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Date:

DISCRETE STRUCTURES

Lecture 1

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Date: 22-8-2022

talk about

Book1 : Discrete Mathematics and Applications
By Susanna S.E 5th Edition

Book2 : Discrete Mathematics and its Applications
By Keath H Rosen 7th Edition

Q. What is mathematics?

Mathematics is all about finding structure and making assumptions and playing around with these assumptions and seeing what happens.

Discrete Mathematics : Study of mathematical structures that can assume distinct and separated values.

Chapter #2 : Logic of compound statements

Statement : a sentence which is either true or false
e.g Today is Sunday → False

open sentence : $x+y > 2$ could be true, could be false

proposition (symbolizing the statement) : The letters can be used to denote statement or proposition. Generally we take p, q, r, w as statement variables.

compound statement :
Logical connectives : There are 3 basic compound ~~con~~ statement.

$\sim$ $\neg$ negative

It is raining outside and it is cloudy
$p \wedge q$

$\wedge$ and , conjunction

$\vee$ or , disjunction

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cartesian product: A (1,2) B (a,b)

A x B = (1a, 1b, 2a, 2b)

Negation: If p is statement variable. Negation of p is "not p" or "this is not the case p". Denoted by ~p.

p	~p
T	F
F	T

The truth value of ~p is different from p. If p is true ~p is false and vice versa.

Conjunction: Let p and q be statement variables. The conjunction of p and q is "p and q". Denoted by p∧q. The conjunction is true when both are true otherwise false.

p	q	p∧q	p∨q
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Disjunction: The disjunction of p and q is "p or q" denoted by p∨q. The disjunction is false when both are false otherwise true.

Lecture # 2

Statement form:

statement form or propositional form is an expression made up of statement variables (such as p, q, r, s) and logical connectives (~, ∧, ∨) that becomes statement when actual statements are substituted for each possible combination of statement variables.

* The truth table for given statement form displays values that corresponds all possible combinations of statement variables.

$(p \vee q) \wedge \sim (p \wedge q)$

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$(p \vee q) \wedge \sim (p \wedge q) \rightarrow$ exclusive OR / XOR $\equiv P$

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	P
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

* Two or more statement forms are logically equivalent if they have identical truth values for each substitution of actual statements for their statement variables.

$P = q \vee p$ and $Q = p \vee q$ then $P \equiv Q$

* $R = \sim (p \wedge q)$ $\quad R \not\equiv S$ $\quad$ * $\sim(p \wedge q) = \sim p \vee \sim q$

$S = \sim p \wedge \sim q$ $\quad \sim(p \vee q) = \sim p \wedge \sim q$

* $\wedge \equiv \cap$

$\sim \equiv /$

$\vee \equiv \cup$

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

* $x \leq 3$ $\quad$ * $\underbrace{x < 3 \quad\quad * \; x \geq 3}_{\text{negation}}$

x is atmost three $\quad$ x is at ... 3

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* $\underbrace{\text{John is 6 feet tall}}_{p}$ and ($\wedge$) $\underbrace{\text{he weighs at least 200 pounds}}_{q}$

$\sim(p \wedge q) = \sim p \vee \sim q$

John is not 6 feet tall or he weighs less than 200 pounds

* but = and , nor = and

* Tautology : which is true for each substitution of actual statements for their statement variables.

e.g

$\forall\ x \in \mathbb{R}, \quad x^2 \geq 0$

proof : $x > 0$

$x \cdot x > x \cdot 0$

$x^2 > 0$

1890

A tautology is a statement form that is always true regardless the truth values of individual statements substituted for its statement variables. A statement whose form is tautology is tautological form.

Lecture 3

Contradiction :

A contradiction is a statement form that is always true regardless the truth values of individual statements substituted for its statement variables. A statement whose form is contradict is contradictive form.

		C	T
p	$\sim p$	$p \wedge \sim p$	$p \vee \sim p$
T	F	F	T
F	T	F	T

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Logical equivalences: Theorem 2.1.1

Given the statement variables p, q, r, a tautology t, and a contradiction c, the following logical equivalence holds.

1. Commutative laws

$p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$

2. Associative laws

$p \vee (q \vee r) \equiv (p \vee q) \vee r$, $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

3. Distributive laws

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

4. Identity laws

$p \wedge t \equiv p$, $p \vee c \equiv p$

5. Negation laws

$p \vee \sim p \equiv t$, $p \wedge \sim p \equiv c$

6. Double negation laws

$\sim(\sim p) \equiv p$

7. Idempotent laws

$p \vee p \equiv p$, $p \wedge p \equiv p$

8. Universal bound law

$p \vee t \equiv t$, $p \wedge c \equiv c$

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9. De Morgan's law

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

10. Absorption law

$p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

11. Negation of t and C

$\sim t \equiv C$, $\sim C \equiv t$

* $p \oplus p \equiv C$

order of operation

$\sim, \vee, \wedge, \rightarrow, \leftrightarrow$

Q

$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

L.H.S

$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$ by De Morgan's law

$\equiv (p \vee \sim q) \wedge (p \vee q)$ Double negation law

$\equiv p \vee (\sim q \wedge q)$ Distributive law

$\equiv p \vee C$ Negation law

$\equiv p$ Identity law

If you show up on Monday morning then you will get a job.

Second part is conditional by first part.

* The statement can be false when 1st part is true and second part does not occur.

* If first part does not occur the statement still can be true.

example,

If a number is divisible by 6, then it is divisible by 3.

$6/N$

$N = 6K$

$= 3.2K \rightarrow$ true

if $N = 15$, first part is not true but statement is still true. In this case statement is true by default.

* If both are false, answer is true.

* answer is only false when 1st statement is true and second state does not occur.

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Conditional statement:

Let p and q are statement variables the conditional of q by p is "p implies q" denoted by $p \to q$. It is false when p is true and q is false otherwise true. We call p hypothesis of conditional statement and q conclusion or consequent.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional statement that is true when its premsis is false (is true by default or vacuously true)

① $p \to q \equiv \sim p \vee q$

$\sim(p \to q) = \sim(\sim p \vee q)$

$= p \wedge \sim q$

② $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r) \equiv (\sim p \vee r) \wedge (\sim q \vee r)$

* "p implies q".
* "p is sufficient for q"
* "q is necessary for p"
* $p \to q$

* q if p
* $p \to q$

Contrapositive:

The contrapositive of conditional statement of the form "p implies q" is "not q implies not p".

Symbolically:

contrapositive of $p \to q$ is $\sim q \to \sim p$

Remark:

The conditional statement is logically equivalent to its contrapositive

$p \sim q \equiv \sim q \sim \sim p$

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Converse and Inverse of conditional statement:

Given the conditional statement of the form "p implies q"

1. The converse is "q implies p"
2. The inverse is not p implies not q.

Symbolically:

The converse of $p \to q$ is $q \to p$

The inverse of $p \to q$ is $\sim p \to \sim q$

Remarks:

1. The conditional statement is not logically equivalent to its converse and inverse.
2. The converse and inverse of conditional statement are logically equivalent.

1. $p \to q \not\equiv q \to p$

$p \to q \not\equiv \sim p \to \sim q$

2. $q \to p \equiv \sim p \to \sim q$

Definition:

Let p and q are statements. p only if q means

$\sim q$ implies $\sim p$

equivalent $p \to q$

Biconditonal: The biconditional if p and q is "p if and only if q" denoted by $p \leftrightarrow q$. It is true when p and q have same truth values and false when they have opposite values.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Quiz 2.1 + 2.2

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Ex 2.2

Q3

"Freeze" or I'll shoot ($\sim p$ $\vee$ q)

If you don't freeze, then I'll shoot (p)

Q12

if $\underbrace{x>2}_{p}$ or ($\vee$) $\underbrace{x<-2}_{q}$ then ($\rightarrow$) $\underbrace{x^2>4}_{r}$

(if $x>2$ then $x^2>4$) and

(if $x<-2$ then $x^2>4$)

Q14

i) $p \rightarrow q \vee r$

ii) $p \wedge \sim q \rightarrow r$

iii) $p \wedge \sim r \rightarrow q$

logically equivalent

$p \rightarrow q \vee r \equiv \sim p \vee (q \vee r)$

$\equiv (\sim p \vee q) \vee r$

$\equiv \sim(\sim p \vee q) \rightarrow r$

$\equiv p \wedge \sim q \rightarrow r$

(i) $\equiv$ (ii)

Q15

$p \rightarrow (q \rightarrow r) \equiv p \rightarrow (\sim q \vee r)$

$\equiv \sim p \vee (\sim q \vee r)$

$(p \rightarrow q) \rightarrow r \equiv (\sim p \vee q) \rightarrow r$

$\equiv \sim(\sim p \vee q) \vee r$

$\equiv p \wedge \sim q \vee r$

so, both are not equivalent

Q16

1. If you paid full prize, you did not buy it at clown books
2. you did not buy it at clown books or you paid full prize

are they logically equivalent

1. $p \rightarrow q \equiv \sim p \vee q$
2. $\sim q \vee p$

Hence, no

Q29

Given

$P \equiv Q$

$P \leftrightarrow Q \equiv t$

Q24

$\underbrace{p \rightarrow (q \vee r)}_{P} \equiv \underbrace{(p \wedge \sim q) \rightarrow r}_{Q}$

$\sim p \vee (q \vee r) \equiv \sim(p \wedge \sim q) \vee r$

$\sim p \vee (q \vee r) \equiv (\sim p \vee q) \vee r$

p	q	r	q∨r	P	~p∨q	Q	P↔Q
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

$P \equiv Q$ if and only if $P \leftrightarrow Q \equiv t$

Q47-50

$p \to q \equiv \sim p \vee q$

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Leture 5

An argument is sequence of statements and argument form is sequence of statement forms. All statements in an argument and all statement form in an argument forms except the final one are called Premises or assumptions or hypothesis. The final statement or statement form is called conclusion. The symbol '∴' which is read 'therefore' is placed just before the conclusion.

An argument form is valid no matter what particular statements are substituted for statement variables, if the premises are true then conclusion is true.

premises $p \to q$
p
$\therefore q$ conclusion

Argument form

If sum of digits of 321 is divisible by 3, then number is divisible by 3. This is an example of valid argument.

Testing the validity of Argument forms:

1. Identify all the premises and conclusion.
2. construct the truth table showing the truth values of all premises and conclusion.
3. In a row in which all premises are true is called critical row.

