National University of Computer and Emerging Sciences, Lahore Campus

	Course:	Discrete Structures	Course Code:	CS-1005
	Program:	BS (Software Engineering)	Semester:	Fall 2022
	Duration:	180 Minutes	Total Marks:	100
	Paper Date:	15-June-22	Weight	NA
	Section:	ALL	Page(s):	21
	Exam:	Final Exam	~~Roll No~~.	*Solution*

Instruction/Notes:

- Use of calculator is allowed; however, ***sharing* of calculator** is **strictly prohibited**.
- Do NOT un-staple your exam, otherwise it might be cancelled.
- You may use rough sheets, but you must write your answers (and the relevant working, where required) on the space provided in the answer sheet. ***No credit would be given for the work done on the rough sheets.***

Question Number	1	2	3	4	5	6	7	8	9	10	Total
Total Marks	10	10	10	10	10	10	10	10	10	10	100
Obtained Marks											

Question 1: **[2+1+2+2+3] Marks**

What is the *negation* of each of these propositions?

The summer in Maine is hot and sunny.

The summer in Maine is either cold (*or not hot*) **or** cloudy (*or not sunny*).

You can get admission in University unless you have failed the entrance exam.

You passed the exam but can't get admission in University.

Translate the given statement into *propositional logic* using the propositions provided.

You can upgrade your operating system only if you have a 32-bit processor running at 1 GHz or faster, at least 1 GB RAM, and 16 GB free hard disk space, or a 64- bit processor running at 2 GHz or faster, at least 2 GB RAM, and at least 32 GB free hard disk space.

Express you answer in terms of

***u*:** "You can upgrade your operating system," ***b*32:** "You have a 32-bit processor," ***b*64:** "You have a 64-bit processor," ***g*1:** "Your processor runs at 1 GHz or faster," ***g*2:** "Your processor runs at 2 GHz or faster," ***r*1:** "Your processor has at least 1 GB RAM," ***r*2:** "Your processor has at least 2 GB RAM," ***h*16:** "You have at least 16 GB free hard disk space," and ***h*32:** "You have at least 32 GB free hard disk space."

$$u \rightarrow (b32 \wedge g1 \wedge r1 \wedge h16) \vee (b64 \wedge g2 \wedge r2 \wedge h32)$$

Express these system specifications using the propositions $\boldsymbol{p}$ "The user enters a valid password", $\boldsymbol{q}$ "Access is granted", and $\boldsymbol{r}$ "The user has paid the subscription fee" and logical connectives (including negations).

"Access is granted whenever the user has paid the subscription fee and enters a valid password."

$$p \wedge r \rightarrow q$$

"If the user has not entered a valid password but has paid the subscription fee, then access is granted."

$$\neg p \wedge r \rightarrow q$$

Write each of these statements in the form **"if *p*, then *q*"** in English.

It is necessary to walk 8 miles to get to the top of Long's Peak.

If one gets to the top of Long's Peak**, then** one has walked 8 miles.

Your guarantee is good only if you bought your CD player less than 90 days ago.

If your guarantee is good**, then** you bought your CD player less than 90 days ago.

Show that $\neg \boldsymbol{p} \rightarrow (\boldsymbol{q} \rightarrow \boldsymbol{r})$ and $\boldsymbol{q} \rightarrow (\boldsymbol{p} \vee \boldsymbol{r})$ are logically equivalent by developing a series of logical equivalences.

Note: You can establish the above equivalence using truth table to get **50% marks**.

$$\neg \boldsymbol{p} \rightarrow (\boldsymbol{q} \rightarrow \boldsymbol{r}) \equiv \big(\neg(\neg p) \vee (q \rightarrow r)\big) \equiv p \vee (\neg q \vee r) \equiv \boldsymbol{p} \vee \neg \boldsymbol{q} \vee \boldsymbol{r} \qquad \textbf{(1)}$$

$$\boldsymbol{q} \rightarrow (\boldsymbol{p} \vee \boldsymbol{r}) \equiv \big(\neg q \vee (p \vee r)\big) \equiv \neg q \vee p \vee r \equiv \boldsymbol{p} \vee \neg \boldsymbol{q} \vee \boldsymbol{r} \qquad \textbf{(2)}$$

From (1) and (2) above, the equivalence is established.

