Differential Equations

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Sequence: → function → Range: set of Natural Numbers

↳ particular arrangement of numbers

first set → set of natural number

2, 4, 6 → finite sequence

2, 4, 6 ... → infinite sequence

function → restrictions

* hr ek element ke liye ek element ho
* unique ho hr element

→ first term

$\{a\} = 2, 4, 6, \ldots\ldots 2n$ (Representation of sequence)

$a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_n$ (Represented By)

n → index

Converge → shrink hona → can be in any form

i.e integers, fraction, decimal.

Kisi particular sequence ki trf move krna

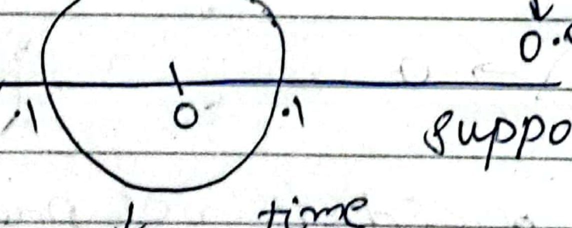
E.g: $1, 1/2, 1/3, 1/4, \ldots \longrightarrow 0$

limit is 0. converging sequence

representation: L

$1/100 = 0.01$ $\quad$ $1/1000 = 0.001$

characteristic of converge sequence



supposed radius = 0.1

after certain ~~value~~ time, ~~valo~~ esi value aajayen gi jo ke radius ε mein aa jayein gi

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Divergent Sequence → no particular limit → no limit → infinite sequence
↓
terms can be in any form i.e fraction, decimal, integer etc.

Characteristics:

$\{a_n\}, \{b_n\}, \{c_n\}$

$\lim_{n \to \infty} \{a_n\} = A$ $\lim_{n \to \infty} \{b_n\} = B$

$\lim_{n \to \infty} \{c_n\} = C$

•) $\lim_{n \to \infty} [\{a_n\} + \{b_n\}] = \lim_{n \to \infty} \{a_n\} + \lim_{n \to \infty} \{b_n\}$

•) $K \lim_{n \to \infty} \{a_n\} = K \cdot A$

Bounded from above sequence:

$\{1, 2, 3, \dots, 16\}, 17, 18$ → upper bounds but not in sequence

↳ greatest lower upper bound

↳ least upper bound

max value → upper bound

min value → lower bound

$\{1, 1/2, 1/3, \dots, 1\} \to 0$

↓

lower bound is not present in the sequence

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•) Sequence in which lower bound or upper bound exists is called Monotonic Sequence

$$\lim_{n\to\infty} \{a_n\} = \Big|_{\infty} \frac{-1}{n}$$

$$= -\lim_{n\to\infty} \frac{1}{n}$$

$$= \frac{-1}{\infty} = -0 = 0$$

→ Ways to check :

•) Make List

•) Check $n \to \infty$ → if ans is definite → converge
indefinite → diverge.

Example:

$$a_n = \left(\frac{n+1}{n}\right)^n \quad \text{(L'Hopital Rule)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \ln \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} n \ln \left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n}\right)}{1/n}$$

$$= \lim_{n\to\infty} \ln \left(\frac{1+1/n}{1/n}\right)$$

$$= \lim_{n\to\infty}{}^{\ln} \frac{\frac{n}{n+1}\left(\frac{n-(n+1)}{(n)^2}\right)}{}$$

$$= \lim_{n\to\infty}{}^{\ln} \frac{\left(\frac{n}{n+1}\right)\left(\cancel{\frac{1}{n^2}}\right)}{-1/n^2}$$

$$= -\lim_{n\to\infty}{}^{\ln} \frac{\left(\frac{n}{n+1}\right)}{\cancel{\frac{-1}{n^2}}}$$

$$= -\lim_{n\to\infty}{}^{\ln} \left(\frac{1}{1}\right)$$

$$= -\lim_{n\to\infty}{}^{\ln} 1 \qquad) = \ln(-1)$$

$$= e^{-1}$$

→ Series : The sum of indicated terms in a sequence

$\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

→ Geometric series:

$a + ar + ar^2 + \ldots + \underbrace{ar^{n-1}}_{\to \text{common ratio}} + \ldots$

•) ek term ko pichli vali se divide kr ke agr ans same aaye to woh geometric series ho gi

$\frac{ar^2}{ar} = \frac{ar}{a} = r$

$\frac{8}{4} = \frac{4}{2} = \frac{16}{8} = 2$ ∵ $2+4+8+16+\ldots$

→ sum

$S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$

$S_n = a_1 + a_2 + \ldots + a_n$ (n^{th} partial sum)

approaches to a particular number → series is convergent

limit of sequ series

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$S_n = a + a + a + \ldots + a$ ∵ $r = 1$

$S_n = n \cdot a$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n \cdot a \longrightarrow \infty$ $\cancel{\downarrow}$

•) definite answer → converge

•) indefinite " → diverge

when $r = -1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(a - a + a - a + \cdots (-)^n a_n\right)$$

•) Limit does not exist bcz there are multiple answers i.e 0 and a. Hence, the series diverge.

$|r| < 1 = -1 < r < 1$ $\quad -1 \longleftrightarrow 1$, r

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}}$$

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots \cancel{ar^{n-1}} + ar^n$$

$$S_n - r \cdot S_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}, \quad r^n \to 0$$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$ $a \to$ first term, $1-r \to$ common ratio

---

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \boxed{\frac{a(1-r^n)}{1-r}} \to \infty$$

$|r| > 1$ or $r < -1$, $r > 1$

-1, r, 1

•) Geometric series converge when $|r| < 1$ and diverge when $r = 1$, $r = -1$, $r >$, $r < -1$ or $|r| \geq 1$ / $r \geq 1$ or $r \leq 1$

Example: (1)

$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{1/9}{1-1/3} = \frac{1/9}{2/3} = \frac{1}{6}$$

(2)

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{4^n} \rightarrow -\Sigma \quad = \frac{(-1)^0 5}{4^0} = \frac{1\cdot 5}{1} = 5$$

$n = 1, \ -\frac{5}{4}$

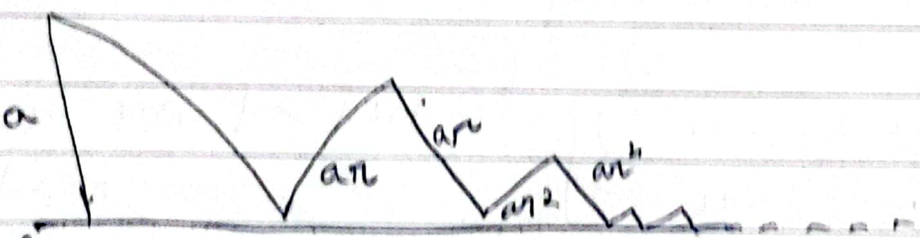
$$G.S = 5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} \cdots$$

r = common ratio $= \frac{-5/4}{5} = \frac{-1}{4}$

$a = 5$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{5}{1-(-1/4)} = \frac{5}{1+1/4} = \frac{5}{5/4}$$
$$= 4$$

(3)



Total Distance $= a + 2ra + 2\cdot r^2 a + \cdots$

$$= a + \frac{2ra}{1-r}$$

$$= a\left(1 + \frac{2r}{1-r}\right)$$

common Ratio $= \frac{2r^2 a}{2ra} = r$

1st term $= a$

$$= a\left(\frac{1+r}{1-r}\right)$$

put, $a=6$, $r=1/3$.