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If in a row in which all premises are true but conclusion is false, the argument row is invalid. If in each critical row, conclusion is true, the argument form is valid.

premises

p	q	$p \to q$	
T	T	T	→ critical row, so valid argument form
T	F	F	
F	T	T	*q is conclusion
F	F	T	

$p \to q \vee \sim r \quad \Rightarrow P$

$q \to p \wedge r \quad \Rightarrow Q$

$p \to r$

p	q	r	$\sim r$	$q \vee \sim r$	✓p	$p \wedge r$	✓Q	conclusion $p \to r$	premises.
T	T	T	F	T	T	F	T	T	✓ critical row
T	T	F	T	T	T	T	F	F	
T	F	T	F	F	F	F	T	T	
T	F	F	T	T	T ✓	T	T ✓	F ×	this is an invalid argument form
F	T	T	F	T	T	F	F	T	
F	T	F	T	T	T	F	F	T	
F	F	T	F	F	T	F	T	T	✓ critical row
F	F	F	T	T	T	F	T	T	✓ critical row

Summary of Rules of Inferences:

1. Modus Ponens

$p \to q$

p

$\therefore q$

2. Modus Tollens

$p \to q$

$\sim q$

$\therefore \sim p$

3. Generalization

i) p

$\therefore p \vee q$

ii) q

$\therefore p \vee q$

4. Specialization

i) $p \wedge q$

$\therefore p$

ii) $p \wedge q$

$\therefore q$

5. Conjunction

p

q

$\therefore p \wedge q$

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6. Elimination

i) $p \vee q$
$\sim q$
$\therefore p$

ii) $p \vee q$
$\sim p$
$\therefore q$

7. Transitivity

$p \to q$
$q \to r$
$\therefore p \to r$

8. Proof division into cases

$p \vee q$
$p \to r$
$q \to r$
$\therefore r$

9. Contradiction Rule

$\sim p \to c$
$\therefore p$

Q31

a) $p \to q$
b) $r \to \sim q$
c) p
d) $r \vee s$
e) $t \to u$

Reasoning

1. $p \to q$ by premises (a)
p by premises (c)
q by premises (a), (c) and M.P

2. $r \to \sim q$ by premises (b)
q by result (1)
$\therefore \sim r$ by premises (b), (1) and M.T

3. $r \vee s$ by premises (d)
$\sim r$ by result (2)
$\therefore s$

$\Rightarrow$ the tresure is buried under the flag pole

Lecture # 6

Q42

a) $p \vee q$
b) $q \to r$
c) $p \wedge s \to t$
d) $\sim r$
e) $\sim q \to u \wedge s$
f) t $\Rightarrow$ we have to prove

i. $q \to r$ (b)
$\sim r$ (c)
$\therefore \sim q$ by (b), (c), Modes Tollens

ii. p by (a) and (1) and elimination

iii. ~~$u \wedge s$ by M. Ponent~~
$\sim q$ by (i)
$\therefore u \wedge s$ by M. Ponens

iv. $u \wedge s$ by (iii)
s by specialization

5. p by (ii)
   s by (iv)
∴ $p \wedge s$ conjunction

6. $p \wedge s \to t$ by (C)
   $p \wedge s$ by (v)
∴ t by M. Ponens

Digital logic circuits :

p	q	r	$p \wedge q$	$(p\wedge q)\wedge \sim r$
T	T	T	T	F
T	T	F	T	T
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

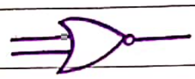
A recognizer is a circuit that outputs 1 for particular combination of input signals and 0 for rest

$\sim$ 

$\wedge$ 

$\vee$ 

NAND  $\sim(p \wedge q)$

NOR  $\sim(p \vee q)$

$P \oplus Q \equiv (P \wedge Q) \sim\wedge (P \vee Q)$

$\underbrace{P \mid Q}_{\text{for NAND}} \equiv \sim(P \wedge Q)$

| ← cheffer's stroke

$P \downarrow Q \equiv \sim(P \vee Q)$

P	Q	R	S	
1	1	1	1	$P \wedge Q \wedge R$
1	1	0	0	$\vee$
1	0	1	1	$P \wedge \sim Q \wedge R$
1	0	0	1	$\vee$ $P \wedge \sim Q \wedge \sim R$
0	1	1	0	
0	1	0	0	
0	0	1	0	
0	0	0	0	

recognizer

Lecture # 8

Chapter # 3

Logic of quantified Statements

Date:

Predicate :
or propositional function

A predicate is statement that contains a finite number of variables and becomes statement when specific values are satisfied for the variables.

The domain of predicate variable is set of all values that may be satisfied for variable.

Example,

$P(x)\ x \geq 1/x \ , x \in \mathbb{R}$

Truth set :

Let $P(x)$ be predicate and x has domain D.

The truth set of $P(x)$ is set of all values of x in D that makes $P(x)$ true. The truth set of $P(x)$ is denoted by

$\{x \in D \mid P(x)\}$

Example,

$P(n) = n$ is factor of 12 , $n \in \mathbb{Z}^+$

P(5) F

P(1) T $\quad T = \{1, 2, 3, 4, 6, 12\}$

P(2) T

P(3) T $\quad$ is $n \in \mathbb{Z}$

P(4) T $\quad T = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$

P(6) T

P(12) T

Universal statement :

Let $P(x)$ is predicate and D is domain of x. The universal statement is statement of the form "$\forall x \in D, P(x)$"

It is true if and only if $Q(x)$ is true for all x in D.

It is false if and only if there is at least one x in D for which $Q(x)$ is false.

The value x for which $Q(x)$ is false is called counter example to the universal statement.

$Q(x) : \forall x \in \mathbb{R} \ , x^2 \geq 0$

~~Q(x) is true for all x belongs to real number where square of x is greater than or equal to 0.~~

Every real number has non negative square.

Existential statement :

$\exists$,s there exists

existential statement

An existential statement is the statement of the form

"$\exists x \in D , P(x)$"

It is defined to be true if and only if there is at least one x in D for which $P(x)$ is true.

It is false if and only if $P(x)$ is false for every x in D.

v.nice

v.v.v nice

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$\frac{2}{3}$

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example,

$P(x) : \exists x \in \mathbb{R}, x \geq 1/x$

$\boxed{x=1}$

$\Rightarrow$ true

* universal statement example se prove ni hoti.

Notation:

Let $P(x)$ and $Q(x)$ be predicates with D as common domain of x.

$P(x) \Rightarrow Q(x)$ means the truth value of $P(x)$ is the truth set of $Q(x)$.

Equivalently:

$\forall x, P(x) \rightarrow Q(x)$

$P(x) \Leftrightarrow Q(x)$ means $P(x)$ and $Q(x)$ has conditional truth set

Equivalently:

$\forall x, P(x) \leftrightarrow Q(x)$

3.1

Q3

$R(m, n)$: if m is factor of n^2 then m is factor of n $\quad m, n \in \mathbb{Z}$

$R(m,n)$ False, when $m=25, n=10$

why?

because $\underset{\overset{\uparrow}{T}}{p} \rightarrow \underset{\overset{\uparrow}{F}}{q}$

Q10

$\forall a \in \mathbb{Z}$ $\frac{a-1}{a}$ is not an integer

Q12

$\forall x, y \in \mathbb{R}$ $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

Q19

$\forall n \in \mathbb{Z}$ if n^2 is even then n is even

Lecture # 9

* $p \rightarrow q \equiv \sim q \rightarrow \sim p$
$\equiv \sim p \vee q$

* $\sim(p \rightarrow q) \equiv p \wedge \sim q$

Negation of universal statement:

The negation of a statement of the form

$\forall x \in D, Q(x)$

is logically equivalent of a statement of the form

$\exists x \in D, \sim Q(x)$

Symbolically

$\sim(\forall x \in D, Q(x)) \equiv \exists x$ in $D, \sim Q(x)$

Negation of existential statement:

The negation of a statement of the form

$\exists x \in D, Q(x)$

$m=3 \quad n=5 \quad m=16 \quad n=8$

Pr

NANDOS ~~$p \to q$~~

$q \to p \equiv \sim p \to q$

$\sim(\sim p \wedge q) \quad p \vee \sim q.$

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Symbolically
$\sim(\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x)$

Relation among $\forall, \exists, \wedge, \vee$

$Q(x)$ is predicate $D: \{x_1, x_2, \dots x_n\}$

relation b/w $\forall$ and $\wedge$

$\forall x \in D, Q(x)$

$Q(x_1) \wedge Q(x_2) \dots \wedge Q(x_n) \to$ universal statement

relation b/w E and V

$Q(x_1) \vee Q(x_2) \dots \vee Q(x_n) \to$ existential statement

Negation of universal conditional statement:

$\sim(\forall x \in D$ if $P(x)$ then $Q(x) \equiv \exists x$ in D $P(x)$ and $\sim Q(x)$

consider the statement of the form
$\forall x \in D$ if $P(x)$ then $Q(x)$
Its contrapositive is
$\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$
Its converse is
$\forall x \in D$, if $Q(x)$ then $P(x)$
Its inverse is
$\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$

* $\forall x \in D, r(x)$ is sufficient for $S(x)$ means
$\forall x \in D \quad r(x) \to S(x)$

* $\forall x \in D, r(x)$ is necessary for $S(x)$ means
$\forall x \in D \quad \sim r(x) \to \sim S(x) \equiv S(x) \to r(x)$

* $\forall x \in D, r(x)$ only if $S(x)$ means
$\forall x \in D, \quad r(x) \to S(x)$

Ex 3.2

Q1

All discrete students are athletic

$\sim(\forall x$, if x is discrete mathematic student then x is an athletic)

$\exists x$, x is discrete mathematic student and is not an athletic

There exists a discrete mathematic student who is not an athletic.

Mid 1 syllabus :
2.1 – 2.4 , 3.1 – 3.2

$\boxed{p \to \sim q}$ $\sim p \vee q$
$q \to \sim p$ $\boxed{p \wedge \sim q}$

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Lecture # 10

Q46

Having a large income is not necessary for a person to be happy.

Formal version:

$\sim$($\forall x$, if a person x has not large income then x is not happy)

$\cong$ ($\forall x$, if a person x is happy then x has large income)

$\equiv \exists x, P(x) \wedge \sim Q(x)$

$\Rightarrow \exists x$, x is happy and x has not large income

$\Rightarrow$ There is person who is happy not having large income.

Ex 3.4

All human beings are mortal,

$\forall x$ if x is human being then x is mortal

Muaaz is human being

$\therefore$ Muaaz is mortal

Universal Instantiation:

If a property is true for all element in the set then it is true for any particular element of the set.

