Fibonacci Closed-Form Solution

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Proposition: $\forall n \in \mathbb{Z}^{\geq 0}$, where $\phi = \frac{1+\sqrt{5}}{2}$, and $\psi = \frac{1-\sqrt{5}}{2}$

1. $\mathrm{Fib}(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$
2. $\mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$

Proof of 1: By induction on n:

Base cases: $n \in \{0, 1\}$

Take $n = 0$: $\mathrm{Fib}(0) = 0$, and $\frac{\phi^0}{\sqrt{5}} = \frac{1}{\sqrt{5}}$, and $[\frac{1}{\sqrt{5}}] = 0$ as desired.

Take $n = 1$: $\mathrm{Fib}(1) = 1$, and $\frac{\phi}{\sqrt{5}} = \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{1}{2\sqrt{5}} + \frac{1}{2} = \frac{1}{2}(\frac{1}{\sqrt{5}} + 1)$, and since $[\frac{1}{\sqrt{5}} + 1] = 1$, it follows that $[\frac{\phi}{\sqrt{5}}] = 1$ as desired.

Inductive step: Take arbitrary $n \in \mathbb{N}$, and assume $[\frac{\phi^n}{\sqrt{5}}] = \mathrm{Fib}(n)$ and $[\frac{\phi^{n-1}}{\sqrt{5}}] = \mathrm{Fib}(n-1)$.

It suffices to show that

$$\mathrm{Fib}(n+1) = [\frac{\phi^{n+1}}{\sqrt{5}}].$$

Given the definition of $\mathrm{Fib}(n+1)$ as $\mathrm{Fib}(n) + \mathrm{Fib}(n-1)$, it follows immediately from the inductive hypothesis that $\mathrm{Fib}(n+1) = [\frac{\phi^n}{\sqrt{5}}] + [\frac{\phi^{n-1}}{\sqrt{5}}]$. Therefore, it suffices to show that

$$[\frac{\phi^{n+1}}{\sqrt{5}}] = [\frac{\phi^n}{\sqrt{5}}] + [\frac{\phi^{n-1}}{\sqrt{5}}]$$

or, equivalently,

$$|\frac{\phi^{n+1} - \phi^n - \phi^{n-1}}{\sqrt{5}}| < \frac{1}{2}$$

Factoring out common terms,

$$\frac{\phi^{n-1}}{\sqrt{5}}|\phi^2 - \phi - 1| < \frac{1}{2} \rightarrow |\phi^2 - \phi - 1| < \frac{\sqrt{5}}{2\phi^{n-1}}$$

By definition, $\phi = \frac{1+\sqrt{5}}{2}$, so the above is equivalently

$$|\frac{1+2\sqrt{5}+5}{4} - \frac{1+\sqrt{5}}{2} - 1| < \frac{\sqrt{5}}{2\phi^{n-1}}$$

Thus,

$$|\frac{6+2\sqrt{5}-2-2\sqrt{5}-4}{4}| < \frac{\sqrt{5}}{2\phi^{n-1}}$$

which is, by arithmetic, $0 < \frac{\sqrt{5}}{2\phi^{n-1}}$, which is obviously true, since $\forall n \in \mathbb{N}, \phi^n > 0$.

Therefore, given $[\frac{\phi^n}{\sqrt{5}}] = \text{Fib}(n)$ and $[\frac{\phi^{n-1}}{\sqrt{5}}] = \text{Fib}(n-1)$, it is true that $\text{Fib}(n+1) = [\frac{\phi^{n+1}}{\sqrt{5}}]$, and therefore, by induction, $\forall n \in \mathbb{N}, \text{Fib}(n) = [\frac{\phi^n}{\sqrt{5}}]$

Q.E.D.

Before proving the second portion of the proposition, let us prove this Lemma which will make the proof of part 2 quite simple.