(3)

$$5.\overline{23} = 5.232323\ldots$$

$$5.2323\ldots = 5+.232323\ldots$$

$$= 5+.23+.0023+.000023+\ldots$$

$$= 5+\frac{23}{100}+\frac{23}{10000}+\frac{23}{1000000}+\ldots$$

.) common ratio r: $\frac{1}{100}$

$$= 5+\frac{23}{100}\left(1+\cancel{\frac{23}{100}}+\frac{1}{10000}+\ldots\right)$$

$$= 5+\frac{23}{100}\left(\frac{1}{1-1/100}\right) \qquad \frac{1}{100}$$

.) First term: $a=1$

$$= 5+\frac{23}{\cancel{100}}\left(\frac{\cancel{100}}{99}\right) = 5+\frac{23}{99}$$

$$= \frac{516}{99}$$

→ Telescoping Series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$S_n = \sum_{k=1}^{n}\left(1/k - 1/k+1\right)$$

$$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}\right.$$

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$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - \frac{1}{\infty+1} = 1 - \frac{1}{\infty}$$

$$= 1 - 0 = 1$$

$\therefore$ converges.

→ Check: Converge / Diverge
- Geometric Series Test
- Divergence Test

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \rightarrow \lim_{n\to\infty} a_n \rightarrow \infty / \neq 0$$

0

(4) $\boxed{\frac{-n}{2n+5}} = \lim_{n\to\infty} \frac{-n}{2n+5}$

$$= \lim_{n\to\infty} \frac{-\cancel{n}}{\cancel{n}(2+5/n)} = \lim_{n\to\infty} \frac{-1}{2+\frac{5}{n}}$$

$$= \frac{-1}{2+5/\infty} = \frac{-1}{2+0} = -\frac{1}{2}$$

→ Integral Test:

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$ where f is a continuous positive and decreasing function $\forall n \geq N$

Example ⟷ $a_n = 1/n$

$f(n) = 1/n$

$$\sum_{n=N}^{\infty} a_n \qquad \int_N^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ → p-series: jb power mein $p^{p>1}$ aaye

→ Conditions:

·) Series should be decreasing, positive and continuous

·) if the answer of integral is definite so the sequence converges

$$\cdot)\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_1^{b} 1/x^p\,dx$$

$$= \int_1^{\infty} 1/x^p\,dx = \lim_{b\to\infty}\int_1^{b} x^{-p}(1)\,dx$$

$$= \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left[b^{-p+1} - 1\right]$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left[\frac{1}{b^{p-1}} - 1\right]$$

$$= \frac{1}{1-p}\left[\frac{1}{(\infty)^{p-1}} - 1\right]$$

$$= \frac{1}{1-p}\left[0 - 1\right] = \frac{-1}{1-p} = \frac{-1}{-(p-1)}$$

$$= \frac{1}{p-1}$$ ∴ convergent for $p \geq 1$

Side working:

$$\left.\frac{x^{-p+1}}{-p+1}\right|_1^b \qquad \frac{1}{-p+1} \qquad -1^{-p+1} \qquad \frac{1}{0} \qquad \frac{1}{1} = 1 \qquad \frac{1}{2-1} \qquad \frac{1}{2}$$

x^{-p} $\quad -p\,x^{-p-1}$ $\quad \frac{p}{x^{-p-2}}$

$p \leq 0$ $\quad 0$ $\quad p > 1$

$$\boxed{\begin{matrix} p+1 > 1 \\ p > 0 \end{matrix}}$$

0 $\quad p > 1$ $\quad \frac{3}{2} > 1$

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$p \le 0$

dgt

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p}$ $\because p \le 0$

The p-series will diverge for $p \le 0$ according to divergent test.

When $p > 0$ and $p < 1$

$\frac{1}{1-p} \lim_{b\to\infty} \left[b^{-p+1} - 1 \right]$

$= \infty$

The p-series diverges.

dgt

- $p > 1 \rightarrow$ convergent
- $p \le 0 \rightarrow$ divergent

- ~~When~~ $p \le 1$

$\rightarrow$ p-series converges only when $p > 1$

Check the convergence or divergence by integral test:

① $\sum_{n=1}^{\infty} 1/n^2+1$

$f(x) = \frac{1}{1+x^2}$, $x \ge 1$

$f'(x) < 0$

$\downarrow$

decreasing function

$\frac{d}{dx}(f(x)) = \frac{d}{dx}(1+x^2)^{-1}$

$= (-1)(1+x^2)^{-2} \, d/dx\,(1+x^2)$

$= \frac{-2x}{(1+x^2)^2} < 0$

Hence proved that $f(x)$ is decreasing.

Antiderivative

$$\int f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \tan^{-1}x \Big|_1^b$$

$$= \lim_{b\to\infty} \tan^{-1}(b) - \tan^{-1}(1)$$

$$= \pi/2 - \pi/4$$

$$= \pi/4$$

As the integral converges, the series also converges.

(2) $\sum_{n=1}^{\infty} n \cdot e^{-n^2}$

$$f(x) = xe^{-x^2}$$

$$= \cancel{xe^{-x^2}}\; e^{-x^2}(1) + x \cdot e^{-x^2} \frac{d}{dx}(-x^2)$$

$$= e^{-x^2} + xe^{-x^2}(-2x)$$

$$= e^{-x^2}(1-2x^2) < 0$$

$$= \lim_{b\to\infty} \int_1^b xe^{-x^2}$$

Let $x^2 = t$

$2x\,dx = dt$

$x\,dx = 1/2\,dt$

$$= \lim_{b\to\infty} \int_1^b e^{-t}\, 1/2\,dt = \frac{1}{2} \lim_{b\to\infty} \frac{e^{-t}}{-1}\Big|_1^b$$

$$= 1/2 \lim_{b\to\infty} \left(e^{-b}/{-1} + e^{-1}\right) = 1/2\left(0 + 1/e\right)$$

$$= \frac{1}{2e}.$$

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Quiz-I on Friday
HomeWork-I

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Direct Comparison Test:

$0 \le a_n \le b_n \rightarrow$ large series

→ If b_n is convergent then a_n is also convergent
→ If a_n is divergent then b_n is also divergent

Limit Comparison Test:

$a_n > 0$ and $b_n > 0$

i) $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \rightarrow a_n$ and b_n both convergent / divergent

↓

Shows same behaviour.

ii) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \rightarrow$ if b_n is convergent then a_n is also convergent

iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \rightarrow$ if b_n is divergent then a_n is also divergent.