Universal Modus Ponens

Informal version
If x makes $P(x)$ true x makes $Q(x)$ true

$\forall x, P(x) \to Q(x)$

$P(a) \to$ a makes $P(x)$ true

$Q(a) \to$ a makes $Q(x)$ true

Universal Modus Tollens

$\forall x, P(x) \to Q(x)$

$\sim Q(a) \to$ a does not make $Q(x)$ true

$\therefore \sim P(a) \to$ a does not make $P(x)$ true

Taimoor is not mortal

Taimoor is not human being

Universal transitivity :

$\forall x, \quad P(x) \to Q(x)$

$\forall x, \quad Q(x) \to R(x)$

$\therefore \forall x, \quad P(x) \to R(x)$

Invalid cases :

$p \to q$	$p \to q$
q	$\sim p$
$\therefore p$	$\therefore \sim q$
converse error	inverse error

All rational numbers are real numbers
(p above "rational numbers", q above "real numbers")

$p \to q$

$\sim q$

$\sim p$ $\therefore$ modus tollens

$\frac{1}{0}$ is not real

$\Rightarrow \frac{1}{0}$ is not rational

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$+m > +n$ is m^2 composite

start stop
stop start
IS

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Date:

Chapter # 4

4.1. Method of direct proof and counter example

Definition:

An integer n is even iff n equals twice some integer. An integer n is odd iff n equals twice some integer plus 1.

Symbolically for each integer n:

- n is even $\Leftrightarrow$ $n = 2K$ for some integer K
- n is odd $\Leftrightarrow$ $n = 2K+1$ for some integer K

Definition:

An integer n is prime iff, $n>1$ and ifor all integer r and s if $n = r.s$ then either r and s equals n. An integer n is composite iff $n>1$ and $n = r.s$ for some integers r and s with $1<r<n$, $1<s<n$

Symbolically for any integer n:

- n is prime $\Leftrightarrow$ $\forall$ positive integers r and s if $n = r.s$ then either $r=n$ and $s=1$ and $r=1$ and $s=n$
- n is composite $\Leftrightarrow$ $\exists$ r and s such that $n = r.s$ with $1<r<n$, $1<s<n$

(2) • Method of exhaustion
(one by one check)

- For each integer n, $1 \le n \le 10$ then n^2-n+11 is prime
- The sum of two odd integers is even

- Generalising from particular generic

To prove that certain property is true for each element of set, suppose x is particular but arbitrary chosen element of the set and show that x satisfies this property

Method of Direct Proof:

I) Express the statement to be proved into the form '$\forall x \in D$ if $P(x)$ then $Q(x)$'

II) Suppose x is particular but arbitrary chosen element of domain for which $P(x)$ is true (this step is often done mentally)

III) [Suppose $x \in D$, $P(x)$]
Prove that $Q(x)$ is true by definition

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- The sum of any two odd integers is even

Following are wrong ways of proving it

1. Arguing from examples

$n = 3$

$n = 5$

$m + n = 8$ Proved

2. Using some letters to mean two different things

Let m and n be odd

$m = 2k+1$

$n = 2k+1$

3. Jumping to the conclusion

$\left.\begin{array}{l} m = 2k+1 \\ n = 2k+1 \end{array}\right\}$ given

$\therefore$ m an n are odd then $m+n$ are even

4. Assuming what is to be present

suppose $m+n$ is even

$\therefore$ $m+n = 2k$

4.4

Divisibility

Definition:

If n and d are integers then n is divisible by d iff n equals d times some integer and $d \neq 0$, instead of "n is divisible by d" we can say that:

n is multiple of d OR

d is factor of n OR

d is divisor of n OR

d divides n

The notation d/n is read as "d divides n"

Symbolically, if d and n are integers

$d/n \Leftrightarrow \exists$ some integer K such that $n = dK$, $d \neq 0$

The notation $d \nmid n$ is read as "d does not divide n"

Theorem 4.4.1: Positive divisors of positive integers

For all integers a, b if a and b are +ve and a/b then $a \leq b$

Theorem 4.4.2: Divisors of 1

The divisors of 1 are -1 and 1

$1 = (1)(1)$

$1 = (-1)(-1)$

Theorem 4.4.3: Transitivity of divisibility

of a/b and b/c then a/c

Proof:

$\because$ a/b and b/c

$\therefore$ $b = aK_1$ —(i)

$c = bK_2$ —(ii)

from (i) and (ii)

$c = (aK_1)K_2$

$c = a(K_1K_2)$

$c = aK$

$\therefore$ a/c

Q15

Proof:

$\because a|b$ and $b|c$

$\therefore b = aK_1$ —(i)

$c = aK_2$ —(ii)

(i) + (ii)

$b+c = a(K_1+K_2)$

$b+c = a(K)$

$\therefore a | b+c$

Q16

Proof:

$\because a|b$ and $b|c$

$\therefore b = aK_1$ —(i)

$c = aK_2$ —(ii)

(i) — (ii)

$b-c = a(K_1-K_2)$

$b-c = a(K)$

$\therefore a | b-c$

Theorem 4.4.4

For any integer $n>1$ is divisible by prime

Proof: H.W

Theorem 4.4.5

Unique factorization of integer theorem

OR

Fundamental theorem of Arithmetics

Given any integer $n>1$, there exist positive integer K, distinct primes $P_1, P_2, \dots P_K$ and positive $e_1, e_2, \dots e_K$ such that $n = P_1^{e_1} \cdot P_2^{e_2} \cdots P_K^{e_K}$

any other factorization of n is identical to this except, perhaps, the order in which factors are written

$10 = 5 \cdot 2$

$10 = 2 \cdot 5 \leftarrow$ standard form (ascending order)

4.5

Quotient Remainder Theorem:

Given any integer n and a positive integer d, there exists unique integers q and r such that:

$\underset{\text{dividend}}{n} = \underset{\text{divisor}}{d}\,\underset{\text{quotient}}{q} + \underset{\text{remainder}}{r}$

$0 \le r < d$

* remainder r lies between 0 and $d-1$.

Definition:

Given any integer n and d, $d>0$

n div d: integer quotient is obtained when n is divided by d

n mod d: integer remainder is obtained when n is divided by d

تخصیص

~~k + k+1~~ $n-1+n+1+n$

$3 \mid n$

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Symbolically, if n and d are integers, $d>0$

n div $d = q$, and $n \bmod d = r \Leftrightarrow n = dq + r$

where q and r are integers and $0 \le r < d$

Parity:

The parity of an integer refers to whether integer is even or odd.

Theorem:

Any two consecutive integers have oppositive parity.

Proof:

$n, n+1$	$n = 2k$	$n = 2k-1$
$n-1, n$	$n+1 = 2k+1$	$n+1 = 2k$

Theorem:

The square of any odd integer has the form $8m+1$ for some integer m.

Proof:

let n is any odd integer

$n = 2k+1$

$n^2 = 4k^2 + 4k + 1$

Show that any integer has one of the following form

$4q, \; 4q+1, \; 4q+2, \; 4q+3$

Proof:

let n is integer, applying Q-R theorem to n with divisor 4

$n = 4q + r \quad , \; 0 \le r < 4 \qquad r = 0,1,2,3$

Let n is any odd integer

$n = 4q+1, \; 4q+3$

$\text{let } n = (4q+1)^2$

$= 16q^2 + 8q + 1$

$= 8[2q^2+q] + 1$

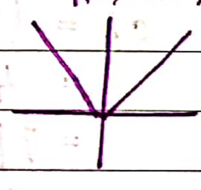
$= 8m + 1$

Absolute value of real numbers:

For any real number x, the absolute value of x denoted by $|x|$ defined as:

$$|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

$f(x) = |x|$



Lemma:

For any real x, $|x| = |-x|$

Proof:

$$|-x| = \begin{cases} -x, & -x > 0 \\ 0, & -x = 0 \\ -(-x), & -x < 0 \end{cases}$$

$$= \begin{cases} -x, & x < 0 \\ 0, & x = 0 \\ x, & x > 0 \end{cases}$$

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Lemma:

For any real r,

$-|r| \le r \le |r|$

$x = \frac{1}{2}$

Triangular and its property:

For any real numbers x and y

$|x+y| \le |x| + |y|$

Ex: 4.5

Q25

Prove that for all integers a and b, a mod 7 = 5 and b mod 7 = 6

a mod 7 = 5 $\rightarrow a = 7q_1 + 5$ —(i)

b mod 7 = 6 $\rightarrow b = 7q_2 + 6$ —(ii)

(ab) mod 7 = 2 $\rightarrow ab = 7k + 2$

From (i) and (ii)

$ab = (7q_1+5)(7q_2+6)$

$= 49q_1q_2 + 42q_1 + 35q_2 + 30$

$= 7[7q_1q_2 + 6q_1 + 5q_2 + 4] + 2$

$= 7k + 2$

Q40

If n is any odd integer then n^4 mod $16 = 1$

Ex: 4.6

Floor:

Given any real number x, the floor of x denoted by $\lfloor x \rfloor$ is defined as follows:

$\lfloor x \rfloor$ = that unique integer n such that $n \le x < n+1$

Symbolically, if x is any real number and n is an integer then

$\lfloor x \rfloor = n \Leftrightarrow n \le x < n+1$

Ceiling:

Given any real number x, the floor of x denoted by is defined as follows:

$\lceil x \rceil$ = that unique integer n such that $n-1 < x \le n$.

