and introduces selected numerical algorithms in pseudo-code. The focus is on algorithms that are essential for the quantification and simulation of the respective quantitative models.

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A Monte, Redding, Rossi-Hansberg (2018)

This section covers a class of models established *Monte, Redding, Rossi-Hansberg (2018): Commuting, Migration, and Local Employment Elasticities*, https://doi.org/10.1257/aer.20151507. The model consists of exogenous parameters exogenous location characteristics, and endogenous objects tabulated in Table 1 (henceforth MRRH2017). We cover a variant of this class models by Seidel and Wickerath (2020): Rush hours and urbanization. https://doi.org/10.1016/j.regsciurbeco.2020.103580 (henceforth WS2020).

Locations are indexed by $i, n \in N$, where n generally refers to a location of consumption and i to a location of production. j indicates varieties. Workers are indexed by ω. Table 1 summarizes the key endogenous and exogenous objects in the model introduced in the baseline version of MMRH2017 as well as the additional objects introduced in SW2020.

Table 1: Codebook for MRRH2018 and SW2020

MRRH	SW	Description
		Structural parameters
α		Expenditure share on land
σ		Elasticity of substitution
$\rho = \frac{\sigma-1}{\sigma}$		Substitution parameter
F	f	Fixed cost of production in units of labour
ψ		Distance elasticity of trade cost
ϕ	$\mu\epsilon$	Commuting decay
-	μ	Travel time elasticity of commuting cost
ϵ		Preference heterogeneity
$-$	δ	Developed land supply elasticity
$-$	ν	Agglomeration elasticity
$\Gamma(.)$		Gamma function
		Exogenous characteristics
H_n	$\bar{H}_n$	Exogenous regional land (MRRH) / housing (WS2020) supply endowment (there is an extension with elastic housing supply)
$\bar{L}$		Exogenous aggregate worker endowment
d_{ni}		Iceberg trade cost of good procured in i and consumed in j
κ_{ni}		Iceberg commuting cost
$dist_{ni}$	τ_{ni}	Comuting distance (MRRH2017) / travel time (SW2020)
B_{ni}		Average amenity from living in n and working in i
$\mathcal{B}_{ni} \equiv B_{ni}\kappa_{ni}^{-\epsilon}$		Ease of commuting
A_i	$\bar{A}_i$	Exogenous location productivity
ι_n		Calibrated land ownership shares
		Endogenous variables and scalars
C_n		Goods consumption index
$c_{ni}(j) = \alpha X_n P_n^{\sigma-1} p_{ni}(j)^{-\sigma}$		Consumption, at n, of variety j produced in i
M_i		Endogenous measure of variety j produced at i
$p_{ni}(j)$		cost inclusive of freight" price of variety j produced in i and consumed in n
P_n	$P_{Q,n}$	Price index at n dual to C_n
π_{ni}		Share of expenditure at location n on goods produced at location i
$l_i(j)$		Labour input to produce variety j at location i
L_i		Workers employed at i
$x_i(j)$		Output of variety j at location i
X_n	E_n	Aggregate expenditure in location n

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Table 1 Continued from previous page

MRRH	SW	Description
w_i		Wage
$\bar{v}_n$		Average labour income of residents living in n
R_n		Residents (residence employment) living in n
Q_n	$P_{H,n}$	Land rent per land unit (sort of housing)
$\lambda_{ni} = \Phi_{ni}/\Phi$		Unconditional location choice probability
$\lambda_{ni\|i}$		Conditional (on residence) workplace choice probability
Φ		Indirect utility shifter
$\bar{U}$		Expected utility
-	H_n	Endogenous housing supply
-	A_n	Endogenous productivity

Note: If not otherwise indicated, notations in SW2020 follow MRRH2018.