Examples:

① $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

$a_n = \frac{5}{5n-1}$

$b_n = \frac{1}{n} \rightarrow$ supposed/asumed

$\frac{5}{5n-1} = \frac{1}{n-1/5} > \frac{1}{n}$

$\sum_{n=1}^{\infty} 1/n$ is divergent because it is Harmonic series

$a_n > b_n$

$\frac{5}{5n-1} > \frac{1}{n}$

$\sum a_n$ is divergent.

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→ infinite geometric series ↓ convergent

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ki kitne baje aaan ha main to $\boxed{2}$ lye the shayd aaon

(2) $$\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{---}$$

$$\leq \quad \leq \quad \leq \quad \leq \quad \leq$$

$$1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \text{---}$$

convergent series.

(3)

a_n: $$\frac{5+2}{3} + \frac{1}{7} + 1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \frac{1}{8+\sqrt{3}} + \text{---}$$

$$\leq \quad \leq \quad \leq \quad \leq \quad \leq \quad \leq \quad \leq$$

b_n: $$\frac{5+2}{3} + \frac{1}{7} + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \text{---}$$

convergent → geometric series.

$a_n \leq b_n$

By comparison Test

$a_n \to$ convergent.

(4) $$\sum \frac{2n+1}{n^2+2n+1}$$

$$a_n = \frac{2n+1}{n^2+2n+1}$$

$$b_n = \frac{n}{n^2} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n+1}{n^2+2n+1}}{1/n}$$

$$= \lim_{n\to\infty} \frac{n(2n+1)}{n^2+2n+1}$$

$\to$ ...

$\sum \frac{1}{n} \leftrightarrow \to$ Harmonic series

$\downarrow$ ~~alt~~ diverge

$\to$ Power series

$\downarrow$

$p > 1 \to$ convergent

$$= \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = \lim_{n\to\infty} \frac{2+1/n}{1+2/n+1/n^2}$$

$$= 2 > 0$$

(5) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

$a_n = \frac{1}{2^n - 1}$, $b_n = \frac{1}{2^n}$ $\to$ should be greater than a_n.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1/2^n - 1}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

$$= \lim_{n\to\infty} \frac{1}{1 - 1/2^n}$$

$$= 1 > 0$$

Convergent $\to$ geometric series.

$\star$ $\lim_{n\to\infty} \frac{\ln n}{n^{3/2}}$ $\to$ Limit Comparison Test

$b_n = \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$

Ex-10.4 → Quiz

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Absolute Series.

Nimra

When we replace the co-responding terms in a series with it's absolute value it is called absolute series.

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

$$r = \frac{-1}{4}$$

General Term: $5 \cdot \left(\frac{-1}{4}\right)^n = \sum a_n$

$$\sum_{n=1}^{\infty}$$

$$\sum_{n=1}^{\infty} \left| 5\left(\frac{-1}{4}\right)^n \right| = \sum |a_n|$$

•) If a series converge absolutely, then the given series is also convergent.

"Ratio Test"

$|a_n|$ $|a_{n+1}|$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} c<1 \to \text{convergent} \\ c>1 \to \text{divergent} \to c=\infty \\ c=1 \to \text{inconclusive} \end{cases}$$

↓

ratio test fails

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$4!$

$1\times2\times3\times4$

$\boxed{a_n = S_n - S_{n-1}}$

$\downarrow$

n^{th} partial sum

Example:

(1) $$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$$

$$|a_n| = \frac{2^n+5}{3^n}$$

$$|a_{n+1}| = \frac{2^{n+1}+5}{3^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}+5}{\cancel{3^{n+1}}} \times \frac{\cancel{3^n}}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{(2^{n+1}+5)}{3(2^n+5)}$$

$$= \lim_{n\to\infty} \frac{2+\cancel{5/2^n}}{3+\cancel{15/2^n}} \quad \because \text{Dividing by } 2^n$$

$$= \frac{2}{3} < 1 \rightarrow \text{convergent}$$

(2) $$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots$$

General Term:

$$\sum_{n=1}^{\infty} \frac{(2n+1)}{(n+1)^2} = a_n$$

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10.5+10.6.

Root Test:

$$\lim_{n\to\infty} (a_n)^{1/n} = c$$

$-\lim_{n\to\infty} (a_n)^{1/n} = c < 1 \to$ cgt
$> 1 \to$ dgt
$= 1 \to$ test inconclusives

"Ratio Test"

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$= c < 1 \to$ convergent
$= c > 1 \to$ divergent
$= 1 \to$ test inconclusive.

Examples:

① $\sum_{n=1}^{\infty} \left(\frac{1}{1+n} \right)^n$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{1}{1+n} \right)^{\cancel{n}} \right]^{1/\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1 \to \text{convergent}$$

② $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$a_n = \frac{n^2}{2^n}$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n^2}{2^n} \right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{(n^{1/n})^2}{2} \quad \because \lim_{n\to\infty} n^{1/n} = 1 \quad 10.5$$

$$= \frac{(1)^2}{2} = \frac{1}{2} < 1$$

convergent

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$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

$\downarrow$

Alternating series

Conditions:

i) All a_n's $\rightarrow$ positive

ii) All a_n's are non-increasing $\rightarrow a_n \geq a_{n+1}$

iii) $a_n \xrightarrow{\text{converge}} 0$

•) If a series converges but absolutely do not converge then the series is conditionally convergent.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^P} = 1 - \frac{1}{2^P} + \frac{1}{3^P} - \frac{1}{4^P} + \cdots$$

$P > 1$

$\downarrow$

convergent

Absolute series $= 1 + \frac{1}{2^P} + \frac{1}{3^P} + \frac{1}{4^P} + \cdots$

$P > 1 \rightarrow$ convergent

$\therefore$ conditionally convergent.