Question 2: **[2+3+1+2+2] Marks**

Let $\boldsymbol{C(x)}$ be the statement "x has a cat", let $\boldsymbol{D(x)}$ be the statement "x has a dog", and let $\boldsymbol{F(x)}$ be the statement "x has a ferret". Express each of these statements in terms of $C(x), D(x), F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

No student in your class has a cat, a dog, and a ferret.

$$\neg\exists x\big(C(x) \land D(x) \land F(x)\big) \equiv \forall x\big(\neg C(x) \lor \neg D(x) \lor \neg F(x)\big)$$

For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

$$\exists x C(x) \land \exists x D(x) \land \exists x F(x)$$

Let $\boldsymbol{P(x)}$ be the predicate "x is a predator," $\boldsymbol{Q(x)}$ be the predicate "x is a prey," and $\boldsymbol{R(x, y)}$ be the predicate "x has killed y," where *the **domain** consists of all animals in a jungle*. Use **quantifiers** to express each of these statements.

Some predator has not killed any prey.

$$\exists x(P(x) \land \forall y(Q(y) \to \neg R(x, y)))$$

There is a prey who has never been killed by a predator.

$$\exists y(Q(y) \land \forall x(P(x) \to \neg R(x, y)))$$

Some predator has killed every prey.

$$\exists x(P(x) \land \forall y(Q(y) \to R(x, y)))$$

$$\forall y(Q(y) \to \exists x(P(x) \land R(x, y)))$$

Express the *negation* of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.

$$\neg(\forall x \exists y(xy = 1)) \equiv \exists x \neg\exists y(xy = 1) \equiv \exists x \forall y \neg(xy = 1) \equiv \exists x \forall y(xy \neq 1)$$

Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators (*the domain consists of* ***all*** *real numbers*).

Every **positive** real number has exactly two square roots.

$$\forall x(x > 0 \rightarrow (\exists a \exists b(a \neq b \land \forall c(c^2 = x \leftrightarrow c = a \lor c = b))))$$

A **negative** real number does not have a square root that is a real number.

$$\forall x(x < 0 \rightarrow \neg \exists y(y^2 = x))$$

Determine the **truth value** of each of these statements if the domain of each variable consists of all real numbers.

$\forall x \exists y(x^2 = y)$ **True** / False

$\forall x \exists y(x = y^2)$ True / **False**

Question 3: **[4+6] Marks**

Use rules of inference to show that the hypotheses "If I do not work hard or if I am not lucky, then I will perform poor and I will get bad grade," "If I will get bad grade, then I will get a warning," and "I did not get warning" imply the conclusion "I worked hard."

Use the following propositional variables in your *argument*:

$\boldsymbol{p} :=$ I work hard, $\boldsymbol{q} :=$ I am lucky, $\boldsymbol{r} :=$ I will perform poor, $\boldsymbol{s} :=$ I will get bad grade, $\boldsymbol{t} :=$ I will get a warning

1	$(\neg p \lor \neg q) \to (r \land s)$	Premise
2	$(\neg p \lor \neg q) \to s$	Simplification using (1)
3	$s \to t$	Premise
4	$\neg t$	Premise
5	$\neg s$	Modus Tollens using (3) & (4)
6	$\neg(\neg p \lor \neg q)$	Modus Tollens using (2) & (5)
7	$\neg\neg p \land \neg\neg q$	Application of De Morgan's Law on (6)
8	$p \land q$	Application of Double negation Law on (7)
9	p	Simplification using (8)

We can break down **Step 2** further as following:

$$(\neg p \lor \neg q) \to (r \land s) \equiv \big((\neg p \lor \neg q) \to r\big) \land \big((\neg p \lor \neg q) \to s\big)$$

and then apply simplification on the R.H.S of the above equivalence resulting in

$$(\neg p \lor \neg q) \to s$$

What rule of inference is used in each of these arguments?