A.1 Equilibrium

For given exogenous parameters and characteristics, the equilibrium of the MRRH2018 model can be reference by the following vector of six variables $\{w_n, \bar{v}_n, Q_n, L_n, R_n, P_n\}_{n=1}^{N}$ and a scalar $\bar{U}$. Given this equilibrium vector and scalar, there is a recursive structure that solves for all values of all other endogenous objects. The vector of six equilibrium variables solves the following six sets of equations:

1. income equals expenditure (7): $w_i L_i = \sum_{n \in N} \pi_{ni} \bar{v}_n R_n$
2. average residential income (14): $\bar{v}_n = \sum_{i \in N} \lambda_{ni}^R w_i$
3. land market clearing (5): $Q_n = (1-\alpha)\,\frac{\bar{v}_n R_n}{H_n}$
4. workplace choice probabilities (11 for L_n):
   $\lambda_n^R = \frac{R_n}{L} = \sum_{i \in N} \lambda_{ni} = \sum_{i \in N} \frac{\Phi_{ni}}{\Phi} = \sum_{i \in N} \frac{B_{ni}\left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}\right)^{-\epsilon} w_i^{\epsilon}}{\Phi}$
5. residence choice probabilities (11 for R_n):
   $\lambda_n^L = \frac{L_n}{L} = \sum_{n \in N} \lambda_{ni} = \sum_{n \in N} \frac{\Phi_{ni}}{\Phi} = \sum_{n \in N} \frac{B_{ni}\left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}\right)^{-\epsilon} w_i^{\epsilon}}{\Phi}$
6. price indices (8):
   $P_n = \frac{\sigma}{\sigma-1}\left(\frac{1}{\sigma F}\right)^{\frac{1}{1-\sigma}} \left[\sum_{i \in N} L_i \left(\frac{d_{ni} w_i}{A_i}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}} = \frac{\sigma}{\sigma-1}\left(\frac{L_n}{\sigma F \pi_{nn}}\right)^{\frac{1}{1-\sigma}} \frac{d_{nn} w_n}{A_n}$

The last condition needed to pin down $\bar{U}$ is

7. the labour market clearing condition $\bar{L} = \sum_{n \in N} R_n = \sum_{n \in N} L_n$

With seven equations for seven endogenous objects, this system of equations is exactly identified.

The SW2020 has the same endogenous variables and, in addition, housing supply H_i and productivity A_i are also endogenous. The equilibrium is pinned down by two additional equations:

8. housing supply (7): $H_n = \bar{H}_n P_{H,n}^{\delta}$

9. productivity: $A_i = \bar{A}_i L_i^{\nu}$,

where $\{\delta, \nu\}$ are two exogenous parameters governing the housing supply elasticity and the agglomeration elasticity and $\{\bar{H}_i, \bar{A}_i\}$ are exogenous components of housing supply and productivity. Therefore, with two additional endogenous variables and two additional equations, the system of equations remains exactly identified.

A.2 Counterfactuals

MRRH2018 show how to use exact hat algebra and the equilibrium conditions in Section A.1 to express counterfactual changes in endogenous variables in terms of changes in exogenous fundamentals, structural parameters, and levels of observed or solved endogenous variables. Values of the counterfactual equilibrium are denoted by prime (x') and the relative change of a variable from the initial to the counterfactual equilibrium is denoted by a hat ($\hat{x} = x'/x$). In the SW2020 variant of the model, the relative changes in endogenous variables $\{\hat{w}_{ni}, \hat{\bar{v}}_{ni}, \hat{L}_n, \hat{R}_n, \hat{\lambda}_{ni}, \hat{P}_{Q,n}, \hat{P}_{H,n}, \hat{\pi}_{ni}\}$ can be expressed as functions of their initial levels, the model's structural parameters $\{\alpha, \sigma, \varepsilon, \delta, \kappa\}$ and the relative changes in fundamentals $\{\hat{A}_n, \hat{B}_n, \hat{\kappa}_{ni}, \hat{d}_{ni}\}$ which are the forcing variables in the counterfactuals. We can solve for the counterfactual changes in the model's endogenous variables from the following system of eight equations that describe the counterfactual state of the endogenous variables.