Symbolically, if x is any real number and n is an integer then

$\lceil x \rceil = n \Leftrightarrow n-1 < x \le n$

observe that:

i) $\lfloor x \rfloor \le x$

ii) $x < \lfloor x \rfloor + 1$

* $\lfloor x \rfloor$ and $\lceil x \rceil$ are equal only when x is an integer

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44 ~~[illegible]~~ 123

52 49 ~~49~~

$n = 2k+1$

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Theorem:

For every real number x and for every integer m

$\lfloor x+m \rfloor = \lfloor x \rfloor + m$

Proof:

Let $\lfloor x \rfloor = n$

then by definition of floor

$n \le x < n+1$ —— (i)

adding m to (i)

$n+m \le x+m < (n+m)+1$ —— (ii)

$\Rightarrow \lfloor x+m \rfloor = n+m$

$= \lfloor x \rfloor + m$

Theorem:

For any integer n

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof:

Let n is even

By definition of integer

$n = 2k$ for some integer k

$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \lfloor k \rfloor$

$= k$

$= \frac{n}{2}$

Rough:

$x = 0.5$

$y = 0.5$

$\frac{2k+1-1}{2}$

$\frac{2(k+\frac{1}{2})k}{2}$

$\lceil 0.5+0.5 \rceil = \lceil 0.5 \rceil + \lceil 0.5 \rceil$

$\lceil 1 \rceil = 0+0$

$1 \neq 0$

Q23

Proof: True

let $\lfloor x \rfloor = n_1$ —— (A)

By definition

$n \le x < n+1$ —— (i)

Multiplying (i) by -1

$-n \ge -x > -(n+1)$

$-(n+1) \le -x < -n$

$\lfloor -x \rfloor = -(n+1)$ —— (B)

(A) + (B)

$\lfloor x \rfloor + \lfloor -x \rfloor = n-(n+1)$

$= -1$

Q27

For every real number x if

$2 - \lfloor x \rfloor \ge \frac{1}{2}$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$

Proof:

$x - \lfloor x \rfloor \ge \frac{1}{2}$ (Given) —— (i)

(i) can be written as

$2x - 2\lfloor x \rfloor \ge 1$

$\boxed{1+2\lfloor x \rfloor \le 2x}$ —— (ii)

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we know from definition

$$x < \lfloor x \rfloor + 1 \quad \text{—— (iii)}$$

$$2x < 2\lfloor x \rfloor + 2 \quad \text{—— (iv)}$$

from (ii) and (iv)

$$\boxed{1 + 2\lfloor x \rfloor \leq 2x < 2 + 2\lfloor x \rfloor}$$

$$\boxed{\lfloor 2x \rfloor = 1 + 2\lfloor x \rfloor}$$

Theorem: There is no greatest integer

Proof:

There is greatest integer N (say)

Then by definition $N \geq n$

for integer n

$$M = N+1$$

is integer being sum of two integers

then

$$M > N$$

$$M+1 > N$$

$\Rightarrow$ N is greatest integer also less than M

$\therefore$ Therefore there is no greatest integer

4.7 Indirect arguments:

Contradiction and Contrapositive:

$\sim p \rightarrow c$

$\therefore p$

Method of proof by contradiction:

1. Suppose statement is false, that is the negation of statement is true
2. Show that this supposition leads logically to contradiction, impossibility, absurdity
3. Conclude that statement is true

Method of proof by contraposition:

1. Express the statement to be proved in the form
   $\forall x \in D$, if $P(x)$ then $Q(x)$
2. Rewrite the statement into contrapositive form
   $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$
3. Prove the contrapositive by direct proof
4. Suppose x is particular but ordinary chosen element of P

i) Show that $Q(x)$ is false

ii) Show that $P(x)$ is false

$P \to Q$

$\sim Q \to \sim P$

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Proposition:

For every integer n if n^2 is even then n is even

Contrapositive:

For every integer n, if n is not even then n^2 is not even

Let n is odd

By definition

$n = 2k+1$, for some integer k

$n^2 = (2k+1)^2 = 4k^2+4k+1$

$= 2[2k^2+2k]+1$

$= 2q+1$

$\therefore$ n^2 is not even

Q3

show that P: for every integer n, $3n+2$ is not divisible by 3

Proof:

$\sim P$: There is some integer n, such that such that $3n+2$ is divisible by 3

Let n is any integer

$3 \mid 3n+2$

By definition of divisibility:

$3n+2 = 3q$

$3(n-q) = -2$

$n-q = -2/3$ contradiction because diff of two integers is integer

Therefore P is true.

R: The square root of irrational number is irrational

$\sim$R: There is some irrational number with rational square root

Proof:

Let x is irrational

$\sqrt{x} = \frac{p}{q}$, $q \neq 0$, $p, q \in \mathbb{Z}$

squaring

$x = \frac{p^2}{q^2}$, $q^2 \neq 0$ contradiction

$\therefore$ R is true

Q14

T: For every integer a if $a \bmod 6 = 3$ then $a \bmod 3 \neq 2$

$\sim$T: There exist some integer a such that $a \bmod 6 = 3$ and $a \bmod 3 = 2$.

Proof:

$a \bmod 6 = 3 \Rightarrow a = 6q_1 + 3$ —(i)

$a \bmod 3 = 2 \Rightarrow a = 3q_2 + 2$ —(ii)

(i) and (ii)

$\Rightarrow 6q_1 + 3 = 3q_2 + 2$

$2q_1 - q_2 = -1/3$ contradiction because it should be an integer

$\therefore$ T is true

Infimum
(greatest lower bound)

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Q 28

For all integers a, b and c if $a|b$ and $a \nmid c$ then $a \nmid b+c$

Proof:

$a|b$ and $a \nmid c$

i) $b = aq_1$ for some integer q_1

ii) $c = aq_2 + r, \quad 0<r<a$

adding (i) and (ii)

$b+c = aq_1 + aq_2 + r, \quad 0<r<a$

$b+c = aq + r, \quad 0<r<a$

Chapter # 5

Sequences, Mathematical inductions and recursions:

Sequences:

A sequence is function with domain set of positive integer.

$f : N \to R$

natural sequence:

$f(n) = n$

$a_1 = 1, \; a_2 = 2 \; \ldots$

$f(x) = x^2$

n^{th} term/ general term $\to$ $a_n = \frac{1}{n+1}, \; n \geq 0$

$1, \frac{1}{2}, \frac{1}{3}, \ldots$

$-2<-1<0<\frac{1}{n+1} \leq 1 \to$ supremum (least upper bound)

Principle of mathematical induction:

Consider the statement of the form "for every $n \geq a$, the property $P(n)$ is true" To prove the statement perform two steps

I Base step: Show that $P(n)$ is true for $n=0$

II Inductive step:

i) Suppose that $P(n)$ is true for $n=k \geq a$

$\to$ Inductive hypothesis

ii) Show/Prove that $P(n)$ is true for $n=a$

$\therefore P(n)$ is true for all $n \geq a$

$P(n) = 1+2+3+\ldots+n = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$P(n) = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} \Rightarrow \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$P(n) = 1^3+2^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Base step:

Take $n=1$

$1^3 = \left[\frac{1(2)}{2}\right]$

$1 = 1$ True

Inductive step:

Suppose $P(n)$ is true for $n=k$

i) $P(k) = 1^3 + 2^3 + \ldots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$ —(A)

(I.H)
inductive hypothesis

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Q12 DVS

ii) Show that $P(n)$ is true for $n=k+1$

$$1^3+2^3+\dots+k^3+(k+1)^3=\left[\frac{(k+1)(k+2)}{2}\right]^2 \text{ --- (B)}$$

L.H.S of (B)

$$= 1^3+2^3+\dots+k^3+(k+1)^3$$

$$= \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3 \quad \text{by (I.H)}$$

$$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$= \frac{(k+1)^2[k^2+4k+4]}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{2^2} = \left[\frac{(k+1)(k+2)}{2}\right]^2 = R.H.S$$

$\because$ $P(n)$ is true for $n=k+1$

$\therefore$ $P(n)$ is true for all $n\geq 1$

Q16

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \quad \forall n\geq 2$$

$$P(n)=\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{(n+1)}{2n} \quad \forall n\geq 2$$

dummy $\rightarrow$ i

Base step:

$n=2$

$$\prod_{i=2}^{2}\left(1-\frac{1}{i^2}\right)=\frac{3}{2.2}$$

$$\left(1-\frac{1}{2^2}\right)=\frac{3}{4}$$

$$\frac{3}{4}=\frac{3}{4}$$

Inductive step:

1) $P(n)$ is true for $n=k$

$$\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)=\frac{(k+1)}{2k} \text{ --- (A)} \quad (I.H)$$

ii) Then show that $P(n)$ is true for $n=k+1$

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)=\frac{k+2}{2(k+1)} \text{ --- (B)}$$

L.H.S of (B):

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{2}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(\frac{k^2+2k+\cancel{1}-\cancel{1}}{(k+1)^2}\right)$$

$$=\frac{k(k+1)(k+2)}{2k(k+1)^2}$$

$$=\frac{(k+2)}{2(k+1)} = R.H.S \text{ of (B)}$$

Q

$P(n)$ = For every integer $n\geq 0$, $2^{2n}-1$ is divisible by 3

Base step: $n=0$

$$2^0-1=0$$

$$0=0 \quad \text{True}$$

Inductive step:

$n=3$

2^6-1

Inductive step:

i) Suppose $P(n)$ is true for $n=k$

i.e.,

$$3 \mid 2^{2k}-1 \Rightarrow 2^{2k}-1 = 3q_1 \quad \text{—(A)}$$

ii) Show that $P(n)$ is true for $n=k+1$

$$3 \mid 2^{2(k+1)}-1 \Rightarrow 2^{2(k+1)}-1 = 3q_2 \quad \text{—(B)}$$

consider:

$$2^{2(k+1)}-1 = 2^{2k+2}-1 = 3[k] = 3q_2$$
$$= 2^2 . 2^k - 1$$
$$= 4 . 2^{2k} - 1$$
$$= 2^{2k}[1+3]-1$$
$$= 2^{2k} + 3.2^{2k} - 1$$
$$= (2^{2k}-1) + 3.2^{2k}$$
$$= 3q_1 + 3.2^{2k}$$

$$3 \mid 2^{2(k+1)}-1$$

$$\therefore 3 \mid 2^{2n}-1 \quad \forall\ n \geq 0$$

EX 5.1

Q21

$P(n)$ $2^n < (n+1)!$ $\forall\, n \geq 0$

$P(0)$ $2^0 < 2!$

$1 < 2$ True

Inductive step: inductive hypothesis

$\forall$ integers if $P(n)$ is true for $n=k$ then $P(n)$ is true for $n=k+1$

i) Suppose $P(n)$ is true for $n=k$ I.H

ii) Prove $P(n)$ is true for $n=k+1$

$$n = k \geq 0$$
$$2^k < (k+2)! \quad (I.H)$$

To prove:

$$2^{k+1} < (k+3)! \quad k \geq 0$$

From (I.H)

$$2^k < (k+2)!$$

multiplying above inequality by 2

$$2^{k+1} < 2(k+2)! \quad \text{—(i)}$$

as

$$2 < 3$$
$$2 < (k+3) \quad \forall\ k \geq 0$$
$$2(k+2)! < (k+3)(k+2)!$$
$$2(k+2)! < (k+3)! \quad \text{—(ii)}$$

from (i) and (ii)

$$2^{k+1} < 2(k+2)! < (k+3)!$$
$$2^{k+1} < (k+3)!$$

hence proved

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Q24

Base step:

$n=1$

$a_1 = 3.7^0 = 3$ True

Inductive step:

$a_k = 3.7^{k-1}$ $\forall$ $k \geq$ (I.H)