Note: Just mention the rule; you are not required to show any working.

1. Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

   Simplification

2. Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

   Modus ponens

3. It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.

   Disjunctive syllogism

4. A convertible car is fun to drive. Isaac's car is convertible. Therefore, Isaac's car is fun to drive.

   Universal *Modus ponens*

5. Quincy likes only action movies. Quincy likes the movie *Eight Men Out*. Therefore, *Eight Men Out* is an action movie.

   Universal *Modus ponens*

6. Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.

   Universal *Modus tollens*

Question 4: **[4+4+2] Marks**

Prove the distributive law $\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{A} \cap \boldsymbol{C})$ without using membership table.

Note: You can prove the above mentioned Law using membership table to get **50% marks**.

$$\begin{aligned}
\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C}) &= \{x | x \in A \land (x \in B \cup C)\} \\
&= \{x | x \in A \land (x \in B \lor x \in C)\} \\
&= \{x | (x \in A \land x \in B) \lor (x \in A \land x \in C)\} \\
&= \{x | x \in A \cap B \lor x \in A \cap C\} \\
&= \left\{x \middle| x \in \big((A \cap B) \cup (A \cap C)\big)\right\} \\
&= (\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{A} \cap \boldsymbol{C})
\end{aligned}$$

Note that besides the definitions of union, intersection, set membership, and set builder notation, this proof uses the distributive law for conjunction over disjunction for logical equivalences.

Find $\boldsymbol{f \circ g}$ and $\boldsymbol{g \circ f}$, where $\boldsymbol{f(x) = x^3 + 1}$ and $\boldsymbol{g(x) = x^2 + 2}$, are functions from $\mathbb{R}$ to $\mathbb{R}$.

$$(\boldsymbol{f \circ g})(\boldsymbol{x}) = f\big(g(x)\big) = f(x^2 + 2) = (x^2 + 2)^3 + 1 = \boldsymbol{x^6 + 6x^4 + 12x^2 + 9}$$

$$(\boldsymbol{g \circ f})(\boldsymbol{x}) = g\big(f(x)\big) = g(x^3 + 1) = (x^3 + 1)^2 + 2 = \boldsymbol{x^6 + 2x^3 + 3}$$

Determine whether the function $f{:}\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is bijective if $f(m, n) = m + n$.

It's not; e.g. 7 has more than one pre-images, (5,2), (4,3) etc.

Question 5: **[3+2+(1+2)+2] Marks**

List the first 5 terms of each of these sequences.

The sequence whose nth term is the sum of the first n positive integers

$a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10, a_5 = 15$

The sequence whose nth term is $n! - 2^n$

$a_1 = -1, a_2 = -2, a_3 = -2, a_4 = 8, a_5 = 88$

The sequence whose nth term is the largest integer whose binary expansion has n bits (Write your answer in decimal notation.)

$a_1 = 1, a_2 = 3, a_3 = 7, a_4 = 15, a_5 = 31$

Using an iterative approach, find the solution to the following recurrence relation and initial condition:

$a_n = na_{n-1}, \qquad a_0 = 5$

$$
\begin{aligned}
&a_1 = 1 \cdot 5 = 5 \\
&a_2 = 2 \cdot (1 \cdot 5) = (2 \cdot 1) \cdot 5 \\
&a_3 = 3 \cdot \big((2 \cdot 1) \cdot 5\big) = (3 \cdot 2 \cdot 1) \cdot 5 \\
&\vdots \\
&a_n = (n \cdot n - 1 \cdot \ldots \cdot 1) \cdot 5 = n! \cdot 5 = 5(n!)
\end{aligned}
$$

A person deposits $1000 in an account that yields 9% interest compounded annually.