1. Counterfactual income equals counterfactual expenditure:
   $w_i \hat{w}_i L_i \hat{L}_i = \sum_{n \in N} \pi_{ni} \hat{\pi}_{ni} \bar{v}_n \hat{\bar{v}}_n R_n \hat{R}_n$

2. Counterfactual residential wage: $\hat{v}_n \bar{v}_n = \sum_{i \in N} \frac{\lambda_{ni} \hat{B}_{ni} (\hat{w}_i / \hat{\kappa}_{ni})^{\epsilon}}{\sum_{s \in N} \lambda_{ns} \hat{B}_{ns} (\hat{w}_s / \hat{\kappa}_{ns})^{\epsilon}} \hat{w}_i w_i$

3. Counterfactual housing price: $\hat{P}_{H,n} P_{H,n} = P_{H,n} \left(\hat{\bar{v}}_n \hat{R}_n \right)^{\frac{1}{1+\delta}}$

4. Counterfactual trade shares: $\hat{\pi}_{ni} \pi_{ni} = \frac{\pi_{ni} \left(\hat{L}_i \right)^{1-(1-\sigma)\nu} \left(\hat{d}_{ni} \hat{w}_i / \hat{A}_i \right)^{1-\sigma}}{\sum_{k \in N} \hat{\pi}_{nk} \left(\hat{L}_k \right)^{1-(1-\sigma)\nu} \left(\hat{d}_{ni} \hat{w}_k / \hat{A}_k \right)^{1-\sigma}}$

5. Counterfactual unconditional commuting probabilities:
   $\hat{\lambda}_{ni} \lambda_{ni} = \frac{\lambda_{ni} \hat{B}_{ni} \left(\left(P_{Q,n} \right)^{\alpha} \left(P_{H,n} \right)^{(1-\alpha)} \right)^{-\epsilon} (w_i K_{ni})^{\epsilon}}{\sum_{r \in N} \sum_{s \in N} \lambda_{rs} \hat{B}_{rs} \left(\left(P_{Q,r} \right)^{\alpha} \left(P_{H,r} \right)^{(1-\alpha)} \right)^{-\epsilon} (w_s K_{rs})^{\epsilon}}$

6. Counterfactual tradable goods price: $\hat{P}_{Q,n} P_{Q,n} = P_{Q,n} \left(\frac{\left(\hat{L}^{(t)} \right)^{1-(1-\sigma)\nu}}{\hat{\pi}_{nn}} \right)^{\frac{1}{1-\sigma}} \frac{\hat{d}_{nn} \hat{w}_n}{\hat{A}_n}$

7. Counterfactual residential population: $\hat{R}_i R_i = \bar{L} \sum_{i \in N} \lambda_{ni} \hat{\lambda}_{ni}$

8. Counterfactual workplace employment: $\hat{L}_i L_i = \bar{L} \sum_{n \in N} \lambda_{ni} \hat{\lambda}_{ni}$

A canonical welfare measure in quantitative spatial models is the expected utility defined in MRRH2018 Eq. (15). Using exact hat algebra, it is possible to derive the relative change in expected utility without taking a stance on the unobserved levels of fundamentals. To this end, define counterfactual expected utility as:

$$\hat{\bar{U}}\bar{U} = \Gamma\left(\frac{\epsilon-1}{\epsilon}\right)\left[\sum_{r\in N}\sum_{s\in N}\hat{B}_{rs}B_{rs}\left(\hat{\kappa}_{rs}\kappa_{rs}\left(\hat{P}_{Q,r}P_{Q,r}\right)^{\alpha}\left(\hat{P}_{H,r}P_{H,r}\right)^{1-\alpha}\right)^{-\epsilon}(\hat{w}_s w_s)^{-\epsilon}\right]^{\frac{1}{\epsilon}}$$

which we can rewrite as

$$\hat{\bar{U}} = \frac{\left[\sum_{r\in N}\sum_{s\in N}\hat{B}_{rs}B_{rs}\left(\hat{\kappa}_{rs}\kappa_{rs}\left(\hat{P}_{Q,r}P_{Q,r}\right)^{\alpha}\left(\hat{P}_{H,r}P_{H,r}\right)^{1-\alpha}\right)^{-\epsilon}(\hat{w}_s w_s)^{-\epsilon}\right]^{\frac{1}{\epsilon}}}{\left[\sum_{r\in N}\sum_{s\in N}B_{rs}\left(\kappa_{rs}\left(P_{Q,r}\right)^{\alpha}\left(P_{H,r}\right)^{1-\alpha}\right)^{-\epsilon}(w_s)^{-\epsilon}\right]^{\frac{1}{\epsilon}}}.$$