Show that

$a_{k+1} = 3.7^k$

Given that

$a_{k+1} = 7.a_k$

$= 7[3.7^{k-1}]$ I.H

$= 3.7.7^{k-1}$

$a_{k+1} = 3.7^k$

i.e., $a_n = 3.7^{n-1}$ $\forall$ $n \geq 1$

Principle of strong mathematical induction:

Let P(n) be property defined for integers n, and let a and b are two fixed integers with $a<b$

Suppose the following two statements are true

1.) P(a), P(a+1), ... P(b) are all true → basis step

2.) For every integer $k \geq b$, if P(n) is true for every i from a through k

then P(n) is true for n = k+1

Then

$\forall$ $n \geq a$ P(n) is true

Q

$S_0, S_1, S_2, \ldots$

$S_0 = 0$, $S_1 = 4$

$S_k = 6S_{k-1} - 5S_{k-2}$ $k \geq 2$

show that

$P(n)$: $S_n = 5^n - 1$ $n \geq 0$

Base step:

P(0), P(1)

$P(0): S_0 = 5^0 - 1 = 1 - 1 = 0$ True

$P(1): S_1 = 5 - 1 = 4$ True

Inductive hypothesis:

$P(i): S_i = 5^i - 1$, $0 \leq i \leq k$ (I.H)

To prove

$P(k+1): S_{k+1} = 5^{k+1} - 1$ $k+1 \geq 2$, $k \geq 1$

From definition

$S_{k+1} = 6S_k - 5S_{k-1}$

$= 6[5^k - 1] - 5[5^{k-1} - 1]$

$= 6.5^k - 6 - 5^k + 5$

$= 5^k(6-1) - 1$

$= 5^k.5 - 1$

$= 5^{k+1} - 1$

Mid II's syllabus:

Ch 3 3.3–3.4

Ch 4 4.1–4.7

Ch 5 5.1–5.4

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Ex 5.4

Q2

Suppose $b_1, b_2 ..$ is sequence defined as follows

$b_1 = 4$, $b_2 = 12$

$b_k = b_{k-2} + b_{k+1}$, $\forall\ k \geq 3$

Prove that b_n is divisible by 4 for every integer $n \geq 1$

Base step:

$b_1 = 4$

$b_2 = 12$

$P(1)$: b_1 is divisible by 4 True $\rightarrow 4 = 4.1$

$P(2)$: b_2 is divisible by 4 True $\rightarrow 12 = 4.3$

$a = 1$

$b = 2$

I.H

For every integer $k \geq 2$, if $P(i)$ is true for each integer i through 1 through 2 then $P(k+1)$ is true

$P(i)$: b_i is divisible by 4, $1 \leq i \leq k$, $k \geq 2$

$P(k)$: b_k is divisible by 4 $i = k \geq 2 \rightarrow b_k = 4q_1$

$P(k-1)$: b_{k-1} is divisible by 4 $i = k-1 \geq 1 \rightarrow b_{k-1} = 4q_2$

To prove:

$P(k+1)$: b_{k+1} is divisible by 4 $\rightarrow b_{k+1} = 4k$

Base step:

$b_{k+1} = b_{k-1} + b_k$

$= 4q_2 + 4q_1 \longrightarrow$ by (I.H)

$b_{k+1} = 4(q_2 + q_1)$

$b_{k+1} = 4q$

$P(n)\ \forall\ n \geq 1$

Q13

Prove that every integer $n > 1$ is either prime or product of prime.

$P(n)$: n is either prime or product of prime

Base step:

Put $n = 2 > 1$

$P(2)$: 2 is either prime or product of prime

Inductive step:

For every integer $k \geq 2$, if $P(i)$ is true for each integer i from 2 to k then $P(k+1)$.

Inductive hypothesis:

$P(i)$: i is either prime or product of prime

$2 \leq i \leq k$, $k \geq 2$ — (a)

$P(k)$: k is either prime or product of prime

we must show that

$P(k+1)$ is true

$k+1$ is either prime or product of prime $\rightarrow$ to prove

i) $k+1$ is prime $\rightarrow P(k+1)$ is proved

ii) $k+1$ is composite

$k+1 = rs$ $1 < r < k+1$, $1 < s < k+1$

$\rightarrow r$ is either prime or product of prime (by a)

$\rightarrow s$ is either prime or product of prime (by a)

$\therefore (k+1)$ is either prime or product of prime

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5.6: Defining sequences recursively

Definition: A recurrance relation for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessor $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$ where i is an integer such that $k-i \geq 0$ & i is fixed integer then initial conditions for this recurrance relation specify the values $a_0, a_1, a_2, \ldots, a_{i-1}$

Recursion: A recursion is one of the central ideas of computer sciences. To solve problem recursively means to find a way to break it down into smaller sub problems each having the same form as the original problem and to do this in such a way that the last of sub-problems are small and easy to solve and solutions of these sub-problems can be woven together to form solutions to the original problem.

Tower of Hanoi: In 1883, the french Mathematician invented this puzzle that he called tower of Hanoi.

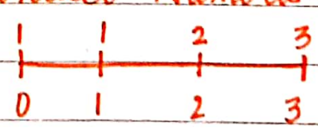
m_n minimum number of moves needed to transfer tower of hanoi of n disks from one pole to another.

1. transfer top $(k-1)$ disks from pole A to pole B. If $k>2$, the execution of this step will require a number of moves for individual disks among the three towers.
2. Move bottom disk from A to C in 1 move.
3. Transfer top $(k-1)$ disks from B to C.

$$m_k = m_{k-1} + 1 + m_{k-1}$$
$$m_k = 2m_{k-1} + 1 \quad \text{—(i)}$$
$$m_1 = 1 \qquad = 2^1 - 1$$
$$m_2 = 2m_1 + 1 = 3 \qquad = 2^2 - 1$$
$$m_3 = 2m_2 + 1 = 7 \qquad = 2^3 - 1$$
$$m_4 = 2m_3 + 1 = 15 \qquad = 2^4 - 1$$
$$m_5 = 2m_4 + 1 = 31 \qquad = 2^5 - 1$$
$$m_n = 2^n - 1, \quad n \geq 1 \quad \text{—(ii)}$$

(ii) is called explicit formula or solution to (i)

Fibonacci Numbers:



F_k = Number of Rabbit pairs alive at the end of month k

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$F_0 = 1$

$F_1 = 2$

$F_2 = F_0 + F_1 = 3$

$F_3 = F_1 + F_1 = 5$

$F_k = \begin{bmatrix} \text{Number of} \\ \text{rabits alive at} \\ \text{the end of } k-2 \end{bmatrix} + \begin{bmatrix} \text{Number of rabits} \\ \text{alive at the end} \\ \text{of month } k-1 \end{bmatrix}$

month

$F_k = F_{k-2} + F_{k-1}$

Methods of defining a sequence

i) Recursive method

ii) General method (iterative method)

Ex 5.7

Q5

$C_k = 3C_{k-1} + 1$, for integer $k \geq 2$, $C_1 = 1$

khud kro

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Exercise 5.7

Q6

$d_k = 2d_{k-1} + 3 \quad \forall\ k \geq 2$

$d_1 = 2$

$k=2, \quad d_2 = 2d_1 + 3$
$= 2.2 + 3$
$= 2^2 + 3$
$= 7$

$k=3, \quad d_3 = 2d_2 + 3$
$= 2(7) + 3$
$= 2(2^2+3)+3$
$= 2^3 + 2.3 + 3$
$= 17$

$k=4, \quad d_4 = 2d_3 + 3$
$= 2[2^3+2.3+3]+3$
$= 2^4 + 2^2.3 + 2.3 + 3$

$k=5, \quad d_5 = 2d_4 + 3$
$= 2[2^4+2^2.3+2.3+3]+3$
$= 2^5 + 2^3+3+2^2.3+2.3+3$

$d_n = 2^n + 2^{n-2}.3 + \ldots + 2^2 3 + 2.3 + 3$
$= 2^n + 3[2^{n-2} + \ldots + 2^2 + 2 + 1]$
$= 2^n + 3\left[\frac{2^{n-1}-1}{2-1}\right]$

$\because 1 + r + \ldots r^{n-1} = \frac{r^n - 1}{r-1}$

$= 2^n + 3.2^{n-1} - 3$

$\boxed{d_n = 2^n + 3.2^{n-1} - 3}$

$n \geq 1$

This is solution of recurrance relation / explicit formula

$y'' + 5y + 6y = 0$

$y = e^{mx}$

$m^2 + 5m + 6 = 0$

$m = -2, \quad m = -3$

$y_1 = e^{-2x}, \quad y_2 = e^{-3x}$

$W(y_1, y_2) \neq 0$

$y = y_c = c_1 e^{-2x} + c_2 e^{-3x}$

Exercise 5.8

Second order linear homogeneous Recurrance solution with constant coefficients

Definition:

A second order linear recurrance relation with constant coefficients is recurrance relation of the form

$$a_k = Aa_{k-1} + Ba_{k-2}$$

for every $k \geq$ some fixed integer where A and B are fixed real numbers with $B \neq 0$.

Lemma 5.8.1

Let A and B are real numbers. A recurrance relation of the form

$a_k = Aa_{k-1} + Ba_{k-2} \qquad k \geq 2$

is satisfied by the sequence $1, t, t^2, \ldots$

where t is non-zero number if and only if t satisfies the eq

$$t - At + B = 0$$

Definition:

Given a second order homogeneous recurrance relation with constant coefficients

$$a_k = Aa_{k-1} + Ba_{k-2} \qquad k \geq 2$$

the characteristic equation of the relation is

$$t^2 - At + B = 0$$

$$\left.\begin{array}{l} F_k = F_{k-1} + F_{k-2} \\ F_0 = 1 \\ F_1 = 1 \end{array}\right\} \qquad t^2 - t\ 1 = 0$$

Lemma 5.8.2

let $r_0, r_1, \ldots$ and $s_0, s_1, \ldots$ satisfy the same second order linear recurrance relation with constant coefficients, and if C and D are numbers then sequence $a_0, a_1, \ldots$ defined by the formula

$$a_n = Cr_n + Ds_n \qquad n \geq 0$$

also satisfies the same recurrance relation

$r_0, r_1, \ldots$

$1, r, r^2, \ldots$

Theorem: Distinct root theorem

Let the sequence $a_0, a_1, \ldots$ satisfies the recurrance relation

$$a_k = Aa_{k-1} + Ba_{k-2} \qquad k \geq 2$$

for some real numbers A and B and every integer $k \geq 2$. If the characteristic equation

$$t^2 - At - B = 0$$

has two distinct real roots r and s then the sequence $a_0, a_1, \ldots$ is defined by explicit formula

$$a_n = Cr^n + Ds^n$$

where C and D are real numbers whose values are determined by the values of a_0 and the values of a_1