Set up a recurrence relation for the amount in the account at the end of n years.

$$P_n = P_{n-1} + 0.09P_{n-1} = 1.09P_{n-1}$$

Find an explicit formula for the amount in the account at the end of n years.

$$\begin{aligned}
&P_0 = 1000 \\
&P_1 = (1.09)P_0 \\
&P_2 = (1.09)P_1 = (1.09)^2 P_0 \\
&P_3 = (1.09)P_2 = (1.09)^3 P_0 \\
&\vdots \\
&P_n = (1.09)P_{n-1} = (1.09)^n P_0
\end{aligned}$$

What is the value of the following sum?

$$\sum_{i=1}^{100} \sum_{j=i}^{100} 5$$

$$\sum_{i=1}^{100} \sum_{j=i}^{100} 5 = 5 \sum_{i=1}^{100} \sum_{j=i}^{100} 1 = 5 \sum_{i=1}^{100} (100 - i + 1) = 5(100 + 99 + \cdots + 1) = \frac{5 \cdot 100 \cdot 101}{2}$$

$$= \mathbf{25250}$$

Question 6: **[5+5] Marks**

Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

Suppose that $m+n$ and $n+p$ are even. Then $m+n = 2s$ for some integer s and $n+p = 2t$ for some integer t. If we add these, we get $m+p+2n = 2s +2t$.

Subtracting $2n$ from both sides and factoring, we have $m+p = 2s+2t - 2n = 2(s + t - n)$. Because we have written $m + p$ as 2 times an integer, we conclude that $m + p$ is even.

Prove that $\sqrt{3}$ is irrational by giving a proof by contradiction. You may use lemma 1 in your proof where needed.

Lemma 1: If n^2 is a multiple of 3 then n is also a multiple of 3.

Let$p$ be the proposition "$\sqrt{3}$ is irrational." To start a proof by contradiction, we suppose that $\neg p$ is true. Note that $\neg p$ is the statement "It is not the case that $\sqrt{3}$ is irrational," which says that $\sqrt{3}$ is rational. **We will show that assuming that $\neg p$ is true leads to a contradiction.**

If $\sqrt{3}$ is rational, there exist integers a and b with $\sqrt{3} = a/b$, where $b \neq 0$ and a and b have no common factors (so that the fraction a/b is in lowest terms.) Because $\sqrt{3} = a/b$, when both sides of this equation are squared, it follows that

$$3 = \frac{a^2}{b^2} \Rightarrow 3b^2 = a^2$$

Hence, a^2 is a multiple of 3. Using the ***Lemma 1***, it follows that a must also be a multiple of 3. Now, if a is a multiple of 3 then a can be written as $3c$ (where c is an integer). Thus

$$3b^2 = 9c^2$$

Dividing both sides of this equation by 3 gives

$$b^2 = 3c^2$$

Hence, b^2 is a multiple of 3. Once again, using the ***Lemma 1***, it follows that b must also be a multiple of 3.

We have now shown that the assumption of $\neg p$ leads to the equation $\sqrt{3} = a/b$, where a and b have no common factors, but both a and b are multiple of 3, that is, 3 divides both a and b. Note that the statement that $\sqrt{3} = a/b$, where a and b have no common factors, means, in particular, that 3 does not divide both a and b. Because our assumption of $\neg p$ leads to the contradiction that 3 divides both a and b and 3 does not divide both a and b, $\neg p$ must be false. That is, the statement p, "$\sqrt{3}$ is irrational," is true. We have proved that $\sqrt{3}$ is irrational.

Question 7: **[5+5] Marks**

An odd number of people stand in a yard at mutually distinct distances. At the same time each person throws a pie at their nearest neighbor, hitting this person. Use **mathematical induction** to show that there is at least one survivor, that is, at least one person who is not hit by a pie.

BASIS STEP: When $n = 1$, there are $2n + 1 = 3$ people in the pie fight. Of the three people, suppose that the closest pair are A and B, and C is the third person. Because distances between pairs of people are different, the distance between A and C and the distance between B and C are both different from, and greater than, the distance between A and B. It follows that A and B throw pies at each other, while C throws a pie at either A or B, whichever is closer. Hence, C is not hit by a pie. This shows that at least one of the three people is not hit by a pie, completing the basis step.