Notice that terms in brackets in the numerator and denominator correspond to the multilateral resistance terms in initial and counterfactual unconditional commuting probabilities in MRRH Eq. (10). Substituting in, we obtain:

$$\hat{\bar{U}} = \frac{\left[\frac{1}{\hat{\lambda}_{ni}\lambda_{ni}}\hat{B}_{ni}B_{ni}\left(\hat{\kappa}_{ni}\kappa_{ni}\left(\hat{P}_{Q,n}P_{Q,n}\right)^{\alpha}\left(\hat{P}_{H,n}P_{H,n}\right)^{1-\alpha}\right)^{-\epsilon}(\hat{w}_i w_i)^{-\epsilon}\right]^{\frac{1}{\epsilon}}}{\left[\frac{1}{\lambda_{ni}}B_{ni}\left(\kappa_{ni}\left(P_{Q,n}\right)^{\alpha}\left(P_{H,n}\right)^{1-\alpha}\right)^{-\epsilon}(w_i)^{-\epsilon}\right]^{\frac{1}{\epsilon}}} = \frac{\hat{B}_{ni}^{\frac{1}{\epsilon}}\hat{\kappa}_{ni}^{-\frac{1}{\epsilon}}}{\hat{\kappa}_{ni}\left(\hat{P}_{Q,n}\right)^{\alpha}\left(\hat{P}_{H,n}\right)^{1-\alpha}\hat{w}_i}$$

A.3 Algorithms

We abstract from the estimation of structural parameters and focus on the algorithms used for the quantification and simulation of the model. We refer to objects whose values we solve numerically as target objects. Within the solver, we refer to guessed values when values of target objects are input into the construction of other objects. We refer to predicted values when the values of target objects are computed as functions of other objects (which typically depend on guesses). Typically, we solve for the values of target objects using iterative procedures in which we update our guesses to weighted combinations of guessed and predicted values. We introduce the key algorithms used by SW2020 and an additional algorithm introduced to facilitate a more general application of the toolkit.

A.3.1 Quantification

MMH2018 and SW2020 quantify the model using observed commuting flows L_{ni}. They rationalize zero commuting flows by setting commuting costs to $\kappa = \infty$. The advantage of this approach is that it allows for arbitrary commuting costs on routes with positive commuting flows.

Depending on the application, this approach can have limitations. Researchers may wish to use commuting cost matrices that are smooth functions of network distance, travel time, or other distance measures. This may be particularly desirable in counterfactuals where researchers wish to allow for changes in commuting flows at the extensive margin. For example, a new road or rail may lead to commuting flows on certain routes changing from zero to positive values. Equivalently importantly, there are instances in which researchers do not observe bilateral commuting flows.

To facilitate the applicability of the model and the toolkit in such instances, Algorithm 1 predicts bilateral commuting flows that are consistent with observed workplace employment, residence population and commuting costs. To this end, it recovers a measure of workplace amenities, B_i that ensures that workplace employment predicted by the model matches data.

Notice that in some contexts wages, w_n may not be observable. In such cases, researchers may feed $w_n = 1$ into Algorithm 1 and interpret the inverted B_i as transformed wages, ω_i, as in ARSW2015. Under the assumption that workplace amenities $B_i = 1$, model-consistent wages can then be recovered as $w = B_i^{1/\epsilon}$.

The toolkit is designed so that it overwrites the original data used by SW2022 upon execution of Algorithm 1. All other algorithms used for the quantification and counterfactuals can be used just like with the original data that accompanies the toolkit.