Lemma: $\forall n \in \mathbb{N}$, with ϕ and ψ defined as in the above proposition,

1. $\phi^n = \phi^{n-1} + \phi^{n-2}$
2. $\psi^n = \psi^{n-1} + \psi^{n-2}$

Proof of 1: By induction on n:

Base case: $n = 2$ By direct computation,

$$\phi^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{1}{4}(1+2\sqrt{5}+5) = \frac{3+\sqrt{5}}{2}$$

And

$$\phi + \phi^0 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = \phi^2$$

Therefore, for $n = 2$, $\phi^n = \phi^{n-1} + \phi^{n-2}$.

Inductive step: Take arbitrary $n \in \mathbb{N}$, where $n > 2$, and assume $\phi^n = \phi^{n-1} + \phi^{n-2}$. We desire to show that

$$\phi^{n+1} = \phi^n + \phi^{n-1}$$

Observe that

$$\phi^{n+1} = \phi\phi^n = \phi(\phi^{n-1} + \phi^{n-2})$$

by assumption, and that by distribution this is equal to

$$\phi^n + \phi^{n-1}$$

as desired. Therefore, $\forall n \in \mathbb{N}$, where $n \geq 2$, $\phi^n = \phi^{n-1} + \phi^{n-2}$.

Q.E.D.

Proof of 2: By induction on n:

Base case: $n = 2$ By direct computation,

$$\psi^2 = \frac{(1-\sqrt{5})^2}{4} = \frac{1}{4}(1 - 2\sqrt{5} + 5) = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2} = \psi^0 + \psi$$

Therefore, for $n = 2$, $\psi^n = \psi^{n-1} + \psi^{n-2}$.

Inductive step: Take arbitrary $n \in \mathbb{N}$, where $n > 2$, and assume $\psi^n = \psi^{n-1} + \psi^{n-2}$. We desire to show that

$$\psi^{n+1} = \psi^n + \psi^{n-1}$$

Observe that

$$\psi^{n+1} = \psi\psi^n = \psi(\psi^{n-1} + \psi^{n-2})$$

by assumption, and that by distribution this is equal to

$$\psi^n + \psi^{n-1}$$

as desired. Therefore, $\forall n \in \mathbb{N}$, where $n \geq 2$, $\psi^n = \psi^{n-1} + \psi^{n-2}$.

Q.E.D.

Now, returning to the second part of the Proposition,

Proof of 2: By induction on n:

Base case: $n = 0$ By direct computation, $\mathrm{Fib}(0) = 0$, and

$$\frac{\phi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$

. Thus, for $n = 0$, $\mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$

Base case: $n = 1$ By direct computation,

$$\mathrm{Fib}(1) = 1 = \frac{1 + \sqrt{5} - 1 + \sqrt{5}}{2\sqrt{5}} = \frac{\phi - \psi}{\sqrt{5}}$$

Base case: $n = 2$ By direct computation, $\mathrm{Fib}(2) = \mathrm{Fib}(1) + \mathrm{Fib}(0) = 1$, and

$$\frac{\phi^2 - \psi^2}{\sqrt{5}} = \frac{1}{\sqrt{5}}(\phi + \phi^0 - \psi - \psi^0) = \frac{1}{\sqrt{5}}(\frac{1 + \sqrt{5} - (1 - \sqrt{5})}{2}) = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$$

Therefore, $\mathrm{Fib}(2) = \frac{\phi^2 - \psi^2}{\sqrt{5}}$, as desired.

Inductive step: Take arbitrary $n \in \mathbb{N}$, where $n > 2$, and assume $\mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$ and $\mathrm{Fib}(n-1) = \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}}$.

Observe that, since $\mathrm{Fib}(n+1) = \mathrm{Fib}(n) + \mathrm{Fib}(n-1)$,

$$\mathrm{Fib}(n+1) = \frac{\phi^n - \psi^n}{\sqrt{5}} + \frac{\phi^{n-1} + \psi^{n-1}}{\sqrt{5}} = \frac{1}{\sqrt{5}}(\phi^n + \phi^{n-1} - \psi^n - \psi^{n-1})$$

Which, by the above Lemma, is equal to

$$\frac{\phi^{n+1} - \psi^{n-1}}{\sqrt{5}}$$

as desired.

Therefore,

$$\forall n \in \mathbb{N}, \mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

Q.E.D.