Exercise 5.8

Q12

$e_k \to t^2$
$e_k \to t$
$e_{k-1} \to t^0$

$$e_k = 9e_{k-2} \quad , k \geq 2$$

$$e_0 = 0$$

$$e_1 = 2$$

the corresponding characteristic eq

$$t^2 - 9 = 0$$

$$(t-3)(t+3) = 0 \Rightarrow t = 3, -3$$

$$r_n = (3)^n$$

$$s_n = (-3)^n$$

$$e_n = C(3)^n + D(-3)^n \quad , n \geq 0$$

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n_0 , $e_0 = C+D=0$ $C = 1/3$

n_1 , $e_1 = 3C-3D=2$ $D = -1/3$

$$e_n = \frac{1}{3}(3)^n - \frac{1}{3}(-3)^n$$

$$e_n = 3^{n-1} + (-3)^{n-1} \quad n \geq 0$$

Theorem: single root theorem

let the sequence $a_0, a_1, \dots$ satisfies the recurrance relation

$$a_k = Aa_{k-1} + Ba_{k-2} \quad , k \geq 2$$

for some real numbers A and B and every integer $k \geq 2$, if the characteristic eq

$$t^2 - At - B = 0$$

has single real root r then the sequence $a_0, a_1, \dots$ is defined by

$$a_n = Cr^n + Dnr^n$$

where C and D are real numbers whose values are determined by the value of a_0 and any known value of the sequence

Q15

$t_k = 6t_{k-1} - 9t_{k-2}$ $k \geq 2$

$t_0 = 1$

$t_1 = 3$

$t_n = 3^n$, $n \geq 0$

$t^2 - 6t + 9 = 0$

$(t-3)^2 = 0$, $t = 3$

$r_n = (3)^n$

$s_n = nr^n$

$t_n = C3^n + Dn3^n$ $n \geq 0$

$n = 0$,

$t_0 = C \Rightarrow C = 1$

$n = 1$,

$t_1 = 3C + 3D$

$3 = 3C + 3D$

$C + D = 1$

$D = 0$

Chapter #8

Properties of relations

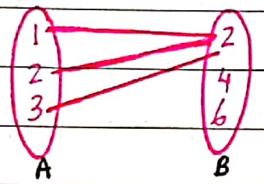
* Every function is a relation but not vice versa.

$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$

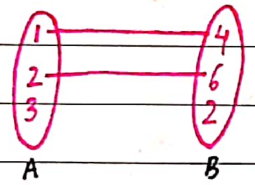
Definition:

Any subset of $A \times B$ is a relation from A to B

$|A \times B| = |A| \times |B|$



function



not a function

- Every element in A has an image.
- The image should be unique

Definition:

Let R is relation from A to B

The inverse relation R^{-1} is defined as

$R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in A \times B\}$

Definition:

A relation on set 'A' is subset of $A \times A$.

For all points x and y in A arrow between x and y is drawn $\Rightarrow xRy$

$\Leftrightarrow (x,y) \in R$

if element is associated to itself a loop is drawn that extends out from the point and goes back to this point.

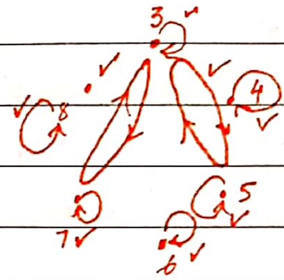
$A = \{3,4,5,6,7,8\}$

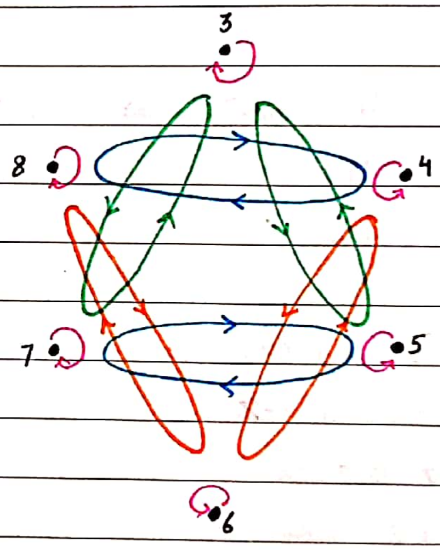
$x, y \in A$

$xRy \Leftrightarrow 2 \mid x-y$

3 R 4 False

3 R 3 True





n-ary relation:

Given the sets $A_1, A_2, \ldots, A_n$ the n-ary relation on $A \times A_2 \times \ldots \times A_n$ is subset of $A_1 \times A_2 \times \ldots \times A_n$. The special cases 2-ary, 3-ary and 4-ary relations are called binary, ternary and quaternary relations respectively.

Ex 8.2

Reflexive, Symmetric and Transitive

Definition:

Let R is relation on A.

1. R is reflexive if and only if, for every $x \in A$, if xRx.
2. R is symmetric if and only if for every $x, y \in A$ if xRy then yRx.
3. R is transitive iff for every x, y and $z \in A$, if xRy and yRz then xRz

$A = \mathbb{Z}!$

$m, n \in \mathbb{Z}! \ \ni \ m E n \Leftrightarrow m-n$ is even.

$(1,2) \in E$, NO

$1 \notin 2 \qquad 1-2$

if $xRy \Rightarrow 2 \mid x-y$ and

$yRz \Rightarrow 2 \mid y-z$

xRz | $x-y = 2k_1$

$y-z = 2k_2$

$x-z = 2q$

$2 \mid x-z$

Transitive closure of Relation:

Definition:

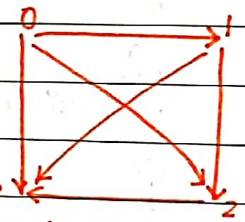
Let A be a set and R be a relation. The transitive closure of R is relation to R^t defined on A with following properties:

i) R^t is transitive

ii) $R \subseteq R^t$

iii) if S is any other transitive relation that contains R then $R^t \subseteq S$

$A = \{0,1,2,3\}$, $R = \{(0,1), (1,2), (2,3)\}$



$R^t = \{(0,1), (1,2), (2,3), (0,2), (1,3), (0,3)\}$

Equivalence Relation:

Definition:

Let A be a set and R be a relation on A. R is equivalence relation if and only if R is reflexive, symmetric and transitive.

Partition of set:

$A \neq 0$

$P = \{A_1, A_2, \ldots, A_n\}$

$A_i \subset A \ \forall i = 1 \ldots n$

i) $A_i \cap A_j = \phi \ (i \neq j)$

ii) $\bigcup_{i=1}^{n} A_i = A$

$A_1 \cup A_2 \cup A_3 \ldots \cup A_n = A$

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Relation induced from the partition of set:

Given a partition of a set A, the relation R induced by the partition of set A defined as follows.

For all $x, y \in A$, xRy if there is any subset A_i of partition such that both x and y are in A_i

$A_0 = \{0, 1, 2, 3, 4\}$

$A_1 = \{0, 3, 4\}$

$A_2 = \{1\}$

$A = \{2\}$

i) $A_1 \cap A_2 \cap A_3 = \emptyset$

ii) $A_1 \cup A_2 \cup A_3 = A$

$R = \{(0,3), (0,4), (3,0), (4,0), (0,0), (1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$

Theorem 8.3.1

let A be a set with partition and R is relation induced by partition of A, then R is reflexive, symmetric and transitive. Therefore R is an equivalence relation.

Equivalence class of equivalence relation:

let A be a set and R an equivalence relation. For every element 'a' in A, the equivalence of a, denoted by $[a]$ is a set of all elements x in A such that x is related to 'a' by R

symbolically

$[a] = \{x \in A \mid xRa\}$

$A = \{0, 1, 2, 3, 4\}$

$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$

$[0] = \{0, 4\}$

$[4] = \{0, 4\}$

$[1] = \{1, 3\}$

$[3] = \{1, 3\}$

$[2] = \{2\}$

distinct classes

$[0] = [4] = \{0, 4\}$

$[1] = [3] = \{1, 3\}$

$[2] = \{2\}$

i) aRb

$[a] = [b]$

ii) The distinct classes of equivalence relation from the partition of A

iii) $[a] = [b]$

or $[a] = [b] = \emptyset$

Theorem 8.3.2
let A be a set, R an equivalence relation on A, and a,b are in A.
If aRb then
$[a] = [b]$
or $[a] = [b] = \emptyset$

Theorem 8.3.4
let A be a set and R an equivalence relation. The distinct equivalence classes of R form a partition of A (union set A, interaction null)

Congruent Modula
let m and n be integers and d is a positive integer we say that m is congruent modulo d and write
$(m \equiv n)$ mod d
symbolically
$m \equiv n (\text{mod}) d \Leftrightarrow d \mid m-n$

congruent modulo 3
$A = Z$
For all $m, n \in Z$
$mRn \Leftrightarrow 3 \mid m-n$
$a \in Z$
$[a] = \{ m \in Z \mid mRa \}$
$= \{ m \in Z \mid 3 \mid m-a \}$
$= \{ m \in Z \mid m-a = 3k \}$ for some integer k

$= \{ m \in Z \mid m = 3k+a \}$ for some integer k

$a = 0,1,2$ (a is remainder)
$[0] = \{ m \in Z \mid m = 3k \}$
$= \{ \ldots, -6, -3, 0, 3, 6, \ldots \}$

Ex 8.3

Q10

$A = \{ 0, \pm 1, \pm 2, \pm 3, \pm 5 \}$
$\forall m, n \in A \quad mRn \Leftrightarrow 3 \mid m^2 - n^2$
1) Reflexive
$\forall m \in A \quad mRm$
$3 \mid m^2 - m^2$
$3 \mid 0 \Rightarrow 0.3 0$
$a \in A$
$[a] = \{ x \in A \mid xRa \}$
$= \{ x \in A \mid 3 \mid x^2 - a^2 \}$
$= \{ x \in A \mid x^2 = 3k + a^2 \}$

$0 \leq a^2 < 3 \qquad r = 0,1,2 \quad d = 1$
$a^2 = 0 \Rightarrow a = 0$
$a^2 = 1 \Rightarrow a = \pm 1 \qquad 0 \leq r < d$
$a^2 \neq 2$
$[0] = \{ x \in A \mid 3 \mid x^2 \} = \{ 0, \pm 3 \}$
$[1] = \{ \pm 1, \pm 2, \pm 4, \pm 3 \}$
better way to solve?
$mRn \Leftrightarrow 3 \mid m^2 - n^2$
$\Rightarrow 3 \mid (m-n)(m+n)$