$\rightarrow$ Ratio Test
$\rightarrow$ Root Test

conditionally convergent
$\downarrow$
imp

imp
Power Series: (whenever variable = x)

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x^1 + C_2 x^2 + \dots + C_n \cdot x^n + \dots$$

(1) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$\cos 180(n)$
-1

$$|a_n| = \frac{|x^n|}{n}$$

$$|a_{n+1}| = \left|\frac{x^{n+1}}{n+1}\right| \rightarrow \text{Ratio Test}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/n+1}{x^n/n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}}{n+1}\right|\cdot\left|\frac{n}{\cancel{x^{n}}}\right|$$

$$= \lim_{n\to\infty} \frac{|x|\cdot n}{n+1}$$

$$= \lim_{n\to\infty} |x| \cdot \frac{1}{1+1/n} \quad \text{(Dividing by } n\text{)}$$

$$= |x|$$

If $|x| < 1 \rightarrow$ convergent
$-1 < x < 1$

Radius of convergence = 1
Interval of convergence = $(-1, 1)$

(2) $$(-1)^{n+1} \frac{x^{2n-1}}{2n-1} \left(\sum_{n=1}^{\infty}\right)$$

$$|a_n| = \left|\frac{x^{2n-1}}{2n-1}\right|$$

$2(n+1)-1$
$2n+2-1$
$2n+1$

$$|a_{n+1}| = \left| \frac{x^{2n+1}}{2n+1} \right|$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{x^{2n+1}}{2n+1} \right| \left| \frac{2n-1}{x^{2n-1}} \right|$$

$$= \lim_{n\to\infty} \frac{|x^2|\, 2n-1}{2n+1}$$

$$= \lim_{n\to\infty} \frac{|x^2|\, 1-1/2n}{1+1/2n}$$

$$= |x^2|$$

If $|x^2| < 1 \rightarrow$ convergent

(3) $\sum_{n=0}^{\infty} n!\, x^n$

dividing by 1

$$\lim_{n\to\infty} \frac{(n+1)!\, x}{n!} = \lim_{n\to\infty} \frac{\left(1 + \frac{1}{n}\right)!\, x}{n!/n}$$

$$= \lim_{n\to\infty} \frac{\left(1 + \frac{1}{n}\right)!\, x}{1} = |x| < 1 \rightarrow \text{convergent}$$

$$\frac{(n+1)!\, |x|}{n!} \qquad (n+1)\, |x|$$

$$\frac{(n+1)\, n!\, |x|}{n!}$$

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Taylor series → $x = a$

$$P_n(x) = f(a) + f'(a)\cdot(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \frac{f^{n}(a)}{n!}(x-a)^n + \cdots \quad —(i)$$

① $f(x) = \frac{1}{x}$; $a = 2$

$f(x) = \frac{1}{x}$ (Given)

$$= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots$$

$f(x) = \frac{1}{x} \Rightarrow f(2) = \frac{1}{2}$

$f'(x) = \frac{-1}{x^2} \Rightarrow f'(2) = \frac{-1}{2^2}$

$f''(x) = \frac{2}{x^3} \Rightarrow f''(2) = \frac{2}{2^3} \Rightarrow \frac{f''(2)}{2!} = \frac{1}{2^3}$

$f'''(x) = \frac{-6}{x^4} \Rightarrow f'''(2) = \frac{-6}{2^4} \Rightarrow \frac{f'''(2)}{3!} = \frac{-1}{2^4}$

$f^{(n)}(2) = \frac{(-1)^n}{n! \cdot x^{n+1}}$

$$= \frac{1}{2} \cancel{(x-2)} + \frac{-1}{2^2}(x-2) + \frac{1}{2^3}(x-2)^2 - \frac{1}{2^4}(x-3)^3 + \cdots$$

$\cancel{\frac{(-1)^n}{x^{n+1}}}$

Bhen Bhai
Ja phir Bache

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Sir ka kehna ha ke "Abhi To mein jwan hoon".

Aho

Common Ratio: $\left|\frac{x-2}{2}\right| < 1$

$$|x-2| < 2$$
$$-2 < x-2 < 2$$
$$0 < x < 4$$

Sir ka bhrm qaim rehne dete hain — Humara kya jata ha.

Radius of convergent $= 2$
interval of convergence $= 0 < x < 4$

Maclaurin Series:
$$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots$$

Hn bai Sir hamare Bhai hain
Bhai chara

Sir bohot ache hain. waise Haan → humari anse hoj achi kr hi aj!

Haan Hm na achi Ik. aj thi

Allah iss bhai chare ko salamat rkhe
Aameen
Ameeem!

Jo bhi chahte hain bai us khud ko yong show kra rahe

"bete" bhi keh "o" rhe n bhai bhi? yeh chahte kya hain

Differential Equation
An equation in which a ~~va~~ derivative of dependent variable is involved.

Seperable variable: (Type of Differential Equation)

$$\frac{dy}{dx} = x \cdot y$$

$$\frac{dy}{y} = x \cdot dx$$

Case-1

$$\frac{dy}{dx} = h(x) \cdot g(y)$$

$$dy/g(y) = h(x)\,dx$$

① $(1+x)\,dy - y\,dx = 0$

$$(1+x)\,dy = y\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$\cancel{\ln y \, dy = \ln(1+x)}$$

$$\ln y = \ln(1+x) + C$$

② $\cancel{\frac{dy}{dx}}\,(e^{2y} - y)\cos x \frac{dy}{dx} = e^{y}\sin 2x.$

$$(e^{2y} - y)\cos x\,dy = e^{y}\sin 2x\,dx.$$

$$\frac{e^{2y} - y}{e^{y}}\,dy = \frac{\sin 2x}{\cos x}\,dx$$

$$\int \frac{e^{2y} - y}{e^{y}}\,dy = \int \frac{2\sin x\,\cancel{\cos x}}{\cancel{\cos x}}\,dx$$

$$\int \frac{e}{e^{y}} - \frac{y}{e^{y}}\,dy = \int 2\sin x$$

$$\int \frac{e^{y}}{e^{y}} \cdot \int y e^{-y} = -2\cos x + C$$

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Integration by parts

$$\int \underset{\mathrm{I}}{y}\cdot \underset{\mathrm{II}}{e^{-y}}\,dy$$

$$= y\cdot\frac{(e^{-y})}{-1} - \int \frac{e^{-y}}{(-1)}\cdot 1\,dy.$$

$$= -ye^{-y} + \int e^{-y}\,dy$$

(3) $$\frac{dy}{dx}\cdot x^2 = y - xy$$

$$y = \frac{e^{-x^{-1}}\cdot c}{x}$$

$$\frac{dy}{dx}x^2 = y(1-x)$$

$$\frac{1}{y}\,dy = \frac{(1-x)}{x^2}\,dx$$

$$\frac{1}{y}\,dy = \frac{1}{x^2} - \frac{\cancel{x}1}{x}\,dx$$

$$\int\frac{1}{y}\,dy = \int \frac{1}{x^2} - \frac{1}{x}\,dx$$

$$\ln y = \frac{x^{-2+1}}{-2+1} - \ln x + C$$

$$\ln y = \frac{x^{-1}}{-1} - \ln x + C$$

$$\ln y = \frac{-1}{x} - \ln x + C.$$

$$\ln y = \ln e^{-x^{-1}} \cancel{e} - \ln x + \ln C$$

$$\cancel{\ln} y = \cancel{\ln}\left(\frac{e^{-x^{-1}}\cdot c}{x}\right)$$

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(4) $\frac{dy}{dx} = e^{3x+2y}$