Algorithm 1: Solving workplace amenities and commuting probabilities: `getBiTK.m`

Data: Given values for structural parameters $\{\mu, \epsilon\}$, observed wages, w_n, bilateral travel times, τ_{ni}, workplace employment L_n and residence employment R_n, and total population L

Guesses of workplace amenity B_i^0

while *Predicted workplace amenity* $B_i^1 \neq B_i^0$ **do**

- Use guesses B_i^0 in MRRH2018 Eq. (12) to compute conditional commuting probability $\lambda^1_{ni|n}$
- Use $\lambda^1_{ni|n}$ to predict workplace employment L^1_n using MRRH2018 Eq. (13)
- Compute new guesses of workplace amenity as $B_i^1 = B_i^0 \frac{L_n}{L_n^1}$; increase workplace amenity to attract more workers if observed employment exceeds predicted employment
- Update guesses B_i^0 to weighted combination of old and new guesses: $\zeta B_i^1 + (1-\zeta) B_i^0$

Compute residential choice probability $\lambda_n^R = \frac{R_n}{\sum_n R_n}$

Compute unconditional location choice probabilities $\lambda_{ni} = \lambda_{ni|n} \lambda_n^R$ using MRRH2018 Eq. (12)

Compute commuting flows $L_{ni} = \lambda_{ni} L$

Result: Workplace amenity, B_i, conditional choice probabilities, $\lambda_{ni|n}$, unconditional choice probabilities, λ_{ni}, commuting flows L_{ni}

Algorithm 2 uses SE2020 Eq. (12) to invert the unobserved exogenous productivity $\bar{A}_n$ that result in trade shares π_{ni} that ensure that firms attract exactly as much expenditure from workers living in n as they spend on wages for workers working in i. Alongside the inverted productivities, it saves trade shares, own trade shares π_{nn}, which it uses to compute and save the price index P_n.

Algorithm 2: Solving for exogenous productivity: `solveProductTK.m`

Data: Given values for structural parameters $\{\sigma, \nu\}$, observed workplace employment L_n and residence employment R_n, wages, w_n, residential wages, $\bar{v}_n$, and bilateral trade cost, d_{ni}

Guesses of exogenous productivity $\bar{A}_i^0$

while *Income not equal expenditure* $w_i L_i \neq \sum_n \pi_{ni} \bar{v}_n R_n$ **do**

- Use guesses $\bar{A}_i^0$ in SW2020 Eq. (10) to compute trade shares π_{ni}
- Compute worker income $w_i L_i$, the left-hand side in SW2020 Eq. (12)
- Use π_{ni} to compute worker expenditure $\sum_n \pi_{ni} \bar{v}_n R_n$, the right-hand side of SW2020 Eq. (12)
- Compute new guess of productivity $\bar{A}_i^1 = \bar{A}_i^0 \frac{\sum_n \pi_{ni}}{\sum_n \pi_{ni} \bar{v}_n R_n}$; increase productivity if firms do not attract enough expenditure
- Update guesses $\bar{A}_i^0$ to weighted combination of old and new guesses: $\zeta \bar{A}_i^1 + (1-\zeta)\bar{A}_i^0$

Use $\bar{A}_n$ and π_{ni} to compute tradable goods price index $P_{Q,n}$ using SW2020 Eq. (11)

Result: Exogenous productivity $\bar{A}_i$, trade shares π_{ni}, tradable goods price index $P_{Q,n}$

A.3.2 Counterfactuals

In the SW2020 variant of the model, the relative changes in endogenous variables $\{\hat{w}_{ni}, \hat{\bar{v}}_{ni}, \hat{L}_n, \hat{R}_n, \hat{\lambda}_{ni}, \hat{P}_{Q,n}, \hat{P}_{H,n}, \hat{\pi}_{ni}\}$ can be expressed as functions of their initial levels, the model's structural parameters $\{\alpha, \sigma, \varepsilon, \delta, \kappa\}$ and the relative changes in fundamentals $\{\hat{A}_n, \hat{B}_n, \hat{\kappa}_{ni}, \hat{d}_{ni}\}$ which are the forcing variables in the counterfactuals.