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$3|(m-n)$ or $3|(m+n)$

Case 1) $m=3k_1$, $n=3k_2$

Case 2) $m=3k_1+1$, $n=3k_2+1$

Case 3) $m=3k_1+2$, $n=3k_2+2$

Case 1) $m=3k_1+1$, $n=3k_2+2$

Case 2) $m=3k_1+2$, $n=3k_2+1$

Case 3) $m=3k_1$ or $m=3k_1+1$ or $n=3k_1+2$

$A=Z'$

$\left.\begin{array}{l}\{m\in Z' \mid m=3k\} \text{ or} \\ \{m\in Z' \mid m=3k_1+1\} \text{ or} \\ \{m\in Z' \mid m=3k_2+2\}\end{array}\right]$ equivalence classes

Q21

$A=Z'$

$\forall m,n\in A=Z'$

$mRn \Leftrightarrow 7m-5n$ is even

① Reflexive:

$\forall\ m\in Z'$ nRm so $n=m$

$7m-5m$

$=2m \rightarrow$ even hence, reflexive

② Symmetric:

$\forall m,n\in Z'$

if mRn then nRm

if $7m-5n$ is even then show $7n-5m$ is also even

$7m-5n=2k$

$7m-5n-2n=2k-2n$

$7m-7n=2k-2n$

$\cancel{7m-7n=2k-2n}$

$7n=7m+2(n-k)$

$7n-5m=2m+2(n-k)$

$7n-5m=2(m+n-k)$

$7n-5m \Rightarrow$ even hence, symmetric

$[a]=\{m\in Z' \mid mRa\}$

$=\{m\in Z' \mid 7m-5a$ is even$\}$

① $\underbrace{7m}_{\text{even}}-\underbrace{5a}_{\text{even}}=2k$ since 7 and 5 are odd for $7m$ and $5a$ to be even m and a should be even.

so m and a are even

② $\underbrace{7m}_{\text{odd}}-\underbrace{5a}_{\text{odd}}=2k$

so m and n are odd.

for $\{m\in Z' \mid 7m-5a$ even$\}$

equivalence classes:

- $\{m\in Z' \mid m$ and a are even$\}$
- $\{m\in Z' \mid m$ and a are odd$\}$

$[0]=\{0,\pm2,\pm4,\pm6,\ldots\}$

$[1]=\{\pm1,\pm3,\pm5,\ldots\}$

Q41

$\forall\ a,b\in A$ if $\underbrace{a\in[b]}_{aRb}$ then $[a]=[b]$

show
1. $[a]\subset[b]$
2. $[b]\subset[a]$

① let $x\in[a]$

then xRa also aRb

by transitivity

xRb

$x\in[b]$

hence $[a]\subset[b]$

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② let $x \in [b]$

then xRb also aRb

since R is equivalence relation

$\therefore$ bRa

then by transitivity

xRa

$x \in [a]$

hence $[b] \subset [a]$

hence, proved $[a] = [b]$

Chapter 1.4

Language of Graphs

Definition:

A graph 'G' consists of two finite sets: a non-empty $V(G)$ of vertices and set $E(G)$ of edges, where each edge is associated to one or two vertices called endpoints.

Edge-Endpoint function:

A correspondence from edges to endpoints is called edge-endpoint function.

Loop:

An edge with one endpoint or vertex is said to be loop.

Parallel edges:

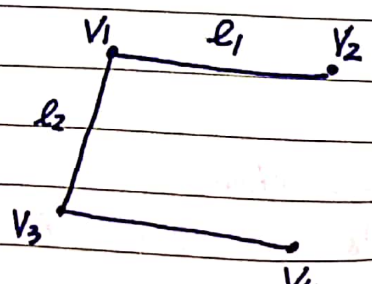
Two or more distinct edges with same set of end points (vertices) are parallel edges.

Adjacent vertices:

An edge is said to connect its end points. Two vertices are adjacent to each other if they are connected by an edge. An edge that is loop has one end point means it is adjacent to itself.

Adjacent edges:

An edge is incident on each of its end points. Two edges are adjacent if they are incident on same endpoint.



V_1 and V_2 are adjacent

V_1 and V_3 are adjacent

V_1 and V_4 are not adjacent

Directed graphs:

A directed graph 'G' consist of two finite sets, a non-empty set V(G) of vertices and set D(G) of directed edges where each edge is associated to ordered pair. If edge e is associated to pair (u,v), it means e is directed edge from u to v.

$u \xrightarrow{e} v$

Degree of vertex:

Let 'G' be a graph and 'V' be a vertex. The degree of V, denoted by deg(v) equals number of edges that are incident on v.

Example

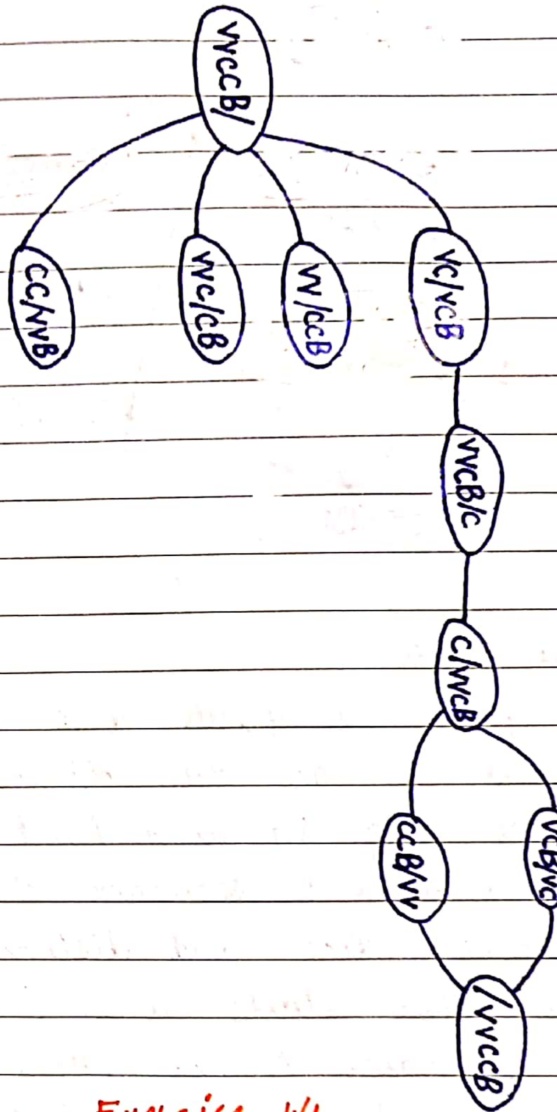
Initially v, v, c, c B

we use notation to indicate all possible arrangements of vegetarian, cannibals, and boat on banks of river.

$\underbrace{\text{vvcc B}}_{\text{land}}$ / $\underbrace{\qquad}_{\text{river}}$

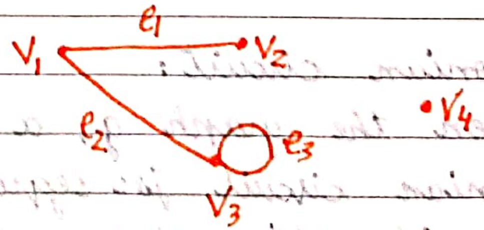
edge ⇒ every legal move

vertex ⇒ possible arrangement



Exercise 1.4

Q1

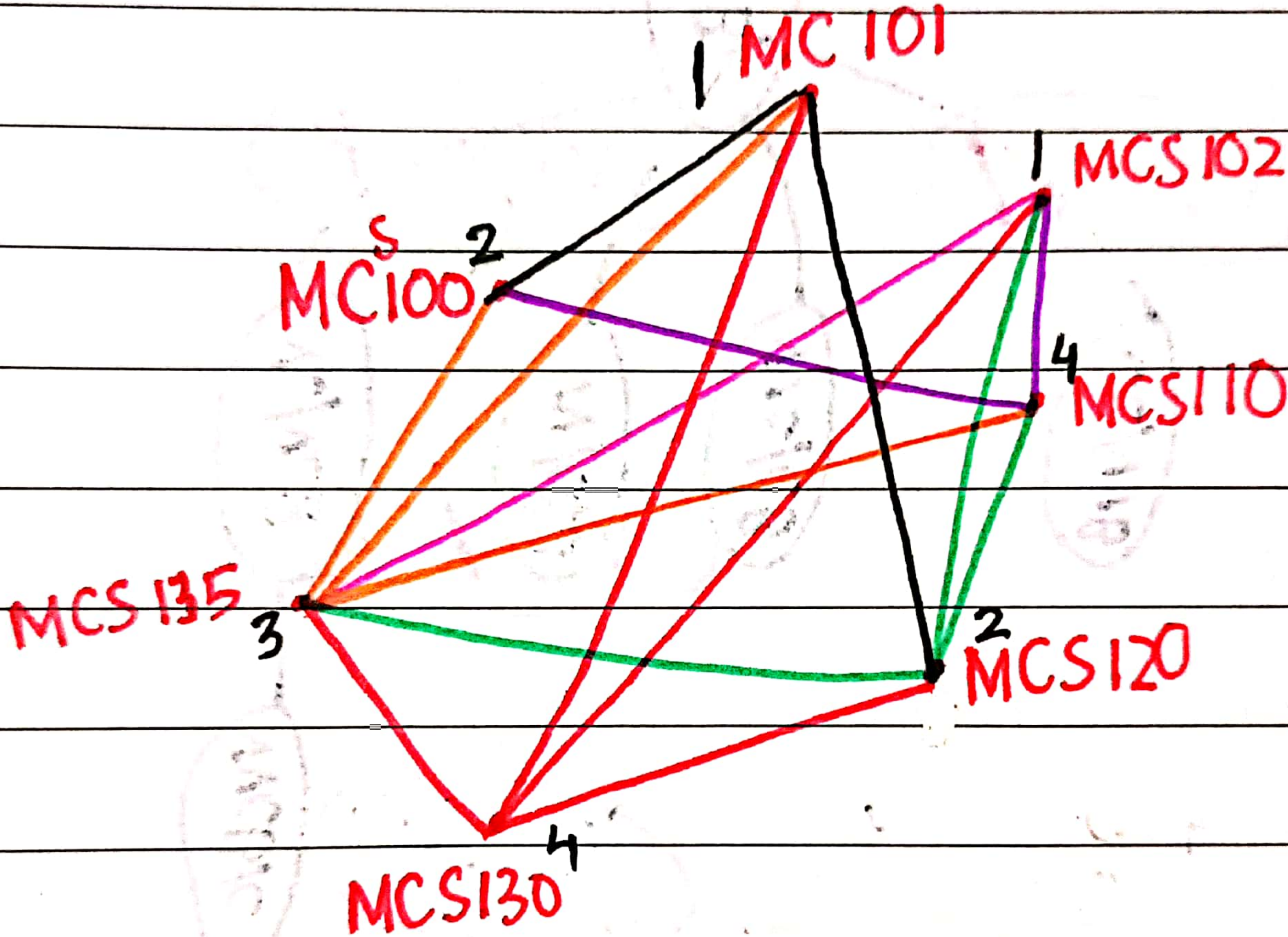


$V(G) = \{v_1, v_2, v_3, v_4\}$

$E(G) = \{e_1, e_2, e\}$

Edge	Endpoint
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

Q 17



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Chapter # 10
Theory of Graphs and Trees

Walk:

Let G be a graph, and v and W are vertices in G.
A walk from vertex v to w is finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form:

$$v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n$$

where v's represent vertices and e's represent the edges, $V_0 = V$, $V_n = W$, for each $i = 1, 2, \dots, n$.
A trivial walk from v to v contains only one vertex v.