$\frac{dy}{dx} = e^{3x} \cdot e^{2y}$

$\frac{1}{e^{2y}} dy = e^{3x} dx$

$\int e^{-2y} dy = \int e^{3x} dx$

$\frac{e^{-2y}}{-2} = \frac{e^{3x}}{3}$

$\frac{-1}{2e^{2y}} = \frac{e^{3x}}{3} + C$

Factorial
$\downarrow$
Ratio Test

Imp Topics:

- conditional convergent
- Power Series
- Ratio and Root Test

(1) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ → Alternative Series

(2) $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n n! n!}$ → Alternative Series.

(3) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ → Alternative Series.

(4) $\sum_{n=0}^{\infty} \frac{(x^2+1)^n}{3}$ → Root Test

(5) $\sum_{n=1}^{\infty} \frac{4^n \cdot x^{2n}}{n}$ → Power Series.

(2) $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n n! n!}$

$|a_n| = \left| \frac{(2n)!}{2^n n! n!} \right|$ $\quad |a_{n+1}| = \left| \frac{(2n+2)!}{2^{n+1} (n+1)! (n+1)!} \right|$

$\cos n\pi$

$180°$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)!}{2^{n+1}(n+1)!\,(n+1)!}\cdot\frac{2^n\, n!\, n!}{(2n)!}$$

$$= \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{2^{n+1}}(n+1)\cancel{n!}\,(n+1)\cancel{n!}}\cdot\frac{\cancel{2^n}\,\cancel{n!}\,\cancel{n!}}{\cancel{(2n)!}}$$

$n!\ (n-1)!$

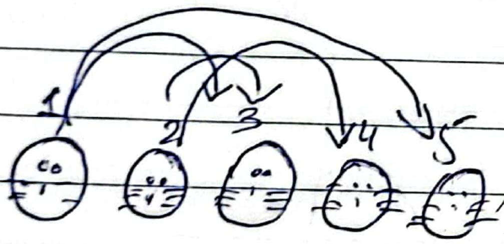
$(n+1)!$

$(n+1)\,n!$

$$= \left|\frac{(2n+2)(2n+1)}{2(n+1)^2}\right|$$

$$= \frac{4n^2+6n+2}{2n^2+2+4n}$$

$$= \frac{4+\frac{6}{n}+\frac{2}{n^2}}{2+\frac{6}{n^2}+\frac{4}{n}}$$



1

$\frac{1}{n^2}$

$\frac{1}{n+1}$

$1\ 1 \ldots$

$1\ 1\ 1\ 1\ 1\ 1$

$(-1)^{n+1}$

$1,\ \frac{1}{2},\ \frac{1}{3},$

$(-1)^n$

$1,\ -1,$

$-1,\ 1,\ -1,\ 1$

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$$\sum_{n=1}^{\infty} a_n$$

a_n — —

$a_n = 1$

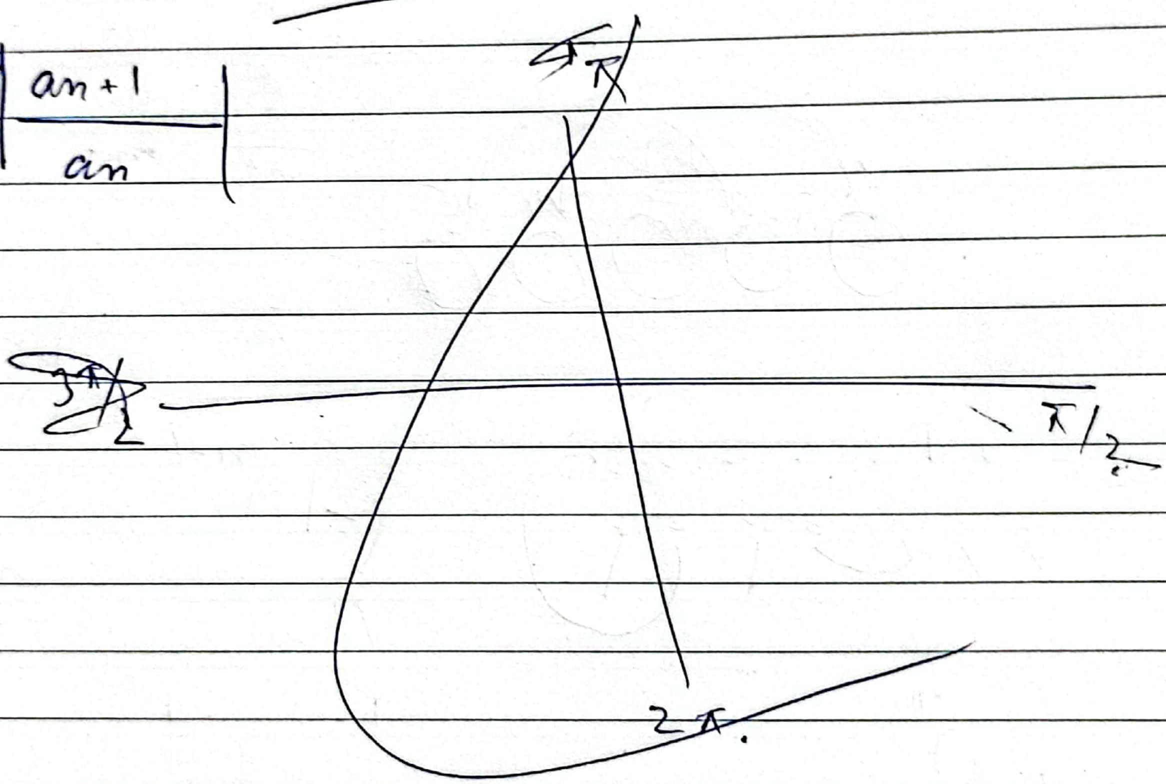
a_{n+1}

$$a_n = \frac{x^n}{n} \qquad = 2n$$

$$= 2(n+1)$$

$$a_{n+1} = \frac{x^{(n+1)}}{n+1}$$

$$\left| \frac{a_{n+1}}{a_n} \right|$$



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→ Removing differential is the solution of every question.