Algorithm 11 uses an iterative procedure to solve for the relative changes in endogenous variables, treating $\{\hat{w}_n, \hat{\lambda}_{ni}\}$ as target variables. Intuitively, the algorithm uses guesses of the target variables and equilibrium conditions listed in Section A.2 to solve for the non-target variables referencing the equilibrium: counterfactual residential wage, counterfactual residential population, counterfactual workplace employment, counterfactual housing price, counterfactual trade shares, counterfactual tradable price index. Then it uses the solutions for the non-target variables to predict the target variables using the following conditions: Counterfactual income equals counterfactual expenditure, and counterfactual unconditional commuting probabilities. Each update of an endogenous variable is executed by one of the dedicated Algorithms 3-10 that are called by Algorithm 11. Using the relative changes and the initial levels of the endogenous variables, it is straightforward to solve for the counterfactual levels of the endogenous variables. We can recursively solve for all other endogenous objects.

Algorithm 3: Update residential wages: `updateResWageTK.m`

Data: Changes in amenities $\hat{B}_{ni}$, wages $\hat{w}_n$, commuting cost $\hat{\kappa}_{ni}$. Initial levels of unconditional commuting probabilities λ_{ni}, residential wages $\bar{v}_n$, wages w_n. Structural parameter ϵ

Compute changes in residential wages $\hat{v}_n^{(t)} = \frac{1}{\bar{v}_n} \sum_{i \in N} \frac{\lambda_{ni} \hat{B}_{ni} \left(\hat{w}_i^{(t)} / \hat{\kappa}_{ni} \right)^{\epsilon}}{\sum_{s \in N} \lambda_{ns} \hat{B}_{ns} \left(\hat{w}_s^{(t)} / \hat{\kappa}_{ns} \right)^{\epsilon}} \hat{w}_i^{(t)} w_i$ (exact hat version of MRRH Eq. (14), see MRRH2018 Appendix Eq. B.17)

Result: Changes in residential wages $\hat{v}_n^{(t)}$

Algorithm 4: Update workplace employment: `updateEmplTK.m`

Data: Changes in unconditional commuting probabilities $\hat{\lambda}_{ni}$. Initial levels of unconditional commuting probabilities λ_{ni}, workplace employment L_n, total employment $\bar{L}$

Compute changes in workplace employment $\hat{L}_i^{(t)} = \frac{\bar{L}}{L_i} \sum_{n \in N} \lambda_{ni} \hat{\lambda}_{ni}^{(t)}$ (exact hat algebra version of MRRH2018 Eq. 11, see MRRH2018 Appendix Eq. B.23)

Result: Changes in workplace employment $\hat{L}_i^{(t)}$

Algorithm 5: Update residence population: `updateResidentsTK.m`

Data: Changes in unconditional commuting probabilities $\hat{\lambda}_{ni}$. Initial levels of unconditional commuting probabilities λ_{ni}, residence population R_n, total employment $\bar{L}$

Compute changes in residential population $\hat{R}_i^{(t)} = \frac{\bar{L}}{R_n} \sum_{i \in N} \lambda_{ni} \hat{\lambda}_{ni}^{(t)}$ (exact hat algebra version of MRRH2018 Eq. 11, see MRRH2018 Appendix Eq. B.22)

Result: Changes in residence population $\hat{R}_n^{(t)}$

Algorithm 6: Update housing price: `updateHousePriceTK.m`

Data: Changes in residential wages $\hat{\bar{w}}_n$ and changes in residence population $\hat{R}_n^{(t)}$. Structural parameter δ

Compute changes in housing price $\hat{P}_{H,n}^{(t)} = \left(\hat{\bar{v}}_n^{(t)} \hat{R}_n^{(t)} \right)^{\frac{1}{1+\delta}}$ (SW2020 Eq. 9 in ratios)