Trail:

A trail from v to w is a walk from v to w that does not contain a repeated edge.

Path:

A path from v to w is a trail that does not contain a represented vertex.

Circuit:

A circuit is a closed walk that contains at least one edge and does not contain a repeated edge.

Simple circuit:

A simple circuit is a circuit that does not have any other repeated vertex ex ept first and last.

Sub-graph:

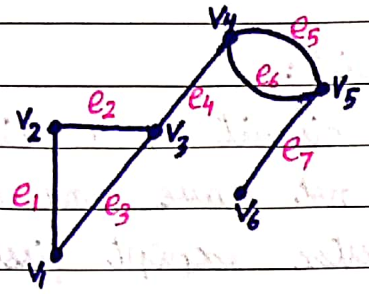
A graph H is subgraph of G if every vertex in H is also vertex in G and every edge in H has same end points as it has in G.

Connectedness:

Let G be a graph. Two distinct vertices v and w are connected if there is a walk from v to w.
The graph G is connected if and only if given any two vertices v and w, there is a walk from v to w.

G is connected $\Leftrightarrow$ [$\forall$ vertices v and w in G, $\exists$ a walk from v to w]

Note: if there are no parallel edges in graph, then walk can be determined by sequence of edges.



$$v_1 \to v_4 \, ; \begin{cases} v_1 e_3 v_3 e_4 v_4 \\ v_1 e_1 v_2 e_2 v_3 e_4 v_4 \end{cases}$$

10.1.1

i) If G is connected, then any two distinct vertices of G can be connected by a walk.

ii) If v and w are part of circuit in G and one edge is removed from circuit then there still exists a trail from v to w.

iii) If G is connected and contains a circuit, then an edge can be removed without disconnecting the graph.

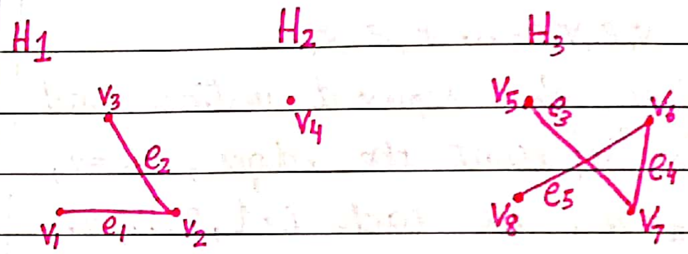
Connected component:

A graph H is connected component of graph G if

i) H is connected

ii) H is subgraph of G

iii) No connected component of G contains H as subgraph and contains vertices of edges that are not in H



$V(H_1) = \{v_1, v_2, v_3\}$, $E(H_1) = \{e_1, e_2\}$

$V(H_2) = \{v_4\}$, $E(H_2) = \{\ \} = \emptyset$

$V(H_3) = \{v_5, v_6, v_7, v_8\}$, $E(H_3) = \{e_3, e_4, e_5\}$

Euler Circuit:

Let G be a graph. An Euler circuit for G is a circuit that contains every vertex and every edge of G. That is, an euler circuit for G is a closed walk that has atleast one edge, starts and ends at same vertex, uses every vertex of G atleast once and uses every edge of G exactly once.

Mon Tue Wed Thu Fri Sat Sun

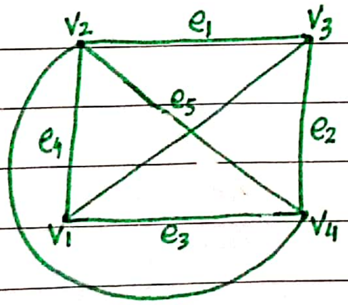
Date:

10.1.2 If a graph has a Euler circuit, then every vertex of the graph has positive even degree.

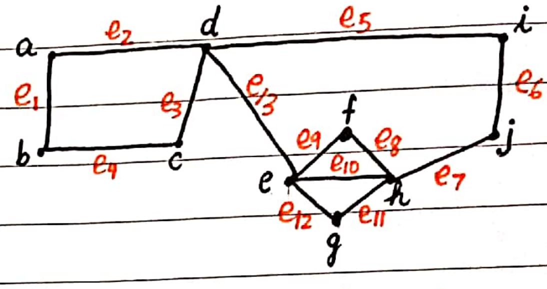
10.1.3 If a graph G is connected and degree of every vertex of G is a positive even integer, then G has an Euler circuit.

10.1.4. The graph G has Euler circuit if and only if G is connected and every vertex of G has a positive even degree.

Example 10.1.6



Example 10.1.7



$a\, e_2\, d\, e_5\, i\, e_6\, j\, e_7\, h\, e_8\, f\, e_9\, e\, e_{10}\, h\, e_{11}\, g$
$e_{12}\, e\, e_{13}\, d\, e_3\, c\, e_4\, b\, e_1\, a$

Chapter 10.

Euler Trail:

Let g be a graph and let v and w are two distinct vertices of g. An euler trail from v to w is sequence of adjacent vertices and distinct edges that starts at v, ends at w, passes through every vertex at least once and traverse every edge of g exactly once.

Hamiltonian circuit:

Given the graph g, a hamiltonian circuit for g is simple circuit that contains every vertex of g. That is, a hamiltonian circuit is sequence adjacent vertices and distinct edges such that vertices are not repeated except the first and last.

Euler Circuit	Hamiltonian circuit
• contains all edges of graph	• does not contain all edges of graph
• vertex rep allowed	• vertex not allowed except for 1st and last one
• contains every vertex	• does not contain every vertex

Lemma 10.1.5:

Let g be a graph and let v and w are two distinct vertices of g. There is an euler trail from v and w if and only if v and w has odd degree and all other vertices of g have positive even degree.

Proposition:

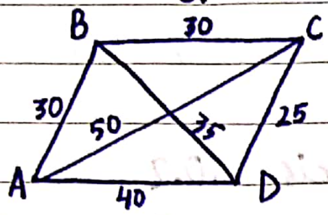
If a graph G has a hamiltonian circuit then G has such graph H with following proposition:

1. H contains all vertices of G
2. H is connected.
3. H has some number of vertices or edges
4. The degree of every vertex of H is 2.

walk → trail → path

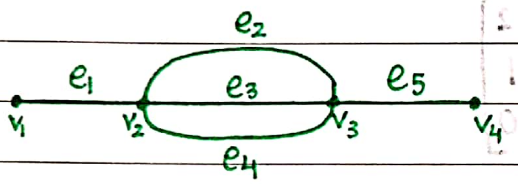
* every path is a trail and a walk.

Q



ABCDA ⇒ 30+30+25+40 = 125 (hamiltonian)

Ex 10.1

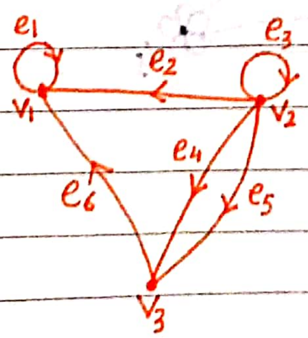


a). $v_1 e_1 v_2 e_2 v_3 e_5 v_4$
$v_1 e_1 v_2 e_3 v_3 e_5 v_4$
$v_1 e_1 v_2 e_4 v_3 e_5 v_4$ } walks

b). $v_1 e_1 v_2 e_3 v_3 e_4 v_2 e_2 v_3 e_5 v_4$ } trail

Q22

AHGBCDGFE
(no doors repeated, only repetition of rooms allowed)



$$
\begin{array}{cccccc}
 & v_1 & v_2 & v_3 & \\
A = & \left[\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}\right. & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \left.\begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix}\right] & \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}
\end{array}
$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

* a_{12} means no. of edges from v_1 to v_2.

* no. of vertices tells order of matrix.

Adjacency Matrix (Directed)

Let G be a directed graph, with ordered vertices $\{v_1, v_2, \dots, v_n\}$, the adjacency matrix of order $n \times n$, $A = (a_{ij})$ such that a_{ij} = no of arrows from v_i to v_j. $\forall i,j = 1-n$

Adjacency Matrix (Undirected)

Let G be an undirected graph, with ordered vertices $\{v_1, v_2, \dots, v_n\}$, the the adjacency matrix of g is a matrix over the set of non-negative integers such that a_{ij} = no of edges from v_i to v_j. $\forall i,j = 1-n$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Theorem 10.2.1

Let G be a graph with connected components $G_1, G_2, \ldots, G_k$. If there are n_i vertices in each connected component G_i and they are numbered consecutively, then the adjacency matrix of G has form.

$$\begin{bmatrix} A_1 & O & O & \cdots & O & O \\ O & A_2 & O & \cdots & O & O \\ O & O & A_3 & & \vdots & O \\ \vdots & \vdots & \vdots & & A_{k-1} & \vdots \\ O & O & O & \cdots & O & A_k \end{bmatrix}$$

where each A_i is $n_i \times n_i$ adjacency matrix of G_i for each $i = 1, 2, \ldots, k$ and the O's represent matrices whose entries are all zeroes.

Walk of length n:

The length of walk is determined by number of edges it contains:

$v_1 e_1 v_2 e_2 v_3$

Theorem 10.2.2

If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and A is adjacency matrix of G, then for each positive integer x and for all integers $i, j = 1, 2, \ldots, n$

The ij^{th} entry of A^n = The number of walks of length n from v_i to v_j

* A graph is complete if there exists an edge between every pair of vertices

Exercise 10.2

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Q10

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 9 & 15 \\ 9 & 5 & 8 \\ 15 & 8 & 8 \end{bmatrix}$$