Linear Differential Equation:

↳ Case-I

$$a_1(x)\frac{dy}{dx} + y a_2(x) = a_3(x)$$ Dividing by $a_1(x)$

$$\frac{dy}{dx} + y \cdot a_4(x) = a_5(x)$$ → standard form of Linear Differential Equation

Non-Homogenus

$$\frac{dy}{dx} + y \cdot a_4(x) = 0$$ → Homogenous

Example:

① $$\frac{dy}{dx} - 3y = 6$$

$$I \cdot F = e^{\int -3dx} = e^{-3x}$$

Integrating Factor

$$\frac{dy}{dx}e^{-3x} - 3ye^{-3x} = 6e^{-3x}$$

$$\frac{d}{dx}\left(y \cdot e^{-3x}\right) = 6e^{-3x}$$

$$\int \frac{d}{dx}\left(y \cdot e^{-3x}\right)dx = 6\int e^{-3x}dx$$

$$y \cdot e^{-3x} = \frac{6e^{-3x}}{-3}$$

$$ye^{-3x} = -2e^{-3x} + c$$

$$ye^{-3x} = -2e^{-3x} + c$$

$$y = -2 + ce^{3x}$$

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Zil

(2) $x\frac{dy}{dx} - 4y = x^6e^x$

$\frac{dy}{dx} - \frac{4y}{x} = x^5e^x.$

$I.F = e^{\int -4/x \, dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4} = \frac{1}{x^4}$

$\cancel{\frac{dy}{dx}e^{-4\ln x} - \frac{4y}{x}e^{-4\ln x} = x^5e^x \cdot e^{-4\ln x}}$

$\frac{d}{dx}\left(y \cdot \frac{1}{x^4}\right) = xe^x$

$\frac{d}{dx}\left(y \cdot \frac{1}{x^4}\right) = xe^x$

$\int \frac{d}{dx}\left(y \cdot \frac{1}{x^4}\right) dx = \int \underset{I}{x}\,\underset{II}{e^x}\, dx$

$y \cdot \frac{1}{x^4} =$

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(3) $(x^2-9)\frac{dy}{dx} + xy = 0$

$\frac{dy}{dx} + \frac{xy}{x^2-9} = 0$

$I.F = e^{\int \frac{x}{x^2-9}} = e^{\frac{1}{2}\int \frac{2x}{x^2-9}} = e^{\frac{1}{2}\ln(x^2-9)}$ $\cancel{= e^{\frac{1}{2}\ln(x^2-9)}}$

$= (x^2-9)^{1/2}$

$(x^2-9)^{1/2}\frac{d}{dx} + (x^2-9)^{1/2}\frac{xy}{(x^2-9)} = 0$

$\cancel{(x^2-9)^{1/2}}$

$\cancel{\int}\frac{d}{dx}\left(y\cdot(x^2-9)^{1/2}\right) = 0$

$y(x^2-9)^{1/2} = 0$

(4) $\frac{dy}{dx} + y = x.$

$I.F = e^{\int 1} = e^x$

$\frac{d}{dx}\left(y\cdot e^x\right) = xe^x$

$\int\frac{d}{dx}(y\cdot e^x) = \int xe^x$

$\cancel{ye^x = \frac{x^2}{2} + c.}$

$ye^x = \int xe^x.$

$ye^x = xe^x - e^x = e^x(x-1)$

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$= e^{x+\ln(x+1)}$

$= e^x \cdot \cancel{e}^{\ln(x+1)} \quad e^x(x+1)$

(5) $(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$

$\frac{dy}{dx} + \frac{(x+2)}{(x+1)}y = \frac{2xe^{-x}}{(x+1)}$

$I \cdot F = e^{\int \frac{x+2}{x+1}}$ $e^{\ln(x+1)}$

$= e^{\int \frac{x+2+1-1}{x+1}} = e^{\int \frac{x+1+1}{x+1}} = e^{\int 1 + \frac{1}{x+1}}$

$= e^{x+\ln(x+1)} = e^x \cdot \cancel{e}^{\ln x+1}(x+1) = (x+1)e^x$

$\frac{d}{dx}\int\left(y \cdot (x+1)e^x\right) = \int \frac{2xe^{-x}}{\cancel{(x+1)}} \cancel{(x+1)}e^x$

$y(x+1)e^x = \int \frac{2x}{\cancel{e^x}} \cancel{e^x}\, dx$

$y(x+1)e^x = \int 2x\, dx$

$y(x+1)e^x = \frac{\cancel{2}x^2}{\cancel{2}} + c$

$y(x+1)e^x = x^2 + C.$

(6) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta.$

$I \cdot F = e^{\int \sec\theta} = e^{r^2/2} + C$

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(6) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$

→ Case- III

Exact Differential Equation:

$(e^{2y} - y\cdot\cos xy)\,dx + (2x\cdot e^{2y} - x\cos xy + 2y)\,dy \;\ldots\; 2xy\,dy + (x^2-1)\,dy = 0$

$M(x,y) = e^{2y} - y\cdot\cos xy \qquad N(x,y) = 2xe^{2y} - x\cos xy + 2y$

$\frac{\partial M}{\partial x} = -y[-\sin xy\cdot y]$

$\cos^2 x$

$-2\sin^2 x$

$N(x,y) = 2xe^{2y} - x\cos xy + 2y$

$M(x,y) = e^{2y} - y\cdot\cos xy$

$\frac{\partial M}{\partial y} = y\cdot e^{2y} - (1\cdot\cos xy + y\cdot(-\sin xy)\cdot x)$

$= 2e^{2y} - \cos xy + xy\sin xy$.

$N(x,y) = 2xe^{2y} - \cancel{y}\cdot x\cdot\cos xy + 2y$

$\frac{\partial N}{\partial x} = 2e^{2y}(1) - [1\cdot\cos xy + x\cdot(-\sin y)\,y] + 0$

$= 2e^{2y} - [\cos xy - xy\sin xy]$

$= 2e^{2y} - \cos xy + xy\sin xy$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ Differential Equation is exact.