INDUCTIVE STEP: For the inductive step, assume that $P(k)$ is true for an arbitrary odd integer k with $k \geq 3$. That is, assume that there is at least one survivor whenever $2k + 1$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor. We must show that if the inductive hypothesis $P(k)$ is true, then $P(k + 1)$, the statement that there is at least one survivor whenever $2(k + 1) + 1 = 2k + 3$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor, is also true.

So suppose that we have $2(k + 1) + 1 = 2k + 3$ people in a yard with distinct distances between pairs of people. Let A and B be the closest pair of people in this group of $2k + 3$ people. When each person throws a pie at the nearest person, A and B throw pies at each other. We have two cases to consider, *(i)* when someone else throws a pie at either A or B and *(ii)* when no one else throws a pie at either A or B.

Case (i): Because A and B throw pies at each other and someone else throws a pie at either A and B, at least three pies are thrown at A and B, and at most $(2k + 3) - 3 = 2k$ pies are thrown at the remaining $2k + 1$ people. This guarantees that at least one person is a survivor, for if each of these $2k + 1$ people was hit by at least one pie, a total of at least $2k + 1$ pies would have to be thrown at them. (The reasoning used in this last step is an example of the pigeonhole principle discussed further in Section 6.2.)

Case (ii): No one else throws a pie at either A and B. Besides A and B, there are $2k + 1$ people. Because the distances between pairs of these people are all different, we can use the inductive hypothesis to conclude that there is at least one survivor S when these $2k + 1$ people each throws a pie at their nearest neighbor. Furthermore, S is also not hit by either the pie thrown by A or the pie thrown by B because A and B throw their pies at each other, so S is a survivor because S is not hit by any of the pies thrown by these $2k + 3$ people.

We have completed both the basis step and the inductive step, using a proof by cases. So by mathematical induction it follows that $P(n)$ is true for all positive integers n. We conclude that whenever an odd number of people located in a yard at distinct mutual distances each throws a pie at their nearest neighbor, there is at least one survivor. ◀

Use mathematical induction to prove that

$$1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3$$

whenever n is a nonnegative integer.

Basis step:

$P(0)$ is true because

$$1^2 = 1 = (0 + 1)(2 \cdot 0 + 1)(2 \cdot 0 + 3)/3.$$

Inductive step:

Assume that $P(k)$ is true. Then

$$\begin{aligned}
1^2 + 3^2 + \cdots + (2k+1)^2 + [2(k+1)+1]^2 &= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 \\
&= (2k+3)\left[\frac{(k+1)(2k+1)}{3} + (2k+3)\right] \\
&= \frac{(2k+3)(2k^2+9k+10)}{3} \\
&= \frac{(2k+3)(2k+5)(k+2)}{3} \\
&= \frac{[(k+1)+1][2(k+1)+1][2(k+1)+3]}{3}
\end{aligned}$$

Question 8: **[3+4+3] Marks**

Find the error in the "proof" of the following "theorem."

> "*Theorem*": Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.
>
> "*Proof* ": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

If no such $b \in A$ exists ***such that*** $(a, b) \in R$, then the proof would collapse.

Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.

{*(0, 0), (1, 1), (2, 2), (3, 3)*}

Equivalence Relation

{*(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)*}

Not **Reflexive** [(1,1) is missing]; Not **Transitive** [(0,2) and (2,3) are in the relation but (0,3) is missing].

{*(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)*}

Not **Transitive** [(1,3) and (3,2) are in the relation but (1,2) is missing; moreover, (2,3) and (3,1) are in the relation but (2,1) is missing]

{*(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)*}

Equivalence Relation

Let R and S be relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad , \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrix representing $R \circ S$

$$M_{R \circ S} = M_S \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Question 9: **[2+3+1.5+1.5+1+1] Marks**

A multiple-choice test contains 10 questions. There are four possible answers for each question.