Result: Changes in residence population $\hat{L}_i^{(t)}$

Algorithm 7: Update trade shares: `updateTradeshTK.m`

Data: Changes in employment $\hat{L}_n$, wages $\hat{w}_n$, exogenous productivities $\hat{\bar{A}}_n$, trade cost $\hat{d}_{ni}$. Initial levels of trade shares π_{ni}. Structural parameters $\{\sigma, \nu\}$

Compute changes in trade shares $\hat{\pi}_{ni}^{(t)} = \frac{\left(\hat{L}_i^{(t)}\right)^{1-(1-\sigma)\nu}\left(\hat{d}_{ni}\hat{w}_i^{(t)}/\hat{A}_i\right)^{1-\sigma}}{\sum_{k\in N}\pi_{nk}\left(\hat{L}_k^{(t)}\right)^{1-(1-\sigma)\nu}\left(\hat{d}_{ni}\hat{w}_k^{(t)}/\hat{A}_k\right)^{1-\sigma}}$ (exact hat algebra version of SW2020 Eq. 10; see MRRH2018 Appendix Eq. B.20, augmented by agglomeration economies introduced by SW2020)

Result: Changes in trade shares $\hat{\pi}_{ni}^{(t)}$

Algorithm 8: Update tradable goods price: `updatePricesTK.m`

Data: Changes in employment $\hat{L}_n$, wages $\hat{w}_n$, trade shares $\hat{\pi}_{ni}$, trade costs $\hat{d}_{ni}$, exogenous productivities $\hat{\bar{A}}_n$. Structural parameters $\{\sigma, \nu\}$

Compute changes in trade shares $\hat{P}_{Q,n}^{(t)} = \left(\frac{\left(\hat{L}^{(t)}\right)^{1-(1-\sigma)\nu}}{\hat{\pi}_{nn}^{(t)}}\right)^{\frac{1}{1-\sigma}} \frac{\hat{d}_{nn}\hat{w}_n^{(t)}}{\hat{A}_n}$ (SW2020 Eq. 11 in ratios)

Result: Changes in tradable goods price $\hat{P}_{n,Q}^{(t)}$

Note: Notice that $\hat{d}_{nn}$ is missing in the description of the algorithms in MRRH2018 and SW2020. This is inconsequential in most cases since trade costs parameterized based on geographic distance won't change within regions. In the toolkit, the equation is generalized in case more sophisticated measures are available in potential applications.

Algorithm 9: Update wage: `updateWageTK.m`

Data: Changes in employment $\hat{L}_n$, trade shares $\hat{\pi}_{ni}$, residential wages $\hat{\bar{v}}_n$, resident population $\hat{R}_n$. Initial levels of workplace employment L_n, workplace wages w_n, trade shares π_{ni}, residential wages $\bar{v}_n$, residential population R_n.

Compute changes in trade shares $\tilde{w}_i^{(t+1)} = \frac{1}{w_i L_i \hat{L}_i^{(t)}} \sum_{n\in N} \pi_{ni}\hat{\pi}_{ni}^{(t)}\bar{v}_n\hat{\bar{v}}_n^{(t)} R_n \hat{R}_n^{(t)}$ (exact hat algebra version of MRRH Eq. 7; see MRRH2018 Appendix Eq. B.16)

Result: Changes in wages $\hat{w}_n^{(t)}$

Algorithm 10: Update unconditional commuting probabilities: `updateLamTK.m`

Data: Changes in amenities $\hat{B}_{ni}$, tradable goods price $\hat{P}_{Q,n}$, housing price $\hat{P}_{H,n}$, workplace wage $\hat{w}_n$, commuting costs $\hat{\kappa}_{ni}$. Initial levels of unconditional commuting probabilities $\hat{\lambda}_{ni}$.