$\int (e^{2y} - y\cos xy)\,dx + \int 2y\,dy = C$

$e^{2y}\int 1\,dx - \int y\cos xy\,dx + 2\int y\,dy = c$ $\Rightarrow y$ is constant

$e^{2y}\cdot x - \cancel{y}\cdot\frac{\sin xy}{\cancel{y}} + \frac{\cancel{2}y^2}{\cancel{2}} = c$

$e^{2y}\cdot x - \sin xy + y^2 = c.$

Example:

(1) $2xy\,dx + (x^2-1)\,dy = 0$

$\int 2xy\,dx + \int (-1)\,dy = c.$

$\int 2xy\,dx - \int 1\,dy = c.$

$\frac{y\cancel{2}x^2}{\cancel{2}} - y = c$

$x^2y - y = c$

$y(x^2-1) = c$

$y(x+1)(x-1) = c$

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(2)

$(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0.$

$\frac{\partial M}{\partial y} = 2xy^2 - 3 = 2xy^2 = 4xy$

$\frac{\partial N}{\partial x} = 2x^2y + 4$
$= 4xy$

$\int 2xy^2 - 3 + \int 4y\,dy$

$= y^2 \cdot \frac{2x^2}{2} - 3x + 4y = y^2x^2 - 3x + 4y.$

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Exact & Non-exact Equation

Example:

1. $(xy^2 + y)\,dx - x\,dy = 0 \quad -(i)$

$M(x,y) = xy^2 + y$ $\qquad$ $N(x,y) = -x.$

$\frac{\partial M}{\partial y} = 2xy + 1 \; (My)$ $\qquad$ $\frac{\partial N}{\partial x} = -1 \; (Nx)$

Given equation is not exact as

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

$\frac{Nx - My}{M}$ = Ans should be the function of y.

$$\frac{Nx - My}{M} = \frac{-1 - 2xy - 1}{xy^2 + y} = \frac{-2(1 + \cancel{2}xy)}{xy^2 + y} = \frac{-2\cancel{(1+xy)}}{y\cancel{(1+xy)}}$$

$$= \frac{-2}{y}$$

$$I.F = e^{\int \frac{-2}{y} dy} = e^{-2\ln y} = \cancel{e^{\ln}} y^{-2} = y^{-2} = 1/y^2$$

Multiplying eq(i) by $1/y^2$

$$\left(\frac{xy^2}{y^2} + \frac{y}{y^2}\right) dx - \frac{x}{y^2}\,dy = 0$$

$$\left(x+\frac{1}{y}\right)dx - \frac{x}{y^2}\,dy = 0$$

The above equation is an exact equation

$$\int\left(x+\frac{1}{y}\right)dx = c$$

$$\int x\,dx + \frac{1}{y}\int 1\,dx = c$$

$$\frac{x^2}{2} + \frac{x}{y} = c \longrightarrow \text{Answer.}$$

(2) $y(2xy+e^x)\,dx - e^x\,dy = 0 \quad -(i)$

$$\frac{\partial M}{\partial y} = y(2xy+e^x) \qquad \frac{\partial N}{\partial x} = -e^x\,dx$$

$$= 2xy^2 + e^x y \qquad = -e^x \,(N_x)$$

$$= 4xy + e^x \,(M_y)$$

$$\frac{N_x - M_y}{M} = \cancel{\frac{-e^x - 4xy - e^x}{y(2xy+e^x)}} = \cancel{\frac{-2e^x - 4xy}{2xy^2+e^x y}}$$

$$= \cancel{-2e^x - 4}$$

$$= \frac{-2e^x - 4xy}{2xy^2 + e^x y}$$

$$= \frac{-2(\cancel{e^x + 2xy})}{y(\cancel{e^x + 2xy})} = \frac{-2}{y}$$

$$\text{I.F.} = e^{\int -\frac{2}{y}} = e^{-2\ln y} = \cancel{e^{\ln y^{-2}}} = \frac{1}{y^2}$$

$\left(\frac{2xy+e^x}{y}\right) dx = 0$

$y(2xy+e^x)$

$(2xy^2+e^x y)\frac{1}{y^2}$

$\frac{2xy}{y} + \frac{e^x}{y} dx = 0$

$2x\cancel{y}\cdot\frac{1}{y^2} + e^x y \cdot \frac{1}{y^2}$

$2x + \frac{e^x}{y} dx = 0$

$\frac{2x+e^x}{y}$

$\int 2x + \int \frac{e^x}{y}$

$2x + \frac{e^x}{y}$

$\frac{2x^2}{2} + \frac{e^x}{y}$

$\frac{2xy+e^x}{y} - \frac{e^x}{y^2} = 0.$

(2) $\int \frac{2xy+e^x}{y} = c$

$= \frac{1}{y}\int 2xy + \frac{1}{y}\int e^x$

$= \frac{x^2 y}{\cancel{2y}} + \frac{e^x}{y}$

$c. = x^2 + \frac{e^x}{y}$

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(3) $\left(y - \frac{\sin x}{x}\right) dx + dy = 0$ (i) $\frac{My - Nx}{N} = P(x)$

$\frac{My - Nx}{N} = P(x)$ (ii)

-1

$\frac{\partial M}{\partial y} = \frac{y}{x} - \frac{\sin x}{x}$

$= \frac{1}{x} - 0$

$\frac{\partial N}{\partial x} = 0.$

$\frac{My - Nx}{N} = \frac{1/x - 0}{1} = \frac{1}{x}$

$I.F = e^{\int 1/x} = \cancel{e^{\ln x}} = x.$

$\int (y - \sin x)\, dx = y \int 1 dx - \int \sin x \, dx.$

$= yx + \cos x.$

(4) $(y^2 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0.$

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(4) $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

$\frac{\partial M}{\partial y} = (y^2+2y) = 2y+2$

$\frac{\partial N}{\partial x} = xy^3+2y^4-4x = y^3+\cancel{2y^4}-4$

$\frac{\partial M}{\partial y} = M = y^4+2y$
$= 4y^3+2$

$\frac{\partial N}{\partial x} = y^3-4$

$= \frac{4y^3+2-(y^3-4)}{xy^3+2y^4-4x}$

$= \frac{4y^3+2-y^3+4}{xy^3+2y^4-4x}$

$= 3y^3+6$

$\frac{y^3-4-4y^3-2}{y^4+2y}$

$\frac{-3y^3-6}{y^4+2y} = \frac{-3(y^3+2)}{y(y^3+2)}$

$= \frac{-3}{y}$

$I\cdot F = e^{\cancel{\ln} y^{-3}} = \frac{1}{y^3}$

$\left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy$

$\int\left(y+\frac{2}{y^2}\right)dx + 2\int y\,dy$

$yx+\frac{2}{y^2}x+y^2 = C.$

Bernoulli's Differential Equation:

$$\frac{dy}{dx} + y \cdot a(x) = b(x) \cdot c(y)$$

Example

① $$\frac{dy}{dx} + \frac{1}{x} y = \frac{\ln x}{x} y^2 .$$

Dividing by y^2

$$y^{-2} \frac{dy}{dx} + \frac{1}{x} y^{-1} = \frac{\ln x}{x}$$

$$-y^{-2} \frac{dy}{dx} - \frac{1}{x} y^{-1} = -\frac{\ln x}{x}$$

Let $y^{-1} = v$

$$-y^{-2} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} - \frac{1}{x} v = -\frac{\ln x}{x} \quad \text{—(i)}$$