In how many ways can a student answer the questions on the test if the student answers every question?

$$4^{10}$$

In how many ways can a student answer the questions on the test if the student can leave answers blank?

$$5^{10}$$

How many bit strings of length seven either begin with two 0s or end with three 1s?

$$2^5 + 2^4 - 2^2 = 32 + 16 - 4 = \mathbf{44}$$

What is the minimum number of students, each of whom comes from one of the 37 states, who must be enrolled in a university to guarantee that there are at least 55 who come from the same state?

$$\min_x \left(\left\lceil \frac{x}{37} \right\rceil = 55\right) \Rightarrow x = \mathbf{1999}$$

There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

18

A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate and the last person in the row is always a man?

$$\boldsymbol{(n!)^2}$$

What is the coefficient of $x^{12}y^9$ in the expansion of $(3x^2 - 2y)^{15}$?

$$\binom{15}{6}(3x^2)^6(-2y)^9 = 5005 \cdot 729x^{12}(-512y^9) = -\mathbf{1868106240}x^{12}y^9$$

Question 10: **[1+2+1.5+1.5+4] Marks**

Can a simple graph exist with 15 vertices each of degree five? Justify your answer with proper reasoning.

Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

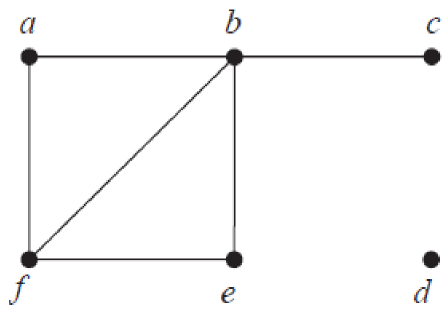
For which values of n are these graphs bipartite?

K_n

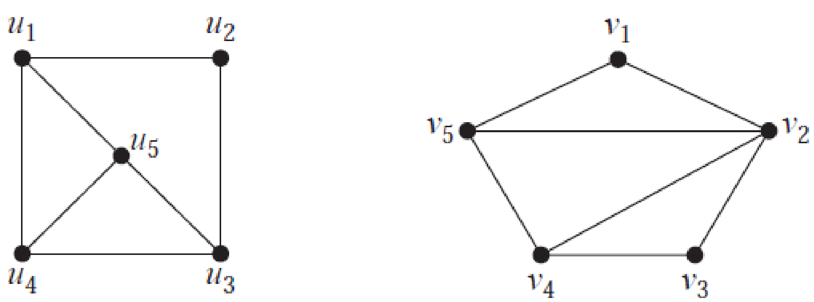
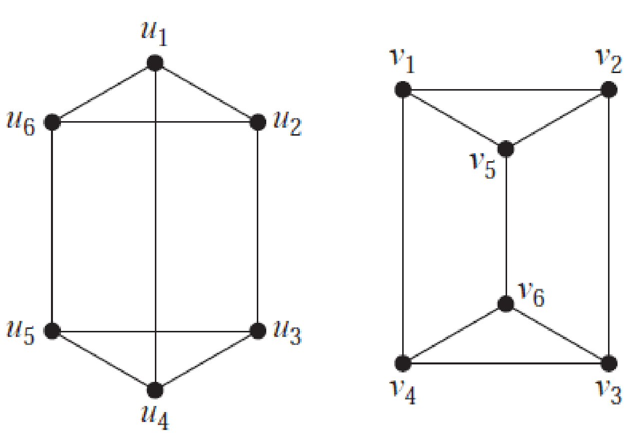
C_n

W_n

For the graph given below find the subgraph induced by the vertices *a*, *b*, *c*, and *f* .

	***Draw your graph here:***

Determine whether the given pairs of graphs are *isomorphic*. Exhibit an *isomorphism* or provide a rigorous argument that none exists.

	***Answer:***
	***Answer:***