1 Compute changes in unconditional commuting probabilities
$\tilde{\lambda}_{ni}^{(t+1)} = \frac{\hat{B}_{ni}\left(\left(P_{Q,n}^{(t)}\right)^{\alpha}\left(P_{H,n}^{(t)}\right)^{(1-\alpha)}\right)^{-\epsilon}\left(w_i^{(t)}K_{ni}\right)^{\epsilon}}{\sum_{r\in N}\sum_{s\in N}\lambda_{rs}\hat{B}_{rs}\left(\left(P_{Q,r}^{(t)}\right)^{\alpha}\left(P_{H,r}^{(t)}\right)^{(1-\alpha)}\right)^{-\epsilon}\left(w_s^{(t)}K_{rs}\right)^{\epsilon}}$ (exact hat algebra version of MRRH Eq. 10; see MRRH2018 Appendix Eq. B.20)

Result: Changes in unconditional commuting probabilities $\hat{\lambda}_{ni}^{(t)}$

Algorithm 11: Solving for relative counterfactual changes: `counterFactsTK.m`

Data: Changes in exogenous productivities $\hat{\bar{A}}_n$, amenities $\hat{B}_{ni}$, commuting costs $\hat{\kappa}_{ni}$, trade costs $\hat{d}_{ni}$. Initial levels of wages w_n, residential wages $\bar{v}_n$, workplace employment L_n, residence population R_n, trade shares π_{ni}. Structural parameters $\{\alpha, \sigma, \epsilon, \delta, \nu\}$.

Initialize guesses of changes in wages, $\hat{w}_n^0$, and unconditional commuting probabilities, $\hat{\lambda_{ni}^0}$

while $\hat{w}_n^0 \neq \hat{w}_n^1$ *or* $\hat{\lambda_{ni}^0} \neq \hat{\lambda_{ni}^1}$ **do**

  Use Algorithm 3`updateResWageTK` and $\hat{w}_n^0$ to compute change in residential wage $\hat{\bar{v}}_n^0$

  Use Algorithm 4 `updateEmplTK` and $\hat{\lambda}_{ni}^0$ to compute change in workplace employment $\hat{L}_n^0$

  Use Algorithm 5 `updateResidentsTK` and $\hat{\lambda}_{ni}^0$ to compute change in residence population $\hat{R}_n^0$

  Use Algorithm 6 `updateHousePriceTK` and $\{\hat{\bar{v}}_n^0, \hat{R}_n^0\}$ to compute change in housing price $\hat{P}_{H,n}^0$

  Use Algorithm 7 `updateTradeshTK` and $\{\hat{w}_n^0, \hat{L}_n^0\}$ to compute change in trade shares $\hat{\pi}_{ni}^0$

  Use Algorithm 8 `updatePricesTK` and $\{\hat{w}_n^0, \hat{L}_n^0, \pi_{ni}^0\}$ to compute change in tradable goods price $\hat{P}_{Q,n}^0$

  Use Algorithm 9 `updateWageTK` and $\{\hat{L}_n^0, \hat{\pi}_{ni}^0, \hat{\bar{v}}_n^0, \hat{R}_n\}$ to predict change in wage $\hat{w}_n^1$

  Normalize $\hat{w}_n^1$ to ensure that w_n' has a unit mean

  Use Algorithm 10 `updateLamTK` and $\{\hat{P}_{Q,n}^0, \hat{P}_{H,n}^0, \hat{w}_n^0\}$ to predict change in unconditional commuting probabilities $\hat{\lambda}_{ni}^1$

  Update guesses of $\hat{w}_n^0$ to convex combination of old guesses and predictions $\zeta \hat{w}_n^1 + (1-\zeta)\hat{w}_n^0$

  Update guesses of $\hat{\lambda}_{ni}^0$ to convex combination of old guesses and predictions $\zeta \hat{\lambda}_{ni}^1 + (1-\zeta)\hat{\lambda}_{ni}^0$

Compute change in expected utility as derived in Section A.1

Result: Changes in wages $\hat{w}_n$, residential wages $\hat{\bar{v}}_{ni}$, tradable goods price $\hat{P}_{Q,n}$, trade shares $\hat{\pi}_{ni}$, unconditional commuting probabilities $\hat{\lambda}_{ni}$, housing price $\hat{P}_{H,n}$, residence population $\hat{R}_n$, workplace employment $\hat{L}_n$, expected utility $\hat{\bar{U}}$