$$I.F = e^{\int -1/x} = e^{-\ln x} = e^{\ln x^{-1}} = 1/x$$

From (i)

$$\frac{d}{dx}\left(v \cdot \frac{1}{x}\right) = -\frac{\ln x}{x} \times \frac{1}{x} \Rightarrow \frac{d}{dx}\left(v \cdot \frac{1}{x}\right) = -\frac{\ln x}{x^2}$$

$$\int \frac{d}{dx}\left(v \cdot \frac{1}{x}\right) = -\int x^{-2} \cdot \ln x \, dx$$

$$v \cdot \frac{1}{x} = -\left[\ln x \left(\frac{x^{-1}}{-1}\right) - \int \left(\frac{x^{-1}}{-1}\right) \cdot \frac{1}{x} \, dx\right]$$

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$$\frac{v \cdot 1}{x} = \frac{\ln x}{x} - \int \frac{1}{x^2}\, dx = \frac{\ln x}{x} - \int x^{-2}\, dx$$

$$\frac{1}{xy} = \frac{\ln x}{x} - \frac{x^{-1}}{-1} + c \qquad \because v = y^{-1} = \frac{1}{y}.$$

$$\frac{1}{xy} = \frac{\ln x}{x} + \frac{1}{x} + c.$$

(2)

$$\frac{dy}{dx} + y = xy^3.$$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x$$

$$y^{-3}\frac{dy}{dx} + y^{-2} = x$$

$$-2y^{-3}\frac{dy}{dx} - 2y^{-2} = -2x$$

Let $y^{-2} = v$

$$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}.$$

$$\frac{dv}{dx} - 2v = -2x \quad -(i)$$

$$I \cdot F = e^{\int -2} = e^{-2x}$$

$$\frac{d}{dx}\left(e^{-2x}\,\cancel{2x}\,v\right) = -2x\,e^{-2x}$$

$$\cancel{\int \frac{d}{dx}\left(\cancel{2x}\,e^{-2x}v\right)} = \int -2x e^{-2x}$$

I (1)
L
A (2)
T
E (3)

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$e^{-2x} v = -2 \int x e^{-2x}$

(2)

$$\frac{dy}{dx} + \frac{1}{2x} y = x \cdot \frac{1}{y^3}$$

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{1}{2x} \cdot \frac{1}{y^2} = x$$

$$y^{-3} \frac{dy}{dx} + y^{-2} \cdot \frac{1}{2x} = x$$

$$-2y^{-3} \frac{dy}{dx} - 2y^{-2} \cdot \frac{1}{2x} = -2x$$

Let $y^{-2} = v$

$$-2y^{-3} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} - 2y^{-2} \cdot \frac{1}{2x} = -2x$$

I.F =

$$\frac{dv}{dx} - 2v \cdot \frac{1}{2x} = -2x$$

$1 - u$ / u

$1 - 1/u$

$1/u$

u / $1 - 1/u$

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Ex 3.1

Rate of change: $\frac{dx}{dt} = kx$ (increasing/decreasing)

Given data: at $t=0$

* In a culture plate initially the no. of bacteria $= P_0$

At $t = 1h \rightarrow$ no. of bacteria $= \frac{3}{2} P_0$

At what time number of bacteria are triple?

$$\frac{dx}{dt} = kx \quad -(i)$$

put $x = p$ in (i)

$$\frac{dp}{dt} = kp$$

$$\frac{dp}{dt} - kp = 0$$

$$I.F = e^{\int -k\,dt} = e^{-kt}$$

$$\frac{d}{dt}(p \cdot e^{-kt}) = 0$$

$$\int \frac{d}{dt}(pe^{-kt}) = c$$

$$pe^{-kt} = c$$

put $t = 0$

$p = P_0$

$\Rightarrow P = ce^{kt}$

$P_0 = ce^{k(0)}$

$P_0 = ce^{0}$

$P_0 = c$

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So, we have
$P = P_0 e^{kt}$ ~~P → no. of bacteria~~

At $t = 1$,
$P = \frac{3}{2} P_0$

$\frac{3}{2} P_0 = P_0 e^{k(1)}$

$\frac{3}{2} = e^{k} \Rightarrow \ln \frac{3}{2} = k$

$k = 0.4055$ (approx).

$P = P_0 e^{0.4055t}$

No. of bacteria, time.

$3 P_0 = P_0 e^{0.4055t}$

$3 = e^{0.4055t}$

$\ln 3 = 0.4055t$

$t = 2.7174$

Rate of change (Related to temperature)

$$\frac{dT}{dt} = K(T - T_m)$$

T_m ↓ Terminating / Required Temperature

★ Initially Temperature = 300°F ⇒ $t=0$, $T=300$

★ After 3 min Temperature = 200°F ⇒ $t=3$, $T=200$

★ At what time, the temperature = 70°F ?

use $\frac{dT}{dt} = K(T-70)$

$$\frac{dT}{T-70} = K\,dt$$

$$\cancel{dT \cdot 1} = \frac{}{T-70}$$

$$\int \frac{1}{T-70}\,dT = \int K\,dt$$

$$\ln(T-70) = Kt + C$$

$$\ln(T-70) = \cancel{k}\ \ln e^{kt} + \ln e^{C}$$

$$\cancel{t} \qquad = \cancel{\ln e^{kt}}\ \cancel{\ln} e^{C}$$

$$= e^{kt}C$$

$$\ln(T-70) = kt + C_1$$

$$\ln(T-70) = \ln e^{kt} + \ln e^{C_1}$$

$$\cancel{\ln}(T-70) = \cancel{\ln} e^{kt}e^{C_1}$$

$$T-70 = e^{kt}\cdot C_1 \longrightarrow T-70 = e^{kt}\cdot 230$$

↓ putting value in C_2

use $\begin{cases} t=0 \\ T=300 \end{cases}$

$$300-70 = e^{k(0)}\cdot C_2$$

$$230 = C_2$$

$T - 70 = e^{kt} \cdot 230 \Rightarrow T = 70 + e^{kt} 230$

$200 - 70 = e^{kt} \cdot 230$

$\frac{130}{230} = e^{kt}$

$0 \cdot 5652 = e^{kt}$

$\cancel{\ln 0 \cdot 5652 = \ln e^{kt}}$

$\ln 0 \cdot 5652 = \ln e^{kt}$

$= kt \Rightarrow t = 3$

$\frac{1}{3} \; -0 \cdot 57057 = k$

$-1 \cdot 669 = k$